an introduction

STABILITY AND STABILIZATION

William J. Terrell

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PRINCETON UNIVERSITY PRESS
PRINCETON AND OXFORD

Published by Princeton University Press, 41 William Street, Princeton, New Jersey 08540

In the United Kingdom: Princeton University Press, 6 Oxford Street, Woodstock, Oxfordshire OX20 1TW

ISBN-13: 978-0-691-13444-4
Library of Congress Control Number: 2008926510

British Library Cataloging-in-Publication Data is available
Printed on acid-free paper. ∞
press.princeton.edu
Typeset by S R Nova Pvt Ltd, Bangalore, India
Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

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Preface

Thanks for turning to the Preface.

This book is a text on stability theory and applications for systems of ordinary differential equations. It covers a portion of the core of mathematical control theory, including the concepts of linear systems theory and Lyapunov stability theory for nonlinear systems, with applications to feedback stabilization of control systems.

The book is written as an introduction for beginning students who want to learn about the mathematics of control through a study of one of its major problems: the problem of stability and feedback stabilization of equilibria. Readers can then explore the concepts within their own areas of scientific, engineering, or mathematical interest.

Previous exposure to control theory is not required. The minimal prerequisite for reading the book is a working knowledge of elementary ordinary differential equations and elementary linear algebra. Introductory courses in each of these areas meet this requirement. Some exposure to undergraduate analysis (advanced calculus) beyond the traditional three-semester calculus sequence will be helpful as the reader progresses through the book, but it is not a strict prerequisite for beginning.

It may be helpful to mention that one or two of the following courses probably provide more than sufficient background, due to the mathematical maturity required by such courses:

- a course in undergraduate analysis (advanced calculus) that covers mappings from $\mathbf{R}^n$ to $\mathbf{R}^m$, continuity, and differentiability
- a course in the theory of ordinary differential equations that covers existence and uniqueness, continuation (extension) of solutions, and continuous dependence of solutions, including discussion of both linear and nonlinear systems
- a second course in linear algebra or matrix theory
- a senior undergraduate or first-year graduate course in numerical analysis or numerical linear algebra

At this point I should note some important omissions. Numerical issues of control analysis and design are not discussed in the book. It is also important to mention right away that the book deals with systems in the state space framework only, with no discussion of transfer function analysis of linear systems. No experience with transfer functions is required, and no properties of transfer functions are invoked. I have included some references for these important areas as pointers to the literature. In addition, there is no systematic coverage of linear optimal control; however, there is a section on the algebraic Riccati equation and its connection with the linear quadratic regulator problem. The book includes an introduction to the state space framework of linear systems theory in Chapters 3–7. Chapters 8–16 are on nonlinear systems. (Detailed chapter descriptions appear in the introductory chapter.)

This book emphasizes basic system concepts, stability, and feedback stabilization for autonomous systems of ordinary differential equations. Thus, it covers a portion of the core of mathematical control theory, and attempts to show the cohesiveness and unity of the mathematical ideas involved.

For deeper foundations and a broader perspective on the wider field of mathematical control theory, as well as extensions and further applications of the ideas, I recommend additional reading. Suggested reading for this purpose appears at the end of each chapter in a Notes and References section and in a brief Further Reading chapter at the end of the book.

Exercise sets appear in separate sections at the end of each chapter, and in total there are over 190 exercises. In general, the exercises are low-to-intermediate hurdles, although there may be a few exceptions to this rule. There are exercises that require showing how something is done, and exercises that require showing why something is true. There are also exercises that ask for an example or counterexample to illustrate certain points. Occasionally, when all the required tools are available, the proof of a labeled result in the text is left as an exercise. Computations in the exercises usually require only a modest amount of calculation by hand, and some standard computations such as matrix rank or phase portraits can be done by software.

The book provides enough material for two academic semesters of coursework for begininng students. It can serve as a text for a second course in ordinary differential equations which provides an introduction to a core area of mathematical control theory. The material is presented at an intermediate level suitable for advanced undergraduates or beginning graduate students, depending on instructor choices and student backgrounds. When teaching this material I have usually covered most of the material of the Mathematical Background chapter (Chapter 2) in detail. Occasionally, I have covered some material from the Appendices; however, the Appendices might be assigned for self-study.

The text is also suitable for self-study by well-motivated readers with experience in ordinary differential equations and linear algebra. In fact, I hope that the book will provide some stimulation for readers to learn more about basic analysis and the theory of ordinary differential equations with the help of a motivating core problem of scientific, engineering, and mathematical significance.

Acknowledgments . The results of this book originate with others. The selection of topics and the organization have been guided by a careful reading of some key papers, texts, and monographs. It is a pleasure to acknowledge my debt to all the authors of works in the Bibliography, Notes and References sections, and Further Reading. I have made selections from a very large literature and attempted to tie the topics together in a coherent way, while remaining aware that other choices might have been made. After sincere efforts to produce a clear and accurate text, any errors that have crept in are my own reponsibility, and I will remain grateful to any readers kind enough to send their comments, suggestions, or notices of correction.

Special thanks go to Stephen L. Campbell at North Carolina State University and Vickie Kearn at Princeton University Press for their interest and encouragement in this project, and to Jan C. Willems at Katholieke Universiteit Leuven for his willingness to read, comment and offer suggestions on many chapters of a draft manuscript. My thanks also to Robert E. Terrell for reading even earlier versions of many chapters and for offering illustration ideas and graphics files.

In addition, I want to thank all the anonymous reviewers of a draft manuscript for their interest and willingness to review portions of the text as it developed.

I also thank the Office of the Dean, College of Humanities and Sciences, Virginia Commonwealth University, for their interest in the project and for financial support during the summer of 2007 toward completion of the book, and the VCU Department of Mathematics and Applied Mathematics for the opportunity to teach topics courses on this material from 2003 to 2007. I also thank the students who participated with interest and patience in those topics courses.

This book is dedicated to: my father, Edgar A. Terrell, Jr., to the memory of my mother, Emmie J. Terrell, and my brothers, Robert E. Terrell and Jon A. Terrell.

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Richmond, Virginia

Chapter One

Introduction

In this short introductory chapter, we introduce the main problem of stability and stabilization of equilibria, and indicate briefly the central role it plays in mathematical control theory. The presentation here is mostly informal. Precise definitions are given later. The chapter serves to give some perspective while stating the primary theme of the text.

We start with a discussion of simple equations from an elementary differential equations course in order to contrast open loop control and feedback control. These examples lead us to a statement of the main problem considered in the book, followed by an indication of the central importance of stability and stabilization in mathematical control theory. We then note a few important omissions. A separate section gives a complete chapter-by-chapter description of the book. The final section of the chapter is a list of suggested collateral reading.

1.1 OPEN LOOP CONTROL

Students of elementary differential equations already have experience with open loop controls. These controls appear as a given time-dependent forcing term in the second order linear equations that are covered in the first course on the subject. A couple of simple examples will serve to illustrate the notion of open loop control and allow us to set the stage for a discussion of feedback control in the next section.

The Forced Harmonic Oscillator. Consider the nonhomogeneous linear mass-spring equation with unit mass and unit spring constant,

$$\ddot{y} + y = u(t).$$

We use $\dot{y}$ and $\ddot{y}$ to denote the first and second derivatives of $y(t)$ with respect to time. The equation involves a known right-hand side, which can be viewed as a preprogrammed, or *open loop*, control defined by $u(t)$. The general real-valued solution for such equations is considered in differential equations courses, and it takes the form

$$y(t) = y_h(t) + y_p(t),$$

where $y_p(t)$ is any particular solution of the nonhomogeneous equation and $y_h(t)$ denotes the general solution of the homogeneous equation, $\ddot{y} + y = 0$.

For this mass-spring equation, we have

$$y_h(t) = c_1 \cos t + c_2 \sin t,$$

where the constants c_1 and c_2 are uniquely determined by initial conditions for $y(0)$ and $\dot{y}(0)$.

Suppose the input signal is $u(t) = \sin t$. This would not be an effective control, for example, if our purpose is to damp out the motion asymptotically or to regulate the motion to track a specified position or velocity trajectory. Since the frequency of the input signal equals the natural frequency of the unforced harmonic oscillator, $\ddot{y} + y = 0$, the sine input creates a resonance that produces unbounded motion of the mass.

On the other hand, the decaying input $u(t) = e^{-t}$ yields a particular solution given by $y_p(t) = \frac{1}{2}e^{-t}$. In this case, every solution approaches a periodic response as $t \to \infty$, given by $y_h(t)$, which depends on the initial conditions $y(0)$ and $\dot{y}(0)$, but not on the input signal.

Suppose we wanted to apply a continuous input signal which would guarantee that all solutions approach the origin defined by zero position and zero velocity. It is not difficult to see that we cannot do this with a continuous open loop control. The theory for second-order linear equations implies that there is no continuous open loop control $u(t)$ such that each solution of $\ddot{y} + y = u(t)$ approaches the origin as $t \to \infty$, independently of initial conditions.

The Double Integrator. An even simpler equation is $\ddot{y} = u(t)$. The general solution has the form $y(t) = c_1 + c_2 t + y_p(t)$, where $y_p(t)$ is a particular solution that depends on $u(t)$. Again, there is no continuous control $u(t)$ that will guarantee that the solutions will approach the origin defined by zero position and zero velocity, independently of initial conditions.

Open loop, or preprogrammed, control does not respond to the state of the system it controls during operation. A standard feature of engineering design involves the idea of injecting a signal into a system to determine the response to an impulse, step, or ramp input signal. Recent work on the active approach to the design of signals for failure detection uses open loop controls as test signals to detect abnormal behavior [22]; an understanding of such open loop controls may enable more autonomous operation of equipment and condition-based maintenance, resulting in less costly or safer operation.

The main focus of this book is on principles of stability and feedback stabilization of an equilibrium of a dynamical system. The next section explains this terminology and gives a general statement of this core problem of dynamics and control.

1.2 THE FEEDBACK STABILIZATION PROBLEM

The main theme of stability and stabilization is focused by an emphasis on time invariant (autonomous) systems of the form

$$\dot{x} = f(x),$$

where $f : \mathcal{D} \subset \mathbf{R}^n \to \mathbf{R}^n$ is a continuously differentiable mapping (a smooth vector field on an open set $\mathcal{D} \subset \mathbf{R}^n$) and $\dot{x} := \frac{dx}{dt}$. If f is continuously differentiable, then f satisfies a local Lipschitz continuity condition in a neighborhood of each point in its domain. From the theory of ordinary differential equations, the condition of local Lipschitz continuity of f guarantees the existence and uniqueness of solutions of initial value problems

$$\dot{x} = f(x), \quad x(0) = x_0,$$

where x_0 is a given point of $\mathcal{D}$.

