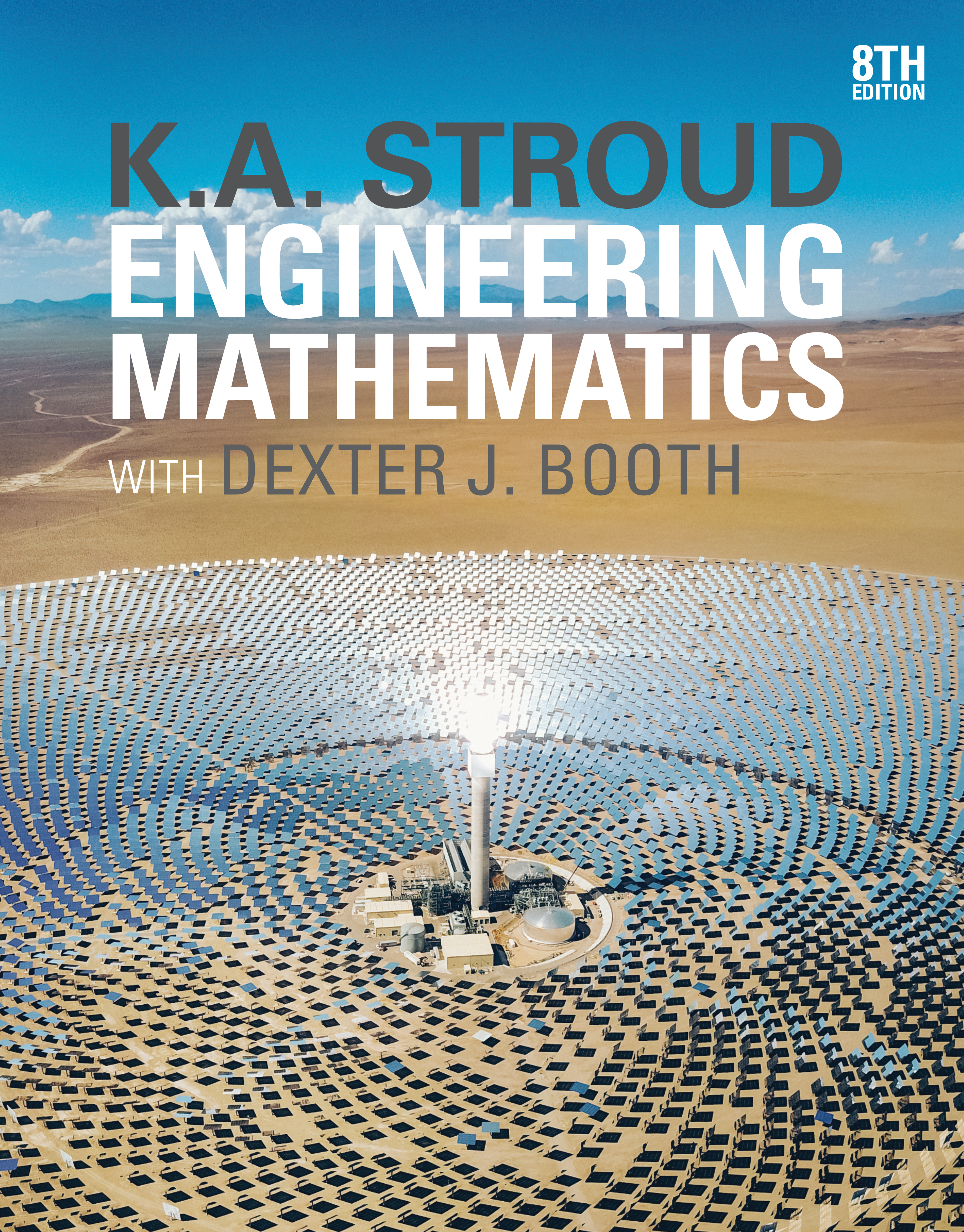
8TH
EDITION

K.A. STROUD
ENGINEERING MATHEMATICS

WITH DEXTER J. BOOTH



ENGINEERING MATHEMATICS

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Preface to the first edition

The purpose of this book is to provide a complete year's course in mathematics for those studying in the engineering, technical and scientific fields. The material has been specially written for courses leading to

(i) Part I of B.Sc. Engineering Degrees,
(ii) Higher National Diploma and Higher National Certificate in technological subjects, and for other courses of a comparable level. While formal proofs are included where necessary to promote understanding, the emphasis throughout is on providing the student with sound mathematical skills and with a working knowledge and appreciation of the basic concepts involved. The programmed structure ensures that the book is highly suited for general class use and for individual self-study, and also provides a ready means for remedial work or subsequent revision.

The book is the outcome of some eight years' work undertaken in the development of programmed learning techniques in the Department of Mathematics at the Lanchester College of Technology, Coventry. For the past four years, the whole of the mathematics of the first year of various Engineering Degree courses has been presented in programmed form, in conjunction with seminar and tutorial periods. The results obtained have proved to be highly satisfactory, and further extension and development of these learning techniques are being pursued.

Each programme has been extensively validated before being produced in its final form and has consistently reached a success level above 80/80, i.e. at least 80% of the students have obtained at least 80% of the possible marks in carefully structured criteria tests. In a research programme, carried out against control groups receiving the normal lectures, students working from programmes have attained significantly higher mean scores than those in the control groups and the spread of marks has been considerably reduced. The general pattern has also been reflected in the results of the sessional examinations.

The advantages of working at one's own rate, the intensity of the student involvement and the immediate assessment of responses, are well known to those already acquainted with programmed learning activities. Programmed learning in the first year of a student's course at a college or university provides the additional advantage of bridging the gap between the rather highly organised aspect of school life and the freer environment and greater personal responsibility for his own progress which faces every student on entry to the realms of higher education.

Acknowledgement and thanks are due to all those who have assisted in any way in the development of the work, including whose who have been actively engaged in validation processes. I especially wish to record my sincere thanks for the continued encouragement and support which I received from my present head of Department at the College, Mr. J.E. Sellars, M.Sc., A.F.R.Ae.S., F.I.M.A., and also from Mr. R. Wooldridge, M.C., B.Sc., F.I.M.A., formerly Head of Department, now Principal of Derby College of Technology. Acknowledgement is also made of the many sources, too numerous to list, from which the selected examples quoted in the programmes have been gleaned over the years. Their inclusion contributes in no small way to the success of the work.

K.A. Stroud

Preface to the second edition

The continued success of *Engineering Mathematics* since its first publication has been reflected in the number of courses for which it has been adopted as the official class text and also in the correspondence from numerous individuals who have welcomed the self-instructional aspects of the work.

Over the years, however, syllabuses of existing courses have undergone some modification and new courses have been established. As a result, suggestions have been received from time to time requesting the inclusion of further programme topics in addition to those already provided as core material for relevant undergraduate and comparable courses. Unlimited expansion of the book to accommodate all the topics requested is hardly feasible on account of the physical size of the book and the commercial aspects of production. However, in the light of these representations and as a result of further research undertaken by the author and the publishers, it has now been found possible to provide a new edition of *Engineering Mathematics* incorporating three of the topics for which there is clearly a wide demand.

The additional programmes cover the following topics:

(a) Matrices: definitions; types of matrices; operations; transpose; inverse; solution of linear equations; eigenvalues and eigenvectors.
(b) Curves and curve fitting: standard curves; asymptotes; systematic curve sketching; curve recognition; curve fitting; method of least squares.
(c) Statistics: discrete and continuous data; grouped data; frequency and relative frequency; histograms; central tendency – mean, mode and median; coding; frequency polygons and frequency curves; dispersion – range, variance and standard deviation; normal distribution and standardised normal curve.

The three new programmes follow the structure of the previous material and each is provided with numerous worked examples and exercises. As before, each programme concludes with a short Test Exercise for self-assessment and set of Further Problems provides valuable extra practice. A complete set of answers is available at the end of the book.

Advantage has also been taken during the revision of the book to amend a small number of minor points in other parts of the text and it is anticipated that, in its new updated form, the book will have an even greater appeal and continue to provide a worthwhile service.

K.A. Stroud

Preface to the third edition

Following the publication of the enlarged second edition of *Engineering Mathematics*, which included a programme on the introduction to Statistics, requests were again received for an associated programme on Probability. This has now been incorporated as Programme XXVIII of the current third edition of the book.

The additional programme follows the established pattern and structure of the previous sections of the text, including the customary worked examples through which the student is guided with progressive responsibility and concluding with the Text Exercise and set of Further Problems for essential practice. Answers to all problems are provided. The opportunity has also been taken to make one or two minor modifications to the remainder of the text.

Engineering Mathematics, originally devised as a first year mathematics course for engineering and science degree undergraduates and students of comparable courses, is widely sought both for general class use and for individual study. A companion volume and sequel, *Further Engineering Mathematics*, dealing with core material of a typical second/third year course, is also now available through the normal channels. The two texts together provide a comprehensive and integrated course of study and have been well received as such.

My thanks are due, once again, to the publishers for their ready cooperation and helpful advice in the preparation of the material for publication.

K.A.S.

Preface to the fourth edition

Since the publication of the third edition of *Engineering Mathematics*, considerable changes in the syllabus and options for A-level qualifications in Mathematics have been introduced nationally, as a result of which numbers of students with various levels of mathematical background have been enrolling for undergraduate courses in engineering and science. In view of the widespread nature of the situation, requests have been received from several universities for the inclusion in the new edition of *Engineering Mathematics* of what amounts to a bridging course of material in relevant topics to ensure a solid foundation on which the main undergraduate course can be established.

Accordingly, the fourth edition now includes ten new programmes ranging from Number Systems and algebraic processes to an introduction to the Calculus. These Foundation Topics constitute Part I of the current book and precede the well-established section of the text now labelled as Part II.

For students already well versed in the contents of the Part I programmes the Test Exercises and Further Problems afford valuable revision and should not be ignored.

With the issue of the new edition, the publishers have undertaken the task of changing the format of the pages and of resetting the whole of the text to provide a more open presentation and improved learning potential for the student.

Once again, I am indebted to the publishers for their continued support and close cooperation in the preparation of the text in its new form and particularly to all those directly involved in the editorial, production and marketing processes both at home and overseas.

K.A.S.

Preface to the eighth edition

Engineering Mathematics by Ken Stroud has been a favoured textbook of science and engineering students for 50 years and to have been asked to contribute to the fifth edition of this remarkable book gave me great pleasure and no little trepidation. And now, twenty years later, we have the eighth edition.

A clear requirement of any additions or changes to such a well established textbook is the retention of the very essence of the book that has contributed to so many students' mathematical abilities over the years. In line with this I have taken great care to preserve the time-tested Stroud format and close attention to technique development throughout the book which have made *Engineering Mathematics* the tremendous success it is. The largest part of my work for previous editions was to re-structure, re-organize and expand the Foundation section as well as the addition of a Programme on Laplace Transforms. For this new edition there has been a general revision of the entire contents with amendments and corrections as required. In the Foundation section a number of frames have been deleted and/or replaced in *Introduction to algebra*, *Expressions and equations* and *Polynomial equations*. In Part II amendments have been made to *Curves and curve fitting* and *Probability*. Further work has been covered in *Matrices* with an emphasis on eigenvalues and eigenvectors and the introduction of the Cayley–Hamilton theorem which has been moved from the sister text *Advanced Engineering Mathematics*. New Review summaries have been introduced and each item in each Review summary has been annotated with the relevant frame number in square braces to make reviewing the subject material more easy to access.

The Personal Tutor, which can be accessed at www.macmillanihe.com/stroud, provides the answers and working of a large number of questions selected from the book for which, in the text, no worked solutions are given. Using the Personal Tutor the reader is closely guided through the solutions to these questions, so confirming and adding to skills in mathematical techniques and knowledge of mathematical ideas. In addition there is a collection of problems set within engineering and scientific contexts available at the website above. Lecturers will also find Powerpoint™ lecture slides which include all the figures and graphs from the book.

The work involved in creating a new edition of an established textbook is always a cooperative, team effort and this book is no exception. I was fortunate enough to be able to meet Ken Stroud in the few months before he died and to be able to discuss with him ideas for the fifth edition. He was very enthusiastic about taking the book forward with new technology and both his eagerness and his concerns have been taken into account in the development of the Personal Tutor. The enormous task which Ken undertook in writing the original book and three subsequent editions cannot be underestimated. Ken's achievement was an extraordinary one and it has been a great privilege for us all to be able to work on such a book. I should like to thank the Stroud family for their support in my work for this new edition and the editorial team for their close attention to detail, their appropriate comments on the text and the assiduous checking of everything that I have written. As with any team the role of the leader is paramount and I should particularly like to thank my Editor Helen Bugler whose continued care over this project inspires us all.

Huddersfield Dexter J. Booth
January 2020

How to use this book

This book contains forty-two lessons called *Programmes*. Each Programme has been written in such a way as to make learning more effective and more interesting. It is like having a personal tutor because you proceed at your own rate of learning and any difficulties you may have are cleared before you have the chance to practise incorrect ideas or techniques.

You will find that each Programme is divided into numbered sections called *frames*. When you start a Programme, begin at Frame 1. Read each frame carefully and carry out any instructions or exercise that you are asked to do. In almost every frame, you are required to make a response of some kind, testing your understanding of the information in the frame, and you can immediately compare your answer with the correct answer given in the next frame. *To obtain the greatest benefit, you are strongly advised to cover up the following frame until you have made your response*. When a series of dots occurs, you are expected to supply the missing word, phrase, number or mathematical expression. At every stage you will be guided along the right path. There is no need to hurry: read the frames carefully and follow the directions exactly. In this way, you must learn.

Each Programme opens with a list of **Learning outcomes** which specify exactly what you will learn by studying the contents of the Programme. The Programme ends with a matching checklist of **Can You?** questions that enables you to rate your success in having achieved the **Learning outcomes**. If you feel sufficiently confident then tackle the short **Test exercise** which follows. This is set directly on what you have learned in the Programme: the questions are straightforward and contain no tricks. To provide you with the necessary practice, a set of **Further problems** is also included: do as many of these problems as you can. Remember, that in mathematics, as in many other situations, practice makes perfect – or more nearly so.

Of the forty-two Programmes, the first thirteen are at Foundation level. Some of these will undoubtedly contain material with which you are already familiar. However, read the Programme's **Learning outcomes** and if you feel confident about them try the **Quiz** that immediately follows – you will soon find out if you need a refresher course. Indeed, even if you feel you have done some of the topics before, it would still be worthwhile to work steadily through the Programme: it will serve as useful revision and fill any gaps in your knowledge that you may have.

When you have come to the end of a Foundation level Programme and have rated your success in achieving the **Learning outcomes** using the **Can You?** checklist, go back to the beginning of the Programme and try the **Quiz** before you complete the Programme with the **Test exercise** and the **Further problems**. This way you will get even more practice.

The Personal Tutor

Alongside this book is an interactive Personal Tutor program that contains a bank of questions for you to answer using your computer. In response to comments from our many readers the Personal Tutor has been re-designed and updated from the previous edition. It is extremely intuitive and easy to navigate and offers **Hints** whenever you need them. If you are having difficulties you can click on either **Check your answers** for a particular **Step** or **Show full working**. There are no scores for the questions and you are guided every inch of the way without having to worry about getting any answers wrong. Using the Personal Tutor will give you more practice and increase your confidence in your learning of the mathematics. As with the exercises in the book, take your time, make mistakes, correct them using all the assistance available to you and you will surely learn.

The bank consists of odd numbered questions from the **Quizzes**, **Test exercises** and **Further problems**. A PERSONAL TUTOR symbol next to an exercise in the book indicates that it is also on the program.

The Personal Tutor is available online through the companion website.

The companion website – www.macmillanihe.com/stroud

You are recommended to visit the book's website at www.macmillanihe.com/stroud. There you will find a growing resource to accompany the text including mathematical questions set within engineering and scientific contexts. From here you can also access the Personal Tutor.

Student review panel

The author and publisher are grateful to students from the following UK universities who acted as chapter reviewers and made many useful suggestions:

From the University of Exeter: Max Bailey, Oliver Chapman, Ben Lawrence and Rhiannon Roberts

From Imperial College London: Matei C. Ignuta-Ciuncanu, Tuan Le, Khoo Hwei Linn, Xinwei Sophie Peng and Eugene Tang Yucheng

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From Liverpool John Moores University: Katie Egerton and Hannah Kowalski

From Sheffield Hallam University: Molly R. Bell, Layla J.M. Graham and Nathan Prentice

Useful background information

Symbols used in the text

$=$	is equal to	$\rightarrow$	tends to
$\simeq$	is approximately equal to	$\neq$	is not equal to
$>$	is greater than	$\equiv$	is identical to
$\geq$	is greater than or equal to	$<$	is less than
$n!$	factorial $n = 1 \times 2 \times 3 \times \ldots \times n$	$\leq$	is less than or equal to
$\lvert k \rvert$	modulus of k, i.e. size of k irrespective of sign	∞	infinity
$\sum$	summation	$\mathop{Lim}\limits_{n\to\infty}$	limiting value as $n \rightarrow \infty$

Useful mathematical information

1 Algebraic identities

$$(a+b)^2 = a^2 + 2ab + b^2 \qquad (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a-b)^2 = a^2 - 2ab + b^2 \qquad (a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$$

$$a^2 - b^2 = (a-b)(a+b) \qquad a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

2 Trigonometrical identities

(a) $\sin^2\theta + \cos^2\theta = 1;\ \sec^2\theta = 1 + \tan^2\theta;\ \mathrm{cosec}^2\theta = 1 + \cot^2\theta$

(b) $\sin(A+B) = \sin A \cos B + \cos A \sin B$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

(c) Let $A = B = \theta \ \therefore \ \sin 2\theta = 2\sin\theta\cos\theta$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

(d) Let $\theta = \frac{\phi}{2}$ $\therefore$ $\sin\phi = 2\sin\frac{\phi}{2}\cos\frac{\phi}{2}$

$$\cos\phi = \cos^2\frac{\phi}{2} - \sin^2\frac{\phi}{2} = 1 - 2\sin^2\frac{\phi}{2} = 2\cos^2\frac{\phi}{2} - 1$$

$$\tan\phi = \frac{2\tan\frac{\phi}{2}}{1 - 2\tan^2\frac{\phi}{2}}$$

(e) $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$

$\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$

$\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

$\cos D - \cos C = 2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$

(f) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$2\cos A\sin B = \sin(A+B) - \sin(A-B)$

$2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$2\sin A\sin B = \cos(A-B) - \cos(A+B)$

(g) Negative angles: $\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta$

(h) Angles having the same trigonometrical ratios:

(i) Same sine: θ and $(180^\circ - \theta)$

(ii) Same cosine: θ and $(360^\circ - \theta)$, i.e. $(-\theta)$

(iii) Same tangent: θ and $(180^\circ + \theta)$

(i) $a\sin\theta + b\cos\theta = A\sin(\theta + \alpha)$

$a\sin\theta - b\cos\theta = A\sin(\theta - \alpha)$

$a\cos\theta + b\sin\theta = A\cos(\theta - \alpha)$

$a\cos\theta - b\sin\theta = A\cos(\theta + \alpha)$

$$\text{where} \begin{cases} A = \sqrt{a^2 + b^2} \\ \alpha = \tan^{-1}\dfrac{b}{a} \quad (0^\circ < \alpha < 90^\circ) \end{cases}$$

3 Standard curves

(a) *Straight line*

Slope, $m = \frac{dy}{dx} = \frac{y_2 - y_1}{x_2 - x_1}$

Angle between two lines, $\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$

For parallel lines, $m_2 = m_1$

For perpendicular lines, $m_1 m_2 = -1$

Equation of a straight line (slope $= m$)

(i) Intercept c on real y-axis: $y = mx + c$

(ii) Passing through (x_1, y_1): $y - y_1 = m(x - x_1)$

(iii) Joining (x_1, y_1) and (x_2, y_2): $\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$

(b) *Circle*

Centre at origin, radius r: $x^2 + y^2 = r^2$

Centre (h, k), radius r: $(x-h)^2 + (y-k)^2 = r^2$

General equation: $x^2 + y^2 + 2gx + 2fy + c = 0$

with centre $(-g, -f)$: radius $= \sqrt{g^2 + f^2 - c}$

Parametric equations: $x = r\cos\theta,\ y = r\sin\theta$

(c) *Parabola*

Vertex at origin, focus $(a, 0)$: $y^2 = 4ax$

Parametric equations : $x = at^2,\ y = 2at$

(d) *Ellipse*

Centre at origin, foci $\left(\pm\sqrt{a^2+b^2},\ 0\right)$: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

where $a =$ semi-major axis, $b =$ semi-minor axis

Parametric equations: $x = a\cos\theta,\ y = b\sin\theta$

(e) *Hyperbola*

Centre at origin, foci $\left(\pm\sqrt{a^2+b^2},\ 0\right)$: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

Parametric equations: $x = a\sec\theta,\ y = b\tan\theta$

Rectangular hyperbola:

Centre at origin, vertex $\pm\left(\dfrac{a}{\sqrt{2}}, \dfrac{a}{\sqrt{2}}\right)$: $xy = \dfrac{a^2}{2} = c^2$

i.e. $xy = c^2$ where $c = \dfrac{a}{\sqrt{2}}$

Parametric equations: $x = ct,\ y = c/t$

4 Laws of mathematics

(a) *Associative laws* – for addition and multiplication

$a + (b + c) = (a + b) + c$

$a(bc) = (ab)c$

(b) *Commutative laws* – for addition and multiplication

$a + b = b + a$

$ab = ba$

(c) *Distributive laws* – for multiplication and division

$a(b + c) = ab + ac$

$\dfrac{b+c}{a} = \dfrac{b}{a} + \dfrac{c}{a}$ (provided $a \neq 0$)

PART I

Foundation topics

Frames 1 to 153

Programme F.1

Arithmetic

Learning outcomes

When you have completed this Programme you will be able to:

☐ Carry out the basic rules of arithmetic with integers
☐ Check the result of a calculation making use of rounding
☐ Write a whole as a product of prime numbers
☐ Find the highest common factor and lowest common multiple of two whole numbers
☐ Manipulate fractions, ratios and percentages
☐ Manipulate decimal numbers
☐ Manipulate powers
☐ Use standard or preferred standard form and complete a calculation to the required level of accuracy
☐ Understand the construction of various number systems and convert from one number system to another

If you already feel confident about these why not try the quiz over the page?
You can check your answers at the end of the book.

Quiz F.1

Questions marked with this icon can be found at www.macmillanihe.com/stroud. There you can go through the question step by step and follow online hints, with full working provided.

