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Trigonometric Delights

Eli Maor

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In memory of my uncles

Ernst C. Stiefel (1907–1997)

Rudy C. Stiefel (1917–1989)

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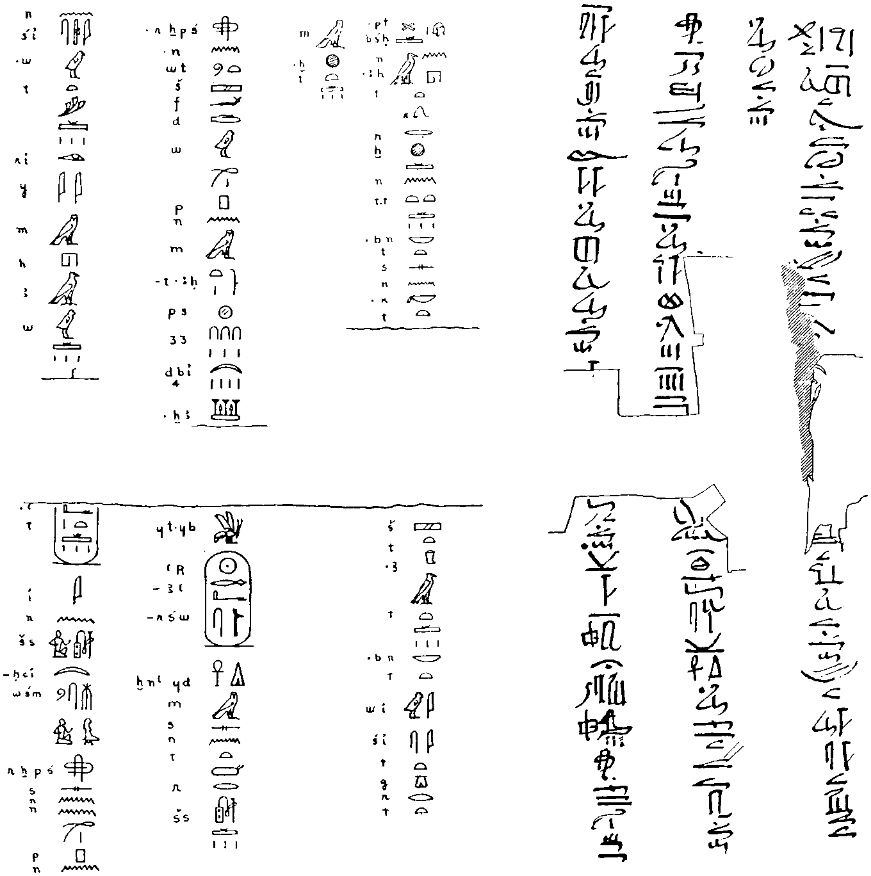

Title page of the Rhind Papyrus.

Preface

There is perhaps nothing which so occupies the middle position of mathematics as trigonometry.
—*J. F. Herbart* (1890)

This book is neither a textbook of trigonometry—of which there are many—nor a comprehensive history of the subject, of which there is almost none. It is an attempt to present selected topics in trigonometry from a historic point of view and to show their relevance to other sciences. It grew out of my love affair with the subject, but also out of my frustration at the way it is being taught in our colleges.

First, the love affair. In the junior year of my high school we were fortunate to have an excellent teacher, a young, vigorous man who taught us both mathematics and physics. He was a no-nonsense teacher, and a very demanding one. He would not tolerate your arriving late to class or missing an exam—and you better made sure you didn't, lest it was reflected on your report card. Worse would come if you failed to do your homework or did poorly on a test. We feared him, trembled when he reprimanded us, and were scared that he would contact our parents. Yet we revered him, and he became a role model to many of us. Above all, he showed us the relevance of mathematics to the real world—especially to physics. And that meant learning a good deal of trigonometry.