The state of the system at time t is described by the vector x. Assuming that $f(0) = 0$, so that the origin is an equilibrium (constant) solution of the system, the core problem is to determine the stability properties of the equilibrium. The main emphasis is on conditions for asymptotic stability of the equilibrium. A precise definition of the term *asymptotic stability* of $x = 0$ is given later. For the moment, we simply state its intuitive meaning: Solutions $x(t)$ with initial condition close to the origin are defined for all forward time $t \geq 0$ and remain close to $x = 0$ for all $t \geq 0$; moreover, initial conditions sufficiently close to the equilibrium yield solutions that approach the equilibrium asymptotically as $t \to \infty$.

We can now discuss the meaning of *feedback stabilization* of an equilibrium. Let $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$ be a continuously differentiable function of $(x, u) \in \mathbf{R}^n \times \mathbf{R}^m$. The introduction of a feedback control models the more complicated process of actually measuring the system state and employing some mechanism to feed the measured state back into the system as a real time control on system operation. The feedback stabilization problems in this book involve autonomous systems with control u, given by

$$\dot{x} = f(x, u).$$

In this framework, the introduction of a smooth (continuously differentiable) state feedback control $u = k(x)$ results in the *closed loop system*

$$\dot{x} = f(x, k(x)),$$

which is autonomous as well. If $f(0, 0) = 0$, then the origin $x_0 = 0$ is an equilibrium of the unforced system, $\dot{x} = f(x, 0)$. If the feedback satisfies $k(0) = 0$, then it preserves the equilibrium; that is, the closed loop system also has an equilibrium at the origin.

We apply stability theory in several different settings to study the question of existence of a continuously differentiable feedback $u = k(x)$ such that the origin $x_0 = 0$ is an asymptotically stable equilibrium of the closed loop system. For certain system classes and conditions, explicit stabilizing feedback controls are constructed. The system classes we consider are not chosen arbitrarily; they are motivated by (i) their relevance in the research activity on stabilization of recent decades, and (ii) their accessibility in an introductory text.

Feedback in the Harmonic Oscillator and Double Integrator Systems. The system corresponding to the undamped and unforced harmonic oscillator, obtained by writing $x_1 = y$ and $x_2 = \dot{y}$, and setting $u = 0$, is given by

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= -x_1.\end{aligned}$$

This system does not have an asymptotically stable equilibrium at the origin $(x_1, x_2) = (0, 0)$. If we had both state components available for feedback, we could define a feedback control of the form $u = k_1 x_1 + k_2 x_2$, producing the closed loop system

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= (k_1 - 1)x_1 + k_2 x_2.\end{aligned}$$

If we can measure only the position variable x_1 and use it for feedback, say $u = k_1 x_1$, then we are not able to make the origin $(0, 0)$ asymptotically stable, no matter what the value of the real coefficient k_1 may be. However, using only feedback from the velocity, if available, say $u = k_2 x_2$, it is possible to make the origin an asymptotically stable equilibrium of the closed loop system. Verification of these facts is straightforward, and to accomplish it, we can even use the second order form for the closed loop system; for position feedback only, $\ddot{y} + (1 - k_1)y = 0$; for velocity feedback only, $\ddot{y} - k_2\dot{y} + y = 0$. For position feedback, the characteristic equation is $r^2 + (1 - k_1) = 0$, and the general real-valued solution for $t \geq 0$ is (i) periodic for $k_1 < 1$, (ii) the sum of an increasing exponential term and a decreasing exponential term for $k_1 > 1$, and (iii) a constant plus an unbounded linear term for $k_1 = 1$. For velocity feedback, choosing $k_2 < 0$ ensures that all solutions that start close to the origin at time $t = 0$ remain close to the origin for all $t \geq 0$, and also satisfy $(x_1(t), x_2(t)) \to (0, 0)$ as $t \to \infty$.

For the simpler double integrator equation, $\ddot{y} = u(t)$, or its equivalent system,

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= u,\end{aligned}$$

one can check that neither position feedback, $u = k_1 x_1$, nor velocity feedback, $u = k_2 x_2$, can make all solutions approach the origin as $t \to \infty$. However, feedback using both position and velocity, $u = k_1 x_1 + k_2 x_2$, will accomplish this if $k_1 > 0$ and $k_2 > 0$.

The study of stability and stabilization of equilibria for ordinary differential equations (ODEs) is a vast area of applications-oriented mathematics. The restriction to smooth feedback still leaves a huge area of results. This area will be explored in selected directions in the pages of this introductory text.

The restriction to smooth feedback avoids some technical issues that arise with discontinuous feedback, or even with merely continuous feedback.

Discontinuous feedback is mathematically interesting and relevant in many applications. For example, the solutions of many optimal control problems (not discussed in this book) involve discontinuous feedback. However, a systematic study of such feedback requires a reconsideration of the type of system under study and the meaning of solution. These questions fall essentially outside the scope of the present book.

Although we consider primarily smooth feedback controls, at several places in the text the admissible open loop controls are piecewise continuous, or at least integrable on any finite interval, that is, *locally integrable.*

The Importance of the Subject

Stability theory provides core techniques for the analysis of dynamical systems, and it has done so for well over a hundred years, at least since the 1892 work of A. M. Lyapunov; see [73]. An earlier feedback control study of a steam engine governor, by J. Clerk Maxwell, was probably the first modern analysis of a control system and its stability. Stability concepts have always been a central concern in the study of dynamical control systems and their applications. The problem of feedback stabilization of equilibria is a core problem of mathematical control theory. Possibly the most important point to make here is that many other issues and problems of control theory depend on concepts and techniques of stability and stabilization for their mathematical foundation and expression. Some of these areas are indicated in the end-of-chapter Notes and References sections.

Some Important Omissions

There are many important topics of stability, stabilization, and, more generally, mathematical control theory which are not addressed in this book. In particular, as mentioned in the Preface, there is no discussion of transfer function analysis for linear time invariant systems, and transfer functions are not used in the text. Also, there is no systematic coverage of optimal control beyond the single section on the algebraic Riccati equation. Since there is no coverage of numerical computation issues in this text, readers interested specifically in numerical methods should be aware of the text by B. N. Datta, *Numerical Methods for Linear Control Systems*, Elsevier Academic Press, London, 2004.

The end-of-chapter Notes and References sections have resources for a few other areas not covered in the text.

1.3 CHAPTER AND APPENDIX DESCRIPTIONS

In general, the chapters follow a natural progression. It may be helpful to mention that readers with a background in the state space framework of linear system theory and a primary interest in nonlinear systems might proceed with Chapter 8 (Stability Theory) after the introductory material of Chapter 2 (Mathematical Background) and Chapter 3 (Linear Systems and Stability). Definitions and examples of stability and instability appear in Chapter 3. For such readers, Chapters 4–7 could be used for reference as needed.

Chapter 2. The Mathematical Background chapter includes material mainly from linear algebra and differential equations. For basic analysis we reference Appendix B or the text [7]. The section on linear and matrix algebra includes some basic notation, linear independence and rank, similarity of matrices, invariant subspaces, and the primary decomposition theorem. The section on matrix analysis surveys ifferentiation and integration of matrix functions, inner products and norms, sequences and series of functions, and quadratic forms. A section on ordinary differential equations states the existence and uniqueness theorem for locally Lipschitz vector fields and defines the Jacobian linearization of a system at a point. The final section has examples of linear and nonlinear mass-spring systems, pendulum systems, circuits, and population dynamics in the phase plane. These examples are familiar from a first course in differential equations. The intention of the chapter is to present only enough to push ahead to the first chapter on linear systems.

Chapter 3. This chapter develops the basic facts for linear systems of ordinary differential equations. It includes existence and uniqueness of solutions for linear systems, the stability definitions that apply throughout the book, stability results for linear systems, and some theory of Lyapunov equations. Jordan forms are introduced as a source of examples and insight into the structure of linear systems. The chapter also includes the Cayley-Hamilton theorem. A few basic facts on linear time varying systems are included as well.

The next four chapters, Chapters 4–7, provide an introduction to the four fundamental structural concepts of linear system theory: controllability, observability, stabilizability, and detectability. We discuss the invariance (or preservation) of these properties under linear coordinate change and certain feedback transformations. All four properties are related to the study of stability and stabilization throughout these four chapters. While there is some focus on single-input single-output (SISO) systems in the examples, we include basic results for multi-input multi-output (MIMO) systems as well. Throughout Chapters 4–7, Jordan form systems are used as examples to help develop insight into each of the four fundamental concepts.

Chapter 4. Controllability deals with the input-to-state interaction of the system. This chapter covers controllability for linear time invariant systems. Single-input controllable systems are equivalent to systems in a special companion form (controller form). Controllability is a strong sufficient condition for stabilization by linear feedback, and it ensures the solvability of transfer-of-state control problems. The chapter includes the eigenvalue placement theorem for both SISO and MIMO systems, a controllability normal form (for uncontrollable systems) and the PBH controllability test. (The PBH controllability test and related tests for observability, stabilizability, and detectability are so designated in recognition of the work of V. M. Popov, V. Belevitch, and M. L. J. Hautus.)

Chapter 5. Observability deals with the state-to-output interaction of the system. The chapter covers the standard rank criteria for observability and the fundamental duality between observability and controllability. Lyapunov equations are considered under some special hypotheses. The chapter includes an observability normal form (for unobservable systems), and a brief discussion of output feedback versus full-state feedback.

Chapter 6. This chapter on stabilizability begins with a couple of standard stabilizing feedback constructions for controllable systems, namely, linear feedback stabilization using the controllability Gramian and Ackermann's formula. We characterize stabilizability with the help of the controllability normal form and note the general limitations on eigenvalue placement by feedback when the system is not controllable. The chapter also includes the PBH stabilizability test and some discussion on the construction of the controllability and observability normal forms.

Chapter 7. Detectability is a weaker condition than observability, but it guarantees that the system output is effective in distinguishing trajectories asymptotically, and this makes the property useful, in particular, in stabilization studies. The chapter begins with an example of an observer system for asymptotic state estimation. We define the detectability property, and establish the PBH detectability test and the duality of detectability and stabilizability. We discuss the role of detectability and stabilizability in defining observer systems, the role of observer systems in observer-based dynamic stabilization, and general linear dynamic controllers and stabilization. The final section provides a brief look at the algebraic Riccati equation, its connection with the linear quadratic regulator problem, and its role in generating stabilizing linear feedback controls.