Frames

1 Place the appropriate symbol < or > between each of the following pairs of numbers:
(a) $-3 \quad -2$ (b) $8 \quad -13$ (c) $-25 \quad 0$
1 to 4

2 Find the value of each of the following:
(a) $13 + 9 \div 3 - 2 \times 5$ (b) $(13 + 9) \div (3 - 2) \times 5$
5 to 12

3 Round each number to the nearest 10, 100 and 1000:
(a) 1354 (b) 2501 (c) −2452 (d) −23 625
13 to 15

4 Write each of the following as a product of prime factors:
(a) 170 (b) 455 (c) 9075 (d) 1140
19 to 22

5 Find the HCF and the LCM of each pair of numbers:
(a) 84, 88 (b) 105, 66
23 to 24

6 Reduce each of the following fractions to their lowest terms:
(a) $\dfrac{12}{18}$ (b) $\dfrac{144}{21}$ (c) $-\dfrac{49}{14}$ (d) $\dfrac{64}{4}$
28 to 36

7 Evaluate the following:
(a) $\dfrac{3}{7} \times \dfrac{2}{3}$ (b) $\dfrac{11}{30} \div \dfrac{5}{6}$ (c) $\dfrac{3}{7} + \dfrac{4}{13}$ (d) $\dfrac{5}{16} - \dfrac{4}{3}$
37 to 46

8 Write the following proportions as ratios:
(a) $\dfrac{1}{2}$ of A, $\dfrac{1}{5}$ of B and $\dfrac{3}{10}$ of C
(b) $\dfrac{1}{4}$ of P, $\dfrac{1}{3}$ of Q, $\dfrac{1}{5}$ of R and the remainder S
47 to 48

9 Complete the following:
(a) $\dfrac{4}{5} = \quad \%$ (b) 48% of $50 =$
(c) $\dfrac{9}{14} = \quad \%$ (d) 15% of $25 =$
49 to 52

10 Round each of the following decimal numbers, first to 3 significant figures and then to 2 decimal places:
(a) 21·355 (b) 0·02456
(c) 0·3105 (d) 5134·555
56 to 65

11 Convert each of the following to decimal form to 3 decimal places:
(a) $\dfrac{4}{15}$ (b) $-\dfrac{7}{13}$ (c) $\dfrac{9}{5}$ (d) $-\dfrac{28}{13}$
66 to 67

12 Convert each of the following to fractional form in lowest terms:
(a) 0·8 (b) 2·8 (c) $3{\cdot}\dot{3}\dot{2}$ (d) −5·5
68 to 73

13 Write each of the following in abbreviated form:
(a) 1·010101 . . . (b) 9·2456456456 . . .
70 to 71

14 Write each of the following as a number raised to a power:
(a) $3^6 \times 3^3$ (b) $4^3 \div 2^5$ (c) $\left(9^2\right)^3$ (d) $\left(7^0\right)^{-8}$
78 to 89

15 Find the value of each of the following to 3 dp:
(a) $15^{\frac{1}{3}}$ (b) $\sqrt[5]{5}$ (c) $(-27)^{\frac{1}{3}}$ (d) $(-9)^{\frac{1}{2}}$
90 to 94

▶

Frames

16 Write each of the following as a single decimal number:
(a) $3{\cdot}2044 \times 10^3$ (b) $16{\cdot}1105 \div 10^{-2}$ 95 to 98

17 Write each of the following in standard form:
(a) 134·65 (b) 0·002401 99 to 101

18 Write each of the following in preferred standard form:
(a) $16{\cdot}1105 \div 10^{-2}$ (b) 9·3304 102 to 104

19 In each of the following the numbers have been obtained by measurement. Evaluate each calculation to the appropriate level of accuracy:
(a) $11{\cdot}4 \times 0{\cdot}0013 \div 5{\cdot}44 \times 8{\cdot}810$
(b) $\dfrac{1{\cdot}01 \div 0{\cdot}00335}{9{\cdot}12 \times 6{\cdot}342}$ 105 to 108

20 Express the following numbers in denary form:
(a) $1011{\cdot}01_2$ (b) $456{\cdot}721_8$
(c) $123{\cdot}\Lambda 29_{12}$ (d) $\mathrm{CA1{\cdot}B22}_{16}$ 112 to 126

21 Convert $15{\cdot}605_{10}$ to the equivalent octal, binary, duodecimal and hexadecimal forms. 127 to 149

Types of number

1

The natural numbers

The first numbers we ever meet are the *whole numbers*. These, together with *zero*, are called the *natural numbers*, and are written down using *numerals*.

Numerals and place value

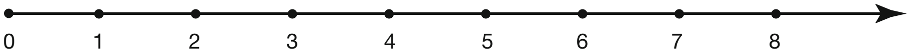
The *natural numbers* are written using the ten numerals 0, 1, . . ., 9 where the position of a numeral dictates the value that it represents. For example:

246 stands for 2 hundreds and 4 tens and 6 units. That is $200 + 40 + 6$

Here the numerals 2, 4 and 6 are called the hundreds, tens and unit *coefficients* respectively. This is the place value principle.

Points on a line and order

The natural numbers can be represented by equally spaced points on a straight line where the first natural number is zero 0.



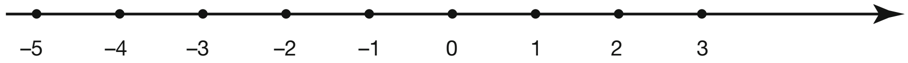
The natural numbers are ordered – they progress from small to large. As we move along the line from left to right the numbers increase as indicated by the arrow at the end of the line. On the line, numbers to the left of a given number are *less than* ($<$) the given number and numbers to the right are *greater than* ($>$) the given number. For example, $8 > 5$ because 8 is represented by a point on the line to the right of 5. Similarly, $3 < 6$ because 3 is to the left of 6.

Now move on to the next frame

2

The integers

If the straight line displaying the natural numbers is extended to the left we can plot equally spaced points to the left of zero.



These points represent *negative* numbers which are written as the natural number preceded by a minus sign, for example -4. These positive and negative whole numbers and zero are collectively called the *integers*. The notion of order still applies. For example, $-5 < 3$ and $-2 > -4$ because the point on the line representing -5 is to the *left* of the point representing 3. Similarly, -2 is to the *right* of -4.

The numbers -10, 4, 0, -13 are of a type called . . . .

You can check your answer in the next frame

3

Integers

They are integers. The natural numbers are all positive. Now try this:

Place the appropriate symbol < or > between each of the following pairs of numbers:

(a) −3   −6
(b) 2   −4
(c) −7   12

Complete these and check your results in the next frame

4

(a) $-3 > -6$
(b) $2 > -4$
(c) $-7 < 12$

The reasons being:

(a) $-3 > -6$ because -3 is represented on the line to the *right* of -6
(b) $2 > -4$ because 2 is represented on the line to the *right* of -4
(c) $-7 < 12$ because -7 is represented on the line to the *left* of 12

Now move on to the next frame

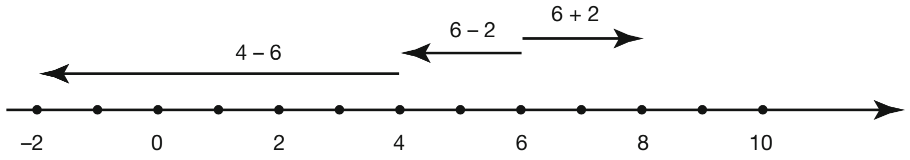
5

Brackets

Brackets should be used around negative numbers to separate the minus sign attached to the number from the arithmetic operation symbol. For example, $5 - -3$ should be written $5 - (-3)$ and 7×-2 should be written $7 \times (-2)$. *Never write two arithmetic operation symbols together without using brackets*.

Addition and subtraction

Adding two numbers gives their *sum* and subtracting two numbers gives their *difference*. For example, $6 + 2 = 8$. Adding moves to the right of the first number and subtracting moves to the left of the first number, so that $6 - 2 = 4$ and $4 - 6 = -2$:



Adding a negative number is the same as subtracting its positive counterpart. For example $7 + (-2) = 7 - 2$. The result is 5. Subtracting a negative number is the same as adding its positive counterpart. For example $7 - (-2) = 7 + 2 = 9$.

So what is the value of:

(a) $8 + (-3)$
(b) $9 - (-6)$
(c) $(-4) + (-8)$
(d) $(-14) - (-7)$?

When you have finished these check your results with the next frame

6

(a)	5
(b)	15
(c)	– 12
(d)	– 7

Move now to Frame 7

7

Multiplication and division

Multiplying two numbers gives their *product* and dividing two numbers gives their *quotient*. Multiplying and dividing two positive or two negative numbers gives a positive number. For example:

$12 \times 2 = 24$ and $(-12) \times (-2) = 24$

$12 \div 2 = 6$ and $(-12) \div (-2) = 6$

Multiplying or dividing a positive number by a negative number gives a negative number. For example:

$12 \times (-2) = -24$, $(-12) \div 2 = -6$ and $8 \div (-4) = -2$

So what is the value of:

(a) $(-5) \times 3$

(b) $12 \div (-6)$

(c) $(-2) \times (-8)$

(d) $(-14) \div (-7)$?

When you have finished these check your results with the next frame

8

(a)	– 15
(b)	– 2
(c)	16
(d)	2

Move on to Frame 9

9

Brackets and precedence rules

Brackets and the precedence rules are used to remove ambiguity in a calculation. For example, $14 - 3 \times 4$ could be either:

$11 \times 4 = 44$ or $14 - 12 = 2$

depending on which operation is performed first.

To remove the ambiguity we rely on the precedence rules:

In any calculation involving all four arithmetic operations we proceed as follows:

(a) Working from the left evaluate divisions and multiplications as they are encountered;

this leaves a calculation involving just addition and subtraction.

(b) Working from the left evaluate additions and subtractions as they are encountered.

For example, to evaluate:

$$4 + 5 \times 6 \div 2 - 12 \div 4 \times 2 - 1$$

a first evaluation from left to right produces:

$$4 + 30 \div 2 - 3 \times 2 - 1$$

a second evaluation from left to right produces:

$$4 + 15 - 6 - 1$$

and a final evaluation produces:

$$19 - 7 = 12$$

If the calculation contains brackets then these are evaluated first, so that:

$$\begin{aligned}(4 + 5 \times 6) \div 2 - 12 \div 4 \times 2 - 1 &= 34 \div 2 - 6 - 1\\ &= 17 - 7\\ &= 10\end{aligned}$$

This means that:

$$\begin{aligned}14 - 3 \times 4 &= 14 - 12\\ &= 2\end{aligned}$$

because, reading from the left we multiply before we subtract. Brackets must be used to produce the alternative result:

$$\begin{aligned}(14 - 3) \times 4 &= 11 \times 4\\ &= 44\end{aligned}$$

because the precedence rules state that brackets are evaluated first.

So that

$$34 + 10 \div (2 - 3) \times 5 = \ldots\ldots\ldots\ldots$$

Result in the next frame

10

$$\boxed{-16}$$

Because

$$\begin{aligned}34 + 10 \div (2 - 3) \times 5 &= 34 + 10 \div (-1) \times 5 && \text{we evaluate the bracket first}\\ &= 34 + (-10) \times 5 && \text{by dividing}\\ &= 34 + (-50) && \text{by multiplying}\\ &= 34 - 50 && \text{finally we subtract}\\ &= -16\end{aligned}$$

Notice that when brackets are used we can omit the multiplication signs and replace the division sign by a line, so that:

$5 \times (6 - 4)$ becomes $5(6 - 4)$

and

$(25 - 10) \div 5$ becomes $(25 - 10)/5$ or $\dfrac{25 - 10}{5}$

▶

When evaluating expressions containing *nested* brackets the innermost brackets are evaluated first. For example:

$$\begin{aligned}
3(4-2[5-1]) &= 3(4-2\times 4) && \text{evaluating the innermost bracket } [\ldots] \text{ first}\\
&= 3(4-8) && \text{multiplication before subtraction inside the } (\ldots) \text{ bracket}\\
&= 3(-4) && \text{subtraction completes the evaluation of the } (\ldots) \text{ bracket}\\
&= -12 && \text{multiplication completes the calculation}
\end{aligned}$$

so that

$5-\{8+7[4-1]-9/3\} = \ldots\ldots\ldots\ldots$

Work this out, the result is in the following frame

11

$$\boxed{-21}$$

Because

$$\begin{aligned}
5-\{8+7[4-1]-9/3\} &= 5-\{8+7\times 3-9\div 3\}\\
&= 5-\{8+21-3\}\\
&= 5-\{29-3\}\\
&= 5-26\\
&= -21
\end{aligned}$$

Now move to Frame 12

12

Basic laws of arithmetic

All the work that you have done so far has been done under the assumption that you know the rules that govern the use of arithmetic operations as, indeed, you no doubt do. However, there is a difference between knowing the rules innately and being consciously aware of them, so here they are. The four basic arithmetic operations are:

addition and subtraction
multiplication and division

where each pair may be regarded as consisting of 'opposites' – in each pair one operation is the reverse operation of the other.

1 Commutativity

Two integers can be added or multiplied in either order without affecting the result. For example:

$5+8=8+5=13$ and $5\times 8=8\times 5=40$

We say that addition and multiplication are commutative operations

The order in which two integers are subtracted or divided *does* affect the result. For example:

$4-2\neq 2-4$ because $4-2=2$ and $2-4=-2$

Notice that $\neq$ means *is not equal to*. Also

$4\div 2\neq 2\div 4$

We say that subtraction and division are not commutative operations

2 Associativity

The way in which three or more integers are associated under addition or multiplication does not affect the result. For example:

$$3 + (4 + 5) = (3 + 4) + 5 = 3 + 4 + 5 = 12$$

and

$$3 \times (4 \times 5) = (3 \times 4) \times 5 = 3 \times 4 \times 5 = 60$$

We say that addition and multiplication are associative operations

The way in which three or more integers are associated under subtraction or division does affect the result. For example:

$3 - (4 - 5) \neq (3 - 4) - 5$ because
$3 - (4 - 5) = 3 - (-1) = 3 + 1 = 4$ and $(3 - 4) - 5 = (-1) - 5 = -6$

Also

$24 \div (4 \div 2) \neq (24 \div 4) \div 2$ because
$24 \div (4 \div 2) = 24 \div 2 = 12$ and $(24 \div 4) \div 2 = 6 \div 2 = 3$

We say that subtraction and division are not associative operations

3 Distributivity

Consider the equations:

$$3 \times (4 + 5) = 3 \times 9 = 27$$

and

$$(3 \times 4) + (3 \times 5) = 12 + 15 = 27$$

From this we can deduce that:

$$3 \times (4 + 5) = (3 \times 4) + (3 \times 5)$$

We say that *multiplication distributes itself over addition from the left*. Multiplication is also distributive over addition from the right. For example:

$$(3 + 4) \times 5 = (3 \times 5) + (4 \times 5) = 35$$

The same can be said of multiplication and subtraction: *multiplication is distributive over subtraction from both the left and the right*. For example:

$3 \times (4 - 5) = (3 \times 4) - (3 \times 5) = -3$ and $(3 - 4) \times 5 = (3 \times 5) - (4 \times 5) = -5$

Division is distributed over addition and subtraction from the right but not from the left. For example:

$(60 + 15) \div 5 = (60 \div 5) + (15 \div 5)$ because

$(60 + 15) \div 5 = 75 \div 5 = 15$ and $(60 \div 5) + (15 \div 5) = 12 + 3 = 15$

However, $60 \div (15 + 5) \neq (60 \div 15) + (60 \div 5)$ because

$60 \div (15 + 5) = 60 \div 20 = 3$ and $(60 \div 15) + (60 \div 5) = 4 + 12 = 16$

Also:

$(20 - 10) \div 5 = (20 \div 5) - (10 \div 5)$ because

$(20 - 10) \div 5 = 10 \div 5 = 2$ and $(20 \div 5) - (10 \div 5) = 4 - 2 = 2$

but $20 \div (10 - 5) \neq (20 \div 10) - (20 \div 5)$ because

$20 \div (10 - 5) = 20 \div 5 = 4$ and $(20 \div 10) - (20 \div 5) = 2 - 4 = -2$

13

Estimating

Arithmetic calculations are easily performed using a calculator. However, by pressing a wrong key, wrong answers can just as easily be produced. Every calculation made using a calculator should at least be checked for the reasonableness of the final result and this can be done by *estimating* the result using *rounding*. For example, using a calculator the sum $39 + 53$ is incorrectly found to be 62 if $39 + 23$ is entered by mistake. If, now, 39 is rounded up to 40, and 53 is rounded down to 50 the reasonableness of the calculator result can be simply checked by adding 40 to 50 to give 90. This indicates that the answer 62 is wrong and that the calculation should be done again. The correct answer 92 is then seen to be close to the approximation of 90.

Rounding

An integer can be rounded to the nearest 10 as follows:

> If the number is less than halfway to the next multiple of 10 then the number is rounded *down* to the previous multiple of 10. For example, 53 is rounded down to 50.
>
> If the number is more than halfway to the next multiple of 10 then the number is rounded *up* to the next multiple of 10. For example, 39 is rounded up to 40.
>
> If the number is exactly halfway to the next multiple of 10 then the number is rounded *up*. For example, 35 is rounded up to 40.

This principle also applies when rounding to the nearest 100, 1000, 10 000 or more. For example, 349 rounds up to 350 to the nearest 10 but rounds down to 300 to the nearest 100, and 2501 rounds up to 3000 to the nearest 1000.

Try rounding each of the following to the nearest 10, 100 and 1000 respectively:

(a) 1846

(b) −638

(c) 445

Finish all three and check your results with the next frame

14

(a) 1850, 1800, 2000

(b) − 640, − 600, − 1000

(c) 450, 400, 0

Because

(a) 1846 is nearer to 1850 than to 1840, nearer to 1800 than to 1900 and nearer to 2000 than to 1000.

(b) −638 is nearer to −640 than to −630, nearer to −600 than to −700 and nearer to −1000 than to 0. The negative sign does not introduce any complications.

(c) 445 rounds to 450 because it is halfway to the next multiple of 10, 445 is nearer to 400 than to 500 and nearer to 0 than 1000.

How about estimating each of the following using rounding to the nearest 10:

(a) $18 \times 21 - 19 \div 11$

(b) $99 \div 101 - 49 \times 8$

Check your results in Frame 15

15

(a) 398
(b) – 499

Because

(a) $18 \times 21 - 19 \div 11$ rounds to $20 \times 20 - 20 \div 10 = 398$
(b) $99 \div 101 - 49 \times 8$ rounds to $100 \div 100 - 50 \times 10 = -499$

At this point let us pause and summarize the main facts so far

16

Review summary

Numbers in square brackets refer you back to the relevant frame for more detail.

1 The integers consist of the positive and negative whole numbers and zero and can be represented by equally spaced points on a line. [1]

2 The integers are ordered so that they range from large negative to small negative through zero to small positive and then large positive. They are written using the ten numerals 0 to 9 according to the principle of place value where the place of a numeral in a number dictates the value it represents. [2]

3 Multiplying or dividing two positive numbers or two negative numbers produces a positive number. Multiplying or dividing a positive number and a negative number produces a negative number. [7]

4 The four arithmetic operations of addition, subtraction, multiplication and division obey specific precedence rules that govern the order in which they are to be executed:

In any calculation involving all four arithmetic operations we proceed as follows:

(a) *working from the left evaluate divisions and multiplications as they are encountered.*

This leaves an expression involving just addition and subtraction:

(b) *working from the left evaluate additions and subtractions as they are encountered.*

Brackets are used to group numbers and operations together. In any arithmetic expression, the contents of brackets are evaluated first. [9]

5 Two basic laws of arithmetic concern the commutativity and associativity of addition and multiplication. [12]

6 Integers can be rounded to the nearest 10, 100 etc. and the rounded values used as estimates for the result of a calculation. [13]

17

Review exercise

1 Place the appropriate symbol < or > between each of the following pairs of numbers:

(a) $-1 \quad -6$
(b) $5 \quad -29$
(c) $-14 \quad 7$

▶

2 Find the value of each of the following:

(a) $16 - 12 \times 4 + 8 \div 2$

(b) $(16 - 12) \times (4 + 8) \div 2$

(c) $9 - 3(17 + 5[5 - 7])$

(d) $8(3[2 + 4] - 2[5 + 7])$

3 Show that:

(a) $6 - (3 - 2) \neq (6 - 3) - 2$

(b) $100 \div (10 \div 5) \neq (100 \div 10) \div 5$

(c) $24 \div (2 + 6) \neq (24 \div 2) + (24 \div 6)$

(d) $24 \div (2 - 6) \neq (24 \div 2) - (24 \div 6)$

4 Round each number to the nearest 10, 100 and 1000:

(a) 2562

(b) 1500

(c) -3451

(d) $-14\,525$

18

1 (a) $-1 > -6$ because -1 is represented on the line to the right of -6

(b) $5 > -29$ because 5 is represented on the line to the right of -29

(c) $-14 < 7$ because -14 is represented on the line to the left of 7

2 (a) $16 - 12 \times 4 + 8 \div 2 = 16 - 48 + 4 = 16 - 44 = -28$
divide and multiply before adding and subtracting

(b) $(16 - 12) \times (4 + 8) \div 2 = (4) \times (12) \div 2 = 4 \times 12 \div 2 = 4 \times 6 = 24$
brackets are evaluated first

(c) $$\begin{aligned} 9 - 3(17 + 5[5 - 7]) &= 9 - 3(17 + 5[-2]) \\ &= 9 - 3(17 - 10) \\ &= 9 - 3(7) \\ &= 9 - 21 = -12 \end{aligned}$$

(d) $$\begin{aligned} 8(3[2 + 4] - 2[5 + 7]) &= 8(3 \times 6 - 2 \times 12) \\ &= 8(18 - 24) \\ &= 8(-6) = -48 \end{aligned}$$

3 (a) Left-hand side (LHS) $= 6 - (3 - 2) = 6 - (1) = 5$
Right-hand side (RHS) $= (6 - 3) - 2 = (3) - 2 = 1 \neq$ LHS

(b) Left-hand side (LHS) $= 100 \div (10 \div 5) = 100 \div 2 = 50$
Right-hand side (RHS) $= (100 \div 10) \div 5 = 10 \div 5 = 2 \neq$ LHS

(c) Left-hand side (LHS) $= 24 \div (2 + 6) = 24 \div 8 = 3$
Right-hand side (RHS) $= (24 \div 2) + (24 \div 6) = 12 + 4 = 16 \neq$ LHS

(d) Left-hand side (LHS) $= 24 \div (2 - 6) = 24 \div (-4) = -6$
Right-hand side (RHS) $= (24 \div 2) - (24 \div 6) = 12 - 4 = 8 \neq$ LHS

4 (a) 2560, 2600, 3000

(b) 1500, 1500, 2000

(c) -3450, -3500, -3000

(d) $-14\,530$, $-14\,500$, $-15\,000$

So now on to Frame 19

Factors and prime numbers

19

Factors

Any pair of whole numbers are called *factors* of their product. For example, the numbers 3 and 6 are factors of 18 because $3 \times 6 = 18$. These are not the only factors of 18. The complete collection of factors of 18 is 1, 2, 3, 6, 9, 18 because

$$\begin{aligned} 18 &= 1 \times 18 \\ &= 2 \times 9 \\ &= 3 \times 6 \end{aligned}$$

So the factors of:

(a) 12

(b) 25

(c) 17 are . . . .

The results are in the next frame

20

(a) 1, 2, 3, 4, 6, 12
(b) 1, 5, 25
(c) 1, 17

Because

(a) $12 = 1 \times 12 = 2 \times 6 = 3 \times 4$

(b) $25 = 1 \times 25 = 5 \times 5$

(c) $17 = 1 \times 17$

Now move to the next frame

21

Prime numbers

If a whole number has only two factors which are itself and the number 1, the number is called a *prime number*. The first six prime numbers are 2, 3, 5, 7, 11 and 13. The number 1 is *not* a prime number for the reason given below.

Prime factorization

Every whole number can be written as a product involving only prime factors. For example, the number 126 has the factors 1, 2, 3, 6, 7, 9, 14, 18, 21, 42, 63 and 126, of which 2, 3 and 7 are prime numbers and 126 can be written as:

$$126 = 2 \times 3 \times 3 \times 7$$

To obtain this *prime factorization* the number is divided by successively increasing prime numbers thus:

$$\begin{array}{r|r} 2 & 126 \\ \hline 3 & 63 \\ \hline 3 & 21 \\ \hline 7 & 7 \\ \hline & 1 \\ \hline \end{array} \quad \text{so that } 126 = 2 \times 3 \times 3 \times 7$$

Notice that a prime factor may occur more than once in a prime factorization.

▶

Fundamental theorem of arithmetic

The fundamental theorem of arithmetic states that the prime factorization of a natural number is unique. For example, we can write the prime factorization of 126 differently by rearranging the prime factors but it will always contain one 2, two 3s and one 7. This explains why 1 is not a prime number because if it were we could write:

$$\begin{aligned} 126 &= 2 \times 3 \times 3 \times 7 \\ &= 1 \times 2 \times 3 \times 3 \times 7 \\ &= 1 \times 1 \times 2 \times 3 \times 3 \times 7 \\ &= \ldots \end{aligned}$$

and the prime factorization would no longer be unique.

Now find the prime factorization of:
(a) 84 (b) 512

Work these two out and check the working in Frame 22

22

(a) $84 = 2 \times 2 \times 3 \times 7$
(b) $512 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$

Because

(a)
$$\begin{array}{r|r} 2 & 84 \\ \hline 2 & 42 \\ \hline 3 & 21 \\ \hline 7 & 7 \\ \hline & 1 \\ \hline \end{array} \quad \text{so that } 84 = 2 \times 2 \times 3 \times 7$$

(b) The only prime factor of 512 is 2 which occurs 9 times. The prime factorization is:

$$512 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$$

Move to Frame 23

23

Highest common factor (HCF)

The *highest common factor* (HCF) of two whole numbers is the largest factor that they have in common. For example, the prime factorizations of 144 and 66 are:

$$\begin{aligned} 144 &= 2 \times 2 \times 2 \times 2 \times 3 \times 3 \\ 66 &= 2 \qquad\qquad\qquad\quad \times 3 \qquad \times 11 \end{aligned}$$

Only the 2 and the 3 are common to both factorizations and so the highest factor that these two numbers have in common (HCF) is $2 \times 3 = 6$.

Lowest common multiple (LCM)

The smallest whole number that each one of a pair of whole numbers divides into a whole number of times is called their *lowest common multiple* (LCM). This is also found from the prime factorization of each of the two numbers. For example:

$$\begin{aligned} 144 &= 2 \times 2 \times 2 \times 2 \times 3 \times 3 \\ 66 &= 2 \qquad\qquad\qquad\quad \times 3 \qquad \times 11 \\ \text{LCM} &= 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 11 = 1584 \end{aligned}$$

The HCF and LCM of 84 and 512 are . . . .