He and I have kept a lively correspondence for many years, and we have met several times. He was very opinionated, and whatever you said about any subject–mathematical or otherwise—he would argue with you, and usually prevail. Years after I finished my university studies, he would let me understand that *he* was still my teacher. Born in China to a family that fled Europe before World War II, he emigrated to Israel and began his education at the Hebrew University of Jerusalem, only to be drafted into the army during Israel's war of independence. Later he joined the faculty of Tel Aviv University and was granted tenure despite not having a Ph.D.—one of only two faculty members so honored. In 1989, while giving his weekly lecture on the history of mathematics, he suddenly collapsed and died instantly. His name was Nathan Elioseph. I miss him dearly.

And now the frustration. In the late 1950s, following the early Soviet successes in space (Sputnik I was launched on October 4, 1957; I remember the date—it was my twentieth birthday) there was a call for revamping our entire educational system, especially science education. New ideas and new programs suddenly proliferated, all designed to close the perceived technological gap between us and the Soviets (some dared to question whether the gap really existed, but their voices were swept aside in the general frenzy). These were the golden years of American science education. If you had some novel idea about how to teach a subject—and often you didn't even need that much—you were almost guaranteed a grant to work on it. Thus was born the "New Math"—an attempt to make students *understand* what they were doing, rather than subject them to rote learning and memorization, as had been done for generations. An enormous amount of time and money was spent on developing new ways of teaching math, with emphasis on abstract concepts such as set theory, functions (defined as sets of ordered pairs), and formal logic. Seminars, workshops, new curricula, and new texts were organized in haste, with hundreds of educators disseminating the new ideas to thousands of bewildered teachers and parents. Others traveled abroad to spread the new gospel in developing countries whose populations could barely read and write.

Today, from a distance of four decades, most educators agree that the New Math did more harm than good. Our students may have been taught the language and symbols of set theory, but when it comes to the simplest numerical calculations they stumble—with or without a calculator. Consequently, many high school graduates are lacking basic algebraic skills, and, not surprisingly, some 50 percent of them fail their first college-level calculus course. Colleges and universities are spending vast resources on remedial programs (usually made more palatable by giving them some euphemistic title like "developmental program" or "math lab"), with success rates that are moderate at best.

Two of the casualties of the New Math were geometry and trigonometry. A subject of crucial importance in science and engineering, trigonometry fell victim to the call for change. Formal definitions and legalistic verbosity—all in the name of mathematical rigor—replaced a real understanding of the subject. Instead of an angle, one now talks of the measure of an angle; instead of defining the sine and cosine in a geometric context—

as ratios of sides in a triangle or as projections of the unit circle on the - and -axes—one talks about the wrapping function from the reals to the interval $[-1\ \ 1]$. Set notation and set language have pervaded all discussion, with the result that a relatively simple subject became obscured in meaningless formalism.

Worse, because so many high school graduates are lacking basic algebraic skills, the level and depth of the typical trigonometry textbook have steadily declined. Examples and exercises are often of the simplest and most routine kind, requiring hardly anything more than the memorization of a few basic formulas. Like the notorious "word problems" of algebra, most of these exercises are dull and uninspiring, leaving the student with a feeling of "so what?" Hardly ever are students given a chance to cope with a really challenging identity, one that might leave them with a sense of accomplishment. For example,

1. Prove that for any number ,

$$\frac{\sin}{\quad} = \cos\frac{}{2}\cos\frac{}{4}\cos\frac{}{8}\cdot$$

This formula was discovered by Euler. Substituting $= \pi\ 2$, using the fact that $\cos\pi\ 4 = \sqrt{2}\ 2$ and repeatedly applying the half-angle formula for the cosine, we get the beautiful formula

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2+\sqrt{2}}}{2}\cdot\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdot$$

discovered in 1593 by François Viète in a purely geometric way.