Chapter 8. Chapter 8 presents the basic concepts and most important Lyapunov theorems on stability in the context of nonlinear systems. We discuss the use of linearization for determining asymptotic stability and instability of equilibria, and we define critical problems of stability and smooth stabilization. We state Brockett's necessary condition for smooth stabilization. This chapter also develops basic properties of limit sets and includes the invariance theorem. There is a discussion of scalar equations which is useful for examples, a section on the basin of attraction for asymptotically stable equilibria, and a statement of converse Lyapunov theorems.

Chapter 9. Chapter 9 develops the stability properties of equilibria for cascade systems. The assumptions are strengthened gradually through the chapter, yielding results on Lyapunov stability, local asymptotic stability, and global asymptotic stability. Two foundational results lead to the main stability results: first, the theorem on total stability of an asymptotically stable equilibrium under a class of perturbations; second, a theorem establishing that the boundedness of certain driven trajectories in a cascade implies the convergence of those trajectories to equilibrium. Cascade systems play a central role in control studies; they arise in control problems directly by design or as a result of attempts to decompose, or to transform, a system for purposes of analysis. The final section shows that cascade forms may also be obtained by appropriate aggregation of state components.

Chapter 10. Center manifold theory provides tools for the study of critical problems of stability, the problems for which Jacobian linearization cannot decide the issue. Many critical problems can be addressed by the theorems of Chapter 8 or Chapter 9. However, center manifold theory is an effective general approach. The chapter begins with examples to show the value of the center manifold concept and the significance of dynamic behavior on a center manifold. Then we state the main results of the theory: (i) the existence of a center manifold; (ii) the reduction of stability analysis to the behavior on a center manifold; and (iii) the approximation of a center manifold to an order sufficient to accomplish the analysis in (ii). Two applications of these ideas are given in this chapter: the preservation of smooth stabilizability when a stabilizable system is augmented by an integrator, and a center manifold proof of a result on asymptotic stability in cascades with a linear driving system. Another application, on the design of a center manifold, appears in Chapter 11.

Chapter 11. In this chapter we consider single-input single-output systems and the zero dynamics concept. We define the relative degree at a point, the normal form, the zero dynamics manifold, and the zero dynamics subsystem on that manifold. Next, we consider asymptotic stabilization by an analysis of the zero dynamics subsystem, including critical cases. A simple model problem of aircraft control helps in contrasting linear and nonlinear problems and their stability analysis. The concept of vector relative degree for multi-input multi-output systems is defined, although it is used within the text only for the discussion of passive systems with uniform relative degree one. (Further developments on MIMO systems with vector relative degree, or on systems without a well-defined relative degree, are available through resources in the Notes and References.) The chapter ends with two applications: the design of a center manifold for the airplane example, and the computation of zero dynamics for low-dimensional controllable linear systems which is useful in Chapter 15.

Chapter 12. We consider feedback linearization only for single-input single-output systems. A single-input control-affine nonlinear system is locally equivalent, under coordinate change and regular feedback transformation in a neighborhood of the origin, to a linear controllable system, if and only if the relative degree at the origin is equal to the dimension of the state space. Feedback linearizable systems are characterized by geometric conditions that involve the defining vector fields of the system. The proof of the main theorem involves a special case of the Frobenius theorem, which appears in an Appendix. Despite the lack of robustness of feedback linearization, there are important areas, for example mechanical systems, where feedback linearization has achieved successes. Most important, the ideas of feedback linearization have played an important role in the development of nonlinear geometric control.

Chapter 13. In the first section of this chapter, we present a theorem on the global stabilization of a special class of nonlinear systems using the feedback construction known as damping control (also known as L_gV *control*, or *Jurdjevic-Quinn feedback*). This theorem provides an opportunity to contrast the strong connections among Lie brackets, controllability, and stabilization for linear systems, with the very different situation of nonlinear systems. Thus, the second section shows that the Lie bracket-based generalization of the controllability rank condition does not imply a local controllability property of the nonlinear system, and even global controllability does not imply stabilizability by smooth feedback. (The definition of controllability used here is the same one used for linear systems: any point can be reached from any other point in finite time along a trajectory corresponding to some admissible open loop control.) We give references for more information on controllability ideas and their application.

Chapter 14. The passivity concept has roots in the study of passive circuit elements and circuit networks. Passivity is defined as an input-output property, but passive systems can be characterized in state-space terms by the KYP property. (The KYP property is so designated in recognition of the work of R. E. Kalman, V. A. Yakubovich, and V. M. Popov.) This chapter develops the stability and stabilization properties of passive systems. It is an exploration of systems having relative degree at the opposite extreme from the feedback linearizable systems: passive systems having a smooth storage function have uniform relative degree one. Moreover, systems that are feedback passive, that is, passive with a smooth positive definite storage function after application of smooth feedback, are characterized by two conditions: they have uniform relative degree one and Lyapunov stable zero dynamics in a neighborhood of the origin. Passivity plays an important role in the feedback stabilization of cascades in Chapter 15.

Chapter 15. Chapter 15 returns to cascade systems. Partial-state feedback, which uses only the states of the driving system, is sufficient for local asymptotic stabilization of a cascade. In general, however, partial-state feedback cannot guarantee global asymptotic stabilization without restrictive growth assumptions on the interconnection term in the driven system of the cascade. This chapter considers an important situation in which global stabilization is assured using full-state feedback. We assume that the driving system is feedback passive with an output function that appears in the interconnection term in an appropriate factored form; global asymptotic stabilization is then achieved with a constructible feedback control.

Chapter 16. This chapter motivates the input-to-state stability concept based on earlier considerations in the text. In particular, input-to-state stability (ISS) addresses the need for a condition on the driven system of a cascade that guarantees not only (i) bounded solutions in response to bounded inputs, but also (ii) converging solutions from converging inputs. This material requires an introduction to the properties of comparison functions from Appendix E. The comparison functions are used in a proof of the basic Lyapunov theorems on stability and asymptotic stability. We give the definition of ISS Lyapunov function and present the main result concerning them: a system is ISS if and only if an ISS Lyapunov function exists for it. This result is applied to establish the ISS property for several examples. We state a result on the use of input-to-state stability in cascade systems and provide some further references.

Chapter 17. This brief chapter collects some additional notes on further reading.

Appendix A. This brief key to notation provides a convenient reference.

Appendix B. This appendix provides a quick reference for essential facts from basic analysis in $\mathbf{R}$ and $\mathbf{R}^n$.

Appendix C. This material on ordinary differential equations is self-contained and includes proofs of basic results on existence and uniqueness of solutions, continuation of solutions, and continuous dependence of solutions on initial conditions and on right-hand sides.

Appendix D. This material on manifolds and the preimage theorem is useful background for the center manifold chapter (Chapter 10) as well as Chapters 11 and 12, which deal with some aspects of geometric nonlinear control. The material on distributions and the Frobenius theorem supports Chapters 11–12 specifically.

Appendix E. The comparison functions are standard tools for the study of ordinary differential equations; they provide a convenient language for expressing basic inequality estimates in stability theory. This material is used explicitly in the text only in Chapter 16 (and in a brief appearance in the proof of Theorem 10.2 (d) on center manifold reduction).

Appendix F. Some hints and answers to selected exercises are included here.

1.4 NOTES AND REFERENCES

For some review of a first course in differential equations, see [21] and [84]. For additional recommended reading in differential equations to accompany this text, see [15] and [40]. In addition, see the text by V. I. Arnold, *Ordinary Differential Equations*, MIT Press, Cambridge, MA, 1973, for its geometric and qualitative emphasis.

The texts [9] and [72] have many examples of control systems described by ordinary differential equations. The material in these books is accessible to an audience having a strong background in upper level undergraduate mathematics. The same is true of the texts [53] and [80]. A senior level course in control engineering is contained in K. Ogata, *Modern Control Engineering*, Prentice-Hall, Upper Saddle River, NJ, third edition, 1997.

For the mathematical foundations of control theory for linear and nonlinear systems, see [91], which is a comprehensive mathematical control theory text on deterministic finite-dimensional systems. It includes material on a variety of model types: continuous and discrete, time invariant and time varying, linear and nonlinear. The presentation in [91] also includes many bibliographic notes and references on stabilization and its development, as well as three chapters on optimal control.

For an interesting and mostly informal article on feedback, see [60].

Chapter Two

Mathematical Background

This chapter contains material from linear algebra and differential equations. It is assumed that the reader has some previous experience with most of these ideas. Thus, in this chapter, in order to allow for ease of reference for items of the most basic importance, as well as a relatively quick reading experience, some terms are defined in labeled definitions and others within the text itself. Careful attention to the notation, definitions, results, and exercises in the background presented here should provide for relatively easy reading in the first chapter on linear systems.

2.1 ANALYSIS PRELIMINARIES

The basic topological concepts in Euclidean space are assumed, as are the concepts of continuity and differentiability of functions mapping a subset of $\mathbf{R}^n$ to $\mathbf{R}^m$. For more background, see Appendix B, or any of the excellent analysis texts available. Most often, we cite the text [7].

2.2 LINEAR ALGEBRA AND MATRIX ALGEBRA

This section includes some basic notation, material on linear independence and rank, similarity of matrices, eigenvalues and eigenvectors, invariant subspaces, and the Primary Decomposition Theorem.

Assumptions and Basic Notation

We assume that the reader is familiar with the algebraic concept of a *field*, and with the axioms that define the concept of a *vector space over a field*. Some experience with the concepts of subspace, basis, and dimension, and how these concepts summarize the solution of systems of linear algebraic equations, is essential. The concepts of eigenvalues and eigenvectors for a linear mapping (or operator) of a vector space V to itself should be familiar, as well as the fact that a linear mapping from a vector space V of dimension n to another vector space W of dimension m (over the same field) may be represented, after a choice of a basis for V and a basis for W, by an $m \times n$ matrix having entries in the common field of scalars.

We write $\mathbf{R}$ for the set of real numbers, $\mathbf{R}_+ := [0, \infty)$, and $\mathbf{R}^n$ for Euclidean n-dimensional space, which is a vector space over the real field $\mathbf{R}$. Vectors are usually column vectors, although they may be written in n-tuple form; when component detail is needed, we may write a vector x in $\mathbf{R}^n$ in any of the following forms:

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^T, \quad (x_1, \ldots, x_n).$$

Similarly, we write $\mathbf{C}$ for the field of complex numbers, and $\mathbf{C}^n$ for complex n-dimensional space, which is a vector space over $\mathbf{C}$.