24

HCF: 4
LCM: 10 752

Because

84 and 512 have the prime factorizations:

$$\begin{array}{rl} 84 = & 2 \times 2 \qquad\qquad\qquad\qquad\qquad\qquad\quad \times 3 \times 7 \\ 512 = & 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \qquad\qquad \text{HCF} = 2 \times 2 = 4 \\ \text{LCM} = & 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 7 = 10\,752 \end{array}$$

At this point let us pause and summarize the main facts on factors and prime numbers

Review summary

25

1 A pair of whole numbers are called factors of their product. [19]

2 If a whole number only has one and itself as factors it is called a prime number. [21]

3 Every whole number can be written as a product of its prime factors, some of which may be repeated. [21]

4 The highest common factor (HCF) is the highest factor that two whole numbers have in common. [23]

5 The lowest common multiple (LCM) is the lowest whole number that two whole numbers will divide into a whole number of times. [23]

Review exercise

26

1 Write each of the following as a product of prime factors:

(a) 429 (b) 1820 (c) 2992 (d) 3185

2 Find the HCF and the LCM of each pair of numbers:

(a) 63, 42 (b) 92, 34

27

1 (a)

$$\begin{array}{r|r} 3 & 429 \\ \hline 11 & 143 \\ \hline 13 & 13 \\ \hline & 1 \\ \hline \end{array} \qquad 429 = 3 \times 11 \times 13$$

(b)

$$\begin{array}{r|r} 2 & 1820 \\ \hline 2 & 910 \\ \hline 5 & 455 \\ \hline 7 & 91 \\ \hline 13 & 13 \\ \hline & 1 \\ \hline \end{array} \qquad 1820 = 2 \times 2 \times 5 \times 7 \times 13$$

(c)

$$\begin{array}{r|r} 2 & 2992 \\ \hline 2 & 1496 \\ \hline 2 & 748 \\ \hline 2 & 374 \\ \hline 11 & 187 \\ \hline 17 & 17 \\ \hline & 1 \\ \hline \end{array} \qquad 2992 = 2 \times 2 \times 2 \times 2 \times 11 \times 17$$

(d)

$$\begin{array}{r|r} 5 & 3185 \\ \hline 7 & 637 \\ \hline 7 & 91 \\ \hline 13 & 13 \\ \hline & 1 \\ \hline \end{array} \qquad 3185 = 5 \times 7 \times 7 \times 13$$

2 (a) The prime factorizations of 63 and 42 are:

$$\begin{aligned} 63 &= \phantom{2 \times{}} 3 \times 3 \times 7 \\ 42 &= 2 \times 3 \phantom{{}\times 3} \times 7 \qquad \text{HCF } 3 \times 7 = 21 \\ \text{LCM} &= 2 \times 3 \times 3 \times 7 = 126 \end{aligned}$$

(b) The prime factorizations of 34 and 92 are:

$$\begin{aligned} 34 &= 2 \phantom{{}\times 2} \times 17 \\ 92 &= 2 \times 2 \phantom{{}\times 17} \times 23 \qquad \text{HCF } 2 \\ \text{LCM} &= 2 \times 2 \times 17 \times 23 = 1564 \end{aligned}$$

Now on to the next topic

Fractions, ratios and percentages

28

Division of integers

A fraction is a number which is represented by one integer – the *numerator* – divided by another integer – the *denominator* (or the *divisor*). For example, $\frac{3}{5}$ is a fraction with numerator 3 and denominator 5. Because fractions are written as one integer divided by another – a *ratio* – they are called *rational* numbers. Fractions are either *proper*, *improper* or *mixed*:

- in a proper fraction the numerator is less than the denominator, for example, $\dfrac{4}{7}$
- in an improper fraction the numerator is greater than the denominator, for example $\dfrac{12}{5}$
- a mixed fraction is in the form of an integer and a fraction, for example $6\frac{2}{3}$

So that $-\dfrac{8}{11}$ is a .... fraction?

The answer is in the next frame

29

$$\boxed{\text{Proper}}$$

Fractions can be either positive or negative.

Now to the next frame

30

Multiplying fractions

Two fractions are multiplied by multiplying their respective numerators and denominators independently. For example:

$$\frac{2}{3}\times\frac{5}{7}=\frac{2\times 5}{3\times 7}=\frac{10}{21}$$

Try this one for yourself.

$$\frac{5}{9}\times\frac{2}{7}=\dots\dots\dots\dots$$

31

$$\boxed{\frac{10}{63}}$$

Because

$$\frac{5}{9}\times\frac{2}{7}=\frac{5\times 2}{9\times 7}=\frac{10}{63}$$

Correct? Then on to the next frame

32

Of

The word ‘of’ when interposed between two fractions means multiply. For example:

Half of half a cake is one-quarter of a cake. That is

$$\frac{1}{2}\ \textit{of}\ \frac{1}{2}=\frac{1}{2}\times\frac{1}{2}=\frac{1\times 1}{2\times 2}=\frac{1}{4}$$

So that, for example:

$$\frac{1}{3}\ \textit{of}\ \frac{2}{5}=\frac{1}{3}\times\frac{2}{5}=\frac{1\times 2}{3\times 5}=\frac{2}{15}$$

So that $\frac{3}{8}\ \textit{of}\ \frac{5}{7}=\dots\dots\dots\dots$

33

$$\boxed{\frac{15}{56}}$$

Because

$$\frac{3}{8}\ \textit{of}\ \frac{5}{7}=\frac{3}{8}\times\frac{5}{7}=\frac{3\times 5}{8\times 7}=\frac{15}{56}$$

On now to the next frame

34

Equivalent fractions

Multiplying the numerator and denominator by the same number is equivalent to multiplying the fraction by unity, that is by 1:

$$\frac{4\times 3}{5\times 3}=\frac{4}{5}\times\frac{3}{3}=\frac{4}{5}\times 1=\frac{4}{5}$$

Now, $\frac{4\times 3}{5\times 3}=\frac{12}{15}$ so that the fraction $\frac{4}{5}$ and the fraction $\frac{12}{15}$ *both represent the same number* and for this reason we call $\frac{4}{5}$ and $\frac{12}{15}$ *equivalent fractions*.

A second fraction, equivalent to a first fraction, can be found by multiplying the numerator and the denominator of the first fraction by the same number.

So that if we multiply the numerator and denominator of the fraction $\frac{7}{5}$ by 4 we obtain the equivalent fraction . . . .

Check your result in Frame 35

35

$$\boxed{\frac{28}{20}}$$

Because

$$\frac{7\times 4}{5\times 4}=\frac{28}{20}$$

We can reverse this process and find the equivalent fraction that has the smallest numerator by *cancelling out* common factors. This is known as reducing the fraction to its *lowest terms*. For example:

$\frac{16}{96}$ can be reduced to its lowest terms as follows:

$$\frac{16}{96}=\frac{4\times 4}{24\times 4}=\frac{4\times\cancel{4}}{24\times\cancel{4}}=\frac{4}{24}$$

by cancelling out the 4 in the numerator and the denominator

The fraction $\frac{4}{24}$ can also be reduced:

$$\frac{4}{24}=\frac{4}{6\times 4}=\frac{\cancel{4}}{6\times\cancel{4}}=\frac{1}{6}$$

Because $\frac{1}{6}$ cannot be reduced further we see that $\frac{16}{96}$ reduced to its lowest terms is $\frac{1}{6}$.

How about this one? The fraction $\frac{84}{108}$ reduced to its lowest terms is . . . .

Check with the next frame

36

$$\boxed{\frac{7}{9}}$$

Because

$$\frac{84}{108}=\frac{7\times 3\times 4}{9\times 3\times 4}=\frac{7\times\cancel{3}\times\cancel{4}}{9\times\cancel{3}\times\cancel{4}}=\frac{7}{9}$$

Now move on to the next frame

Dividing fractions

37

The expression $6 \div 3$ means the number of 3's in 6, which is 2. Similarly, the expression $1 \div \dfrac{1}{4}$ means the number of $\frac{1}{4}$'s in 1, which is, of course, 4. That is:

$1 \div \dfrac{1}{4} = 4 = 1 \times \dfrac{4}{1}$. Notice how the numerator and the denominator of the divisor are switched and the division replaced by multiplication.

Two fractions are divided by switching the numerator and the denominator of the divisor and multiplying the result. For example:

$$\frac{2}{3} \div \frac{5}{7} = \frac{2}{3} \times \frac{7}{5} = \frac{14}{15}$$

So that $\dfrac{7}{13} \div \dfrac{3}{4} =$ . . . .

38

$$\boxed{\frac{28}{39}}$$

Because

$$\frac{7}{13} \div \frac{3}{4} = \frac{7}{13} \times \frac{4}{3} = \frac{28}{39}$$

In particular:

$$1 \div \frac{3}{5} = 1 \times \frac{5}{3} = \frac{5}{3}$$

The fraction $\dfrac{5}{3}$ is called the *reciprocal* of $\dfrac{3}{5}$

So that the reciprocal of $\dfrac{17}{4}$ is . . . .

39

$$\boxed{\frac{4}{17}}$$

Because

$$1 \div \frac{17}{4} = 1 \times \frac{4}{17} = \frac{4}{17}$$

And the reciprocal of -5 is . . . .

40

$$\boxed{-\frac{1}{5}}$$

Because

$$1 \div (-5) = 1 \div \left(-\frac{5}{1}\right) = 1 \times \left(-\frac{1}{5}\right) = -\frac{1}{5}$$

Move on to the next frame

41

Adding and subtracting fractions

Two fractions can only be added or subtracted immediately if they both possess the same denominator, in which case we add or subtract the numerators and divide by the common denominator. For example:

$$\frac{2}{7}+\frac{3}{7}=\frac{2+3}{7}=\frac{5}{7}$$

If they do not have the same denominator they must be rewritten in equivalent form so that they do have the same denominator – called the *common denominator*. For example:

$$\frac{2}{3}+\frac{1}{5}=\frac{10}{15}+\frac{3}{15}=\frac{10+3}{15}=\frac{13}{15}$$

The common denominator of the equivalent fractions is the LCM of the two original denominators. That is:

$$\frac{2}{3}+\frac{1}{5}=\frac{2\times 5}{3\times 5}+\frac{1\times 3}{5\times 3}=\frac{10}{15}+\frac{3}{15} \text{ where 15 is the LCM of 3 and 5}$$

So that $\frac{5}{9}+\frac{1}{6}=\ldots\ldots\ldots\ldots$

The result is in Frame 42

42

$$\boxed{\frac{13}{18}}$$

Because

The LCM of 9 and 6 is 18 so that
$$\begin{aligned}\frac{5}{9}+\frac{1}{6}&=\frac{5\times 2}{9\times 2}+\frac{1\times 3}{6\times 3}=\frac{10}{18}+\frac{3}{18}\\&=\frac{10+3}{18}=\frac{13}{18}\end{aligned}$$

There's another one to try in the next frame

43

Now try $\frac{11}{15}-\frac{2}{3}=\ldots\ldots\ldots\ldots$

44

$$\boxed{\frac{1}{15}}$$

Because

$$\begin{aligned}\frac{11}{15}-\frac{2}{3}&=\frac{11}{15}-\frac{2\times 5}{3\times 5}=\frac{11}{15}-\frac{10}{15}\\&=\frac{11-10}{15}=\frac{1}{15}\qquad\text{(15 is the LCM of 3 and 15)}\end{aligned}$$

Correct? Then on to Frame 45

Fractions on a calculator

45

The calculator we shall use is the Casio *fx-85GT PLUS* and on this calculator the fraction key is denoted by the symbol:

$\frac{\blacksquare}{\square}$

on the key and the symbol for a mixed fraction:

$\blacksquare\frac{\square}{\square}$

above the key – accessed by using in combination with the SHIFT key. The REPLAY key is used to move the flashing cursor around the screen to enable functions to be entered and manipulated with the results given in fractional form. For example, to evaluate $\frac{2}{3} \times 2\frac{3}{4}$ using this calculator [*note*: your calculator may not have the identical display in what follows, indeed the symbol $a^b\!/_c$ is often used for the fraction key]:

- Press the fraction key
  - the fraction symbol is then displayed on the screen
- Enter the number 2
  - the numerator in the display
- Press the down arrow on the REPLAY key
  - the flashing cursor moves to the denominator
- Enter the number 3
  - the denominator in the display
- Press the right arrow on the REPLAY key
  - the flashing cursor moves to the right of the display
- Press the × key
- Press the SHIFT key and then the fraction key
  - the mixed fraction symbol is then displayed on the screen
- Enter the number 2
- Press the right arrow on the REPLAY key
- Enter the number 3
- Press the down arrow on the REPLAY key
- Enter the number 4

The display then should look just like:

$\frac{2}{3} \times 2\frac{3}{4}$

Press the $=$ key and the result appears in the bottom right-hand corner of the screen:

$$\frac{11}{6}$$

Now press the SHIFT key and then the S⇔D key to change the display to the mixed fraction:

$1\frac{5}{6}$

That is:

$$\begin{aligned}\tfrac{2}{3} \times 2\tfrac{3}{4} &= \frac{2}{3} \times \frac{11}{4} \\ &= \frac{11}{6} \\ &= 1\tfrac{5}{6}\end{aligned}$$

▶

Now use your calculator to evaluate each of the following:

(a) $\frac{5}{7}+3\frac{2}{3}$

(b) $\dfrac{8}{3}-\dfrac{5}{11}$

(c) $\dfrac{13}{5}\times\dfrac{4}{7}-\dfrac{2}{9}$

(d) $4\frac{1}{11}\div\left(-\dfrac{3}{5}\right)+\dfrac{1}{8}$

Check your answers in Frame 46

46

(a) $\frac{5}{7}+3\frac{2}{3}=\frac{92}{21}=4\frac{8}{21}$

(b) $\dfrac{8}{3}-\dfrac{5}{11}=\dfrac{73}{33}=2\frac{7}{33}$

(c) $\dfrac{13}{5}\times\dfrac{4}{7}-\dfrac{2}{9}=\dfrac{398}{315}=1\frac{83}{315}$

(d) $4\frac{1}{11}\div\left(-\dfrac{3}{5}\right)+\dfrac{1}{8}=-\dfrac{589}{88}=-6\frac{61}{88}$

On now to the next frame

47

Ratios

If a whole number is separated into a number of fractional parts where each fraction has the same denominator, the numerators of the fractions form a *ratio*. For example, if a quantity of brine in a tank contains $\dfrac{1}{3}$ salt and $\dfrac{2}{3}$ water, the salt and water are said to be in the ratio 'one-to-two' – written 1 : 2.

What ratio do the components A, B and C form if a compound contains $\dfrac{3}{4}$ of A, $\dfrac{1}{6}$ of B and $\dfrac{1}{12}$ of C?

Take care here and check your results with Frame 48

48

$\boxed{9 : 2 : 1}$

Because the LCM of the denominators 4, 6 and 12 is 12, then:

$\dfrac{3}{4}$ of A is $\dfrac{9}{12}$ of A, $\dfrac{1}{6}$ of B is $\dfrac{2}{12}$ of B and the remaining $\dfrac{1}{12}$ is of C. This ensures that the components are in the ratio of their numerators. That is: 9 : 2 : 1

Notice that the sum of the numbers in the ratio is equal to the common denominator.

On now to the next frame

49

Percentages

A percentage is a fraction whose denominator is equal to 100. For example, if 5 out of 100 people are left-handed then the fraction of left-handers is $\dfrac{5}{100}$ which is written as 5%, that is 5 *per cent* (%).

So if 13 out of 100 cars on an assembly line are red, the percentage of red cars on the line is . . . .

50

$$\boxed{13\%}$$

Because

The fraction of cars that are red is $\dfrac{13}{100}$ which is written as 13%.

Try this. What is the percentage of defective resistors in a batch of 25 if 12 of them are defective?

51

$$\boxed{48\%}$$

Because

The fraction of defective resistors is $\dfrac{12}{25} = \dfrac{12 \times 4}{25 \times 4} = \dfrac{48}{100}$ which is written as 48%.

Notice that this is the same as:

$$\left(\frac{12}{25} \times 100\right)\% = \left(\frac{12}{25} \times 25 \times 4\right)\% = (12 \times 4)\% = 48\%$$

A fraction can be converted to a percentage by multiplying the fraction by 100.

To find the percentage part of a quantity we multiply the quantity by the percentage written as a fraction. For example, 24% of 75 is:

$$24\%\ of\ 75 = \frac{24}{100}\ of\ 75 = \frac{24}{100} \times 75 = \frac{6 \times 4}{25 \times 4} \times 25 \times 3 = \frac{6 \times \cancel{4}}{\cancel{25} \times \cancel{4}} \times \cancel{25} \times 3$$
$$= 6 \times 3 = 18$$

So that 8% of 25 is ....

Work it through and check your results with the next frame

52

$$\boxed{2}$$

Because

$$\frac{8}{100} \times 25 = \frac{2 \times 4}{25 \times 4} \times 25 = \frac{2 \times \cancel{4}}{\cancel{25} \times \cancel{4}} \times \cancel{25} = 2$$

At this point let us pause and summarize the main facts on fractions, ratios and percentages

Review summary

53

1 A fraction is a number represented as one integer (the numerator) divided by another integer (the denominator or divisor). [28]

2 A fraction with no common factors other than unity in its numerator and denominator is said to be in its lowest terms. [30]

3 The same number can be represented by different but equivalent fractions. [34]

4 Two fractions are multiplied by multiplying the numerators and denominators independently. [35]

5 Two fractions can only be added or subtracted immediately when their denominators are equal. [41]

6 A ratio consists of the numerators of fractions with identical denominators. [47]

7 The numerator of a fraction whose denominator is 100 is called a percentage. [49]

54

Review exercise

1 Reduce each of the following fractions to their lowest terms:

(a) $\frac{24}{30}$ (b) $\frac{72}{15}$ (c) $-\frac{52}{65}$ (d) $\frac{32}{8}$

2 Evaluate the following:

(a) $\frac{5}{9}\times\frac{2}{5}$ (b) $\frac{13}{25}\div\frac{2}{15}$ (c) $\frac{5}{9}+\frac{3}{14}$ (d) $\frac{3}{8}-\frac{2}{5}$

(e) $\frac{12}{7}\times\left(-\frac{3}{5}\right)$ (f) $\left(-\frac{3}{4}\right)\div\left(-\frac{12}{7}\right)$ (g) $\frac{19}{2}+\frac{7}{4}$ (h) $\frac{1}{4}-\frac{3}{8}$

3 Write the following proportions as ratios:

(a) $\frac{1}{2}$ of A, $\frac{2}{5}$ of B and $\frac{1}{10}$ of C

(b) $\frac{1}{3}$ of P, $\frac{1}{5}$ of Q, $\frac{1}{4}$ of R and the remainder S

4 Complete the following:

(a) $\frac{2}{5}=\quad\%$ (b) 58% of 25 =

(c) $\frac{7}{12}=\quad\%$ (d) 17% of 50 =

55

1 (a) $\frac{24}{30}=\frac{2\times2\times2\times3}{2\times3\times5}=\frac{2\times2}{5}=\frac{4}{5}$

(b) $\frac{72}{15}=\frac{2\times2\times2\times3\times3}{3\times5}=\frac{2\times2\times2\times3}{5}=\frac{24}{5}$

(c) $-\frac{52}{65}=-\frac{2\times2\times13}{5\times13}=-\frac{2\times2}{5}=-\frac{4}{5}$

(d) $\frac{32}{8}=\frac{2\times2\times2\times2\times2}{2\times2\times2}=4$

2 (a) $\frac{5}{9}\times\frac{2}{5}=\frac{5\times2}{9\times5}=\frac{2}{9}$

(b) $\frac{13}{25}\div\frac{2}{15}=\frac{13}{25}\times\frac{15}{2}=\frac{13\times15}{25\times2}=\frac{13\times3\times5}{5\times5\times2}=\frac{39}{10}$

(c) $\frac{5}{9}+\frac{3}{14}=\frac{5\times14}{9\times14}+\frac{3\times9}{14\times9}=\frac{70}{126}+\frac{27}{126}=\frac{70+27}{126}=\frac{97}{126}$

(d) $\frac{3}{8}-\frac{2}{5}=\frac{3\times5}{8\times5}-\frac{2\times8}{5\times8}=\frac{15}{40}-\frac{16}{40}=\frac{15-16}{40}=-\frac{1}{40}$

(e) $\frac{12}{7}\times\left(-\frac{3}{5}\right)=\frac{12\times(-3)}{7\times5}=\frac{-36}{35}=-\frac{36}{35}$

(f) $\left(-\frac{3}{4}\right)\div\left(-\frac{12}{7}\right)=\left(-\frac{3}{4}\right)\times\left(-\frac{7}{12}\right)=\frac{(-3)\times(-7)}{4\times12}=\frac{3\times7}{4\times3\times4}=\frac{7}{16}$

(g) $\frac{19}{2}+\frac{7}{4}=\frac{38}{4}+\frac{7}{4}=\frac{45}{4}$

(h) $\frac{1}{4}-\frac{3}{8}=\frac{2}{8}-\frac{3}{8}=-\frac{1}{8}$

▶

3 (a) $\frac{1}{2}, \frac{2}{5}, \frac{1}{10} = \frac{5}{10}, \frac{4}{10}, \frac{1}{10}$ so ratio is 5 : 4 : 1

(b) $\frac{1}{3}, \frac{1}{5}, \frac{1}{4} = \frac{20}{60}, \frac{12}{60}, \frac{15}{60}$ and $\frac{20}{60} + \frac{12}{60} + \frac{15}{60} = \frac{47}{60}$

so the fraction of S is $\frac{13}{60}$

so P, Q , R and S are in the ratio 20 : 12 : 15 : 13

4 (a) $\frac{2}{5} = \frac{2 \times 20}{5 \times 20} = \frac{40}{100}$ that is 40% or $\frac{2}{5} = \frac{2}{5} \times 100\% = 40\%$

(b) $\frac{58}{100} \times 25 = \frac{58}{4} = \frac{29}{2} = 14\frac{1}{2}$

(c) $\frac{7}{12} = \frac{7}{12} \times 100\% = \frac{700}{12}\% = \frac{58 \times 12 + 4}{12}\% = 58\frac{4}{12}\% = 58\frac{1}{3}\%$

(d) $\frac{17}{100} \times 50 = \frac{17}{2} = 8\frac{1}{2}$

Now let's look at decimal numbers

Decimal numbers

Division of integers

56

If one integer is divided by a second integer that is not one of the first integer's factors the result will not be another integer. Instead, the result will lie between two integers. For example, using a calculator it is seen that:

$25 \div 8 = 3{\cdot}125$

which is a number greater than 3 but less than 4. As with integers, the position of a numeral within the number indicates its value. Here the number 3·125 represents

3 units + 1 tenth + 2 hundredths + 5 thousandths.

That is $3 + \frac{1}{10} + \frac{2}{100} + \frac{5}{1000}$

where the decimal point shows the separation of the units from the tenths. Numbers written in this format are called *decimal numbers*.

On to the next frame

Rounding

57

All the operations of arithmetic that we have used with the integers apply to decimal numbers. However, when performing calculations involving decimal numbers it is common for the end result to be a number with a large quantity of numerals after the decimal point. For example:

$15{\cdot}11 \div 8{\cdot}92 = 1{\cdot}6939461883\ldots$

To make such numbers more manageable or more reasonable as the result of a calculation, they can be rounded either to a specified number of *significant figures* or to a specified number of *decimal places*.

Now to the next frame

58

Significant figures

Significant figures are counted from the first non-zero numeral encountered starting from the left of the number. When the required number of significant figures has been counted off, the remaining numerals are deleted with the following proviso:

> If the first of a group of numerals to be deleted is a 5 or more, the last significant numeral is increased by 1. For example:
>
> 9·4534 to two significant figures is 9·5, to three significant figures is 9·45, and 0·001354 to two significant figures is 0·0014

Try this one for yourself. To four significant figures the number 18·7249 is . . . .

Check your result with the next frame

59

$\boxed{18{\cdot}72}$

Because

> The first numeral deleted is a 4 which is less than 5.

There is one further proviso. If the only numeral to be dropped is a 5 then the last numeral retained is rounded up. So that 12·235 to four significant figures (abbreviated to *sig fig*) is 12·24 and 3·465 to three sig fig is 3·47.

So 8·1265 to four sig fig is . . . .

Check with the next frame

60

$\boxed{8{\cdot}127}$

Because

> The only numeral deleted is a 5 and the last numeral is rounded up.