2. Prove that in any triangle,

$$\sin\alpha+\sin\beta+\sin\gamma = 4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$$

$$\sin 2\alpha+\sin 2\beta+\sin 2\gamma = 4\sin\alpha\ \sin\beta\ \sin\gamma$$

$$\sin 3\alpha+\sin 3\beta+\sin 3\gamma = -4\cos\frac{3\alpha}{2}\cos\frac{3\beta}{2}\cos\frac{3\gamma}{2}$$

$$\tan\alpha+\tan\beta+\tan\gamma = \tan\alpha\ \tan\beta\ \tan\gamma$$

(The last formula has some unexpected consequences, which we will discuss in chapter 12.) These formulas are remarkable for their symmetry; one might even call them "beautiful"—a kind word for a subject that has undeservedly gained a reputation of being dry and technical. In Appendix 3, I have collected some additional beautiful formulas, recognizing of course that "beauty" is an entirely subjective trait.

"Some students," said Edna Kramer in *The Nature and Growth of Modern Mathematics*, consider trigonometry "a glorified geometry with superimposed computational torture." The present book is an attempt to dispel this view. I have adopted a historical approach, partly because I believe it can go a long way to endear mathematics–and science in general—to the students. However, I have avoided a strict chronological presentation of topics, selecting them instead for their aesthetic appeal or their relevance to other sciences. Naturally, my choice of subjects reflects my own preferences; numerous other topics could have been selected.

The first nine chapters require only basic algebra and trigonometry; the remaining chapters rely on some knowledge of calculus (no higher than Calculus II). Much of the material should thus be accessible to high school and college students. Having this audience in mind, I limited the discussion to plane trigonometry, avoiding spherical trigonometry altogether (although historically it was the latter that dominated the subject at first). Some additional historical material–often biographical in nature—is included in eight "sidebars" that can be read independently of the main chapters. If even a few readers will be inspired by these chapters, I will consider myself rewarded.

My dearest thanks go to my son Eyal for preparing the illustrations; to William Dunham of Muhlenberg College in Allentown, Pennsylvania, and Paul J. Nahin of the University of New Hampshire for their very thorough reading of the manuscript; to the staff of Princeton University Press for their meticulous care in preparing the work for print; to the Skokie Public Library, whose staff greatly helped me in locating rare and out-of-print sources; and last but not least to my dear wife Dalia for constantly encouraging me to see the work through. Without their help, this book would have never seen the light of day.

Note: frequent reference is made throughout this book to the *Dictionary of Scientific Biography* (16 vols.; Charles Coulston Gillispie, ed.; New York: Charles Scribner's Sons, 1970–1980). To avoid repetition, this work will be referred to as *DSB*.

Skokie, Illinois
February 20, 1997

Trigonometric Delights

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PROLOGUE

Ahmes the Scribe, 1650 B.C.

Soldiers: from the summit of yonder pyramids forty centuries look down upon you.
—*Napoleon Bonaparte* in Egypt, July 21, 1798

In 1858 a Scottish lawyer and antiquarian, A. Henry Rhind (1833–1863), on one of his trips to the Nile valley, purchased a document that had been found a few years earlier in the ruins of a small building in Thebes (near present-day Luxor) in Upper Egypt. The document, known since as the Rhind Papyrus, turned out to be a collection of 84 mathematical problems dealing with arithmetic, primitive algebra, and geometry.[1] After Rhind's untimely death at the age of thirty, it came into the possession of the British Museum, where it is now permanently displayed. The papyrus as originally found was in the form of a scroll 18 feet long and 13 inches wide, but when the British Museum acquired it some fragments were missing. By a stroke of extraordinary luck these were later found in the possession of the New-York Historical Society, so that the complete text is now available again.