The systems of differential equations in this book involve real vector fields. In particular, linear systems or Jacobian linearizations of nonlinear systems involve real coefficient matrices. However, eigenvalues and corresponding eigenvectors of these matrices may be complex, so we must work with matrices with either real or complex number entries. We write $\mathbf{R}^{n\times n}$ for the set of $n \times n$ matrices with real entries. Elements of $\mathbf{R}^{n\times n}$ are also called real $n \times n$ matrices. $\mathbf{R}^{n\times n}$ is a vector space over $\mathbf{R}$. We write $\mathbf{C}^{n\times n}$ for the set of $n \times n$ matrices with complex entries. Elements of $\mathbf{C}^{n\times n}$ are also called complex $n \times n$ matrices. $\mathbf{C}^{n\times n}$ is a vector space over $\mathbf{C}$. As sets, $\mathbf{R}^{n\times n} \subset \mathbf{C}^{n\times n}$, so every real matrix is a complex matrix of the same size.

Linear Independence and Rank

We first recall the definition of linear independence of vectors in a vector space.

Definition 2.1 *Let V be a vector space over the field $\mathbf{F}$ ($\mathbf{F} = \mathbf{R}$ or $\mathbf{F} = \mathbf{C}$).*

(a) *The vectors $v_1, \ldots, v_k$ in V are* linearly dependent *over $\mathbf{F}$ if at least one of these vectors can be written as a linear combination of the others. That is, there are a j and scalars $c_1, c_2, \ldots, c_{j-1}, c_{j+1}, \ldots, c_k$ such that $v_j = c_1v_1 + \cdots + c_{j-1}v_{j-1} + c_{j+1}v_{j+1} + \cdots + c_kv_k$.*

(b) *The vectors $v_1, \ldots, v_k$ in V are* linearly independent *over $\mathbf{F}$ if they are not linearly dependent over $\mathbf{F}$.*

Equivalently, $v_1, \ldots, v_k$ are linearly independent if and only if none of the vectors can be written as a linear combination of the remaining vectors in the collection, which is equivalent to saying that for scalars $c_1, \ldots, c_k$,

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0 \quad \Longrightarrow \quad c_1 = c_2 = \cdots = c_k = 0.$$

The $n\times n$ identity matrix, I_n, is the matrix with ones on the main diagonal and zeros elsewhere. Equivalently, we write $I = I_n = [e_1 \cdots e_n]$, where the j-th column e_j is the j-th *standard basis vector*. A matrix $A \in \mathbf{C}^{n\times n}$ is *invertible* if there exists a matrix $B \in \mathbf{C}^{n\times n}$ such that $BA = AB = I_n$.

Then B, if it exists, is unique and we write B as A^{-1} for the inverse of A. An invertible matrix is also called a *nonsingular* matrix.

If A is real and invertible, then A^{-1} is also real. To see this, write the inverse as $B = X + iY$ with X and Y real. Then $I = AB = AX + iAY$, and therefore $AY = 0$; hence, $Y = 0$ and $B = X$ is real. The idea of invertibility is to solve $Ax = y$ in the form $x = A^{-1}y$. This solution can be carried out simultaneously for the standard basis vectors in place of y, by the row reduction of the augmented matrix $[A \quad I_n]$ to the form $[I_n \quad A^{-1}]$, assuming that A is indeed invertible. Since elementary row operations preserve the property of nonzero value for the determinant of a square matrix, we have the criterion that a matrix A is invertible if and only if the determinant of A is nonzero, $\det A \neq 0$.

A *subspace* of a vector space V is a nonempty subset X of V that is closed under vector addition and scalar multiplication. That is, for any vectors $v, w \in X$ and any scalar $c \in \mathbf{F}$, we have $v + w \in X$ and $cv \in X$. A *basis* for a subspace X is a set $\{v_1, \ldots, v_k\}$ such that $v_1, \ldots, v_k$ are linearly independent and $X = \operatorname{span}\{v_1, \ldots, v_k\}$, where $\operatorname{span}\{v_1, \ldots, v_k\}$ is the *set of all linear combinations* of $v_1, \ldots, v_k$. A basis for X is a minimal spanning set for X and a maximal linearly independent subset of X. All bases for X must have the same number of elements, and the *dimension* of the subspace X, $\dim X$, is defined to be the number of elements in any basis for X. Two useful subspaces associated with an $m \times n$ matrix A are the range space and the nullspace. The *range* of A is denoted $R(A)$ and is the span of the columns of A. (If A is real and considered as a linear operator on $\mathbf{R}^n$, then this span is defined using real scalars, and if A is complex and considered as a linear operator on $\mathbf{C}^n$, then the span is defined using complex scalars.) The *nullspace* of A is denoted $N(A)$ or $\ker(A)$; it is the solution space of the linear system $Ax = 0$. (If A is real, we are usually interested in $N(A)$ as a subspace of $\mathbf{R}^n$.)

The *rank* of a matrix $A \in \mathbf{C}^{m\times n}$ is the dimension of the column space or range $R(A)$, and is written $\operatorname{rank} A$, so $\operatorname{rank} A = \dim R(A)$. (The rank is also equal to the number of basic columns in the row echelon form of the matrix. The basic columns are the columns which contain the pivots used in the row reduction.) The rank is always the rank over the field of complex numbers. In particular, the rank of a real matrix $A \in \mathbf{R}^{m\times n}$ is the rank over $\mathbf{C}$, but this is also the rank of A over the field of real numbers $\mathbf{R}$. (See Exercise 2.2) The *nullity* of A equals $\dim N(A)$. For any $A \in \mathbf{C}^{m\times n}$, we have

$$\dim R(A) + \dim N(A) = n.$$

Similarity of Matrices

Let $A, B \in \mathbf{C}^{n\times n}$. A is *similar* to B if there exists a nonsingular matrix $S \in \mathbf{C}^{n\times n}$ such that $S^{-1}AS = B$. This relation of similarity of matrices is an equivalence relation on $\mathbf{C}^{n\times n}$. An important similarity invariant of a matrix

A is the *characteristic polynomial* of A, defined by $p(\lambda) = \det(\lambda I - A)$. For later reference it is worth noting that, by this definition, the characteristic polynomial of a matrix is a *monic polynomial*, meaning that the coefficient of the highest power of λ in $\det(\lambda I - A)$ is equal to one. To see the invariance of this polynomial under a similarity transformation S, first recall that $\det(AB) = \det A \det B$ for square matrices A, B of the same size. Then compute

$$\det(\lambda I - S^{-1}AS) = \det S^{-1}(\lambda I - A)S = \det S^{-1}\det(\lambda I - A)\det S,$$

and finally, use the fact that $\det S^{-1} \det S = \det I_n = 1$, to conclude that $\det(\lambda I - S^{-1}AS) = \det(\lambda I - A)$. The set of eigenvalues of A is also a similarity invariant, since the *eigenvalues* are the roots of the *characteristic equation* of A, which is $p(\lambda) = \det(\lambda I - A) = 0$. The relation of similarity is also an equivalence relation on $\mathbf{R}^{n\times n}$. Let $A, B \in \mathbf{R}^{n\times n}$. We say that A and B are *similar via a real similarity* if and only if there exists a nonsingular matrix $S \in \mathbf{R}^{n\times n}$ such that $S^{-1}AS = B$. As we will see later in this chapter, the complex Jordan form of a (real or complex) $n \times n$ matrix A is an especially simple representative of the similarity equivalence class of A using complex similarities. The real Jordan form of a *real* matrix A is an especially simple representative of the similarity equivalence class of A using real similarities.

An $n \times n$ matrix $A = [a_{ij}]$ is *diagonal* if every entry off the main diagonal is zero. Thus, A is diagonal if and only if $a_{ij} = 0$ for $i \neq j$. An $n \times n$ matrix A is *diagonalizable* if A is similar to a diagonal matrix; that is, there exist a diagonal matrix, denoted $D = \operatorname{diag}[d_1, \ldots, d_n]$, where the d_i are the diagonal entries, and a nonsingular matrix S such that $S^{-1}AS = D$. The diagonal entries $d_1, \ldots, d_n$ are necessarily the eigenvalues of A, because the eigenvalues of a diagonal matrix are the main diagonal entries.

Invariant Subspaces

Let V be a vector space over $\mathbf{C}$. A linear mapping $A : V \to V$ is a function such that $A(v + w) = A(v) + A(w)$ for all $v, w \in V$, and $A(cv) = cA(v)$ for all $v \in V$ and $c \in \mathbf{C}$. We usually write $Av := A(v)$ when A is linear. A subspace $W \subset V$ is an *invariant subspace* for a linear mapping $A : V \to V$ if $A(W) \subset W$, that is, $Aw \in W$ for all $w \in W$. We also say that the subspace W is invariant under A.

Important examples of invariant subspaces of $\mathbf{C}^n$ are given by the eigenspaces associated with the eigenvalues of a matrix $A \in \mathbf{C}^{n\times n}$. First, we recall the definitions of eigenvalues and eigenvectors of a square matrix.

Definition 2.2 (Eigenvalues and Eigenvectors of $A \in \mathbf{C}^{n\times n}$)
Let $A \in \mathbf{C}^{n\times n}$. The complex number λ is an eigenvalue *of A if there exists a nonzero vector v such that $Av = \lambda v$. Such a vector v is called an* eigenvector *for the eigenvalue λ.*

Thus, if $W = N(A - \lambda I)$, where λ is an eigenvalue of A, then W is the *eigenspace* asssociated with λ, and W is invariant under A. Note that $v, w \in W$ implies $A(v + w) = Av + Aw = \lambda v + \lambda w = \lambda(v + w)$, hence $v + w \in W$. Also, $v \in W$ and $c \in \mathbf{C}$ imply $A(cv) = cAv = c\lambda v = \lambda(cv)$, so $cv \in W$. By definition, the nonzero vectors in $W = N(A - \lambda I)$ are the eigenvectors of A associated with λ. The *geometric multiplicity* of an eigenvalue λ is the dimension of $N(A - \lambda I)$, that is, the number of linearly independent eigenvectors for λ.