Now on to the next frame

61

Decimal places

Decimal places are counted to the right of the decimal point and the same rules as for significant figures apply for rounding to a specified number of decimal places (abbreviated to *dp*). For example:

> 123·4467 to one decimal place is 123·4 and to two dp is 123·45

So, 47·0235 to three dp is . . . .

62

$\boxed{47{\cdot}024}$

Because

> The only numeral dropped is a 5 and the last numeral retained is rounded up.

Now move on to the next frame

63

Trailing zeros

Sometimes zeros must be inserted within a number to satisfy a condition for a specified number of either significant figures or decimal places. For example:

12 645 to two significant figures is 13 000, and 13·1 to three decimal places is 13·100.

These zeros are referred to as *trailing zeros*.

So that 1515 to two sig fig is ....

64

$$\boxed{1500}$$

And 25·13 to four dp is ....

65

$$\boxed{25{\cdot}1300}$$

On to the next frame

66

Fractions as decimals

Because a fraction is one integer divided by another it can be represented in decimal form simply by executing the division. For example:

$$\frac{7}{4} = 7 \div 4 = 1{\cdot}75$$

So that the decimal form of $\frac{3}{8}$ is ....

67

$$\boxed{0{\cdot}375}$$

Because

$$\frac{3}{8} = 3 \div 8 = 0{\cdot}375$$

Now move on to the next frame

68

Decimals as fractions

A decimal can be represented as a fraction. For example:

$$1{\cdot}224 = \frac{1224}{1000} \text{ which in lowest terms is } \frac{153}{125}$$

So that 0·52 as a fraction in lowest terms is ....

69

$$\boxed{\frac{13}{25}}$$

Because

$$0{\cdot}52 = \frac{52}{100} = \frac{13}{25}$$

Now move on to the next frame

70

Unending decimals

Converting a fraction into its decimal form by performing the division always results in an infinite string of numerals after the decimal point. This string of numerals may contain an infinite sequence of zeros or it may contain an infinitely repeated pattern of numerals. A repeated pattern of numerals can be written in an abbreviated format. For example:

$$\frac{1}{3} = 1 \div 3 = 0{\cdot}3333\ldots$$

Here the pattern after the decimal point is of an infinite number of 3's. We abbreviate this by placing a dot over the first 3 to indicate the repetition, thus:

$0{\cdot}3333\ldots = 0{\cdot}\dot{3}$ (described as zero point 3 recurring)

For other fractions the repetition may consist of a sequence of numerals, in which case a dot is placed over the first and last numeral in the sequence. For example:

$$\frac{1}{7} = 0{\cdot}142857142857142857\ldots = 0{\cdot}\dot{1}4285\dot{7}$$

So that we write $\dfrac{2}{11} = 0{\cdot}181818\ldots$ as ....

71

$$\boxed{0{\cdot}\dot{1}\dot{8}}$$

Sometimes the repeating pattern is formed by an infinite sequence of zeros, in which case we simply omit them. For example:

$\dfrac{1}{5} = 0{\cdot}20000\ldots$ is written as $0{\cdot}2$

Next frame

72

Unending decimals as fractions

Any decimal that displays an unending repeating pattern can be converted to its fractional form. For example:

To convert $0{\cdot}181818\ldots = 0{\cdot}\dot{1}\dot{8}$ to its fractional form we note that because there are two repeating numerals we multiply by 100 to give:

$$100 \times 0{\cdot}\dot{1}\dot{8} = 18{\cdot}\dot{1}\dot{8}$$

Subtracting $0{\cdot}\dot{1}\dot{8}$ from both sides of this equation gives:

$$100 \times 0{\cdot}\dot{1}\dot{8} - 0{\cdot}\dot{1}\dot{8} = 18{\cdot}\dot{1}\dot{8} - 0{\cdot}\dot{1}\dot{8}$$

That is:

$$99 \times 0{\cdot}\dot{1}\dot{8} = 18{\cdot}0$$

This means that:

$$0{\cdot}\dot{1}\dot{8} = \frac{18}{99} = \frac{2}{11}$$

Similarly, the fractional form of $2{\cdot}0\dot{3}1\dot{5}$ is found as follows:

$2{\cdot}0\dot{3}1\dot{5} = 2{\cdot}0 + 0{\cdot}0\dot{3}1\dot{5}$ and, because there are three repeating numerals:

$$1000 \times 0{\cdot}0\dot{3}1\dot{5} = 31{\cdot}5\dot{3}1\dot{5}$$

Subtracting $0{\cdot}0\dot{3}1\dot{5}$ from both sides of this equation gives:

$$1000 \times 0{\cdot}0\dot{3}1\dot{5} - 0{\cdot}0\dot{3}1\dot{5} = 31{\cdot}5\dot{3}1\dot{5} - 0{\cdot}0\dot{3}1\dot{5} = 31{\cdot}5$$

That is:

$$999 \times 0{\cdot}0\dot{3}1\dot{5} = 31{\cdot}5 \text{ so that } 0{\cdot}0\dot{3}1\dot{5} = \frac{31{\cdot}5}{999} = \frac{315}{9990}$$

This means that:

$$2{\cdot}0\dot{3}1\dot{5} = 2{\cdot}0 + 0{\cdot}0\dot{3}1\dot{5} = 2 + \frac{315}{9990} = 2\tfrac{35}{1110} = 2\tfrac{7}{222}$$

What are the fractional forms of $0{\cdot}\dot{2}\dot{1}$ and $3{\cdot}2\dot{1}$?

The answers are in the next frame

73

$$\boxed{\tfrac{7}{33} \text{ and } 3\tfrac{19}{90}}$$

Because

$100 \times 0{\cdot}\dot{2}\dot{1} = 21{\cdot}\dot{2}\dot{1}$ so that $99 \times 0{\cdot}\dot{2}\dot{1} = 21$ giving

$$0{\cdot}\dot{2}\dot{1} = \frac{21}{99} = \frac{7}{33}$$

and

$3{\cdot}2\dot{1} = 3{\cdot}2 + 0{\cdot}0\dot{1}$ and $10 \times 0{\cdot}0\dot{1} = 0{\cdot}1\dot{1}$ so that $9 \times 0{\cdot}0\dot{1} = 0{\cdot}1$ giving

$0{\cdot}0\dot{1} = \dfrac{0{\cdot}1}{9} = \dfrac{1}{90}$, hence

$$3{\cdot}2\dot{1} = \frac{32}{10} + \frac{1}{90} = \frac{289}{90} = 3\tfrac{19}{90}$$

Rational, irrational and real numbers

74

A number that can be expressed as a fraction is called a *rational* number. An *irrational* number is one that *cannot* be expressed as a fraction and has a decimal form consisting of an infinite string of numerals that does not display a repeating pattern. As a consequence it is not possible either to write down the complete decimal form or to devise an abbreviated decimal format. Instead, we can only round them to a specified number of significant figures or decimal places. Alternatively, we may have a numeral representation for them, such as $\sqrt{2}$, e or π. The complete collection of rational and irrational numbers is called the collection of *real* numbers.

At this point let us pause and summarize the main facts so far on decimal numbers

75

Review summary

1 A decimal number can be rounded to a specified number of significant figures (sig fig) by counting from the first non-zero numeral on the left. [58]

2 A decimal number can be rounded to a specified number of decimal places (dp) by counting from the decimal point. [61]

3 Every fraction can be written as a decimal number by performing the division. [66]

4 The decimal number obtained will consist of an infinitely repeating pattern of numerals to the right of one of its digits. [70]

5 Other decimals, with an infinite, non-repeating sequence of numerals after the decimal point are the irrational numbers. [79]

76

Review exercise

1 Round each of the following decimal numbers, first to 3 significant figures and then to 2 decimal places:

(a) $12{\cdot}455$ (b) $0{\cdot}01356$ (c) $0{\cdot}1005$ (d) $1344{\cdot}555$

2 Write each of the following in abbreviated form:

(a) $12{\cdot}110110110\ldots$ (b) $0{\cdot}123123123\ldots$
(c) $-3{\cdot}11111\ldots$ (d) $-9360{\cdot}936093609360\ldots$

3 Convert each of the following to decimal form to 3 decimal places:

(a) $\dfrac{3}{16}$ (b) $-\dfrac{5}{9}$ (c) $\dfrac{7}{6}$ (d) $-\dfrac{24}{11}$

4 Convert each of the following to fractional form in lowest terms:

(a) $0{\cdot}6$ (b) $1{\cdot}\dot{4}$ (c) $1{\cdot}\dot{2}\dot{4}$ (d) $-7{\cdot}3$

77

1 (a) $12{\cdot}5$, $12{\cdot}46$ (b) $0{\cdot}0136$, $0{\cdot}01$ (c) $0{\cdot}101$, $0{\cdot}10$ (d) 1340, $1344{\cdot}56$

2 (a) $12{\cdot}\dot{1}1\dot{0}$ (b) $0{\cdot}\dot{1}2\dot{3}$ (c) $-3{\cdot}\dot{1}$ (d) $-9360{\cdot}\dot{9}36\dot{0}$

3 (a) $\dfrac{3}{16} = 0{\cdot}1875 = 0{\cdot}188$ to 3 dp

(b) $-\dfrac{5}{9} = -0{\cdot}555\ldots = -0{\cdot}556$ to 3 dp

(c) $\dfrac{7}{6} = 1{\cdot}1666\ldots = 1{\cdot}167$ to 3 dp

(d) $-\dfrac{24}{11} = -2{\cdot}1818\ldots = -2{\cdot}182$ to 3 dp

4 (a) $0{\cdot}6 = \dfrac{6}{10} = \dfrac{3}{5}$

(b) $1{\cdot}\dot{4} = 1 + \dfrac{4}{9} = \dfrac{13}{9}$

(c) $1{\cdot}\dot{2}\dot{4} = 1 + \dfrac{24}{99} = \dfrac{123}{99} = \dfrac{41}{33}$

(d) $-7{\cdot}3 = -\dfrac{73}{10}$

Now move on to the next topic

Powers

78

Raising a number to a power

The arithmetic operation of raising a number to a *power* is devised from repetitive multiplication. For example:

$10 \times 10 \times 10 \times 10 = 10^4$ that is, 4 number 10s multiplied together

The power is also called an *index* and the number to be raised to the power is called the *base*. Here the number 4 is the power (index) and 10 is the base.

So $5 \times 5 \times 5 \times 5 \times 5 \times 5 =$ .... (in the form of 5 raised to a power)

Compare your answer with the next frame

79

$$\boxed{5^6}$$

Because the number 5 (the base) is multiplied by itself 6 times (the power or index).

Now to the next frame

80

The laws of powers

The laws of powers are contained within the following set of rules:

- **Power unity**

  Any number raised to the power 1 equals itself.

  $3^1 = 3$

So $99^1 =$ ....

On to the next frame

81

$$\boxed{99}$$

Because any number raised to the power 1 equals itself.

- **Multiplication of numbers and the addition of powers**

  If two numbers are each written as a given base raised to some power then the *product of the two numbers* is equal to the same base raised to the *sum of the powers*. For example, $16 = 2^4$ and $8 = 2^3$ so:

$$\begin{aligned} 16 \times 8 &= 2^4 \times 2^3 \\ &= (2 \times 2 \times 2 \times 2) \times (2 \times 2 \times 2) \\ &= 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \\ &= 2^7 \\ &= 2^{4+3} \\ &= 128 \end{aligned}$$

  Multiplication requires powers to be added.

So $8^3 \times 8^5 =$ .... (in the form of 8 raised to a power)

Next frame

82

$\boxed{8^8}$

Because multiplication requires powers to be added.

Notice that we cannot combine different powers with different bases. For example:

$2^2 \times 4^3$ cannot be written as 8^5

but we can combine different bases to the same power. For example:

$3^4 \times 5^4$ can be written as 15^4 because

$$\begin{aligned} 3^4 \times 5^4 &= (3 \times 3 \times 3 \times 3) \times (5 \times 5 \times 5 \times 5) \\ &= 15 \times 15 \times 15 \times 15 \\ &= 15^4 \\ &= (3 \times 5)^4 \end{aligned}$$

So that $2^3 \times 4^3$ can be written as . . . .

(in the form of a number raised to a power)

83

$\boxed{8^3}$

Next frame

84

- **Division of numbers and the subtraction of powers**

  If two numbers are each written as a given base raised to some power then the *quotient of the two numbers* is equal to the same base raised to the *difference of the powers*. For example:

$$\begin{aligned} 15\,625 \div 25 &= 5^6 \div 5^2 \\ &= (5 \times 5 \times 5 \times 5 \times 5 \times 5) \div (5 \times 5) \\ &= \frac{5 \times 5 \times 5 \times 5 \times 5 \times 5}{5 \times 5} \\ &= 5 \times 5 \times 5 \times 5 \\ &= 5^4 \\ &= 5^{6-2} \\ &= 625 \end{aligned}$$

  Division requires powers to be subtracted.

So $12^7 \div 12^3 =$ . . . . (in the form of 12 raised to a power)

Check your result in the next frame

85

$\boxed{12^4}$

Because division requires the powers to be subtracted.

- **Power zero**

  Any number raised to the power 0 equals unity. For example:

$$\begin{aligned} 1 &= 3^1 \div 3^1 \\ &= 3^{1-1} \\ &= 3^0 \end{aligned}$$

So $193^0 =$ . . . .

86

$$\boxed{1}$$

Because any number raised to the power 0 equals unity.

- **Negative powers**

  A number raised to a negative power denotes the reciprocal. For example:

$$\begin{aligned} 6^{-2} &= 6^{0-2} \\ &= 6^0 \div 6^2 \qquad \text{subtraction of powers means division} \\ &= 1 \div 6^2 \qquad \text{because } 6^0 = 1 \\ &= \frac{1}{6^2} \end{aligned}$$

Also $6^{-1} = \dfrac{1}{6}$

A negative power denotes the reciprocal.

So $3^{-5} = \ldots\ldots\ldots\ldots$

87

$$\boxed{\frac{1}{3^5}}$$

Because

$$3^{-5} = 3^{0-5} = 3^0 \div 3^5 = \frac{1}{3^5}$$

A negative power denotes the reciprocal.

Now to the next frame

88

- **Multiplication of powers**

  If a number is written as a given base raised to some power then that number *raised to a further power* is equal to the base raised to the *product of the powers*. For example:

$$\begin{aligned} (25)^3 &= \left(5^2\right)^3 \\ &= 5^2 \times 5^2 \times 5^2 \\ &= 5 \times 5 \times 5 \times 5 \times 5 \times 5 \\ &= 5^6 \\ &= 5^{2\times 3} \\ &= 15\,625 \qquad \text{Notice that } \left(5^2\right)^3 \neq 5^{(2^3)} \text{ because } 5^{(2^3)} = 5^8 = 390\,625. \end{aligned}$$

Raising to a power requires powers to be multiplied.

So $\left(4^2\right)^4 = \ldots\ldots\ldots\ldots$ (in the form of 4 raised to a power)

89

$$\boxed{4^8}$$

Because raising to a power requires powers to be multiplied.

Now to the next frame

90

Powers on a calculator

Powers on a calculator can be evaluated by using the $x^{\blacksquare}$ key (again, on other calculators this is often annotated as x^y). For example, enter the number 4, press the $x^{\blacksquare}$ key, enter the number 3 and press $=$. The result is 64 which is 4^3.

Try this one for yourself. To two decimal places, the value of $1{\cdot}3^{3{\cdot}4}$ is . . . .

The result is in the following frame

91

$$\boxed{2{\cdot}44}$$

Because

Enter the number 1·3
Press the $x^{\blacksquare}$ key
Enter the number 3·4
Press the $=$ key

The number displayed is 2·44 to 2 dp.

Now try this one using the calculator:

$8^{\frac{1}{3}} =$ . . . . The 1/3 is a problem, use the fraction key $\frac{\blacksquare}{\square}$

Check your answer in the next frame

92

$$\boxed{2}$$

Because

Enter the number 8
Press the $x^{\blacksquare}$ key
Enter the number 1
Press the fraction key
Enter the number 3
Press $=$ the number 2 is displayed.

Now move on to the next frame

93

Fractional powers and roots

We have just seen that $8^{\frac{1}{3}} = 2$. We call $8^{\frac{1}{3}}$ the *third root* or, alternatively, the *cube root* of 8 because:

$\left(8^{\frac{1}{3}}\right)^3 = 8$ *the number 8 is the result of raising the 3rd root of 8 to the power 3*

Roots are denoted by such fractional powers. For example, the 5th root of 6 is given as $6^{\frac{1}{5}}$ because:

$$\left(6^{\frac{1}{5}}\right)^5 = 6$$

and by using a calculator $6^{\frac{1}{5}}$ can be seen to be equal to 1·431 to 3 dp. Odd roots are unique in the real number system but even roots are not. For example, there are two 2nd roots – *square roots* – of 4, namely:

$4^{\frac{1}{2}} = 2$ and $4^{\frac{1}{2}} = -2$ because $2 \times 2 = 4$ and $(-2) \times (-2) = 4$

Similarly:

$$81^{\frac{1}{4}} = \pm 3$$

▶

Odd roots of negative numbers are themselves negative. For example:

$$(-32)^{\frac{1}{5}} = -2 \text{ because } \left[(-32)^{\frac{1}{5}}\right]^5 = (-2)^5 = -32$$

Even roots of negative numbers, however, pose a problem. For example, because

$$\left[(-1)^{\frac{1}{2}}\right]^2 = (-1)^1 = -1$$

we conclude that the square root of -1 is $(-1)^{\frac{1}{2}}$. Unfortunately, we cannot write this as a decimal number – we cannot find its decimal value because there is no decimal number which when multiplied by itself gives -1. We decide to accept the fact that, for now, we cannot find the even roots of a negative number. We shall return to this problem in a later programme when we introduce complex numbers.

Surds

An alternative notation for the square root of 4 is the surd notation $\sqrt{4}$ and, by convention, this is always taken to mean the positive square root. This notation can also be extended to other roots, for example, $\sqrt[7]{9}$ is an alternative notation for $9^{\frac{1}{7}}$.

Use your calculator to find the value of each of the following roots to 3 dp:

(a) $16^{\frac{1}{7}}$ (b) $\sqrt{8}$ (c) $19^{\frac{1}{4}}$ (d) $\sqrt{-4}$

Answers in the next frame

94

(a) 1·486	use the fraction key
(b) 2·828	the positive value only
(c) ±2·088	there are two values for even roots
(d) We cannot find the square root of a negative number	

On now to Frame 95

95

Multiplication and division by integer powers of 10

If a decimal number is multiplied by 10 raised to an integer power, the decimal point moves the integer number of places to the right if the integer is positive and to the left if the integer is negative. For example:

$1{\cdot}2345 \times 10^3 = 1234{\cdot}5$ (3 places to the right) and

$1{\cdot}2345 \times 10^{-2} = 0{\cdot}012345$ (2 places to the left).

Notice that, for example:

$1{\cdot}2345 \div 10^3 = 1{\cdot}2345 \times 10^{-3}$ and

$1{\cdot}2345 \div 10^{-2} = 1{\cdot}2345 \times 10^2$

So now try these:

(a) $0{\cdot}012045 \times 10^4$

(b) $13{\cdot}5074 \times 10^{-3}$

(c) $144{\cdot}032 \div 10^5$

(d) $0{\cdot}012045 \div 10^{-2}$

Work all four out and then check your results with the next frame

96

(a) 120·45
(b) 0·0135074
(c) 0·00144032
(d) 1·2045

Because

(a) multiplying by 10^4 moves the decimal point 4 places to the right
(b) multiplying by 10^{-3} moves the decimal point 3 places to the left
(c) $144{\cdot}032 \div 10^5 = 144{\cdot}032 \times 10^{-5}$ move the decimal point 5 places left
(d) $0{\cdot}012045 \div 10^{-2} = 0{\cdot}012045 \times 10^2$ move the decimal point 2 places right

Now move on to the next frame

97

Precedence rules

With the introduction of the arithmetic operation of raising to a power we need to amend our earlier precedence rules – *evaluating powers is performed before dividing and multiplying*. For example:

$$\begin{aligned} 5\left(3 \times 4^2 \div 6 - 7\right) &= 5(3 \times 16 \div 6 - 7) \\ &= 5(48 \div 6 - 7) \\ &= 5(8 - 7) \\ &= 5 \end{aligned}$$

So that $14 \div \left(125 \div 5^3 \times 4 + 3\right) =$ . . . .

Check your result in the next frame

98

$\boxed{2}$

Because

$$\begin{aligned} 14 \div \left(125 \div 5^3 \times 4 + 3\right) &= 14 \div (125 \div 125 \times 4 + 3) \\ &= 14 \div (4 + 3) \\ &= 2 \end{aligned}$$

99

Standard form

Any decimal number can be written as a decimal number greater than or equal to 1 and less than 10 (called the *mantissa*) multiplied by the number 10 raised to an appropriate power (the power being called the *exponent*). For example:

$$\begin{aligned} 57{\cdot}3 &= 5{\cdot}73 \times 10^1 \\ 423{\cdot}8 &= 4{\cdot}238 \times 10^2 \\ 6042{\cdot}3 &= 6{\cdot}0423 \times 10^3 \\ \text{and} \quad 0{\cdot}267 &= 2{\cdot}67 \div 10 = 2{\cdot}67 \times 10^{-1} \\ 0{\cdot}000485 &= 4{\cdot}85 \div 10^4 = 4{\cdot}85 \times 10^{-4} \text{ etc.} \end{aligned}$$

So, written in standard form:

(a) 52 674 = ...
(b) 0·00723 = ...
(c) 0·0582 = ...
(d) 1 523 800 = ...

100

(a) $5{\cdot}2674 \times 10^{4}$ (c) $5{\cdot}82 \times 10^{-2}$
(b) $7{\cdot}23 \times 10^{-3}$ (d) $1{\cdot}5238 \times 10^{6}$

Working in standard form

Numbers written in standard form can be multiplied or divided by multiplying or dividing the respective mantissas and adding or subtracting the respective exponents. For example:

$$\begin{aligned} 0{\cdot}84 \times 23\,000 &= \left(8{\cdot}4 \times 10^{-1}\right) \times \left(2{\cdot}3 \times 10^{4}\right) \\ &= (8{\cdot}4 \times 2{\cdot}3) \times 10^{-1} \times 10^{4} \\ &= 19{\cdot}32 \times 10^{3} \\ &= 1{\cdot}932 \times 10^{4} \end{aligned}$$

Another example:

$$\begin{aligned} 175{\cdot}4 \div 6340 &= \left(1{\cdot}754 \times 10^{2}\right) \div \left(6{\cdot}34 \times 10^{3}\right) \\ &= (1{\cdot}754 \div 6{\cdot}34) \times 10^{2} \div 10^{3} \\ &= 0{\cdot}2767 \times 10^{-1} \\ &= 2{\cdot}767 \times 10^{-2} \text{ to 4 sig fig} \end{aligned}$$

Where the result obtained is not in standard form, the mantissa is written in standard number form and the necessary adjustment made to the exponent.

In the same way, then, giving the results in standard form to 4 dp:

(a) $472{\cdot}3 \times 0{\cdot}000564 =$ . . . .
(b) $752\,000 \div 0{\cdot}862 =$ . . . .

101

(a) $2{\cdot}6638 \times 10^{-1}$
(b) $8{\cdot}7239 \times 10^{5}$

Because

(a) $$\begin{aligned} 472{\cdot}3 \times 0{\cdot}000564 &= \left(4{\cdot}723 \times 10^{2}\right) \times \left(5{\cdot}64 \times 10^{-4}\right) \\ &= (4{\cdot}723 \times 5{\cdot}64) \times 10^{2} \times 10^{-4} \\ &= 26{\cdot}638 \times 10^{-2} = 2{\cdot}6638 \times 10^{-1} \end{aligned}$$

(b) $$\begin{aligned} 752\,000 \div 0{\cdot}862 &= \left(7{\cdot}52 \times 10^{5}\right) \div \left(8{\cdot}62 \times 10^{-1}\right) \\ &= (7{\cdot}52 \div 8{\cdot}62) \times 10^{5} \times 10^{1} \\ &= 0{\cdot}87239 \times 10^{6} = 8{\cdot}7239 \times 10^{5} \end{aligned}$$

For *addition and subtraction in standard form* the approach is slightly different.

Example 1

$4{\cdot}72 \times 10^{3} + 3{\cdot}648 \times 10^{4}$

Before these can be added, the powers of 10 must be made the same:

$$\begin{aligned} 4{\cdot}72 \times 10^{3} + 3{\cdot}648 \times 10^{4} &= 4{\cdot}72 \times 10^{3} + 36{\cdot}48 \times 10^{3} \\ &= (4{\cdot}72 + 36{\cdot}48) \times 10^{3} \\ &= 41{\cdot}2 \times 10^{3} = 4{\cdot}12 \times 10^{4} \text{ in standard form} \end{aligned}$$

Similarly in the next example.