Ancient Egypt, with its legendary shrines and treasures, has always captivated the imagination of European travelers. Napoleon's military campaign in Egypt in 1799, despite its ultimate failure, opened the country to an army of scholars, antiquarians, and adventurers. Napoleon had a deep interest in culture and science and included on his staff a number of scholars in various fields, among them the mathematician Joseph Fourier (about whom we will have more to say later). These scholars combed the country for ancient treasures, taking with them back to Europe whatever they could lay their hands on. Their most famous find was a large basalt slab unearthed near the town of Rashid—known to Europeans as Rosetta—at the western extremity of the Nile Delta.

The Rosetta Stone, which like the Rhind Papyrus ended up in the British Museum, carries a decree issued by a council of Egyptian priests during the reign of Ptolemy V (195 *b.c.*) and is recorded in three languages: Greek, demotic, and hieroglyphic (picture script). The English physicist Thomas Young (1773–1829), a man of many interests who is best known for his wave theory of light, was the first to decipher the inscription on the stone. By comparing the recurrence of similar groups of signs in the three scripts, he was able to compile a primitive dictionary of ancient Egyptian words. His work was completed in 1822 by the famous French Egyptologist, Jean François Champollion (1790–1832), who identified the name Cleopatra in the inscription. Champollion's epochal work enabled scholars to decipher numerous Egyptian texts written on papyri, wood, and stone, among them several scrolls dealing with mathematics. The longest and most complete of the mathematical texts is the Rhind Papyrus.

August Eisenlohr, a German scholar, was the first to translate the Rhind Papyrus into a modern language (Leipzig, 1877); an English translation by Thomas Eric Peet appeared in London in 1923.[2] But the most extensive edition of the work was completed in 1929 by Arnold Buffum Chase (1845–1932), an American businessman whose trip to Egypt in 1910 turned him into an Egyptologist. It is through this edition that the Rhind Papyrus became accessible to the general public.[3]

The papyrus is written from right to left in hieratic (cursive) script, as opposed to the earlier hieroglyphic or pictorial script. The text is in two colors—black and red—and is accompanied by drawings of geometric shapes. It is written in the hand of a scribe named A'h-mose, commonly known to modern writers as Ahmes. But it is not his own work; he copied it from an older manuscript, as we know from his own introduction:

> This book was copied in the year 33, in the fourth month of the inundation season, under the majesty of the king of Upper and Lower Egypt, 'A-user-Re', endowed with life, in likeness to writings of old made in the time of the king of Upper and Lower Egypt, Ne-ma'et-Re'. It is the scribe A'h-mose who copies this writing.[4]

The first king mentioned, 'A-user-Re', has been identified as a member of the Hyksos dynasty who lived around 1650 *b.c.*; the second king, Ne-ma'et-Re', was Amenem-het III, who reigned from 1849 to 1801 *b.c.* during what is known as the Middle Kingdom. Thus we can fix the dates of both the original work and its copy with remarkable accuracy: it was written nearly four thousand years ago and is one of the earliest, and by far the most extensive, ancient mathematical document known to us.[5]

The work opens with a grand vision of what the author plans to offer: a "complete and thorough study of all things, insight into all that exists, knowledge of all secrets."[6] Even if these promises are not quite fulfilled, the work gives us an invaluable insight into early Egyptian mathematics. Its 84 problems deal with arithmetic, verbal algebra (finding an unknown quantity), mensuration (area and volume calculations), and even arithmetic and geometric progressions. To anyone accustomed to the formal structure of Greek mathematics—definitions, axioms, theorems, and proofs—the content of the Rhind Papyrus must come as a disappointment: there are no general rules that apply to an entire *class* of problems, nor are the results derived logically from previously established facts. Instead, the problems are in the nature of specific examples using particular numbers. Mostly they are "story problems" dealing with such mundane matters as finding the area of a field or the volume of a granary, or how to divide a number of loaves of bread among so many men. Apparently the work was intended as a collection of exercises for use in a school of scribes, for it was the class of royal scribes to whom all literary tasks were assigned—reading, writing, and arithmetic, our modern "three R's."[7] The papyrus even contains a recreational problem of no apparent practical use, obviously meant to challenge and entertain the reader (see p. 11).