Additional examples of invariant subspaces of A are provided by the following construction. If $f(t)$ is a polynomial, then we write $f(A)$ for the polynomial expression obtained by substituting A for t in $f(t)$. If $f(t)$ is a polynomial, let $W := N(f(A))$. Then W is invariant under A. For if $w \in W$, then $f(A)w = 0$, and therefore $f(A)Aw = Af(A)w = A0 = 0$, which says that $Aw \in W$. In particular, let $f(t) = (t - \lambda)^r$, where λ is an eigenvalue of A and $r \geq 1$. The subspace $N(f(A)) = N((A - \lambda I)^r)$ is invariant under A, and, when $r \geq 2$, this subspace may contain vectors which are not true eigenvectors of A, that is, vectors not in $N(A - \lambda I)$. Any vector in $N((A - \lambda I)^r)$ for some r is called a *generalized eigenvector* of A associated with the eigenvalue λ. It is a consequence of the Primary Decomposition Theorem below that we do not generate any new generalized eigenvectors by considering $N((A - \lambda I)^r)$ for r greater than the algebraic multiplicity of λ. The *algebraic multiplicity* of an eigenvalue λ is the algebraic multiplicity of λ as a root of the characteristic polynomial of A. The subspace $N((A - \lambda I)^m)$, where m is the algebraic multiplicity of the eigenvalue λ, is called the *generalized eigenspace* associated with λ. This generalized eigenspace includes all true eigenvectors for λ as well as all generalized eigenvectors (and the zero vector). The algebraic multiplicity of an eigenvalue must be greater than or equal to the geometric multiplicity.

Example 2.1 The matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 3$, as is easily seen from the triangular structure of A. The eigenspace for $\lambda_1 = 2$, which is the solution space of $(A - 2I)v = 0$, is spanned by the vector $e_1 = [1\ 0\ 0]^T$, the first standard basis vector. The eigenspace for $\lambda_2 = 3$ is spanned by the vector $e_2 = [0\ 1\ 0]^T$. Thus, A has only two linearly independent eigenvectors and is therefore not diagonalizable. However, we can find a nonzero solution of the system $(A - 3I)^2 v = 0$, which is linearly independent of the eigenvectors e_1 and e_2. The vector $e_3 = [0\ 0\ 1]^T$ satisfies $(A - 3I)e_3 = e_2$; therefore $(A - 3I)^2 e_3 = (A - 3I)e_2 = 0$. Thus, the generalized eigenspace for $\lambda_2 = 3$ is span $\{e_2, e_3\}$. $\triangle$

The Primary Decomposition Theorem

The Primary Decomposition Theorem is used later in the text to deduce a general solution formula for systems of linear differential equations. The resulting solution formula is then used to deduce norm bounds for the solutions.

In order to state the Primary Decomposition Theorem, we need the next definition.

Definition 2.3 *Let V be a vector space, and let $W_1, \ldots, W_k$ be subspaces of V. We say that V is the* direct sum *of the subspaces W_i, $i = 1, \ldots, k$ if every vector v in V can be written uniquely as a sum*

$$v = w_1 + \cdots + w_k$$

with $w_i \in W_i$, $i = 1, \ldots, k$.

When V is a direct sum of subspaces W_i, we write

$$V = W_1 \oplus \cdots \oplus W_k \,.$$

If V is the direct sum of the subspaces W_i, and $\mathcal{B}_i$ is a basis for W_i, $1 \le i \le k$, then a basis $\mathcal{B}$ for V is obtained by setting

$$\mathcal{B} = \bigcup_{i=1}^{k} \mathcal{B}_i \,.$$

We can now state the Primary Decomposition Theorem.

Theorem 2.1 (The Primary Decomposition Theorem)
Let A be an $n \times n$ real or complex matrix, and write $p(\lambda) = \prod_{i=1}^{k}(\lambda - \lambda_i)^{m_i}$ for the characteristic polynomial of A, where $\lambda_1, \ldots, \lambda_k$ are the distinct eigenvalues of A, having algebraic multiplicities $m_1, \ldots, m_k$, respectively.

Then C^n is the direct sum of the generalized eigenspaces of A, and the dimension of each generalized eigenspace equals the algebraic multiplicity of the corresponding eigenvalue. That is,

$$C^n = N((A - \lambda_1 I)^{m_1}) \oplus \cdots \oplus N((A - \lambda_k I)^{m_k}).$$

Proof. See [15] (pp. 278–279) or [40] (pp. 331–333). □

2.3 MATRIX ANALYSIS

In a quick survey, this section covers differentiation and integration of matrix functions, and norms, sequences and series of functions, and quadratic forms.

Differentiation and Integration of Matrix Functions

If $A(t) \in \mathbf{C}^{n\times n}$ for each t in a real interval $\mathcal{I}$, then we say that $A(t)$ is a *matrix function* on $\mathcal{I}$. A matrix function $A(t) = [a_{ij}(t)]$ is *continuous* on $\mathcal{I}$ if each entry $a_{ij}(t)$ is a continuous function on $\mathcal{I}$. Similarly, $A(t)$ is *differentiable* (or *smooth*) of class C^k if each entry $a_{ij}(t)$ is differentiable (or smooth) of class C^k on $\mathcal{I}$ (has k continuous derivatives on $\mathcal{I}$). If $A(t) = [a_{ij}(t)]$ is a C^1 matrix function, we write $\dot{A}(t)$ for the matrix with, ij-entry equal to $\frac{da_{ij}}{dt}(t)$.

Example 2.2 Given that

$$A(t) = \begin{bmatrix} t & t^2 \\ 2 & \frac{1}{t} \end{bmatrix},$$

then $A(t)$ is smooth of class C^∞ on any interval not containing $t = 0$, and, in particular, for $t \neq 0$,

$$\dot{A}(t) = \begin{bmatrix} 1 & 2t \\ 0 & -\frac{1}{t^2} \end{bmatrix}. \qquad \triangle$$

See Exercise 2.1 for a few important facts about differentiation.

A matrix function $A(t) = [a_{ij}(t)]$ is *integrable* on the closed and bounded interval $\mathcal{I} = [\alpha, \beta]$ if each entry is integrable on $\mathcal{I}$. If $A(t)$ is integrable on $\mathcal{I} = [\alpha, \beta]$, then we define

$$\int_\alpha^\beta A(t)\,dt := \left[\int_\alpha^\beta a_{ij}(t)\,dt\right].$$

Example 2.3 For the 2×2 $A(t)$ considered in Example 2.2, we have

$$\int_\alpha^\beta A(t)\,dt = \begin{bmatrix} \displaystyle\int_\alpha^\beta t\,dt & \displaystyle\int_\alpha^\beta t^2\,dt \\ \displaystyle\int_\alpha^\beta 2\,dt & \displaystyle\int_\alpha^\beta \frac{1}{t}\,dt \end{bmatrix} = \begin{bmatrix} \dfrac{\beta^2-\alpha^2}{2} & \dfrac{\beta^3-\alpha^3}{3} \\ 2(\beta-\alpha) & \ln|\beta| - \ln|\alpha| \end{bmatrix},$$

if the interval $[\alpha, \beta]$ does not contain zero. On the other hand, if $0 \in [\alpha, \beta]$, then the definite integral of this $A(t)$ from α to β is an improper matrix integral because of the lower right entry. In this case, the improper matrix integral does not converge because the improper real integral in the lower right entry does not converge. $\triangle$

Inner Products and Norms

Suppose that $A \in \mathbf{C}^{m\times n}$ with $A = [a_{ij}]$, $1 \leq i \leq m$, $1 \leq j \leq n$. The *transpose* of A is written A^T and is the matrix with i,j-entry equal to a_{ji}, for $1 \leq j \leq n$, $1 \leq i \leq m$. That is, the rows of A^T are the columns of A, taken in the same order.

Example 2.4 Let

$$A = \begin{bmatrix} 2 & 3 \\ -3i & 4i \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}.$$

Then the transposes of these matrices are given by

$$A^T = \begin{bmatrix} 2 & -3i \\ 3 & 4i \end{bmatrix} \quad \text{and} \quad B^T = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}. \qquad \triangle$$

The dot product of vectors in $\mathbf{R}^n$ is a pairing of vectors which is bilinear in each vector argument. Formally, the dot product is a mapping $B : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ often written $B(u, v) := u \cdot v$. It is an *inner product* according to the following definition.

Definition 2.4 *Let V be a vector space over $\mathbf{R}$. A function $B : V \times V \mapsto \mathbf{R}$ is a* real inner product *on V if*

(a) $B(u, u) > 0$ *for all* $u \neq 0 \in V$, *and* $B(0, 0) = 0$;
(b) $B(v, u) = B(u, v)$ *for all* $u, v \in V$;
(c) $B(u, \alpha v + \beta w) = \alpha B(u, v) + \beta B(u, w)$ *for all* $u, v, w \in V$ *and all* $\alpha, \beta \in \mathbf{R}$.

Definition 2.5 *The* standard inner product (*Euclidean inner product*) *on* $\mathbf{R}^n$ *is defined by*

$$u^T v = u_1 v_1 + \cdots + u_n v_n,$$

for any vectors u and v in $\mathbf{R}^n$.

The verification that the product $(u, v) \mapsto u^T v$ satisfies Definition 2.4 is left as an exercise.

We also need complex-valued inner products. The formal definition only requires a change in item (b) of Definition 2.4. First, recall that the *conjugate* of a complex number, written in standard form $z = a + ib$ with a, b real, is $\bar{z} := a - ib$.

Definition 2.6 *Let V be a vector space over $\mathbf{C}$. A function $H : V \times V \mapsto \mathbf{C}$ is a* Hermitian (complex) inner product *on V if*

(a) $B(u, u) > 0$ *for all* $u \neq 0 \in V$, *and* $B(0, 0) = 0$;
(b) $B(v, u) = \overline{B(u, v)}$ *for all* $u, v \in V$;
(c) $B(u, \alpha v + \beta w) = \alpha B(u, v) + \beta B(u, w)$ *for all* $u, v, w \in V$ *and all* $\alpha, \beta \in \mathbf{C}$.

Note that (b) and (c) of Definition 2.6 imply that $B(\alpha u + \beta v, w) = \bar{\alpha} B(u, w) + \bar{\beta} B(v, w)$ for all choices of the arguments.

The bar notation is also used to indicate the componentwise conjugate of a vector $v = [v_1 \cdots v_n]^T$ in $\mathbf{C}^n$; thus, $\bar{v} = [\bar{v}_1 \cdots \bar{v}_n]^T$. The *conjugate transpose* of the column vector v is the row vector $v^* := [\bar{v}_1 \cdots \bar{v}_n]$.