▶

Example 2

$13{\cdot}26 \times 10^{-3} - 1{\cdot}13 \times 10^{-2}$

Here again, the powers of 10 must be equalized:

$$\begin{aligned}13{\cdot}26 \times 10^{-3} - 1{\cdot}13 \times 10^{-2} &= 1{\cdot}326 \times 10^{-2} - 1{\cdot}13 \times 10^{-2}\\ &= (1{\cdot}326 - 1{\cdot}13) \times 10^{-2}\\ &= 0{\cdot}196 \times 10^{-2} = 1{\cdot}96 \times 10^{-3} \text{ in standard form}\end{aligned}$$

Using a calculator

Numbers given in standard form can be manipulated on the Casio calculator by making use of the Mode key.

Press SHIFT followed by MODE	
Select option 7 : Sci	This is the standard form display
Enter 5	This is the number of significant figures

Now enter the number 12345 and press the = key to give the display:

$1{\cdot}2345 \times 10^4$

Numbers can be entered in standard form as well.

Press AC to clear the screen
Enter the number 1·234
Press ×
Press SHIFT followed by the log key (this accesses $10^{\blacksquare}$)
Enter the number 3
Press =

The number displayed is then $1{\cdot}234 \times 10^3$. We can also manipulate numbers in standard form on the calculator. For example:

Enter $1{\cdot}234 \times 10^3 + 2{\cdot}6 \times 10^2$ and press =

This results in the display:

$1{\cdot}4940 \times 10^3$

Therefore, working in standard form:

(a) $43{\cdot}6 \times 10^2 + 8{\cdot}12 \times 10^3$ = ...
(b) $7{\cdot}84 \times 10^5 - 12{\cdot}36 \times 10^3$ = ...
(c) $4{\cdot}25 \times 10^{-3} + 1{\cdot}74 \times 10^{-2}$ = ...

102

(a) $1{\cdot}248 \times 10^4$
(b) $7{\cdot}7164 \times 10^5$
(c) $2{\cdot}165 \times 10^{-2}$

Preferred standard form

In the SI system of units, it is recommended that when a number is written in standard form, the power of 10 should be restricted to powers of 10^3, i.e. 10^3, 10^6, 10^{-3}, 10^{-6}, etc. Therefore in this *preferred standard form* up to three figures may appear in front of the decimal point.

In practice it is best to write the number first in standard form and to adjust the power of 10 to express this in preferred standard form.

Example 1

$$
\begin{aligned}
&5{\cdot}2746 \times 10^4 \text{ in standard form}\\
&= 5{\cdot}2746 \times 10 \times 10^3\\
&= 52{\cdot}746 \times 10^3 \text{ in preferred standard form}
\end{aligned}
$$

Example 2

$$
\begin{aligned}
&3{\cdot}472 \times 10^8 \text{ in standard form}\\
&= 3{\cdot}472 \times 10^2 \times 10^6\\
&= 347{\cdot}2 \times 10^6 \text{ in preferred standard form}
\end{aligned}
$$

Example 3

$$
\begin{aligned}
&3{\cdot}684 \times 10^{-2} \text{ in standard form}\\
&= 3{\cdot}684 \times 10 \times 10^{-3}\\
&= 36{\cdot}84 \times 10^{-3} \text{ in preferred standard form}
\end{aligned}
$$

If you are using the Casio calculator then all this can be achieved by using the ENG key whose action is to alter the display of a number in standard form to preferred standard form. For example,

Enter the number 123456 and press the = key
Press the ENG key and the display changes to $123{\cdot}456 \times 10^3$

So, rewriting the following in preferred standard form, we have

(a) $8{\cdot}236 \times 10^7 =$ ....
(b) $1{\cdot}624 \times 10^{-4} =$ ....
(c) $4{\cdot}827 \times 10^4 =$ ....
(d) $6{\cdot}243 \times 10^5 =$ ....
(e) $3{\cdot}274 \times 10^{-2} =$ ....
(f) $5{\cdot}362 \times 10^{-7} =$ ....

103

(a) $82{\cdot}36 \times 10^6$
(b) $162{\cdot}4 \times 10^{-6}$
(c) $48{\cdot}27 \times 10^3$
(d) $624{\cdot}3 \times 10^3$
(e) $32{\cdot}74 \times 10^{-3}$
(f) $536{\cdot}2 \times 10^{-9}$

One final exercise on this piece of work:

Example 4

The product of $(4{\cdot}72 \times 10^2)$ and $(8{\cdot}36 \times 10^5)$

(a) in standard form = ...
(b) in preferred standard form = ...

104

(a) $3{\cdot}9459 \times 10^8$
(b) $394{\cdot}59 \times 10^6$

Because

$$
\begin{aligned}
\text{(a)}\quad (4{\cdot}72 \times 10^2) \times (8{\cdot}36 \times 10^5) &= (4{\cdot}72 \times 8{\cdot}36) \times 10^2 \times 10^5\\
&= 39{\cdot}459 \times 10^7\\
&= 3{\cdot}9459 \times 10^8 \text{ in standard form}\\
\text{(b)}\quad (4{\cdot}72 \times 10^2) \times (8{\cdot}36 \times 10^5) &= 3{\cdot}9459 \times 10^2 \times 10^6\\
&= 394{\cdot}59 \times 10^6 \text{ in preferred standard form}
\end{aligned}
$$

Now move on to the next frame

105

Checking calculations

When performing a calculation involving decimal numbers it is always a good idea to check that your result is reasonable and that an arithmetic blunder or an error in using the calculator has not been made. This can be done using standard form. For example:

$$\begin{aligned}59{\cdot}2347 \times 289{\cdot}053 &= 5{\cdot}92347 \times 10^1 \times 2{\cdot}89053 \times 10^2\\ &= 5{\cdot}92347 \times 2{\cdot}89053 \times 10^3\end{aligned}$$

This product can then be estimated for reasonableness as:

$6 \times 3 \times 1000 = 18\,000$ (see Frames 22–24)

The answer using the calculator is 17 121·968 to three decimal places, which is 17 000 when rounded to the nearest 1000. This compares favourably with the estimated 18 000, indicating that the result obtained could be reasonably expected.

So, the estimated value of $800{\cdot}120 \times 0{\cdot}007953$ is ....

Check with the next frame

106

$$\boxed{6{\cdot}4}$$

Because

$$\begin{aligned}800{\cdot}120 \times 0{\cdot}007953 &= 8{\cdot}00120 \times 10^2 \times 7{\cdot}953 \times 10^{-3}\\ &= 8{\cdot}00120 \times 7{\cdot}9533 \times 10^{-1}\end{aligned}$$

This product can then be estimated for reasonableness as:

$8 \times 8 \div 10 = 6{\cdot}4$

The exact answer is 6·36 to two decimal places.

Now move on to the next frame

107

Accuracy

Many calculations are made using numbers that have been obtained from measurements. Such numbers are only accurate to a given number of significant figures but using a calculator can produce a result that contains as many figures as its display will permit. Because any calculation involving measured values will not be accurate to *more significant figures than the least number of significant figures in any measurement*, we can justifiably round the result down to a more manageable number of significant figures. For example:

> The base length and height of a rectangle are measured as 114·8 mm and 18 mm respectively. The area of the rectangle is given as the product of these lengths. Using a calculator this product is 2066·4 mm^2. Because one of the lengths is only measured to 2 significant figures, the result cannot be accurate to more than 2 significant figures. It should therefore be read as 2100 mm^2.

Assuming the following contains numbers obtained by measurement, use a calculator to find the value to the correct level of accuracy:

$19{\cdot}1 \times 0{\cdot}0053 \div 13{\cdot}345$

108

$\boxed{0{\cdot}0076}$

Because

> The calculator gives the result as 0·00758561 but because 0·0053 is only accurate to 2 significant figures the result cannot be accurate to more than 2 significant figures, namely 0·0076.

At this point let us pause and summarize the main facts so far on powers

109

Review summary

1 Powers are devised from repetitive multiplication of a given number. [78]

2 Negative powers denote reciprocals and any number raised to the power 0 is unity. [85]

3 Multiplication of a decimal number by 10 raised to an integer power moves the decimal point to the right if the power is positive and to the left if the power is negative. [95]

4 A decimal number written in standard form is in the form of a mantissa (a number between 1 and 10 but excluding 10) multiplied by 10 raised to an integer power, the power being called the exponent. [99]

5 In preferred standard form the powers of 10 in the exponent are restricted to multiples of 3. [102]

6 Writing decimal numbers in standard form permits an estimation of the reasonableness of a calculation. [105]

7 If numbers used in a calculation are obtained from measurement, the result of the calculation is a number accurate to no more than the least number of significant figures in any measurement. [107]

110

Review exercise

1 Write each of the following as a number raised to a power:

(a) $5^8 \times 5^2$ (b) $6^4 \div 6^6$ (c) $\left(7^4\right)^3$ (d) $\left(19^{-8}\right)^0$

2 Find the value of each of the following to 3 dp:

(a) $16^{\frac{1}{4}}$ (b) $\sqrt[3]{3}$ (c) $(-8)^{\frac{1}{5}}$ (d) $(-7)^{\frac{1}{4}}$

3 Write each of the following as a single decimal number:

(a) $1{\cdot}0521 \times 10^3$ (b) $123{\cdot}456 \times 10^{-2}$

(c) $0{\cdot}0135 \div 10^{-3}$ (d) $165{\cdot}21 \div 10^4$

4 Write each of the following in standard form:

(a) 125·87 (b) 0·0101 (c) 1·345 (d) 10·13

5 Write each of the following in preferred standard form:

(a) $1{\cdot}3204 \times 10^5$ (b) 0·0101 (c) 1·345 (d) $9{\cdot}5032 \times 10^{-8}$

▶

6 In each of the following the numbers have been obtained by measurement. Evaluate each calculation to the appropriate level of accuracy:

(a) $13{\cdot}6 \div 0{\cdot}012 \times 7{\cdot}63 - 9015$ (b) $\dfrac{0{\cdot}003 \times 194}{13{\cdot}6}$

(c) $19{\cdot}3 \times 1{\cdot}04^{2{\cdot}00}$ (d) $\dfrac{18 \times 2{\cdot}1 - 3{\cdot}6 \times 0{\cdot}54}{8{\cdot}6 \times 2{\cdot}9 + 5{\cdot}7 \times 9{\cdot}2}$

111

1 (a) $5^8 \times 5^2 = 5^{8+2} = 5^{10}$ (b) $6^4 \div 6^6 = 6^{4-6} = 6^{-2}$ (c) $\left(7^4\right)^3 = 7^{4\times3} = 7^{12}$

(d) $\left(19^{-8}\right)^0 = 1$ as any number raised to the power 0 equals unity

2 (a) $16^{\frac{1}{4}} = \pm 2{\cdot}000$ (b) $\sqrt[3]{3} = 1{\cdot}442$ (c) $5(-8)^{\frac{1}{5}} = -1{\cdot}516$

(d) $(-7)^{\frac{1}{4}}$ You cannot find the even root of a negative number

3 (a) $1{\cdot}0521 \times 10^3 = 1052{\cdot}1$ (b) $123{\cdot}456 \times 10^{-2} = 1{\cdot}23456$

(c) $0{\cdot}0135 \div 10^{-3} = 0{\cdot}0135 \times 10^3 = 13{\cdot}5$

(d) $165{\cdot}21 \div 10^4 = 165{\cdot}21 \times 10^{-4} = 0{\cdot}016521$

4 (a) $125{\cdot}87 = 1{\cdot}2587 \times 10^2$ (b) $0{\cdot}0101 = 1{\cdot}01 \times 10^{-2}$

(c) $1{\cdot}345 = 1{\cdot}345 \times 10^0$ (d) $10{\cdot}13 = 1{\cdot}013 \times 10^1 = 1{\cdot}013 \times 10$

5 (a) $1{\cdot}3204 \times 10^5 = 132{\cdot}04 \times 10^3$ (b) $0{\cdot}0101 = 10{\cdot}1 \times 10^{-3}$

(c) $1{\cdot}345 = 1{\cdot}345 \times 10^0$ (d) $9{\cdot}5032 \times 10^{-8} = 95{\cdot}032 \times 10^{-9}$

6 (a) $13{\cdot}6 \div 0{\cdot}012 \times 7{\cdot}63 - 9015 = -367{\cdot}\dot{6} = -370$ to 2 sig fig

(b) $\dfrac{0{\cdot}003 \times 194}{13{\cdot}6} = 0{\cdot}042794\ldots = 0{\cdot}04$ to 1 sig fig

(c) $19{\cdot}3 \times 1{\cdot}04^{2{\cdot}00} = 19{\cdot}3 \times 1{\cdot}0816 = 20{\cdot}87488 = 20{\cdot}9$ to 3 sig fig

(d) $\dfrac{18 \times 2{\cdot}1 - 3{\cdot}6 \times 0{\cdot}54}{8{\cdot}6 \times 2{\cdot}9 + 5{\cdot}7 \times 9{\cdot}2} = \dfrac{35{\cdot}856}{77{\cdot}38} = 0{\cdot}46337554\ldots = 0{\cdot}46$ to 2 sig fig

On now to the next topic

Number systems

112

Denary (or decimal) system

This is our basic system in which quantities large or small can be represented by use of the symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 together with appropriate place values according to their positions.

For example	2	7	6	5	·	3	2	1_{10}
has place values	10^3	10^2	10^1	10^0		10^{-1}	10^{-2}	10^{-3}
	1000	100	10	1		$\dfrac{1}{10}$	$\dfrac{1}{100}$	$\dfrac{1}{1000}$

In this case, the place values are powers of 10, which gives the name *denary* (or *decimal*) to the system. The denary system is said to have a *base* of 10. You are, of course, perfectly familiar with this system of numbers, but it is included here as it leads on to other systems which have the same type of structure but which use different place values.

So let us move on to the next system

113

Binary system

This is widely used in all forms of switching applications. The only symbols used are 0 and 1 and the place values are powers of 2, i.e. the system has a base of 2.

$$\begin{array}{lcccccccc} \text{For example} & 1 & 0 & 1 & 1 & \cdot & 1 & 0 & 1_2 \\ \text{has place values} & 2^3 & 2^2 & 2^1 & 2^0 & & 2^{-1} & 2^{-2} & 2^{-3} \\ \text{i.e.} & 8 & 4 & 2 & 1 & & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} \end{array}$$

$$\begin{array}{lccccccccccccccl} \text{So:} & 1 & & 0 & & 1 & & 1 & \cdot & 1 & & 0 & & 1 & & \text{in the binary system} \\ = & 1\times 8 & & 0\times 4 & & 1\times 2 & & 1\times 1 & & 1\times\frac{1}{2} & & 0\times\frac{1}{4} & & 1\times\frac{1}{8} & & \\ = & 8 & + & 0 & + & 2 & + & 1 & + & \frac{1}{2} & + & 0 & + & \frac{1}{8} & & \text{in denary} \end{array}$$

$= 11\frac{5}{8} = 11{\cdot}625$ in the denary system.

Therefore $1011{\cdot}101_2 = 11{\cdot}625_{10}$

The small subscripts 2 and 10 indicate the bases of the two systems. In the same way, the denary equivalent of $1\ 1\ 0\ 1\ \cdot\ 0\ 1\ 1_2$ is:

. . . . to 3dp.

114

$$\boxed{13{\cdot}375_{10}}$$

Because

$$\begin{array}{lccccccccccccc} & 1 & & 1 & & 0 & & 1 & \cdot & 0 & & 1 & & 1_2 \\ = & 8 & + & 4 & + & 0 & + & 1 & + & 0 & + & \frac{1}{4} & + & \frac{1}{8} \end{array}$$

$= 13\frac{3}{8} = 13{\cdot}375_{10}$

Octal system (base 8)

This system uses the symbols

0, 1, 2, 3, 4, 5, 6, 7

with place values that are powers of 8.

$$\begin{array}{lccccccl} \text{For example} & 3 & 5 & 7 & \cdot & 3 & 2 & 1 \quad \text{in the octal system} \\ \text{has place values} & 8^2 & 8^1 & 8^0 & & 8^{-1} & 8^{-2} & 8^{-3} \\ \text{i.e.} & 64 & 8 & 1 & & \frac{1}{8} & \frac{1}{64} & \frac{1}{512} \end{array}$$

$$\begin{array}{lccccccccccc} \text{So} & 3 & & 5 & & 7 & \cdot & 3 & & 2 & & 1_8 \\ = & 3\times 64 & & 5\times 8 & & 7\times 1 & & 3\times\frac{1}{8} & & 2\times\frac{1}{64} & & 1\times\frac{1}{512} \\ = & 192 & + & 40 & + & 7 & + & \frac{3}{8} & + & \frac{1}{32} & + & \frac{1}{512} \\ = & 239\frac{209}{512} & & = 239{\cdot}4082_{10} & & & & & & & & \end{array}$$

That is

$357{\cdot}321_8 = 239{\cdot}408_{10}$ to 3 dp

As you see, the method is very much as before: the only change is in the base of the place values.

In the same way then, $263{\cdot}452_8$ expressed in denary form is

. . . . to 3 dp.

115

$$\boxed{179{\cdot}582_{10}}$$

Because

$$\begin{aligned}
&\ 2\ 6\ 3\cdot 4\ 5\ 2_8\\
&= 2\times 8^2+6\times 8^1+3\times 8^0+4\times 8^{-1}+5\times 8^{-2}+2\times 8^{-3}\\
&= 2\times 64+6\times 8+3\times 1+4\times \tfrac{1}{8}+5\times \tfrac{1}{64}+2\times \tfrac{1}{512}\\
&= 128+48+3+\tfrac{1}{2}+\tfrac{5}{64}+\tfrac{1}{256}\\
&= 179\tfrac{149}{256}\\
&= 179{\cdot}582_{10} \text{ to 3 dp}
\end{aligned}$$

Now we come to the duodecimal system, which has a base of 12.

So move on to the next frame

116

Duodecimal system (base 12)

With a base of 12, the units column needs to accommodate symbols up to 11 before any carryover to the second column occurs. Unfortunately, our denary symbols go up to only 9, so we have to invent two extra symbols to represent the values 10 and 11. Several suggestions for these have been voiced in the past, but we will adopt the symbols X and Λ for 10 and 11 respectively. The first of these calls to mind the Roman numeral for 10 and the Λ symbol may be regarded as the two strokes of 11 tilted together $\overset{\curvearrowright}{1}\ \overset{\curvearrowleft}{1}$ to join at the top.

The duodecimal system, therefore, uses the symbols

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, X, Λ

and has place values that are powers of 12.

For example	2	X	5	·	1	3	6_{12}
has place values	12^2	12^1	12^0		12^{-1}	12^{-2}	12^{-3}
i.e.	144	12	1		$\dfrac{1}{12}$	$\dfrac{1}{144}$	$\dfrac{1}{1728}$

$$\begin{array}{lcccccccccccc}
\text{So} & 2 & & \text{X} & & 5 & \cdot & 1 & & 3 & & 6_{12}\\
= & 2\times 144 & + & 10\times 12 & + & 5\times 1 & + & 1\times\tfrac{1}{12} & + & 3\times\tfrac{1}{144} & + & 6\times\tfrac{1}{1728}
\end{array}$$

$$= \ldots\ldots\ldots\ldots_{10} \text{ to 3 dp}$$

Finish it off

117

$$\boxed{413{\cdot}108_{10}}$$

Because

$$\begin{aligned}
&\ 2\ \text{X}\ 5\cdot 1\ 3\ 6_{12}\\
&= 288+120+5+\tfrac{1}{12}+\tfrac{1}{48}+\tfrac{1}{288}\\
&= 413\tfrac{31}{288}
\end{aligned}$$

Therefore $2\text{X}5{\cdot}136_{12} = 413{\cdot}108_{10}$ to 3 dp.

Hexadecimal system (base 16)

This system has computer applications. The symbols here need to go up to an equivalent denary value of 15, so, after 9, letters of the alphabet are used as follows:

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F

The place values in this system are powers of 16.

For example	2	A	7	·	3	E	2_{16}
has place values	16^2	16^1	16^0		16^{-1}	16^{-2}	16^{-3}
i.e.	256	16	1		$\frac{1}{16}$	$\frac{1}{256}$	$\frac{1}{4096}$

Therefore $2\ A\ 7 \cdot 3\ E\ 2_{16} =$ .... to 3 dp.

118

$$\boxed{679{\cdot}243_{10}}$$

Here it is: $2\ \ A\ \ 7\ \cdot\ 3\ \ E\ \ 2_{16}$

$$= 2 \times 256 + 10 \times 16 + 7 \times 1 + 3 \times \frac{1}{16} + 14 \times \frac{1}{256} + 2 \times \frac{1}{4096}$$
$$= 679\tfrac{497}{2048} = 679{\cdot}243_{10} \text{ to 3 dp.}$$

And now, two more by way of practice.

Express the following in denary form:

(a) $3\ \Lambda\ 4 \cdot 2\ 6\ 5_{12}$
(b) $3\ C\ 4 \cdot 2\ 1\ F_{16}$

Finish both of them and check the working with the next frame

119

Here they are:

(a)

	3	Λ	4	·	2	6	5_{12}
Place values	144	12	1		$\frac{1}{12}$	$\frac{1}{144}$	$\frac{1}{1728}$

So $3\ \Lambda\ 4 \cdot 2\ 6\ 5_{12}$

$$= 3 \times 144 + 11 \times 12 + 4 \times 1 + 2 \times \frac{1}{12} + 6 \times \frac{1}{144} + 5 \times \frac{1}{1728}$$
$$= 432 + 132 + 4 + \frac{1}{6} + \frac{1}{24} + \frac{5}{1728}$$
$$= 568\tfrac{365}{1728} \quad = 568{\cdot}211_{10}$$

Therefore $3\ \Lambda\ 4 \cdot 265_{12} = 568{\cdot}211_{10}$ to 3 dp.

(b)

	3	C	4	·	2	1	F_{16}
Place values	256	16	1		$\frac{1}{16}$	$\frac{1}{256}$	$\frac{1}{4096}$

So $3\ C\ 4 \cdot 2\ 1\ F_{16}$

$$= 3 \times 256 + 12 \times 16 + 4 \times 1 + 2 \times \frac{1}{16} + 1 \times \frac{1}{256} + 15 \times \frac{1}{4096}$$
$$= 768 + 192 + 4 + \frac{1}{8} + \frac{1}{256} + \frac{15}{4096}$$
$$= 964\tfrac{543}{4096} \quad = 964{\cdot}133_{10}$$

Therefore $3\ C\,4 \cdot 2\,1\,F_{16} = 964{\cdot}133_{10}$ to 3 dp.

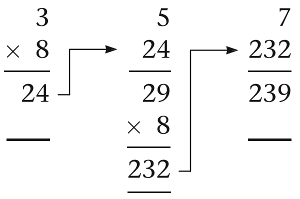
120

An alternative method

So far, we have changed numbers in various bases into the equivalent denary numbers from first principles. Another way of arriving at the same results is by using the fact that two adjacent columns differ in place values by a factor which is the base of the particular system. An example will show the method.

Express the octal $357{\cdot}121_8$ in denary form.

First of all, we will attend to the whole-number part 357_8. Starting at the left-hand end, multiply the first column by the base 8 and add the result to the entry in the next column (making 29).



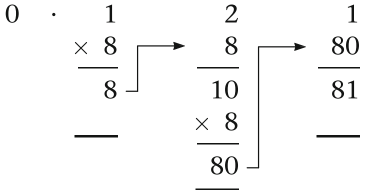
Now repeat the process. Multiply the second column total by 8 and add the result to the next column. This gives 239 in the units column.

So $357_8 = 239_{10}$

Now we do much the same with the decimal part of the octal number

121

$\boxed{\text{The decimal part is } 0{\cdot}121_8}$



Starting from the left-hand column immediately following the decimal point, multiply by 8 and add the result to the next column. Repeat the process, finally getting a total of 81 in the end column.

But the place value of this column is ....

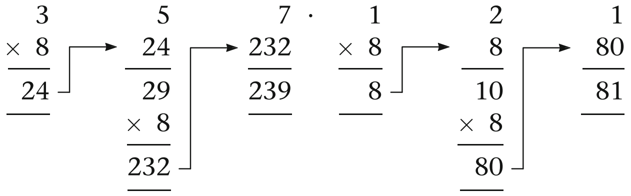
122

$\boxed{8^{-3}}$

The denary value of $0{\cdot}121_8$ is 81×8^{-3} i.e. $81 \times \dfrac{1}{8^3} = \dfrac{81}{512} = 0{\cdot}1582_{10}$

Collecting the two partial results together, $357{\cdot}121_8 = 239{\cdot}1582_{10}$ to 4 dp.