The work begins with two tables: a division table of 2 by all odd integers from 3 to 101, and a division table of the integers 1 through 9 by 10. The answers are given in *unit fractions*—fractions whose numerator is 1. For some reason this was the only way the Egyptians knew of handling fractions; the one exception was 2/3, which was regarded as a fraction in its own right. A great amount of effort and ingenuity was spent in decomposing a fraction into a sum of unit fractions. For example, the result of dividing 6 by 10 is given as 1 2 + 1 10, and that of 7 by 10 as 2 3 + 1 30.[8] The Egyptians, of course, did not use our modern notation for fractions; they indicated the reciprocal of an integer by placing a dot (or an oval in hieroglyphic script) over the symbol for that integer. There was no symbol for addition; the unit fractions were simply written next to each other, their summation being implied.[9]

The work next deals with arithmetic problems involving subtraction (called "completion") and multiplication, and problems where an unknown quantity is sought; these are known as *aha* problems because they often begin with the word "h" (pronounced "aha" or "hau"), which probably means "the quantity" (to be found).[10] For example, Problem 30 asks: "If the scribe says, What is the quantity of which 2 3 + 1 10 will make 10, let him hear." The answer is given as 13 + 1 23, followed by a proof (today we would say a "check") that this is indeed the correct answer.

In modern terms, Problem 30 amounts to solving the equation (2 3 + 1 10) = 10. Linear equations of this kind were solved by the so-called "rule of false position": assume some convenient value for , say 30, and substitute it in the equation; the left side then becomes 23, instead of the required 10. Since 23 must be multiplied by 10/23 to get 10, the correct solution will be 10/23 times the assumed value, that is, = 300 23 = 13 + 1 23. Thus, some 3,500 years before the creation of modern symbolic algebra, the Egyptians were already in possession of a method that allowed them, in effect, to solve linear equations.[11]

Problems 41 through 60 are geometric in nature. Problem 41 simply says: "Find the volume of a cylindrical granary of diameter 9 and height 10." The solution follows: "Take away 1/9 of 9, namely, 1; the remainder is 8. Multiply 8 times 8; it makes 64. Multiply 64 times 10; it makes 640 cubed cubits." (The author then multiplies this result by 15/2 to convert it to *hekat*, the standard unit of volume used for measuring grain; one hekat has been determined to equal 292.24 cubic inches or 4.789 liters.)[12] Thus, to find the area of the circular base, the scribe replaced it by a square of side 8/9 of the diameter. Denoting the diameter by , this amounts to the formula $= [(8\ 9)\ ]^2 = (64\ 81)\ ^2$. If we compare this to the formula $= \pi\ ^2\ 4$, we find that the Egyptians used the value $\pi = 256\ 81 = 3\ 16049$, in error of only 0.6 percent of the true value. A remarkable achievement![13]

✧ ✧ ✧

Of particular interest to us are Problems 56–60. They deal with that most famous of Egyptian monuments, the pyramids, and all use the word *seked* (see fig. 1).[14] What this word means we shall soon find out.

Problem 56 says: "If a pyramid is 250 cubits high and the side of its base 360 cubits long, what is its seked?" Ahmes's solution follows:

> Take 1/2 of 360; it makes 180. Multiply 250 so as to get 180; it makes 1/2 1/5 1/50 of a cubit. A cubit is 7 palms. Multiply 7 by 1/2 1/5 1/50:

1	7		
1/2	3	1/2	
1/5	1	1/3	1/15
1/50		1/10	1/25

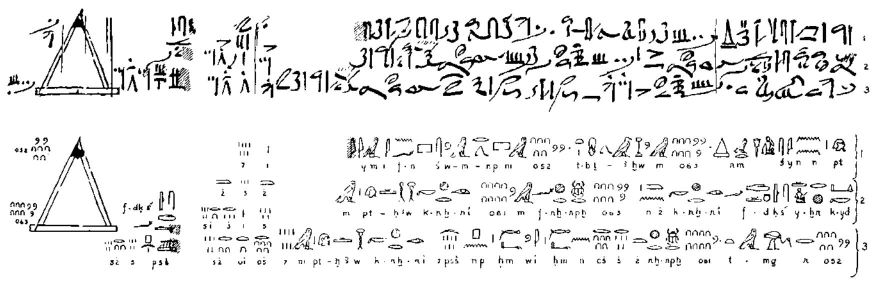

FIG. 1. Problem 56 of the Rhind Papyrus.

The seked is $5\frac{1}{25}$ palms [that is, $(3 + 1\ 2) + (1 + 1\ 3 + 1\ 15) + (1\ 10 + 1\ 25) = 5\frac{1}{25}$].[15]

Let us analyze the solution. Clearly 1/2 of 360, or 180, is half the side of the square base of the pyramid (fig. 2). "Multiply 250 so as to get 180" means to find a number   such that 250 times   equals 180. This gives us   = 180 250 = 18 25. But Egyptian mathematics required that all answers be given in unit fractions; and the sum of the unit fractions 1/2, 1/5, and 1/50 is indeed 18/25. This number, then, is the ratio of half the side of the base of the pyramid to its height, or the run-to-rise ratio of its face. In effect, the quantity that Ahmes found, the seked, is the *cotangent of the angle between the base of the pyramid and its face*.[16]

Two questions immediately arise: First, why didn't he find the *reciprocal* of this ratio, or the rise-to-run ratio, as we would do today? The answer is that when building a vertical structure, it is natural to measure the *horizontal* deviation from the vertical line for each unit increase in height, that is, the run-to-rise ratio. This indeed is the practice in architecture, where one uses the

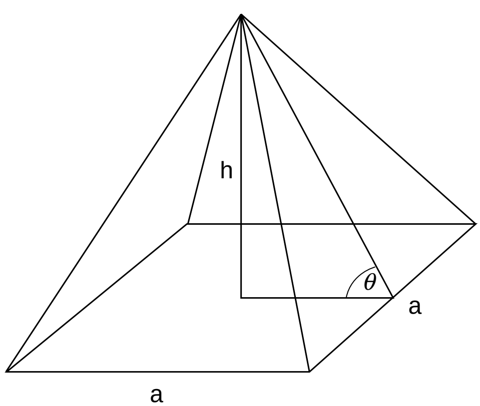

FIG. 2. Square-based pyramid.

term *batter* to measure the inward slope of a supposedly vertical wall.

Second, why did Ahmes go on to multiply his answer by 7? For some reason the pyramid builders measured horizontal distances in "palms" or "hands" and vertical distances in cubits. One cubit equals 7 palms. Thus the required seked, $5\frac{1}{25}$, gives the run-to-rise ratio in units of palms per cubit. Today, of course, we think of these ratios as a pure numbers.

Why was the run-to-rise ratio considered so important as to deserve a special name and four problems devoted to it in the papyrus? The reason is that it was crucial for the pyramid builders to maintain a constant slope of each face relative to the horizon. This may look easy on paper, but once the actual construction began, the builders constantly had to check their progress to ensure that the required slope was maintained. That is, the seked had to be the same for each one of the faces.

Problem 57 is the inverse problem: we are given the seked and the side of a base and are asked to find the height. Problems 58 and 59 are similar to Problem 56 and lead to a seked of $5\frac{1}{4}$ palms (per cubit), except that the answer is given as 5 palms and 1 "finger" (there being 4 fingers in a palm). Finally, Problems 60 asks to find the seked of a pillar 30 cubits high whose base is 15 cubits. We do not know if this pillar had the shape of a pyramid or a cylinder (in which case 15 is the diameter of the base); in either case the answer is 1/4.