The next definition features the most important complex inner product for our purposes.

Definition 2.7 *The* standard complex inner product (*Hermitian inner product*) *on* $\mathbf{C}^n$ *is defined by*

$$u^* v = \bar{u}_1 v_1 + \cdots + \bar{u}_n v_n,$$

for any vectors $u = [u_1 \ \cdots \ u_n]^T$, $v = [v_1 \ \cdots \ v_n]^T$ *in* $\mathbf{C}^n$.

The verification that the product $(u, v) \mapsto u^* v$ satisfies Definition 2.6 is left as an exercise. The conjugation of one of the factors guarantees that $u^* u \geq 0$. We use conjugation on the left-hand factor so that we can use the conjugate transpose operation on the left-hand vector; this choice is consistent with [75]. Without the conjugation, we may have $u^T u < 0$ for a complex vector u. For example, if $u = [i \ \ 0]^T$, then $u^T u = -1$.

Suppose that $A \in \mathbf{C}^{m \times n}$ and $A = [a_{ij}]$, $1 \leq i \leq m$, $1 \leq j \leq n$. The *conjugate* of A is given by

$$\bar{A} := [\bar{a}_{ij}], \quad 1 \leq i \leq m, 1 \leq j \leq n.$$

The *conjugate transpose* of A is written A^* and is given by

$$A^* := (\bar{A})^T.$$

Note that we also have $A^* = \overline{(A^T)}$. (The operations of conjugation and transposition commute.) Matrix A^* is also called the *Hermitian transpose* of A. If the matrix product AB is defined, then $(AB)^* = B^* A^*$. If A and B are in $\mathbf{R}^{m \times n}$, then the last property reads $(AB)^T = B^T A^T$. The conjugate transpose A^* has the property that

$$(Ax)^* y = x^* A^* y$$

for all vectors x and y in $\mathbf{C}^n$. In general, $(Ax)^* y \neq x^*(Ay)$, unless $A^* = A$, that is, A is a *Hermitian* matrix.

Example 2.5 Suppose that

$$A = \begin{bmatrix} i & 1 \\ 3 & i \end{bmatrix}, \quad x = \begin{bmatrix} 2 \\ i \end{bmatrix}, \quad y = \begin{bmatrix} i \\ 2i \end{bmatrix}.$$

Then we have

$$x^* A^* y = [2 \quad -i] \begin{bmatrix} -i & 3 \\ 1 & -i \end{bmatrix} \begin{bmatrix} i \\ 2i \end{bmatrix} = [2 \quad -i] \begin{bmatrix} 1 + 6i \\ 2 + i \end{bmatrix} = 3 + 10i,$$

and

$$x^*(Ay) = [2 \quad -i] \begin{bmatrix} i & 1 \\ 3 & i \end{bmatrix} \begin{bmatrix} i \\ 2i \end{bmatrix} = [2 \quad -i] \begin{bmatrix} -1 + 2i \\ -2 + 3i \end{bmatrix} = 1 + 6i. \qquad \triangle$$

The next result is a useful fact that emerges from a careful development of the Gaussian elimination process.

Lemma 2.1 (Transposes and Rank)
If $A \in \mathbf{C}^{m\times n}$, then

$$\operatorname{rank} A = \operatorname{rank} A^T \quad \textit{and} \quad \operatorname{rank} A = \operatorname{rank} A^*.$$

Proof. See [75] (page 139). □

If V is a subspace of $\mathbf{C}^n$, then the *orthogonal complement* of V in $\mathbf{C}^n$ is the subspace defined by

$$V^\perp := \{w \in \mathbf{C}^n \; : \; w^* v = 0 \text{ for all } v \in V\}.$$

Similarly, if V is a subspace of $\mathbf{R}^n$, then the *orthogonal complement* of V in $\mathbf{R}^n$ is the subspace defined by

$$V^\perp := \{w \in \mathbf{R}^n \; : \; w^T v = 0 \text{ for all } v \in V\}.$$

In either context, the set $V^\perp$ is closed under vector addition and appropriate scalar multiplication, so $V^\perp$ is indeed a subspace. And, in either context, we have $(V^\perp)^\perp = V$.

The next theorem is often called the Fundamental Theorem of Linear Algebra.

Theorem 2.2 (Fundamental Theorem of Linear Algebra)
Let A be in $\mathbf{R}^{m\times n}$ with $\operatorname{rank} A = r$. Then the following statements are true:

- $\dim R(A) = r$;
- $\dim N(A) = n - r$;
- $\dim R(A^T) = r$;
- $\dim N(A^T) = m - r$;
- $R(A) = N(A^T)^\perp$: *equivalently,* $N(A^T) = R(A)^\perp$;
- $R(A^T) = N(A)^\perp$: *equivalently,* $N(A) = R(A^T)^\perp$.

If A is in $\mathbf{C}^{m\times n}$, with $\operatorname{rank} A = r$, then the statements of Theorem 2.2 remain true when A^T is replaced in each instance with A^*.

We now discuss vector norms.

Definition 2.8 *Let V be a vector space. A function $\nu : V \mapsto \mathbf{R}$ is a* norm *on V, written $\|v\| := \nu(v)$, if*

(a) $\|v\| \geq 0$ *and* $\|v\| = 0$ *only if* $v = 0$;
(b) $\|cv\| = |c| \, \|v\|$ *for every v in V and scalar c*;
(c) $\|v + w\| \leq \|v\| + \|w\|$ *for every v, w in V.*

An example is the vector space of real numbers, which is normed by the absolute value function. An inner product $B(u, v)$ always induces a norm defined by $\|x\| := B(x, x)^{\frac{1}{2}}$. The Euclidean norm on $\mathbf{R}^n$ is induced in this way by the standard (Euclidean) inner product: $\|x\|_2^2 = x^T x$ for $x \in \mathbf{R}^n$. In $\mathbf{C}^n$ we have the standard norm defined by $\|x\|_2^2 = x^* x$ for $x \in \mathbf{C}^n$.

Definition 2.9 *Let V be a normed vector space, with norm $\|\cdot\|$. A sequence of vectors v_n in V is a* Cauchy sequence *if for every $\epsilon > 0$ there is an $N(\epsilon) > 0$ such that*

$$m, n \geq N(\epsilon) \implies \|v_m - v_n\| < \epsilon.$$

A sequence of vectors v_n in V converges *with limit $w \in V$ if for every $\epsilon > 0$ there is an $N(\epsilon) > 0$ such that*

$$n \geq N(\epsilon) \implies \|v_n - w\| < \epsilon.$$

It is a good exercise to show that a convergent sequence (i) has a unique limit and (ii) must be a Cauchy sequence.

Definition 2.10 *A normed vector space V is* complete *if every Cauchy sequence in V converges to a vector in V. A complete normed vector space is also called a* Banach space.

A closed subset of a Banach space need not be a subspace (a vector space in its own right); however, a closed subset of a Banach space is itself complete in the sense that it contains the limit of every Cauchy sequence of elements from the subset. This follows from the definition of closed set, since, by definition, a closed set contains all its accumulation points.

The most important examples of complete normed spaces for this book are the spaces $\mathbf{R}^n$ and $\mathbf{C}^n$. Their completeness depends on the completeness of the set of real numbers $\mathbf{R}$: every Cauchy sequence of real numbers converges to a real number. In contrast, the field Q of rational numbers is a vector space over Q, but there are Cauchy sequences of rationals which do not converge to a rational number. The completeness of $\mathbf{R}$ follows from the least upper bound property of the ordered field $\mathbf{R}$. For more information on the foundations of the real numbers and their completeness, see Appendix B or [7].

The next lemma states that a norm on a vector space V is a continuous function on that space.

Lemma 2.2 (Continuity of a Norm)
Let $\|\cdot\|$ be a norm on a vector space V. If x_n is a sequence in V that converges to x in V, then

$$\lim_{n\to\infty} \|x_n\| = \|\lim_{n\to\infty} x_n\| = \|x\|.$$

Proof. From the triangle inequality, we have

$$\Big| \|v\| - \|w\| \Big| \leq \|v - w\|$$

for any vectors v, w in V. Thus,

$$\Big|\, \|x_n\| - \|x\| \,\Big| \le \|x_n - x\|$$

for all n. Letting $n \to \infty$, the lemma follows. □

It is useful to know that all norms on a finite-dimensional vector space V are equivalent in the sense that they define the same notion of convergence of sequences in V. Thus, a sequence converges with respect to one norm if and only if it converges with respect to every other possible norm that might be defined on V. Formally, two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ are *equivalent* if there exist numbers $\alpha > 0$ and $\beta > 0$ such that

$$\alpha\|x\|_a \le \|x\|_b \le \beta\|x\|_a \quad \text{for all } x \in V.$$

It is a good exercise to check that this defines an equivalence relation. In $\mathbf{C}^n$, $\|x\|_\infty := \max_{1\le k\le n} |x_k|$ gives a norm which satisfies

$$\max_k |x_k| \le \sqrt{|x_1|^2 + \cdots + |x_n|^2} \le \sqrt{n}\, \max_k |x_k| \ ;$$

that is,

$$\|x\|_\infty \le \|x\|_2 \le \sqrt{n}\, \|x\|_\infty \ ,$$

where $\|x\|_2$ is the Euclidean norm on $\mathbf{R}^n$. The proof of the next proposition shows that any norm $\|\cdot\| : \mathbf{R}^n \to \mathbf{R}$ is continuous in the Euclidean norm on $\mathbf{R}^n$, and the equivalence of any two norms on $\mathbf{R}^n$ follows from this fact.