In fact, we can set this out across the page to save space, thus:



$$81 \times \frac{1}{8^3} = \frac{81}{512} = 0{\cdot}1582_{10} \quad \text{Therefore } 357{\cdot}121_8 = 239{\cdot}158_{10}$$

Now you can set this one out in similar manner.

Express the duodecimal $245{\cdot}136_{12}$ in denary form.

Space out the duodecimal digits to give yourself room for the working:

$2 \quad 4 \quad 5 \quad \cdot \quad 1 \quad 3 \quad 6_{12}$

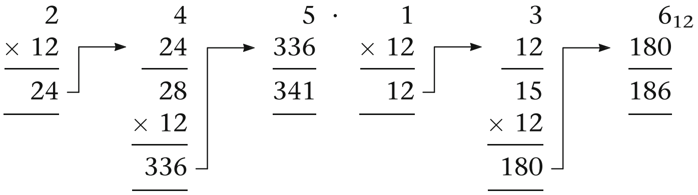
Then off you go.

$245{\cdot}136_{12} = \ldots\ldots\ldots\ldots$ to 4 dp.

123

$\boxed{341{\cdot}1076_{10}}$

Here is the working as a check:



Place value of last column is 12^{-3}, therefore

$$0{\cdot}136_{12} = 186 \times 12^{-3} = \frac{186}{1728} = 0{\cdot}1076_{10}$$

So $245{\cdot}136_{12} = 341{\cdot}1076_{10}$ to 4 dp.

On to the next

124

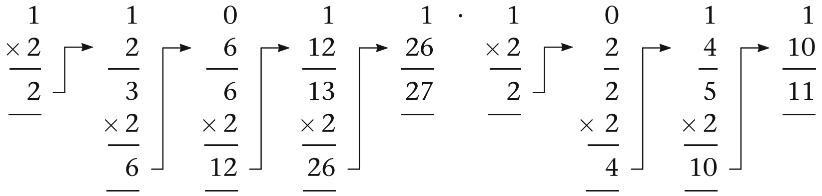
Now for an easy one. Find the denary equivalent of the binary number

$1\,1\,0\,1\,1 \cdot 1\,0\,1\,1_2$.

Setting it out in the same way, the result is $\ldots\ldots\ldots\ldots$ to 4 dp.

125

$\boxed{27{\cdot}6875_{10}}$



$$11 \times 2^{-4} = \frac{11}{16} = 0{\cdot}6875_{10}$$

Therefore $1\,1\,0\,1\,1 \cdot 1\,0\,1\,1_2 = 27{\cdot}6875_{10}$ to 4 dp.

And now a hexadecimal. Express $4\,C\,5 \cdot 2\,B\,8_{16}$ in denary form. Remember that $C = 12$ and $B = 11$. There are no snags.

$4\,C\,5 \cdot 2\,B\,8_{16} = \ldots\ldots\ldots\ldots$ to 4 dp.

126

$$\boxed{1221{\cdot}1699_{10}}$$

Here it is:

$$\begin{array}{rrrrrr}
4 & C & 5\,\cdot & 2 & B & 8\\
\times\ 16 & 64 & 1216 & \times\ 16 & 32 & 688\\
\hline
64 & 76 & 1221 & 32 & 43 & 696\\
 & \times\ 16 & & & \times\ 16 & \\
 & 456 & & & 258 & \\
 & 76 & & & 43 & \\
 & 1216 & & & 688 &
\end{array}$$

Place value 16^{-3}

So $\dfrac{696}{4096} = 0{\cdot}16992_{10}$

Therefore $4\ C\ 5 \cdot 2\ B\ 8_{16} = 1221{\cdot}1699_{10}$ to 4 dp.

They are all done in the same way.

By now then, we can change any binary, octal, duodecimal or hexadecimal number – including a decimal part – into the equivalent denary number. So here is a brief review summary followed by a short review exercise for valuable practice. Complete the set and check your results with Frame 129.

127 Review summary

1 *Denary (or decimal) system*: Base 10. Place values powers of 10. Symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. [112]

2 *Binary system*: Base 2. Place values powers of 2. Symbols 0, 1. [113]

3 *Octal system*: Base 8. Place values powers of 8. Symbols 0, 1, 2, 3, 4, 5, 6, 7. [114]

4 *Duodecimal system*: Base 12. Place values powers of 12. Symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, X, Λ. [116]

5 *Hexadecimal system*: Base 16. Place values powers of 16. Symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F. [117]

Now move on to the next frame

128 Review exercise

Express each of the following in denary form to 3 dp.

(a) $1\ 1\ 0\ 0\ 1 \cdot 1\ 1_2$

(b) $7\ 7\ 6 \cdot 1\ 4\ 3_8$

(c) $4\ X\ 9 \cdot 2\ \Lambda\ 5_{12}$

(d) $6\ F\ 8 \cdot 3\ D\ 5_{16}$

129

(a) $25{\cdot}750_{10}$
(b) $510{\cdot}193_{10}$
(c) $705{\cdot}246_{10}$
(d) $1784{\cdot}240_{10}$

Just in case you have made a slip anywhere, here is the working.

(a)
$$\begin{array}{ccccccccccccc}
1 & & 1 & & 0 & & 0 & & 1 & \cdot & 1 & & 1 \\
\underline{\times 2} & \rightarrow & \underline{2} & \rightarrow & \underline{6} & \rightarrow & \underline{12} & \rightarrow & \underline{24} & & \underline{\times 2} & \rightarrow & \underline{2} \\
\underline{2} & & 3 & & 6 & & 12 & & \underline{25_{10}} & & \underline{2} & & \underline{3} \times 2^{-2} = \frac{3}{4} = 0{\cdot}75_{10} \\
 & & \underline{\times 2} & & \underline{\times 2} & & \underline{\times 2} & & & & & & \\
 & & \underline{6} & & \underline{12} & & \underline{24} & & & & & &
\end{array}$$

$$11001{\cdot}11_2 = 25{\cdot}750_{10}$$

(b)
$$\begin{array}{ccccccccccc}
7 & & 7 & & 6 & \cdot & 1 & & 4 & & 3 \\
\underline{\times 8} & \rightarrow & \underline{56} & \rightarrow & \underline{504} & & \underline{\times 8} & \rightarrow & \underline{8} & \rightarrow & \underline{96} \\
\underline{56} & & 63 & & \underline{510_{10}} & & \underline{8} & & 12 & & \underline{99} \times 8^{-3} = \frac{99}{512} \\
 & & \underline{\times 8} & & & & & & \underline{\times 8} & & = 0{\cdot}19336_{10} \\
 & & \underline{504} & & & & & & \underline{96} & &
\end{array}$$

$$776{\cdot}143_8 = 510{\cdot}193_{10}$$

(c)
$$\begin{array}{ccccccccccc}
4 & & X & & 9 & \cdot & 2 & & \Lambda & & 5 \\
\underline{\times 12} & \rightarrow & \underline{48} & \rightarrow & \underline{696} & & \underline{\times 12} & \rightarrow & \underline{24} & \rightarrow & \underline{420} \\
\underline{48} & & 58 & & \underline{705_{10}} & & \underline{24} & & 35 & & \underline{425} \times 12^{-3} = \frac{425}{1728} \\
 & & \underline{\times 12} & & & & & & \underline{\times 12} & & = 0{\cdot}2459_{10} \\
 & & \underline{696} & & & & & & \underline{420} & &
\end{array}$$

$$4X9{\cdot}2\Lambda5_{12} = 705{\cdot}246_{10}$$

(d)
$$\begin{array}{ccccccccccc}
6 & & F & & 8 & \cdot & 3 & & D & & 5 \\
\underline{\times 16} & \rightarrow & \underline{96} & \rightarrow & \underline{1776} & & \underline{\times 16} & \rightarrow & \underline{48} & \rightarrow & \underline{976} \\
\underline{96} & & 111 & & \underline{1784_{10}} & & \underline{48} & & 61 & & \underline{981} \times 16^{-3} = \frac{981}{4096} \\
 & & \underline{\times 16} & & & & & & \underline{\times 16} & & = 0{\cdot}2395_{10} \\
 & & 666 & & & & & & 366 & & \\
 & & \underline{111} & & & & & & \underline{61} & & \\
 & & \underline{1776} & & & & & & \underline{976} & &
\end{array}$$

$$6F8{\cdot}3D5_{16} = 1784{\cdot}240_{10}$$

In all the previous examples, we have changed binary, octal, duodecimal and hexadecimal numbers into their equivalent denary forms. The reverse process is also often required, so we will now see what is involved.

So on then to the next frame

Change of base from denary to a new base

130

Binary form

The simplest way to do this is by repeated division by 2 (the new base), noting the remainder at each stage. Continue dividing until a final zero quotient is obtained.

For example, to change 245_{10} to binary:

$$\begin{array}{r|c l c}
2 & 245_{10} & & \\ \hline
2 & 122 & -1 & \uparrow \\ \hline
2 & 61 & -0 & \\ \hline
2 & 30 & -1 & \\ \hline
2 & 15 & -0 & \\ \hline
2 & 7 & -1 & \\ \hline
2 & 3 & -1 & \\ \hline
2 & 1 & -1 & \\ \hline
 & 0 & -1 & \bot \\ \hline
\end{array}$$

Now write all the remainders in the reverse order, i.e. from bottom to top.

Then $245_{10} = 11110101_2$

Octal form

The method here is exactly the same except that we divide repeatedly by 8 (the new base). So, without more ado, changing 524_{10} to octal gives ....

131

$$\boxed{1014_8}$$

For:

$$\begin{array}{r|c l c}
8 & 524_{10} & & \\ \hline
8 & 65 & -4 & \uparrow \\ \hline
8 & 8 & -1 & \\ \hline
8 & 1 & -0 & \\ \hline
 & 0 & -1 & \bot
\end{array}$$

As before, write the remainders in order, i.e. from bottom to top.

$\therefore\ 524_{10} = 1014_8$

Duodecimal form

Method as before, but this time we divide repeatedly by 12.

So $897_{10} =$ ....

132

$$\boxed{629_{12}}$$

Because

$$\begin{array}{r|c l c}
12 & 897_{10} & & \\ \hline
12 & 74 & -9 & \uparrow \\ \hline
12 & 6 & -2 & \\ \hline
 & 0 & -6 & \bot \\ \hline
\end{array}$$

$\therefore\ 897_{10} = 629_{12}$

Now move to the next frame

133

The method we have been using is quick and easy enough when the denary number to be changed is a whole number. When it contains a decimal part, we must look further.

A denary decimal in octal form

To change $0{\cdot}526_{10}$ to octal form, we multiply the decimal repeatedly by the new base, in this case 8, but on the second and subsequent multiplication, we do not multiply the whole-number part of the previous product.

$$\begin{array}{rcrl}
0 & \cdot & 526_{10} & \\
 & & 8 & \\ \hline
4 & \cdot & 208 & \text{Now multiply again by 8, but treat only the decimal part} \\
 & & 8 & \\ \cline{2-3}
1 & \cdot & 664 & \\
 & & 8 & \\ \cline{2-3}
5 & \cdot & 312 & \\
 & & 8 & \\ \cline{2-3}
2 & \cdot & 496 & \text{and so on} \\ \cline{2-3}
\end{array}$$

Finally, we write the whole-number numerals downwards to form the required octal decimal.

Be careful *not* to include the zero unit digit in the original denary decimal. In fact, it may be safer simply to write the decimal as $\cdot 526_{10}$ in the working.

So $$0{\cdot}526_{10} = 0{\cdot}4152_8$$

Converting a denary decimal into any new base is done in the same way. If we express $0{\cdot}306_{10}$ as a duodecimal, we get . . . .

Set it out in the same way: there are no snags

134

$$\boxed{0{\cdot}3809_{12}}$$

$$\begin{array}{rcrl}
 & \cdot & 306_{10} & \\
 & & 12 & \\ \hline
3 & \cdot & 672 & \\
 & & 12 & \\ \cline{2-3}
8 & \cdot & 064 & \\
 & & 12 & \\ \cline{2-3}
0 & \cdot & 768 & \text{There is no carryover into the units column,} \\
 & & 12 & \text{so enter a zero.} \\ \cline{2-3}
9 & \cdot & 216 & \\
 & & 12 & \\ \cline{2-3}
2 & \cdot & 592 & \text{etc.} \\ \cline{2-3}
\end{array}$$

$$\therefore\ 0{\cdot}306_{10} = 0{\cdot}3809_{12}$$

Now let us go one stage further – so on to the next frame

135

If the denary number consists of both a whole-number and a decimal part, the two parts are converted separately and united in the final result. The example will show the method.

Express $492{\cdot}731_{10}$ in octal form:

$$\begin{array}{r|l l} 8 & 492_{10} & \\ \cline{2-2} 8 & 61 & -4 \\ \cline{2-2} 8 & 7 & -5 \\ \cline{2-2} \multicolumn{1}{r}{} & 0 & -7 \\ \cline{2-2} \end{array} \uparrow \qquad \downarrow \begin{array}{r c r} & \cdot & 731_{10} \\ & & 8 \\ \hline 5 & \cdot & 848 \\ & & 8 \\ \cline{2-3} 6 & \cdot & 784 \\ & & 8 \\ \cline{2-3} 6 & \cdot & 272 \\ & & 8 \\ \cline{2-3} 2 & \cdot & 176 \\ \cline{2-3} \end{array}$$

Then $\quad 492{\cdot}731_{10} = 754{\cdot}5662_8$

In similar manner, $384{\cdot}426_{10}$ expressed in duodecimals becomes ....

Set the working out in the same way

136

$$\boxed{280{\cdot}5142_{12}}$$

Because we have:

$$\begin{array}{r|l l} 12 & 384_{10} & \\ \cline{2-2} 12 & 32 & -0 \\ \cline{2-2} 12 & 2 & -8 \\ \cline{2-2} \multicolumn{1}{r}{} & 0 & -2 \\ \cline{2-2} \end{array} \uparrow \qquad \downarrow \begin{array}{r c r} & \cdot & 426_{10} \\ & & 12 \\ \hline 5 & \cdot & 112 \\ & & 12 \\ \cline{2-3} 1 & \cdot & 344 \\ & & 12 \\ \cline{2-3} 4 & \cdot & 128 \\ & & 12 \\ \cline{2-3} 1 & \cdot & 536 \\ & & 12 \\ \cline{2-3} 6 & \cdot & 432 \\ \cline{2-3} \end{array}$$

$$\therefore \; 384{\cdot}426_{10} = 280{\cdot}5142_{12}$$

That is straightforward enough, so let us now move on to see a very helpful use of octals in the next frame.

Use of octals as an intermediate step

137

This gives us an easy way of converting denary numbers into binary and hexadecimal forms. As an example, note the following.

Express the denary number $348{\cdot}654_{10}$ in octal, binary and hexadecimal forms.

(a) First we change $348{\cdot}654_{10}$ into octal form by the usual method.

This gives $348{\cdot}654_{10}$ = ...

138

$$\boxed{534{\cdot}517_8}$$

(b) Now we take this octal form and write the binary equivalent of each digit in groups of three binary digits, thus:

$$101 \quad 011 \quad 100 \quad \cdot \quad 101 \quad 001 \quad 111$$

Closing the groups up we have the binary equivalent of $534{\cdot}517_8$

i.e. $348{\cdot}654_{10} = 534{\cdot}517_8$

$= 101011100{\cdot}101001111_2$

(c) Then, starting from the decimal point and working in each direction, regroup the same binary digits in groups of four. This gives . . . .

139

$$\boxed{0001 \quad 0101 \quad 1100 \quad \cdot \quad 1010 \quad 0111 \quad 1000}$$

completing the group at either end by addition of extra zeros, as necessary.

(d) Now write the hexadecimal equivalent of each group of four binary digits, so that we have . . . .

140

$$\boxed{1 \quad 5 \quad (12) \quad \cdot \quad (10) \quad 7 \quad 8}$$

Replacing (12) and (10) with the corresponding hexadecimal symbols, C and A, this gives $1\ 5\ C \cdot A\ 7\ 8_{16}$

So, collecting the partial results together:

$$\begin{aligned} 348{\cdot}654_{10} &= 534{\cdot}517_8 \\ &= 101011100{\cdot}101001111_2 \\ &= 15C{\cdot}A78_{16} \end{aligned}$$

Next frame

141

We have worked through the previous example in some detail. In practice, the method is more concise. Here is another example.

Change the denary number $428{\cdot}371_{10}$ into its octal, binary and hexadecimal forms.

(a) First of all, the octal equivalent of $428{\cdot}371_{10}$ is . . . .

142

$$\boxed{654{\cdot}276_8}$$

(b) The binary equivalent of each octal digit in groups of three is . . . .

143

$$\boxed{110\quad 101\quad 100\quad \cdot\quad 010\quad 111\quad 110_2}$$

(c) Closing these up and rearranging in groups of four in each direction from the decimal point, we have ....

144

$$\boxed{0001\quad 1010\quad 1100\quad \cdot\quad 0101\quad 1111_2}$$

(d) The hexadecimal equivalent of each set of four binary digits then gives ....

145

$$\boxed{1AC{\cdot}5F_{16}}$$

$$\begin{aligned}\text{So } 428{\cdot}371_{10} &= 654{\cdot}276_8\\ &= 110101100{\cdot}010111110_2\\ &= 1AC{\cdot}5F_{16}\end{aligned}$$

This is important, so let us do one more.

Convert $163{\cdot}245_{10}$ into octal, binary and hexadecimal forms.

You can do this one entirely on your own. Work right through it and then check with the results in the next frame

146

$$\boxed{\begin{aligned}163{\cdot}245_{10} &= 243{\cdot}175_8\\ &= 010100011{\cdot}001111101_2\\ &= 1010\ 0011\ \cdot\ 0011\ 1110\ 1000_2\\ &= A3{\cdot}3E8_{16}\end{aligned}}$$

And that is it.

On to the next frame

Reverse method

147

Of course, the method we have been using can be used in reverse, i.e. starting with a hexadecimal number, we can change it into groups of four binary digits, regroup these into groups of three digits from the decimal point, and convert these into the equivalent octal digits. Finally, the octal number can be converted into denary form by the usual method.

Here is one you can do with no trouble.

Express the hexadecimal number $4B2{\cdot}1A6_{16}$ in equivalent binary, octal and denary forms.

(a) Rewrite $4B2{\cdot}1A6_{16}$ in groups of four binary digits.
(b) Regroup into groups of three binary digits from the decimal point.
(c) Express the octal equivalent of each group of three binary digits.
(d) Finally convert the octal number into its denary equivalent.

Work right through it and then check with the solution in the next frame

148

$4B2{\cdot}1A6_{16} =$
(a) $0100\ 1011\ 0010 \cdot 0001\ 1010\ 0110_2$
(b) $010\ 010\ 110\ 010 \cdot 000\ 110\ 100\ 110_2$
(c) $2\ \ 2\ \ 6\ \ 2 \cdot 0\ \ 6\ \ 4\ \ 6_8$
(d) $1202{\cdot}103_{10}$

Now one more for good measure.

Express $2E3{\cdot}4D_{16}$ in binary, octal and denary forms.

Check results with the next frame

149

$$\begin{aligned} 2E3{\cdot}4D_{16} &= 0010\ 1110\ 0011 \cdot 0100\ 1101_2 \\ &= 001\ 011\ 100\ 011 \cdot 010\ 011\ 010_2 \\ &= 1\ \ 3\ \ 4\ \ 3 \cdot 2\ \ 3\ \ 2_8 \\ &= 739{\cdot}301_{10} \end{aligned}$$

Let us pause and summarize the main facts on number systems

150

Review summary

1 Converting a whole number from denary to binary requires remainders to be collated after dividing through by 2. [130]

2 Converting a whole number from denary to octal requires remainders to be collated after dividing through by 8. [130]

3 Converting a whole number from denary to duodecimal requires remainders to be collated after dividing through by 12. [131]

4 Converting a decimal number from denary to octal requires repeated multiplication by 8 to the desired level of accuracy. [133]

5 Converting a decimal number from denary to duodecimal requires repeated multiplication by 12 to the desired level of accuracy. [136]

6 Converting a decimal number from denary to hexadecimal requires a conversion to octal where each octal digit is then written in binary. Each quadruple group of binary digits can then be converted into an individual hexadecimal digit. [137]

151

Review exercise

1 Express the following numbers in denary form:
(a) $1110{\cdot}11_2$ (b) $507{\cdot}632_8$ (c) $345{\cdot}2\Lambda 7_{12}$ (d) $2B4{\cdot}CA3_{16}$

2 Express the denary number $427{\cdot}362_{10}$ as a duodecimal number.

3 Convert $139{\cdot}825_{10}$ to the equivalent octal, binary and hexadecimal forms.

152

1 (a) $14{\cdot}75_{10}$ (b) $327{\cdot}801_{10}$ (c) $485{\cdot}247_{10}$ (d) $692{\cdot}790_{10}$ to 3 dp

2 $2\Lambda 7{\cdot}442_{12}$

3 $213{\cdot}646_8$, $10\ 001\ 011 \cdot 110\ 100\ 110_2$ and $8B{\cdot}D3_{16}$

153 You have now come to the end of this Programme. A list of **Can you?** questions follows for you to gauge your understanding of the material in the Programme. You will notice that these questions match the **Learning outcomes** listed at the beginning of the Programme so go back and try the **Quiz** that follows them. After that try the **Test exercise**. *Work through these at your own pace, there is no need to hurry*. A set of **Further problems** provides additional valuable practice.

Can you?

Checklist F.1

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can: **Frames**

- Carry out the basic rules of arithmetic with integers? 1 to 12

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Check the result of a calculation making use of rounding? 13 to 15

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Write a whole number as a product of prime numbers? 19 to 22

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Find the highest common factor and lowest common multiple of two whole numbers? 23 to 24

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Manipulate fractions, ratios and percentages? 28 to 52

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Manipulate decimal numbers? 56 to 74

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Manipulate powers? 78 to 98

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Use standard or preferred standard form and complete a calculation to the required level of accuracy? 99 to 108

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Understand the construction of various number systems and convert from one number system to another? 112 to 149

  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise F.1

Questions marked with can be found online at www.macmillanihe.com/stroud.

There you can go through the question step by step and follow online hints, with full working provided.