The seked found in Problem 56, namely 18/25 (in dimensionless units) corresponds to an angle of $54^\circ\ 15'$ between the base and face. The seked found in Problems 58–59, when converted back to dimensionless units, is $(5\frac{1}{4}) : 7$ or 3/4, corresponding to an angle of $53^\circ\ 8'$. It is interesting to compare these figures to the actual angles of some of pyramids at Giza:[17]

Cheops:	$51^\circ 52'$
Chephren:	$52^\circ 20'$
Mycerinus:	$50^\circ 47'$

The figures are in close agreement. As for the pillar in Problem 60, its angle is much larger, as of course we expect of such a structure: $\phi = \cot^{-1}(1\ \ 4) = 75^\circ\ 58'$.

It would be ludicrous, of course, to claim that the Egyptians invented trigonometry. Nowhere in their writings does there appear the concept of an angle, so they were in no position to formulate quantitative relations between the angles and sides of a triangle. And yet (to quote Chase) "at the beginning of the 18th century *b.c.*, and probably a thousand years earlier, when the great pyramids were built, the Egyptian mathematicians had some notion of referring a right triangle to a similar triangle, one of whose sides was a unit of measure, as a standard." We may therefore be justified in crediting the Egyptians with a crude knowledge of practical trigonometry—perhaps "proto-trigonometry" would be a better word—some two thousand years before the Greeks took up this subject and transformed it into a powerful tool of applied mathematics.

NOTES AND SOURCES

1. The papyrus also contains three fragmentary pieces of text unrelated to mathematics, which some authors number as Problems 85, 86, and 87. These are described in Arnold Chase, *The Rhind Mathematical Papyrus: Free Translation and Commentary with Selected Photographs, Transcriptions, Transliterations and Literal Translations* (Reston, VA: National Council of Teachers of Mathematics, 1979), pp. 61–62.

2. *The Rhind Mathematical Papyrus, British Museum 10057 and 10058: Introduction, Transcription, Translation and Commentary* (London, 1923).

3. Chase, *Rhind Mathematical Papyrus*. This extensive work is a reprint, with minor changes, of the same work published by the Mathematical Association of America in two volumes in 1927 and 1929. It contains detailed commentary and an extensive bibliography, as well as numerous color plates of text material. For a biographical sketch of Chase, see the article "Arnold Buffum Chase" in the *American Mathematical Monthly*, vol. 40 (March 1933), pp. 139–142. Other good sources on Egyptian mathematics are Richard J. Gillings, *Mathematics in the Time of the Pharaohs* (1972; rpt. New York: Dover, 1982); George Gheverghese Joseph, *The Crest of the Peacock: Non-European Roots of Mathematics* (Harmondsworth, U.K.: Penguin Books, 1991), chap. 3; Otto Neugebauer, *The Exact Sciences in Antiquity* (1957; rpt. New York: Dover, 1969), chap. 4; and Baertel L. van der Waerden, *Science Awakening*, trans. Arnold Dresden (New York: John Wiley, 1963), chap. 1.

4. Chase, *Rhind Mathematical Papyrus*, p. 27. The royal title "Re" is pronounced "ray."

5. Another important document from roughly the same period is the Golenishchev or Moscow Papyrus, a scroll about the same length as the Rhind Papyrus but only three inches wide. It contains 25 problems and is of poorer quality than the Rhind Papyrus. See Gillings, *Mathematics*, pp. 246–247; Joseph, *Crest of the Peacock*, pp. 84–89; van der Waerden, *Science Awakening*, pp. 33–35; and Carl B. Boyer, *A History of Mathematics* (1968; rev. ed. New York: John Wiley, 1989), pp. 22–24. References to other Egyptian mathematical documents can be found in Chase, *Rhind Mathematical Papyrus*, p. 67; Gillings, *Mathematics*, chaps. 9, 14, and 22; Joseph, *Crest of the Peacock*, pp. 59–61, 66–67 and 78–79; and Neugebauer, *Exact Sciences*, pp. 91–92;

6. As quoted by van der Waerden, *Science Awakening*, p. 16, who apparently quoted from Peet. This differs slightly from Chase's free translation (*Rhind Mathematical Papyrus*, p. 27).