Proposition 2.1 *Any two norms on $\mathbf{R}^n$ are equivalent.*

Proof. We show that an arbitrary norm, denoted $\|\cdot\|$, is equivalent to the Euclidean norm, $\|x\|_2 = (\sum_{i=1}^n |x_i|^2)^{\frac{1}{2}}$. Let e_i be the i-th standard basis vector. For each vector x there exist unique real numbers x_i such that $x = \sum_{i=1}^n x_i e_i$. Then

$$\|x\| \le \sum_{i=1}^n \|x_i e_i\| = \sum_{i=1}^n |x_i|\ \|e_i\| \le \|x\|_2 \sum_{i=1}^n \|e_i\| = \beta\|x\|_2,$$

where $\beta = \sum_{i=1}^n \|e_i\|$. This inequality shows that the norm $\|\cdot\|$ is continuous in the Euclidean norm, since

$$\Big|\, \|x\| - \|y\| \,\Big| \le \|x - y\| \le \beta\|x - y\|_2$$

for any two vectors x and y. Let S be the unit sphere in $\mathbf{R}^n$ defined by the Euclidean norm: $S = \{x : \|x\|_2 = 1\}$. Since S is compact, the continuous function $x \mapsto \|x\|$ achieves its minimum value on S at some point $x_0 \in S$. Then we have

$$\|x\| \ge \|x_0\| =: \alpha > 0 \quad \text{for all } x \in S.$$

Any nonzero vector x can be written in the form $x = cu$ for some $u \in S$ and $c = \|x\|_2$. It follows that

$$\|x\| = c\,\|u\| \geq c\alpha = \alpha\|x\|_2.$$

This completes the proof of the equivalence of $\|\cdot\|$ and $\|\cdot\|_2$. □

If D is a nonempty subset of $\mathbf{R}^n$, and $\|\cdot\|$ is a norm on $\mathbf{R}^n$, the distance from a point x to the set D is given by

$$\text{dist}(x, D) := \inf\{\|x - a\| : a \in D\}.$$

The infimum, or greatest lower bound, exists because the set $\{\|x - a\| : a \in D\}$ is bounded below (by zero, for instance). The distance from x to D depends on the norm. A curve defined by a function $x : [0, \infty) \to \mathbf{R}^n$ *approaches* the set D if

$$\lim_{t\to\infty} \text{dist}(x(t), D) = 0.$$

This concept is well defined, being independent of the norm used. If $\epsilon > 0$, we define the *ϵ-neighborhood* of a nonempty set $D \subset \mathbf{R}^n$ by

$$B_\epsilon(D) := \{y \in \mathbf{R}^n : \text{dist}(y, D) < \epsilon\}.$$

For a single point $a \in \mathbf{R}^n$, we simply write $B_\epsilon(a)$ rather than $B_\epsilon(\{a\})$; thus,

$$B_\epsilon(a) := \{x : \|x - a\| < \epsilon\}$$

is the open ball of radius ϵ centered at the point a.

A *matrix norm* is a function $\|\cdot\|$ on the vector space of $n \times n$ matrices which satisfies, in addition to the properties of Definition 2.8, the property that

(d) $\|AB\| \leq \|A\|\,\|B\|$ for any two $n \times n$ matrices A and B.

A *matrix norm compatible with a given vector norm* $\|x\|$ is a matrix norm $\|\cdot\|$ on the $n \times n$ matrices which satisfies, in addition, the compatibility property

(e) $\|Ax\| \leq \|A\|\,\|x\|$ for any $n \times n$ matrix A and vector x.

The *matrix norm induced by a given vector norm* is defined as follows. Let $\|x\|$ denote a given vector norm. This norm induces a matrix norm on $\mathbf{C}^{n\times n}$, given by

$$\|A\| := \max_{\|x\|\leq 1} \|Ax\|, \tag{2.1}$$

where $\|Ax\|$ is the given vector norm of the image vector Ax. This does indeed define a norm on the space $\mathbf{C}^{n\times n}$ compatible with the given vector norm. For a matrix norm, it is straightforward to show by induction that $\|A^k\| \leq \|A\|^k$ for every positive integer k.

The Absolute Sum Norms. For some estimates needed later on, we choose to work with the vector norm defined by

$$\|x\| = \sum_{j=1}^{n} |x_j| = |x_1| + \cdots + |x_n|. \tag{2.2}$$

In some references this norm is denoted $\|x\|_1$, but we will not use the subscript. We will also work with the matrix norm defined by

$$\|A\| = \sum_{i,j=1}^{n} |a_{ij}|, \quad \text{where } A = [a_{ij}]. \tag{2.3}$$

We leave as an exercise the verification that (2.2) defines a vector norm and (2.3) defines a matrix norm. The matrix norm (2.3) is not induced by the absolute sum norm (2.2); however, we now show the compatibility of (2.3) with (2.2). We have

$$\begin{aligned} \|Ax\| &= \sum_{i=1}^{n} |(Ax)_i| \\ &= \sum_{i=1}^{n} \Big| \sum_{j=1}^{n} a_{ij} x_j \Big| \\ &\le \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}| \; |x_j|. \end{aligned}$$

Since $|x_j| \le \|x\|$ for each j, we have $\|Ax\| \le \|A\| \; \|x\|$, as we wanted to show.

Finally, we note that, using the absolute sum norms for vector functions and matrix functions, for any real interval $[a, b]$ we have

$$\left\| \int_a^b f(t)\, dt \right\| \le \int_a^b \|f(t)\|\, dt$$

for a vector function $f : [a, b] \to \mathbf{R}^n$, and

$$\left\| \int_a^b A(t)\, dt \right\| \le \int_a^b \|A(t)\|\, dt$$

for a matrix function $A : [a, b] \to \mathbf{R}^n$. These estimates can be useful in establishing the convergence of certain improper integrals of matrix functions.

Sequences and Series of Functions

The Cauchy-Schwartz inequality states that $|v^T w| \le \|v\|_2 \; \|w\|_2$ for any vectors $v, w \in \mathbf{R}^n$. If e_i denotes the i-th standard basis vector in $\mathbf{R}^n$, and $A = [a_{ij}]$ for $A \in \mathbf{R}^{n \times n}$, then

$$|a_{ij}| = |e_i^T A e_j| \le \|e_i\|_2 \; \|A e_j\|_2 \le \|A\|_2, \tag{2.4}$$

where $\|A\|_2$ is the matrix norm induced by the Euclidean vector norm. It follows from (2.4) that every Cauchy sequence of matrices A_k in $\mathbf{R}^{n\times n}$ must converge to a matrix $A \in \mathbf{R}^{n\times n}$. This last fact is also transparent from the matrix norm in (2.3). That is, if $A_k \in \mathbf{R}^{n\times n}$ is a sequence with the property that for every $\epsilon > 0$, there exists an $N(\epsilon) > 0$ such that $m, n > N(\epsilon)$ implies $\|A_m - A_n\| < \epsilon$, then there is a matrix $A \in \mathbf{R}^{n\times n}$ such that $\lim_{k\to\infty} A_k = A$, that is, $\lim_{k\to\infty} \|A_k - A\| = 0$. Thus, with any matrix norm, the space $\mathbf{R}^{n\times n}$ is a complete normed vector space.

Definition 2.11 (Absolute Convergence)
Let V be a normed vector space with norm $\|\cdot\|$. The infinite series $\sum_{k=1}^{\infty} a_k$, $a_k \in V$, is absolutely convergent *if the series $\sum_{k=1}^{\infty} \|a_k\|$ of nonnegative real numbers converges.*

The next lemma is used later on in the discussion of the matrix exponential.

Lemma 2.3 (Absolute Convergence Implies Convergence)
Let V be a complete normed vector space. If the infinite series $\sum_{k=1}^{\infty} a_k$, $a_k \in V$, is absolutely convergent, then it converges in the norm on V to a limit $s \in V$, that is,

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = s.$$

Proof. See Exercise 2.7. □

It is useful to recall the definition of uniform convergence of sequences and series of functions. This property is important because it allows us to interchange limit processes of calculus in certain situations.

Definition 2.12 (Uniform Convergence of Sequences and Series)
Let the functions s_j, f_j, $j \geq 1$, be defined on a common domain $D \subseteq \mathbf{R}^n$, and suppose that s_j, f_j all take values in $\mathbf{R}^m$.

(a) *The sequence s_j* converges uniformly on D *to a function $s : D \to \mathbf{R}^m$ if, given any $\epsilon > 0$, there is an $N(\epsilon) > 0$ such that*

$$j \geq N(\epsilon) \implies \|s_j(x) - s(x)\| < \epsilon \quad \text{for all } x \in D.$$

(b) *The series $\sum_{j=1}^{\infty} f_j$* converges uniformly on D *to a function $f : D \to \mathbf{R}^m$ if the sequence of partial sums*

$$s_n := \sum_{j=1}^{n} f_j$$

converges uniformly on D to f.

The Weierstrass M-test is useful in showing that a series of functions converges uniformly, when the terms in the series satisfy appropriate bounds.

Lemma 2.4 (Weierstrass M-Test for Uniform Convergence)
Let the sequence of functions f_j, $j \geq 1$, be defined on a common domain D in $\mathbf{R}^n$, with common range space $\mathbf{R}^m$. Suppose that each f_j satisfies a bound of the form

$$\|f_j(x)\| \leq M_j \quad \text{for all } x \in D,$$

where the M_j are fixed numbers. If the series of the M_j converges, that is,

$$\sum_{j=1}^{\infty} M_j < \infty,$$

then the series of functions

$$\sum_{j=1}^{\infty} f_j(x)$$

converges uniformly on D.

Proof. By the boundedness hypothesis, we can invoke Lemma 2.3 and conclude that the series $\sum_{j=1}^{\infty} f_j$ converges pointwise to a limit function f which is defined on D. It remains to show that the series converges uniformly to f on D. Define the sequence S_n of partial sums of the series by

$$S_n(x) = \sum_{j=1}^{n} f_j(x), \quad x \in D.$$

We want to show that the sequence S_n converges uniformly on D, and to do so we may work with any norm on the range space $\mathbf{R}^m$. Let T_n be the sequence of partial sums of the series $\sum_{j=1}^{\infty} M_j$, and note that $T_n \geq 0$ for each n. Given $\epsilon > 0$, there is a number $N(\epsilon) > 0$ such that

$$m > n > N(\epsilon) \implies \sum_{j=n+1}^{m} M_j = T_m - T_n < \frac{\epsilon}{2}.$$

Thus, for $m > n > N(\epsilon)$ and all $x \in D$, the partial sums $S_n(x)$ satisfy

$$\begin{aligned}
\|S_m(x) - S_n(x)\| &= \left\| \sum_{j=n+1}^{m} f_j(x) \right\| \\
&\leq \sum_{j=n+1}^{m} \|f_j(x)\| \\
&\leq \sum_{j=n+1}^{m} M_j \\
&< \frac{\epsilon}{2}.
\end{aligned}$$

Fix $x \in D$ and let $m \to \infty$. By the continuity of the norm (Lemma 2.2), for any fixed $n > N(\epsilon)$ we have

$$\lim_{m\to\infty} \|S_m(x) - S_n(x)\| = \|f(x) - S_n(x)\| \leq \frac{\epsilon}{2} < \epsilon.$$

Since x in D was fixed but arbitrary, we conclude that if $n > N(\epsilon)$, then $\|f(x) - S_n(x)\| < \epsilon$ for all $x \in D$. Thus the sequence S_n converges uniformly to f on D. This completes the proof. □

Quadratic Forms

The Euclidean norm $\|x\|_2$ is associated with the quadratic form $x_1^2 + \cdots + x_n^2$, because

$$\|x\|_2 = (x_1^2 + \cdots + x_n^2)^{1/2}.$$

Vector norms defined by more general positive definite quadratic forms are very convenient in discussions of stability of equilibria.