Frames

1 Place the appropriate symbol < or > between each of the following pairs of numbers:
(a) $-12 \quad -15$ (b) $9 \quad -17$ (c) $-11 \quad 10$ 1 to 4

2 Find the value of each of the following:
(a) $24 - 3 \times 4 + 28 \div 14$ (b) $(24 - 3) \times (4 + 28) \div 14$ 5 to 12

3 Write each of the following as a product of prime factors:
(a) 156 (b) 546 (c) 1445 (d) 1485 19 to 22

4 Round each number to the nearest 10, 100 and 1000:
(a) 5045 (b) 1100 (c) -1552 (d) -4995 13 to 15

5 Find (i) the HCF and (ii) the LCM of:
(a) 1274 and 195 (b) 64 and 18 23 to 24

6 Reduce each of the following fractions to their lowest terms:
(a) $\dfrac{8}{14}$ (b) $\dfrac{162}{36}$ (c) $-\dfrac{279}{27}$ (d) $-\dfrac{81}{3}$ 28 to 36

7 Evaluate each of the following, giving your answer as a fraction:
(a) $\dfrac{1}{3} + \dfrac{3}{5}$ (b) $\dfrac{2}{7} - \dfrac{1}{9}$ (c) $\dfrac{8}{3} \times \dfrac{6}{5}$ (d) $\dfrac{4}{5}$ *of* $\dfrac{2}{15}$
(e) $\dfrac{9}{2} \div \dfrac{3}{2}$ (f) $\dfrac{6}{7} - \dfrac{4}{5} \times \dfrac{3}{2} \div \dfrac{7}{5} + \dfrac{9}{4}$ 37 to 46

8 In each of the following the proportions of a compound are given. Find the ratios of the components in each case:
(a) $\dfrac{3}{4}$ of A and $\dfrac{1}{4}$ of B
(b) $\dfrac{2}{3}$ of P, $\dfrac{1}{15}$ of Q and the remainder of R
(c) $\dfrac{1}{5}$ of R, $\dfrac{3}{5}$ of S, $\dfrac{1}{6}$ of T and the remainder of U 47 to 48

9 What is:
(a) $\dfrac{3}{5}$ as a percentage?
(b) 16% as a fraction in its lowest terms?
(c) 17·5% of £12·50? 49 to 52

10 Evaluate each of the following (i) to 4 sig fig and (ii) to 3 dp:
(a) $13{\cdot}6 \times 25{\cdot}8 \div 4{\cdot}2$
(b) $13{\cdot}6 \div 4{\cdot}2 \times 25{\cdot}8$
(c) $9{\cdot}1(17{\cdot}43 + 7{\cdot}2(8{\cdot}6 - 4{\cdot}1^2 \times 3{\cdot}1))$
(d) $-8{\cdot}4((6{\cdot}3 \times 9{\cdot}1 + 2{\cdot}2^{1\cdot3}) - (4{\cdot}1^{-3\cdot1} \div 3{\cdot}3^3 - 5{\cdot}4))$ 56 to 65

Frames

11 Convert each of the following to decimal form to 3 decimal places:

(a) $\frac{3}{17}$ (b) $-\frac{2}{15}$ (c) $\frac{17}{3}$ (d) $-\frac{24}{11}$ 66 to 67

12 Write each of the following in abbreviated form:

(a) 6·7777 . . . (b) 0·01001001001 . . . 70 to 71

13 Convert each of the following to fractional form in lowest terms:

(a) 0·4 (b) 3·68 (c) $1{\cdot}\dot{4}$ (d) – 6·1 68 to 73

14 Write each of the following as a number raised to a power:

(a) $2^9 \times 2^2$ (b) $6^2 \div 5^2$ (c) $((-4)^4)^{-4}$ (d) $(3^{-5})^0$ 78 to 89

15 Find the value of each of the following to 3 dp:

(a) $11^{\frac{1}{4}}$ (b) $\sqrt[7]{3}$ (c) $(-81)^{\frac{1}{5}}$ (d) $(-81)^{\frac{1}{4}}$ 90 to 94

16 Express in standard form:

(a) 537·6 (b) 0·364 (c) 4902 (d) 0·000125 95 to 101

17 Convert to preferred standard form:

(a) $6{\cdot}147 \times 10^7$ (b) $2{\cdot}439 \times 10^{-4}$ (c) $5{\cdot}286 \times 10^5$

(d) $4{\cdot}371 \times 10^{-7}$ 102 to 104

18 Determine the following product, giving the result in both standard form and preferred standard form:

$(6{\cdot}43 \times 10^3)(7{\cdot}35 \times 10^4)$ 95 to 104

19 Each of the following contains numbers obtained by measurement. Evaluate each to the appropriate level of accuracy:

(a) $18{\cdot}4^{1{\cdot}6} \times 0{\cdot}01$ (b) $\dfrac{7{\cdot}632 \times 2{\cdot}14 - 8{\cdot}32 \div 1{\cdot}1}{16{\cdot}04}$ 105 to 108

20 Express the following numbers in denary form:

(a) $1111{\cdot}11_2$ (b) $777{\cdot}701_8$ (c) $3\Lambda3{\cdot}9\Lambda1_{12}$

(d) $\text{E02}{\cdot}\text{FAB}_{16}$ 112 to 129

21 Convert $19{\cdot}872_{10}$ to the equivalent octal, binary, duodecimal and hexadecimal forms. 130 to 152

Further problems F.1

1 Place the appropriate symbol < or > between each of the following pairs of numbers:

(a) – 4 – 11 (b) 7 – 13 (c) – 15 13

2 Find the value of each of the following:

(a) $6 + 14 \div 7 - 2 \times 3$ (b) $(6 + 14) \div (7 - 2) \times 3$

3 Round each number to the nearest 10, 100 and 1000:

(a) 3505 (b) 500 (c) – 2465 (d) – 9005

4 Calculate each of the following to:

(i) 5 sig fig;

(ii) 4 dp;

(iii) The required level of accuracy given that each number, other than those indicated in brackets, has been obtained by measurement.

(a) $\dfrac{3{\cdot}21^{2{\cdot}33} + (5{\cdot}77 - 3{\cdot}11)}{8{\cdot}32 - 2{\cdot}64 \times \sqrt{2{\cdot}56}}$

(b) $\dfrac{3{\cdot}142 \times 1{\cdot}95}{6}(3 \times 5{\cdot}44^2 + 1{\cdot}95^2)$ (power 2, divisor 6, multiplier 3)

(c) $\dfrac{3{\cdot}142 \times 1{\cdot}234}{12}(0{\cdot}424^2 + 0{\cdot}424 \times 0{\cdot}951 + 0{\cdot}951^2)$ (power 2, divisor 12)

(d) $\sqrt{\dfrac{2 \times 0{\cdot}577}{3{\cdot}142 \times 2{\cdot}64} + \dfrac{2{\cdot}64^2}{3}}$ (power 2, divisor 3, multiplier 2)

(e) $\dfrac{3{\cdot}26 + \sqrt{12{\cdot}13}}{14{\cdot}192 - 2{\cdot}4 \times 1{\cdot}63^2}$ (power 2)

(f) $\dfrac{4{\cdot}62^2 - (7{\cdot}16 - 2{\cdot}35)}{2{\cdot}63 + 1{\cdot}89 \times \sqrt{73{\cdot}24}}$ (power 2)

5 Find the prime factorization for each of the following:

(a) 924 (b) 825 (c) 2310 (d) 35 530

6 Find the HCF and LCM for each of the following pairs of numbers:

(a) 9, 21 (b) 15, 85 (c) 66, 42 (d) 64, 360

7 Reduce each of the following fractions to their lowest terms:

(a) $\dfrac{6}{24}$ (b) $\dfrac{104}{48}$ (c) $-\dfrac{120}{15}$ (d) $-\dfrac{51}{7}$

8 Evaluate:

(a) $\dfrac{9}{2} - \dfrac{4}{5} \div \left(\dfrac{2}{3}\right)^2 \times \dfrac{3}{11}$

(b) $\dfrac{\dfrac{3}{4} + \dfrac{7}{5} \div \dfrac{2}{9} \times \dfrac{1}{3}}{\dfrac{7}{3} - \dfrac{11}{2} \times \dfrac{2}{5} + \dfrac{4}{9}}$

(c) $\left(\dfrac{3}{4} + \dfrac{7}{5}\right)^2 \div \left(\dfrac{7}{3} - \dfrac{11}{5}\right)^2$

(d) $\dfrac{\left(\dfrac{5}{2}\right)^3 - \dfrac{2}{9} \div \left(\dfrac{2}{3}\right)^2 \times \dfrac{3}{2}}{\dfrac{3}{11} + \left(\dfrac{11}{2} \times \dfrac{2}{5}\right)^2 - \dfrac{7}{5}}$

9 Express each of the following as a fraction in its lowest terms:

(a) 36% (b) 17·5% (c) 8·7% (d) 72%

10 Express each of the following as a percentage accurate to 1 dp:

(a) $\dfrac{4}{5}$ (b) $\dfrac{3}{11}$ (c) $\dfrac{2}{9}$ (d) $\dfrac{1}{7}$

(e) $\dfrac{9}{19}$ (f) $\dfrac{13}{27}$ (g) $\dfrac{7}{101}$ (h) $\dfrac{199}{200}$

11 Find:

(a) 16% *of* 125 (b) 9·6% *of* 5·63

(c) 13·5% *of* (−13·5) (d) 0·13% *of* 92·66

12 In each of the following the properties of a compound are given. In each case find A : B : C.

(a) $\frac{1}{5}$ of A, $\frac{2}{3}$ of B and the remainder of C;

(b) $\frac{3}{8}$ of A with B and C in the ratio 1 : 2;

(c) A, B and C are mixed according to the ratios A : B = 2 : 5 and B : C = 10 : 11;

(d) A, B and C are mixed according to the ratios A : B = 1 : 7 and B : C = 13 : 9.

13 Write each of the following in abbreviated form:

(a) 8·767676 . . . (b) 212·211211211 . . .

14 Convert each of the following to fractional form in lowest terms:

(a) 0·12 (b) 5·25 (c) $5{\cdot}\dot{3}0\dot{6}$ (d) – 9·3

15 Write each of the following as a number raised to a power:

(a) $8^4 \times 8^3$ (b) $2^9 \div 8^2$ (c) $(5^3)^5$ (d) $3^4 \div 9^2$

16 Find the value of each of the following to 3 dp:

(a) $17^{\frac{2}{5}}$ (b) $\sqrt[4]{13}$ (c) $(-5)^{\frac{2}{3}}$ (d) $\sqrt{(-5)^4}$

17 Convert each of the following to decimal form to 3 decimal places:

(a) $\frac{5}{21}$ (b) $-\frac{2}{17}$ (c) $\frac{8}{3}$ (d) $-\frac{32}{19}$

18 Express in standard form:

(a) 52·876 (b) 15 243 (c) 0·08765
(d) 0·0000492 (e) 436·2 (f) 0·5728

19 Rewrite in preferred standard form:

(a) 4285 (b) 0·0169 (c) $8{\cdot}526 \times 10^{-4}$
(d) $3{\cdot}629 \times 10^5$ (e) $1{\cdot}0073 \times 10^7$ (f) $5{\cdot}694 \times 10^8$

20 Evaluate the following, giving the result in standard form and in preferred standard form:

$$\frac{(4{\cdot}26 \times 10^4)(9{\cdot}38 \times 10^5)}{3{\cdot}179 \times 10^2}$$

21 Convert each of the following decimal numbers to (i) binary, (ii) octal, (iii) duodecimal and (iv) hexadecimal format:

(a) 1·83 (b) $3{\cdot}425 \times 10^2$

22 Convert each of the following octal numbers to (i) binary, (ii) decimal, (iii) duodecimal and (iv) hexadecimal format:

(a) 0·577 (b) 563

23 Convert each of the following duodecimal numbers to (i) binary, (ii) octal, (iii) decimal and (iv) hexadecimal format:

(a) 0·ΛX (b) 9Λ1

24 Convert each of the following binary numbers to (i) decimal, (ii) octal, (iii) duodecimal and (iv) hexadecimal format:

(a) 0·10011 (b) 111001100

25 Convert each of the following hexadecimal numbers to (i) binary, (ii) octal, (iii) duodecimal and (iv) decimal format:

(a) 0·F4B (b) 3A5

Now visit the companion website at www.macmillanihe.com/stroud for more questions applying this mathematics to engineering and science.

Frames 1 to 72

Programme F.2

Introduction to algebra

Learning outcomes

When you have completed this Programme you will be able to:

☐ Use alphabetic symbols to supplement the numerals and to combine these symbols using all the operations of arithmetic

☐ Simplify algebraic expressions by collecting like terms and by abstracting common factors from similar terms

☐ Remove brackets and so obtain alternative algebraic expressions

☐ Manipulate expressions involving powers and logarithms

☐ Multiply and divide algebraic expressions

☐ Manipulate algebraic fractions

☐ Factorize algebraic expressions using standard factorizations

☐ Factorize quadratic algebraic expressions

If you already feel confident about these why not try the quiz over the page? You can check your answers at the end of the book.

Quiz F.2

Questions marked with this icon can be found at www.macmillanihe.com/stroud. There you can go through the question step by step and follow online hints, with full working provided.

Frames

1 Simplify each of the following:

(a) $3pq + 5pr - 2qr + qp - 6rp$

(b) $5l^2mn + 2nl^2m - 3mln^2 + l^2nm + 4n^2ml - nm^2$

(c) $w^4 \div w^{-a} \times w^{-b}$

(d) $\dfrac{\left(s^{\frac{1}{3}}\right)^{\frac{3}{4}} \times \left(t^{\frac{1}{4}}\right)^{-1} \div \left(s^{\frac{2}{7}}\right)^{-\frac{7}{4}}}{\left(s^{-\frac{1}{4}}\right)^{-1} \div \left(t^{\frac{1}{2}}\right)^{4}}$ 1 to 22

2 Remove the brackets in each of the following:

(a) $-4x(2x - y)(3x + 2y)$

(b) $(a - 2b)(2a - 3b)(3a - 4b)$

(c) $-\{-2[x - 3(y - 4)] - 5(z + 6)\}$ 12 to 17

3 Evaluate by calculator or by change of base where necessary (to 3 dp):

(a) $\log 0{\cdot}0101$

(b) $\ln 3{\cdot}47$

(c) $\log_2 3{\cdot}16$ 23 to 32

4 Express $\log F = \log G + \log m - \log\left(\dfrac{1}{M}\right) - 2\log r$ without logs.

5 Rewrite $T = 2\pi\sqrt{\dfrac{l}{g}}$ in log form. 33 to 35

6 Perform the following multiplications and simplify your results:

(a) $(n^2 + 2n - 3)(4n + 5)$

(b) $(v^3 - v^2 - 2)(1 - 3v + 2v^2)$ 39 to 43

7 Perform the following divisions:

(a) $(2y^2 - y - 10) \div (y + 2)$

(b) $\dfrac{q^3 - 8}{q - 2}$

(c) $\dfrac{2r^3 + 5r^2 - 4r - 3}{r^2 + 2r - 3}$ 44 to 48

8 Simplify each of the following:

(a) $\dfrac{p}{q^3} \div \dfrac{q}{p^3}$

(b) $\dfrac{a^2b}{2c} \times \dfrac{ac^2}{2b} \div \dfrac{b^2c}{2a}$ 52 to 57

9 Factorize the following:

(a) $18x^2y - 12xy^2$

(b) $x^3 + 4x^2y - 3xy^2 - 12y^3$

(c) $4(x - y)^2 - (x - 3y)^2$

(d) $12x^2 - 25x + 12$ 61 to 71

Algebraic expressions

1

Think of a number
Add 15 to it
Double the result
Add this to the number you first thought of
Divide the result by 3
Take away the number you first thought of

The answer is 10

Why?

Check your answer in the next frame

2

Symbols other than numerals

A letter of the alphabet can be used to represent a number when the specific number is unknown and because the number is unknown (except, of course, to the person who thought of it) we shall represent the number by the letter a:

Think of a number	a
Add 15 to it	$a + 15$
Double the result	$2 \times (a + 15) = (2 \times a) + (2 \times 15)$ $= (2 \times a) + 30$
Add the result to the number you first thought of	$a + (2 \times a) + 30 = (3 \times a) + 30$
Divide the result by 3	$[(3 \times a) + 30] \div 3 = a + 10$
Take away the number you first thought of	$a + 10 - a = 10$

The result is 10

Next frame

3

This little puzzle has demonstrated how:

an unknown number can be represented by a letter of the alphabet which can then be manipulated just like an ordinary numeral within an arithmetic expression.

So that, for example:

$a + a + a + a = 4 \times a$
$3 \times a - a = 2 \times a$
$8 \times a \div a = 8$

and

$a \times a \times a \times a \times a = a^5$

▶

If a and b represent two unknown numbers then we speak of the:

sum of a and b	$a + b$
difference of a and b	$a - b$
product of a and b	$a \times b$, $a.b$ or simply ab (*we can and do suppress the multiplication sign*)
quotient of a and b	$a \div b$, a/b or $\frac{a}{b}$ provided $b \neq 0$

and

raising a to the power b	a^b

Using letters and numerals in this way is referred to as *algebra*.

Now move to the next frame

4

Constants

In the puzzle of Frame 1 we saw how to use the letter a to represent an unknown number – we call such a symbol a *constant*.

In many other problems we require a symbol that can be used to represent not just one number but any one of a collection of numbers. Such a symbol is called a **variable**.

Next frame

5

Variables

We have seen that the operation of addition is commutative. That is, for example:

$$2 + 3 = 3 + 2$$

To describe this rule as applying to any pair of numbers and not just 2 and 3 we resort to the use of alphabetic characters x and y and write:

$$x + y = y + x$$

where x and y represent any two numbers. Used in this way, the letters x and y are referred to as *variables* because they each represent, not just one number, but any one of a collection of numbers.

So how would you write down the fact that multiplication is an associative operation? (Refer to Frame 12 of Programme F.1.)

You can check your answer in the next frame

6

$$\boxed{x(yz) = (xy)z = xyz}$$

where x, y and z represent numbers. Notice the suppression of the multiplication sign.

While it is not a hard and fast rule, it is generally accepted that letters from the beginning of the alphabet, i.e. $a, b, c, d, \ldots$ are used to represent constants and letters from the end of the alphabet, i.e. $\ldots v, w, x, y, z$ are used to represent variables. In any event, when a letter of the alphabet is used it should be made clear whether the letter stands for a constant or a variable.

Now move on to the next frame

Rules of algebra

The rules of arithmetic that we met in the previous Programme for integers also apply to any type of number and we express this fact in the *rules of algebra* where we use variables rather than numerals as specific instances. The rules are:

Commutativity

Two numbers x and y can be added or multiplied in any order without affecting the result. That is:

$x + y = y + x$ and
$xy = yx$

Addition and multiplication are commutative operations

The order in which two numbers are subtracted or divided *does* affect the result. That is:

$x - y \neq y - x$ unless $x = y$ and

$x \div y \neq y \div x$, $\left(\dfrac{x}{y} \neq \dfrac{y}{x}\right)$ unless $x = y$ and neither equals 0

Subtraction and division are not commutative operations except in very special cases

Associativity

The way in which the numbers x, y and z are associated under addition or multiplication *does not* affect the result. That is:

$x + (y + z) = (x + y) + z = x + y + z$ and
$x(yz) = (xy)z = xyz$

Addition and multiplication are associative operations

The way in which the numbers are associated under subtraction or division *does* affect the result. That is:

$x - (y - z) \neq (x - y) - z$ unless $z = 0$ and
$x \div (y \div z) \neq (x \div y) \div z$ unless $z = 1$ and $y \neq 0$

Subtraction and division are not associative operations except in very special cases

Distributivity

Multiplication is distributed over addition and subtraction from both the left and the right. For example:

$x(y + z) = xy + xz$ and $(x + y)z = xz + yz$
$x(y - z) = xy - xz$ and $(x - y)z = xz - yz$

Division is distributed over addition and subtraction from the right but not from the left. For example:

$(x + y) \div z = (x \div z) + (y \div z)$ but
$x \div (y + z) \neq (x \div y) + (x \div z)$

that is:

$$\frac{x+y}{z} = \frac{x}{z} + \frac{y}{z} \text{ but } \frac{x}{y+z} \neq \frac{x}{y} + \frac{x}{z}$$

Take care here because it is a common mistake to get this wrong

▶

Rules of precedence

The familiar rules of precedence continue to apply when algebraic expressions involving mixed operations are to be manipulated.

Next frame

8

Terms and coefficients

An algebraic expression consists of alphabetic characters and numerals linked together with the arithmetic operators. For example:

$8x - 3xy$

is an algebraic expression in the two variables x and y. Each component of this expression is called a *term* of the expression. Here there are two terms, namely:

the x term and the xy term.

The numerals in each term are called the *coefficients* of the respective terms. So that:

8 is the coefficient of the x term and -3 is the coefficient of the xy term.

Collecting like terms

Terms which have the same variables are called *like* terms and like terms can be collected together by addition or subtraction. For example:

$4x + 3y - 2z + 5y - 3x + 4z$ can be rearranged as $4x - 3x + 3y + 5y - 2z + 4z$ and simplified to:

$x + 8y + 2z$

Similarly, $4uv - 7uz - 6wz + 2uv + 3wz$ can be simplified to ....

Check your answer with the next frame

9

$$\boxed{6uv - 7uz - 3wz}$$

Next frame

10

Similar terms

In the algebraic expression:

$ab + ac$

both terms contain the letter a and for this reason these terms, though not like terms, are called *similar* terms. Common symbols such as this letter a are referred to as *common factors* and by using brackets these common factors can be *factored out*. For example, the common factor a in this expression can be factored out to give:

$ab + ac = a(b + c)$ This process is known as *factorization.*

Numerical factors are factored out in the same way. For example, in the algebraic expression:

$3pq - 3qr$

the terms are similar, both containing the letter q. They also have a common coefficient 3 and this, as well as the common letter q, can be factored out to give:

$$\begin{aligned}3pq - 3qr &= 3qp - 3qr\\ &= 3q(p - r)\end{aligned}$$

So the algebraic expression $9st - 3sv - 6sw$ simplifies to ....

The answer is in the next frame

11

$$\boxed{3s(3t - v - 2w)}$$

Because

$$\begin{aligned} 9st - 3sv - 6sw &= 3s \times 3t - 3s \times v - 3s \times 2w \\ &= 3s(3t - v - 2w) \end{aligned}$$

Next frame

Expanding brackets

12

Sometimes it will be desired to reverse the process of factorizing an expression by *removing* the brackets. This is done by:

(a) multiplying or dividing each term inside the bracket by the term outside the bracket, but

(b) if the term outside the bracket is negative then each term inside the bracket changes sign.

For example, the brackets in the expression:

$3x(y - 2z)$ are removed to give $3xy - 6xz$

and the brackets in the expression:

$-2y(2x - 4z)$ are removed to give $-4yx + 8yz$.

As a further example, the expression:

$$\frac{y+x}{8x} - \frac{y-x}{4x}$$

is an alternative form of $(y + x) \div (8x) - (y - x) \div (4x)$ and the brackets can be removed as follows:

$$\begin{aligned} \frac{y+x}{8x} - \frac{y-x}{4x} &= \frac{y}{8x} + \frac{x}{8x} - \frac{y}{4x} + \frac{x}{4x} \\ &= \frac{y}{8x} + \frac{1}{8} - \frac{y}{4x} + \frac{1}{4} \\ &= \frac{3}{8} - \frac{y}{8x} \end{aligned}$$

which can be written as $\frac{1}{8}\left(3 - \frac{y}{x}\right)$ or as $\frac{1}{8x}(3x - y)$

Nested brackets

13

Whenever an algebraic expression contains brackets nested within other brackets the innermost brackets are removed first. For example:

$$\begin{aligned} 7(a - [4 - 5(b - 3a)]) &= 7(a - [4 - 5b + 15a]) \\ &= 7(a - 4 + 5b - 15a) \\ &= 7a - 28 + 35b - 105a \\ &= 35b - 98a - 28 \end{aligned}$$

So that the algebraic expression $4(2x + 3[5 - 2(x - y)])$ becomes, after the removal of the brackets ....

Next frame

14

$$\boxed{24y - 16x + 60}$$

Because

$$\begin{aligned} 4(2x + 3[5 - 2(x - y)]) &= 4(2x + 3[5 - 2x + 2y]) \\ &= 4(2x + 15 - 6x + 6y) \\ &= 8x + 60 - 24x + 24y \\ &= 24y - 16x + 60 \end{aligned}$$

At this point let us pause and summarize the main facts so far on algebraic expressions

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Review summary

1 Alphabetic characters can be used to represent numbers and then be subjected to the arithmetic operations in much the same way as numerals. [2]

2 An alphabetic character that represents a single number is called a *constant*. [4]

3 An alphabetic character that represents any one of a collection of numbers is called a *variable*. [5]

4 Some algebraic expressions contain terms multiplied by numerical coefficients. [8]

5 Like terms contain identical alphabetic characters. [8]

6 Similar terms have some but not all alphabetic characters in common. [10]

7 Similar terms can be factorized by identifying their common factors and using brackets. [10]

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Review exercise

1 Simplify each of the following by collecting like terms:

(a) $4xy + 3xz - 6zy - 5zx + yx$

(b) $-2a + 4ab + a - 4ba$

(c) $3rst - 10str + 8ts - 5rt + 2st$

(d) $2pq - 4pr + qr - 2rq + 3qp$

(e) $5lmn - 6ml + 7lm + 8mnl - 4ln$

2 Simplify each of the following by collecting like terms and factorizing:

(a) $4xy + 3xz - 6zy - 5zx + yx$

(b) $3rst - 10str + 8ts - 5rt + 2st$

(c) $2pq - 4pr + qr - 2rq + 3qp$

(d) $5lmn - 6ml + 7lm + 8mnl - 4ln$

3 Expand the following and then refactorize where possible:

(a) $8x(y - z) + 2y(7x + z)$ (b) $(3a - b)(b - 3a) + b^2$

(c) $-3(w - 7[x - 8(3 - z)])$ (d) $\dfrac{2a - 3}{4b} + \dfrac{3a + 2}{6b}$

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1 (a) $$\begin{aligned} 4xy + 3xz - 6zy - 5zx + yx &= 4xy + xy + 3xz - 5xz - 6yz \\ &= 5xy - 2xz - 6yz \end{aligned}$$

Notice that the characters are written in alphabetic order.