7. Van der Waerden, *Science Awakening*, pp. 16–17.

8. Note that the decomposition is not unique: 7/10 can also be written as $1\ 5 + 1\ 2$.

9. For a more detailed discussion of the Egyptians' use of unit fractions, see Boyer, *History of Mathematics*, pp. 15–17; Chase, *Rhind Mathematical Papyrus*, pp. 9–17; Gillings, *Mathematics*, pp. 20–23; and van der Waerden, *Science Awakening*, pp. 19–26.

10. Chase, *Rhind Mathematical Papyrus*, pp. 15–16; van der Waerden, *Science Awakening*, pp. 27–29.

11. See Gillings, *Mathematics*, pp. 154–161.

12. Chase, *Rhind Mathematical Papyrus*, p. 46. For a discussion of Egyptian measures, see ibid., pp. 18–20; Gillings, *Mathematics*, pp. 206–213.

13. The Egyptian value can be conveniently written as $(4\ 3)^4$. Gillings (*Mathematics*, pp. 139–153) gives a convincing theory as to how Ahmes derived the formula $= [(8\ 9)\ ]^2$ and credits him as being "the first authentic circle-squarer in recorded history!" See also Chase, *Rhind Mathematical Papyrus*, pp. 20–21, and Joseph, *Crest of the Peacock*, pp. 82–84 and 87–89. Interestingly the Babylonians, whose mathematical skills generally exceeded those of the Egyptians, simply equated the area of a circle to the area of the inscribed regular hexagon, leading to $\pi = 3$; see Joseph, *Crest of the Peacock*, p. 113.

14. Pronounced "saykad" or "sayket."

15. Chase, *Rhind Mathematical Papyrus*, p. 51.

16. See, however, ibid., pp. 21–22 for an alternative interpretation.

17. Gillings, *Mathematics*, p. 187.

Recreational Mathematics in Ancient Egypt

Problem 79 of the Rhind Papyrus says (fig. 3):[1]

A house inventory:		houses	7
1	2,801	cats	49
2	5,602	mice	343
4	11,204	spelt	2,301
		hekat	16,807
Total	19,607	Total	19,607

The Egyptian word for "cat" is *myw*; when the missing vowels are inserted, this becomes *meey'a uw*.
Obviously Ahmes made a mistake here. The correct entry should be 2,401.

What is the meaning behind this cryptic verse? Clearly we have before us a geometric progression whose initial term and common ratio are both 7, and the scribe shows us how to find its sum. But as any good teacher would do to break the monotony of a routine math class, Ahmes embellishes the exercise with a little story which might be read like this: There are seven houses; in each house there are seven cats; each cat eats seven mice; each mouse eats seven ears of spelt; each ear of spelt produces seven hekat of grain. Find the total number of items involved.

The right hand column clearly gives the terms of the progression $7\ 7^2\ 7^3\ 7^4\ 7^5$ followed by their sum, 19,607 (whether the mistaken entry 2,301 was Ahmes's own error in copying or whether it had already been in the original document, we shall never know). But now Ahmes plays his second card: in the left-hand column he shows us how to obtain the answer in a shorter, "clever" way; and in following it we can see the Egyptian method of multiplication at work. The Egyptians knew that any integer can be represented as a sum of terms of the geometric progression 1 2 4 8 and that the representation is unique (this is precisely the representation of an integer in terms of the base 2, the coefficients, or "binary digits," being 0 and 1). To multiply, say, 13 by 17, they only had to write one of the multipliers, say 13, as a sum of powers of 2, $13 = 1 + 4 + 8$,