Definition 2.13 *Let P be a symmetric matrix in $\mathbf{R}^{n\times n}$, that is, $P^T = P$.*

(a) P *is* positive definite *if $x^T Px > 0$ for every nonzero vector x in $\mathbf{R}^n$.*
(b) P *is* positive semidefinite *if $x^T Px \geq 0$ for every x in $\mathbf{R}^n$.*
(c) P *is* negative definite *if $x^T Px < 0$ for every nonzero vector x in $\mathbf{R}^n$.*
(d) P *is* negative semidefinite *if $x^T Px \leq 0$ for every x in $\mathbf{R}^n$.*
(e) P *is* indefinite *if none of the conditions (a)–(d) hold.*

Note that a matrix Q is negative definite if and only if $-Q$ is positive definite, and that Q is negative semidefinite if and only if $-Q$ is positive semidefinite.

Suppose P is symmetric positive definite, and define

$$\|x\|_P := (x^T Px)^{\frac{1}{2}}.$$

It is straightforward to show that this defines a norm on $\mathbf{R}^n$. In fact, $B(u, v) = u^T Pv$ defines an inner product on $\mathbf{R}^n$. There is no loss of generality in specifying symmetric matrices in Definition 2.13, when defining quadratic forms; see Exercise 2.4.

Recall that every real symmetric matrix P can be diagonalized by a real *orthogonal* matrix S, that is $S^T S = I$, hence $S^T = S^{-1}$, and

$$S^T PS = \text{diag}\,[\lambda_1, \ldots, \lambda_n],$$

where λ_i, $1 \leq i \leq n$, are the eigenvalues of P. (See [75] (page 549).) The diagonalizing transformation S is given by $S = [v_1 \cdots v_n]$, where $\{v_1, \ldots, v_n\}$ is an *orthornormal basis* of $\mathbf{R}^n$ consisting of eigenvectors of P. By definition of orthonormal basis, $v_1, \ldots, v_n$ are linearly independent and satisfy $\|v_i\|_2^2 = v_i^T v_i = 1$, $1 \leq i \leq n$, and $v_i^T v_j = 0$ for $i \neq j$.

There are two criteria for positive definiteness that are useful to remember, especially when dealing with small size matrices. Recall that the *leading* *principal minors* of an $n \times n$ matrix P are the determinants of the $k \times k$ upper left submatrices of P, for $k = 1, \ldots, n$.

Proposition 2.2 *Let P be a real symmetric matrix. The following are equivalent:*

(a) *P is positive definite.*
(b) *All eigenvalues of P are positive.*
(c) *All leading principal minors of P are positive.*

Proof. We prove only the equivalence of (a) and (b). There is a real orthogonal S such that $S^{-1}PS = \operatorname{diag}[\lambda_1, \ldots, \lambda_n]$, with the eigenvalues of P, which are necessarily real, on the main diagonal. If $x = Sz$, then the quadratic form $x^T P x$ is, in z coordinates, $\sum_k \lambda_k z_k^2$, and the equivalence of (a) and (b) follows. A proof of the equivalence of (a) and (c) may be found in [75] (pages 558–559), or in K. Hoffman and R. Kunze, *Linear Algebra*, Prentice-Hall, Englewood Cliffs, NJ, second edition, 1971 (pages 328–329). □

Example 2.6 Consider the matrix

$$P = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 4 & -2 \\ 0 & -2 & 5 \end{bmatrix}.$$

The three principal minors of P are

$$\det[3] = 3 > 0, \quad \det \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix} = 8 > 0, \quad \det P = 3(16) - 2(10) = 28 > 0,$$

so P is positive definite. On the other hand, the matrix

$$Q = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -4 & 2 \\ 0 & 2 & -5 \end{bmatrix}$$

has a negative 2×2 principal minor, so Q is not positive definite. Notice that Q cannot be negative definite either, because of the first principal minor. Q is indefinite. As an alternative argument, we might observe that Q has eigenvalues 3, λ_2, and λ_3, with $\lambda_2 \lambda_3 = 16$ and $\lambda_2 + \lambda_3 = -9$. Therefore the symmetric Q must have a negative eigenvalue, and therefore cannot be positive definite. △

It is useful to know that a real symmetric matrix P is positive semidefinite if and only if all eigenvalues of P are nonnegative. Suppose that R is a real $n \times n$ matrix. Then $R^T R$ is symmetric positive semidefinite: For every real vector x, $x^T R^T R x = \|Rx\|^2 \geq 0$. The next lemma provides a converse.

Lemma 2.5 *Let* $Q \in \mathbf{R}^{n\times n}$ *be symmetric positive semidefinite. Then there exists a positive semidefinite matrix* $R \in \mathbf{R}^{n\times n}$ *such that*

$$Q = R^T R.$$

If Q *is positive definite, then* R *can be chosen positive definite.*

Proof. Since Q is symmetric, it is diagonalizable by a real orthogonal matrix S. Write $S^T Q S = D = \operatorname{diag}[\lambda_1, \ldots, \lambda_n]$, and define

$$R = S \operatorname{diag}[\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n}]\, S^T =: S D^{\frac{1}{2}} S^T.$$

Then R is symmetric and

$$\begin{aligned} R^T R &= (SD^{\frac{1}{2}}S^T)(SD^{\frac{1}{2}}S^T) \\ &= SDS^T \\ &= Q. \end{aligned}$$

Clearly, $z^T R z \geq 0$ for all vectors z, so R is positive semidefinite. By this construction, it is clear that if Q is positive definite then so is R. □

A matrix $Q \geq 0$ can have other factorizations $Q = C^T C$, where C need not be positive semidefinite. For example, consider the factorization

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} =: C^T C.$$

2.4 ORDINARY DIFFERENTIAL EQUATIONS

In this section we state the basic theorem on the existence and uniqueness of solutions of initial value problems for ordinary differential equations and give examples to illustrate the theorem. We define the Jacobian linearization of a system at a point. The section concludes with examples of linear and nonlinear systems in the plane.

Existence and Uniqueness Theorem

Consider the initial value problem for a system of ordinary differential equations,

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{2.5}$$

where $f : D \mapsto \mathbf{R}^n$ is a C^1 (continuously differentiable) vector field defined on an open set $D \subset \mathbf{R}^{n+1}$ and $\dot{x} = \frac{dx}{dt}$. A *solution* is a differentiable function $x(t)$ that satisfies (2.5) on some real interval J containing t_0. We say that system (2.5) is *autonomous* or *time invariant* if the vector field f does not depend explicitly on time t; otherwise the system is nonautonomous (time varying). For autonomous systems we usually take the initial time to be $t_0 = 0$.

In order to guarantee that initial-value problems have a unique solution, some local growth restriction must be imposed on the vector field f in (2.5).

Definition 2.14 (Locally Lipschitz Vector Field)
Let D be an open set in $\mathbf{R}^{n+1}$. A function $f : D \mapsto \mathbf{R}^n$, denoted $f(t,x)$ with $t \in \mathbf{R}$ and $x \in \mathbf{R}^n$, is locally Lipschitz *in x on D if for any point $(t_0, x_0) \in D$, there are an open ball $B_r(t_0, x_0)$ about (t_0, x_0) and a number L such that*

$$\|f(t,x_1) - f(t,x_2)\| \leq L\|x_1 - x_2\|$$

for all (t, x_1), (t, x_2) in $B_r(t_0, x_0) \cap D$.

If f has continuous first order partial derivatives with respect to all components of x at every point of $D \subseteq \mathbf{R}^{n+1}$, then f is locally Lipschitz in x on D. (See Theorem B.5 and the discussion that follows it.)

Theorem 2.3 (Existence and Uniqueness)
Let D be an open set in $\mathbf{R}^{n+1}$. If $f : D \mapsto \mathbf{R}^n$ is locally Lipschitz on D, then, given any $x_0 \in D$ and any $t_0 \in \mathbf{R}$, there exists a $\delta > 0$ such that the initial value problem (2.5) *has a unique solution $x(t, t_0, x_0)$ defined on the interval $[t_0 - \delta, t_0 + \delta]$; that is, if $z(t) := x(t, t_0, x_0)$, then*

$$\frac{d}{dt} z(t) = f(t, z(t)), \quad \text{for } t \in [t_0 - \delta, t_0 + \delta],$$

and $z(t_0) = x(t_0, t_0, x_0) = x_0$.

Proof. See Appendix C on ordinary differential equations. □

Solutions of locally Lipschitz systems can always be extended to a maximal interval of existence $(t_{\min}(x_0), t_{\max}(x_0))$, which depends on x_0 and f. In some cases, a final extension may be made to include one or both endpoints $t_{\min}(x_0)$, $t_{\max}(x_0)$, when these are finite numbers. A solution will normally mean the unique solution determined by an initial condition, with domain given by the maximal interval assured by the extension of solutions. (See the discussion of extension of solutions and Theorem C.3 in the Appendix.) If all solutions exist for all forward times $t \geq 0$, then the system is said to be *forward complete*. If all solutions are defined for $t \in (-\infty, \infty)$, then the system is *complete*.

We consider a few scalar autonomous differential equations in order to illustrate Theorem 2.3. We often write $\phi_t(x_0) = x(t, t_0, x_0)$ for the solution of an initial value problem.

Example 2.7 The linear initial value problem $\dot{x} = ax$, $x(0) = x_0 \in \mathbf{R}$, has unique solution $x(t) = e^{at} x_0$ defined on the whole real line. △

Example 2.8 Consider the initial value problem $\dot{x} = x^2$, $x(0) = x_0 \in \mathbf{R}$. After separating the variables, a direct integration gives the following unique and maximally defined solutions:

$$\phi_t(x_0) = \frac{x_0}{1 - x_0 t} \quad \text{for} \quad -\infty < t < \frac{1}{x_0} \quad \text{if} x_0 > 0,$$