(b) $$\begin{aligned} -2a + 4ab + a - 4ba &= -2a + a + 4ab - 4ab \\ &= -a \end{aligned}$$

(c) $$\begin{aligned} 3rst - 10str + 8ts - 5rt + 2st &= 3rst - 10rst + 8st + 2st - 5rt \\ &= -7rst + 10st - 5rt \end{aligned}$$

(d) $$\begin{aligned} 2pq - 4pr + qr - 2rq + 3qp &= 2pq + 3pq - 4pr + qr - 2qr \\ &= 5pq - 4pr - qr \end{aligned}$$

(e) $$\begin{aligned} 5lmn - 6ml + 7lm + 8mnl - 4ln &= 5lmn + 8lmn - 6lm + 7lm - 4ln \\ &= 13lmn + lm - 4ln \end{aligned}$$

2 (a) $$\begin{aligned} 4xy + 3xz - 6zy - 5zx + yx &= 5xy - 2xz - 6yz \\ &= x(5y - 2z) - 6yz \text{ or} \\ &= 5xy - 2z(x + 3y) \text{ or} \\ &= y(5x - 6z) - 2xz \end{aligned}$$

(b) $$\begin{aligned} 3rst - 10str + 8ts - 5rt + 2st &= -7rst + 10st - 5rt \\ &= st(10 - 7r) - 5rt \text{ or} \\ &= 10st - rt(7s + 5) \text{ or} \\ &= t(10s - 7rs - 5r) = t(s[10 - 7r] - 5r) \end{aligned}$$

(c) $$\begin{aligned} 2pq - 4pr + qr - 2rq + 3qp &= 5pq - 4pr - qr \\ &= p(5q - 4r) - qr \text{ or} \\ &= q(5p - r) - 4pr \text{ or} \\ &= 5pq - r(4p + q) \end{aligned}$$

(d) $$\begin{aligned} 5lmn - 6ml + 7lm + 8mnl - 4ln &= 13lmn + lm - 4ln \\ &= l(13mn + m - 4n) \\ &= l(m[13n + 1] - 4n) \text{ or} \\ &= l(n[13m - 4] + m) \end{aligned}$$

3 (a) $$\begin{aligned} 8x(y - z) + 2y(7x + z) &= 8xy - 8xz + 14xy + 2yz \\ &= 22xy - 8xz + 2yz \\ &= 2(x[11y - 4z] + yz) \end{aligned}$$

(b) $$\begin{aligned} (3a - b)(b - 3a) + b^2 &= 3a(b - 3a) - b(b - 3a) + b^2 \\ &= 3ab - 9a^2 - b^2 + 3ab + b^2 \\ &= 6ab - 9a^2 \\ &= 3a(2b - 3a) \end{aligned}$$

(c) $$\begin{aligned} -3(w - 7[x - 8(3 - z)]) &= -3(w - 7[x - 24 + 8z]) \\ &= -3(w - 7x + 168 - 56z) \\ &= -3w + 21x - 504 + 168z \end{aligned}$$

(d) $$\begin{aligned} \frac{2a - 3}{4b} + \frac{3a + 2}{6b} &= \frac{2a}{4b} - \frac{3}{4b} + \frac{3a}{6b} + \frac{2}{6b} \\ &= \frac{a}{2b} - \frac{3}{4b} + \frac{a}{2b} + \frac{1}{3b} \\ &= \frac{a}{b} - \frac{5}{12b} \\ &= \frac{1}{12b}(12a - 5) \end{aligned}$$

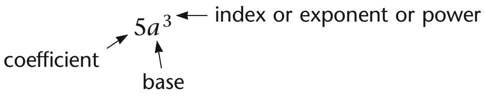
So now on to the next topic

Powers and logarithms

18

Powers

The use of *powers* (also called *indices* or *exponents*) provides a convenient form of algebraic shorthand. Repeated factors of the same base, for example $a \times a \times a \times a$ can be written as a^4, where the number 4 indicates the number of factors multiplied together. In general, the product of n such factors a, where a and n are positive integers, is written a^n, where a is called the *base* and n is called the *index* or *exponent* or *power*. Any number multiplying a^n is called the *coefficient* (as described in Frame 8).



From the definitions above a number of rules of indices can immediately be established.

Rules of indices

1 $a^m \times a^n = a^{m+n}$ e.g. $a^5 \times a^2 = a^{5+2} = a^7$

2 $a^m \div a^n = a^{m-n}$ e.g. $a^5 \div a^2 = a^{5-2} = a^3$

3 $(a^m)^n = a^{mn}$ e.g. $(a^5)^2 = a^5 \times a^5 = a^{10}$

These three basic rules lead to a number of important results.

4 $a^0 = 1$ because $a^m \div a^n = a^{m-n}$ and also $a^m \div a^n = \dfrac{a^m}{a^n}$

Then if $n = m$, $a^{m-m} = a^0$ and $\dfrac{a^m}{a^m} = 1$. So $a^0 = 1$

5 $a^{-m} = \dfrac{1}{a^m}$ because $a^{-m} = \dfrac{a^{-m} \times a^m}{a^m} = \dfrac{a^0}{a^m} = \dfrac{1}{a^m}$. So $a^{-m} = \dfrac{1}{a^m}$

6 $a^{\frac{1}{m}} = \sqrt[m]{a}$ because $\left(a^{\frac{1}{m}}\right)^m = a^{\frac{m}{m}} = a^1 = a$. So $a^{\frac{1}{m}} = \sqrt[m]{a}$

From this it follows that $a^{\frac{n}{m}} = \sqrt[m]{a^n}$ or $\left(\sqrt[m]{a}\right)^n$.

Make a note of any of these results that you may be unsure about.

Then move on to the next frame

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So we have:

(a) $a^m \times a^n = a^{m+n}$

(b) $a^m \div a^n = a^{m-n}$

(c) $(a^m)^n = a^{mn}$

(d) $a^0 = 1$

(e) $a^{-m} = \dfrac{1}{a^m}$

(f) $a^{\frac{1}{m}} = \sqrt[m]{a}$

(g) $a^{\frac{n}{m}} = (\sqrt[m]{a})^n$
or $\sqrt[m]{a^n}$

Now try to apply the rules:

$$\frac{6x^{-4} \times 2x^3}{8x^{-3}} = \ldots\ldots\ldots\ldots$$

20

$$\boxed{\tfrac{3}{2}x^2}$$

Because $\dfrac{6x^{-4} \times 2x^3}{8x^{-3}} = \dfrac{12}{8}.\dfrac{x^{-4+3}}{x^{-3}} = \dfrac{12}{8}.\dfrac{x^{-1}}{x^{-3}} = \dfrac{3}{2}x^{-1+3} = \dfrac{3}{2}x^2$

That was easy enough. In the same way, simplify:

$$\begin{aligned} E &= (5x^2y^{-\frac{3}{2}}z^{\frac{1}{4}})^2 \times (4x^4y^2z)^{-\frac{1}{2}} \\ &= \ldots\ldots\ldots\ldots \end{aligned}$$

21

$$\boxed{\dfrac{25x^2}{2y^4}}$$

$$\begin{aligned} E &= 25x^4y^{-3}z^{\frac{1}{2}} \times 4^{-\frac{1}{2}}x^{-2}y^{-1}z^{-\frac{1}{2}} \\ &= 25x^4y^{-3}z^{\frac{1}{2}} \times \frac{1}{2}x^{-2}y^{-1}z^{-\frac{1}{2}} \\ &= \frac{25}{2}x^2y^{-4}z^0 = \frac{25}{2}x^2y^{-4}.1 = \frac{25x^2}{2y^4} \end{aligned}$$

And one more:

Simplify $F = \sqrt[3]{a^6b^3} \div \sqrt{\dfrac{1}{9}a^4b^6} \times \left(4\sqrt{a^6b^2}\right)^{-\frac{1}{2}}$ giving the result without fractional indices.

$F = \ldots\ldots\ldots\ldots$

22

$$\boxed{\dfrac{3}{2ab^2\sqrt{ab}}}$$

$$\begin{aligned} F &= a^2b \div \frac{1}{3}a^2b^3 \times \frac{1}{(4a^3b)^{\frac{1}{2}}} = a^2b \times \frac{3}{a^2b^3} \times \frac{1}{2a^{\frac{3}{2}}b^{\frac{1}{2}}} \\ &= \frac{3}{b^2} \times \frac{1}{2a^{\frac{3}{2}}b^{\frac{1}{2}}} = \frac{3}{b^2.2a(ab)^{\frac{1}{2}}} = \frac{3}{2ab^2\sqrt{ab}} \end{aligned}$$

Logarithms

23

Any real number can be written as another number raised to a power. For example:

$9 = 3^2$ and $27 = 3^3$

By writing numbers in the form of a number raised to a power some of the arithmetic operations can be performed in an alternative way. For example:

$$\begin{aligned} 9 \times 27 &= 3^2 \times 3^3 \\ &= 3^{2+3} \\ &= 3^5 \\ &= 243 \end{aligned}$$

Here the process of multiplication is replaced by the process of relating numbers to powers and then adding the powers.

▶

If there were a simple way of relating numbers such as 9 and 27 to powers of 3 and then relating powers of 3 to numbers such as 243, the process of multiplying two numbers could be converted to the simpler process of adding two powers. In the past a system based on this reasoning was created. It was done using tables that were constructed of numbers and their respective powers.

In this instance:

Number	*Power of 3*
1	0
3	1
9	2
27	3
81	4
243	5
...	...

They were not called tables of powers but tables of *logarithms*. Nowadays, calculators have superseded the use of these tables but the logarithm remains an essential concept.

Let's just formalize this

---

24 If a, b and c are three real numbers where:

$a = b^c$ and where $a, b > 0$ and $b \neq 1$

the power c is called the *logarithm* of the number a to the base b and is written:

$c = \log_b a$ spoken as *c is the log to the base b of a*

For example, because

$25 = 5^2$

the power 2 is the logarithm of 25 to the base 5. That is:

$2 = \log_5 25$

So in each of the following what is the value of x, remembering that if $a = b^c$ then $c = \log_b a$?

(a) $x = \log_2 16$
(b) $4 = \log_x 81$
(c) $2 = \log_7 x$

The answers are in the next frame

25

(a) $x = 4$
(b) $x = 3$
(c) $x = 49$

Because

(a) If $x = \log_2 16$ then $2^x = 16 = 2^4$ and so $x = 4$
(b) If $4 = \log_x 81$ then $x^4 = 81 = 3^4$ and so $x = 3$
(c) If $2 = \log_7 x$ then $7^2 = x = 49$

Move on to the next frame

Rules of logarithms

26

Since logarithms are powers, the rules that govern the manipulation of logarithms closely follow the rules of powers.

(a) If $x = a^b$ so that $b = \log_a x$ and
$y = a^c$ so that $c = \log_a y$ then:

$xy = a^b a^c = a^{b+c}$ hence $\log_a(xy) = b + c = \log_a x + \log_a y$. That is:

$\log_a xy = \log_a x + \log_a y$ *The log of a product equals the sum of the logs*

(b) Similarly $x \div y = a^b \div a^c = a^{b-c}$ so that $\log_a(x \div y) = b - c = \log_a x - \log_a y$ That is:

$\log_a(x \div y) = \log_a x - \log_a y$ *The log of a quotient equals the difference of the logs*

(c) Because $x^n = (a^b)^n = a^{bn}$, $\log_a(x^n) = bn = n \log_a x$. That is:

$\log_a(x^n) = n \log_a x$ *The log of a number raised to a power is the product of the power and the log of the number*

The following important results are also obtained from these rules:

(d) $\log_a 1 = 0$ because, from the laws of powers $a^0 = 1$. Therefore, from the definition of a logarithm $\log_a 1 = 0$

(e) $\log_a a = 1$ because $a^1 = a$ so that $\log_a a = 1$

(f) $\log_a a^x = x$ because $\log_a a^x = x \log_a a = x.1$ so that $\log_a a^x = x$

(g) $a^{\log_a x} = x$ because if we take the log of the left-hand side of this equation:

$\log_a a^{\log_a x} = \log_a x \log_a a = \log_a x$ so that $a^{\log_a x} = x$

(h) $\log_a b = \dfrac{1}{\log_b a}$ because, if $\log_b a = c$ then $b^c = a$ and so $b = \sqrt[c]{a} = a^{\frac{1}{c}}$

Hence, $\log_a b = \dfrac{1}{c} = \dfrac{1}{\log_b a}$. That is $\log_a b = \dfrac{1}{\log_b a}$

So, cover up the results above and complete the following

(a) $\log_a(x \times y) =$ . . . .

(b) $\log_a(x \div y) =$ . . . .

(c) $\log_a(x^n) =$ . . . .

(d) $\log_a 1 =$ . . . .

(e) $\log_a a =$ . . . .

(f) $\log_a a^x =$ . . . .

(g) $a^{\log_a x} =$ . . . .

(h) $\dfrac{1}{\log_b a} =$ . . . .

27

(a) $\log_a x + \log_a y$
(b) $\log_a x - \log_a y$
(c) $n \log_a x$
(d) 0
(e) 1
(f) x
(g) x
(h) $\log_a b$

Now try it with numbers. Complete the following:

(a) $\log_a(6{\cdot}788 \times 1{\cdot}043) = \ldots\ldots\ldots\ldots$
(b) $\log_a(19{\cdot}112 \div 0{\cdot}054) = \ldots\ldots\ldots\ldots$
(c) $\log_a(5{\cdot}889^{1\cdot2}) = \ldots\ldots\ldots\ldots$
(d) $\log_8 1 = \ldots\ldots\ldots\ldots$
(e) $\log_7 7 = \ldots\ldots\ldots\ldots$
(f) $\log_3 27 = \ldots\ldots\ldots\ldots$
(g) $12^{\log_{12} 4} = \ldots\ldots\ldots\ldots$
(h) $\dfrac{1}{\log_3 4} = \ldots\ldots\ldots\ldots$

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(a) $\log_a 6{\cdot}788 + \log_a 1{\cdot}043$
(b) $\log_a 19{\cdot}112 - \log_a 0{\cdot}054$
(c) $1{\cdot}2 \log_a 5{\cdot}889$
(d) 0
(e) 1
(f) 3
(g) 4
(h) $\log_4 3$

Next frame

29

Base 10 and base *e*

On a typical calculator there are buttons that provide access to logarithms to two different bases, namely 10 and the exponential number $e = 2{\cdot}71828\ldots$.

Logarithms to base 10 were commonly used in conjunction with tables for arithmetic calculations – they are called *common logarithms* and are written without indicating the base. For example:

$\log_{10} 1{\cdot}2345$ is normally written simply as $\log 1{\cdot}2345$

You will see it on your calculator as .

The logarithms to base e are called *natural logarithms* and are important for their mathematical properties. These also are written in an alternative form:

$\log_e 1{\cdot}2345$ is written as $\ln 1{\cdot}2345$

You will see it on your calculator as .

So, use your calculator and complete the following (to 3 dp):

(a) $\log 5{\cdot}321 = \ldots\ldots\ldots\ldots$
(b) $\log 0{\cdot}278 = \ldots\ldots\ldots\ldots$
(c) $\log 1 = \ldots\ldots\ldots\ldots$
(d) $\log(-1{\cdot}005) = \ldots\ldots\ldots\ldots$
(e) $\ln 13{\cdot}45 = \ldots\ldots\ldots\ldots$
(f) $\ln 0{\cdot}278 = \ldots\ldots\ldots\ldots$
(g) $\ln 0{\cdot}00001 = \ldots\ldots\ldots\ldots$
(h) $\ln(-0{\cdot}001) = \ldots\ldots\ldots\ldots$

The answers are in the next frame

30

(a) 0·726	(e) 2·599
(b) – 0·556	(f) – 1·280
(c) 0	(g) – 11·513
(d) ERROR	(h) ERROR

Notice that for any base larger than 1 the:

logarithm of 1 is zero
logarithm of 0 is not defined
logarithm of a number greater than 1 is positive
logarithm of a number between 0 and 1 is negative
logarithm of a negative number cannot be evaluated as a real number.

Move to the next frame

Change of base

31

In the previous two frames you saw that $\log 0{\cdot}278 \neq \ln 0{\cdot}278$, i.e. logarithms with different bases have different values. The different values are, however, related to each other as can be seen from the following:

Let $a = b^c$ so that $c = \log_b a$ and let $x = a^d$ so that $d = \log_a x$. Now:

$x = a^d = (b^c)^d = b^{cd}$ so that $cd = \log_b x$. That is:

$$\log_b a \log_a x = \log_b x$$

This is the change of base formula which relates the logarithms of a number relative to two different bases. For example:

$\log_e 0{\cdot}278 = -1{\cdot}280$ to 3 dp and
$\log_e 10 \times \log_{10} 0{\cdot}278 = 2{\cdot}303 \times (-0{\cdot}556) = -1{\cdot}280$ which confirms that:

$$\log_e 10 \log_{10} 0{\cdot}278 = \log_e 0{\cdot}278$$

Now, use your calculator to complete each of the following (to 3 dp):

(a) $\log_2 3{\cdot}66 = \ldots\ldots\ldots\ldots$
(c) $\log_{9{\cdot}9} 6{\cdot}35 = \ldots\ldots\ldots\ldots$
(b) $\log_{3{\cdot}4} 0{\cdot}293 = \ldots\ldots\ldots\ldots$
(d) $\log_{7{\cdot}34} 7{\cdot}34 = \ldots\ldots\ldots\ldots$

32

(a) 1·872	(b) –1·003	(c) 0·806	(d) 1

Because

(a) $(\log_{10} 2) \times (\log_2 3{\cdot}66) = \log_{10} 3{\cdot}66$ so that

$$\log_2 3{\cdot}66 = \frac{\log_{10} 3{\cdot}66}{\log_{10} 2} = \frac{0{\cdot}563\ldots}{0{\cdot}301\ldots} = 1{\cdot}872$$

(b) $(\log_{10} 3{\cdot}4) \times (\log_{3{\cdot}4} 0{\cdot}293) = \log_{10} 0{\cdot}293$ so that

$$\log_{3{\cdot}4} 0{\cdot}293 = \frac{\log_{10} 0{\cdot}293}{\log_{10} 3{\cdot}4} = \frac{-0{\cdot}533\ldots}{0{\cdot}531\ldots} = -1{\cdot}003$$

(c) $(\log_{10} 9{\cdot}9) \times (\log_{9{\cdot}9} 6{\cdot}35) = \log_{10} 6{\cdot}35$ so that

$$\log_{9{\cdot}9} 6{\cdot}35 = \frac{\log_{10} 6{\cdot}35}{\log_{10} 9{\cdot}9} = \frac{0{\cdot}802\ldots}{0{\cdot}995\ldots} = 0{\cdot}806$$

(d) $\log_{7{\cdot}34} 7{\cdot}34 = 1$ because for any base a, $\log_a a = 1$.

33 Logarithmic equations

The following four examples serve to show you how logarithmic expressions and equations can be manipulated.

Example 1

Simplify the following:

$$\log_a x^2 + 3\log_a x - 2\log_a 4x$$

Solution

$$\begin{aligned}\log_a x^2 + 3\log_a x - 2\log_a 4x &= \log_a x^2 + \log_a x^3 - \log_a (4x)^2\\ &= \log_a\left(\frac{x^2x^3}{16x^2}\right)\\ &= \log_a\left(\frac{x^3}{16}\right)\end{aligned}$$

Example 2

Solve the following for x:

$$2\log_a x - \log_a(x-1) = \log_a(x+3)$$

Solution

$$\begin{aligned}\text{LHS} &= 2\log_a x - \log_a(x-1)\\ &= \log_a x^2 - \log_a(x-1)\\ &= \log_a\left(\frac{x^2}{x-1}\right)\\ &= \log_a(x+3) \text{ so that } \frac{x^2}{x-1} = x+3. \text{ That is}\end{aligned}$$

$$x^2 = (x+3)(x-1) = x^2 + 2x - 3 \text{ so that } 2x - 3 = 0 \text{ giving } x = \frac{3}{2}$$

Example 3

Find y in terms of x:

$$5\log_a y - 2\log_a(x+4) = 2\log_a y + \log_a x$$

Solution

$5\log_a y - 2\log_a(x+4) = 2\log_a y + \log_a x$ so that

$5\log_a y - 2\log_a y = \log_a x + 2\log_a(x+4)$ that is

$\log_a y^5 - \log_a y^2 = \log_a x + \log_a(x+4)^2$ that is

$\log_a\left(\dfrac{y^5}{y^2}\right) = \log_a y^3 = \log_a x(x+4)^2$ so that $y^3 = x(x+4)^2$ hence

$$y = \sqrt[3]{x(x+4)^2}$$

Example 4

For what values of x is $\log_a(x-3)$ valid?

Solution

$\log_a(x-3)$ is valid for $x - 3 > 0$, that is $x > 3$

Now you try some

34

1 Simplify $2\log_a x - 3\log_a 2x + \log_a x^2$

2 Solve the following for x:

$$4\log_a \sqrt{x} - \log_a 3x = \log_a x^{-2}$$

3 Find y in terms of x where:

$$2\log_a y - 3\log_a(x^2) = \log_a \sqrt{y} + \log_a x$$

Next frame for the answers

35

1 $2\log_a x - 3\log_a 2x + \log_a x^2 = \log_a x^2 - \log_a(2x)^3 + \log_a x^2$

$$= \log_a\left(\frac{x^2x^2}{8x^3}\right)$$

$$= \log_a\left(\frac{x}{8}\right)$$

2 $\text{LHS} = 4\log_a \sqrt{x} - \log_a 3x$

$$= \log_a(\sqrt{x})^4 - \log_a 3x$$

$$= \log_a x^2 - \log_a 3x$$

$$= \log_a\left(\frac{x^2}{3x}\right)$$

$$= \log_a\left(\frac{x}{3}\right)$$

$= \log_a x^{-2}$ the right-hand side of the equation.

So that:

$x^{-2} = \dfrac{x}{3}$, that is $x^3 = 3$ giving $x = \sqrt[3]{3}$

3 $2\log_a y - 3\log_a(x^2) = \log_a \sqrt{y} + \log_a x$, that is

$\log_a y^2 - \log_a(x^2)^3 = \log_a y^{\frac{1}{2}} + \log_a x$ so that

$\log_a y^2 - \log_a y^{\frac{1}{2}} = \log_a(x^2)^3 + \log_a x$ giving

$\log_a \dfrac{y^2}{y^{\frac{1}{2}}} = \log_a x^6.x$. Consequently $y^{\frac{3}{2}} = x^7$ and so $y = \sqrt[3]{x^{14}}$

At this point let us pause and summarize the main facts so far on powers and logarithms

Review summary

36

1 Rules of powers: [18]

(a) $a^m \times a^n = a^{m+n}$ (b) $a^{-m} = \dfrac{1}{a^m}$

(c) $a^m \div a^n = a^{m-n}$ (d) $a^{\frac{1}{m}} = \sqrt[m]{a}$

(e) $(a^m)^n = a^{mn}$ (f) $a^{\frac{n}{m}} = \sqrt[m]{a^n}$ or $a^{\frac{n}{m}} = (\sqrt[m]{a})^n$

(g) $a^0 = 1$

▶

2 Any real number can be written as another number raised to a power called the logarithm of the number. [23]

3 Rules of logarithms: [26]

(a) $\log_a xy = \log_a x + \log_a y$	*The log of a product equals the sum of the logs*
(b) $\log_a(x \div y) = \log_a x - \log_a y$	*The log of a quotient equals the difference of the logs*
(c) $\log_a(x^n) = n\log_a x$	*The log of a number raised to a power is the product of the power and the log of the number*

(d) $\log_a 1 = 0$

(e) $\log_a a = 1$ and $\log_a a^x = x$

(f) $a^{\log_a x} = x$

(g) $\log_a b = \dfrac{1}{\log_b a}$

4 Logarithms to base 10 are called *common logarithms* and are written as $\log x$. [29]

5 Logarithms to base $e = 2{\cdot}71828\ldots$ are called *natural logarithms* and are written as $\ln x$. [29]

6 The change of base formula which relates the logarithms of a number relative to two different bases is given as $\log_b a \times_a x = \log_b x$. [31]

37 Review exercise

1 Simplify each of the following:

(a) $a^6 \times a^5$

(b) $x^7 \div x^3$

(c) $(w^2)^m \div (w^m)^3$

(d) $s^3 \div t^{-4} \times (s^{-3}t^{-2})^3$

(e) $\dfrac{8x^{-3} \times 3x^2}{6x^{-4}}$

(f) $(4a^3b^{-1}c)^2 \times (a^{-2}b^4c^{-2})^{\frac{1}{2}} \div \left[64(a^6b^4c^2)^{-\frac{1}{2}}\right]$

(g) $\sqrt[3]{8a^3b^6} \div \sqrt{\dfrac{1}{25}a^4b^7} \times \left(16\sqrt{a^4b^6}\right)^{-\frac{1}{2}}$

2 Express the following without logs:

(a) $\log K = \log P - \log T + 1{\cdot}3\log V$

(b) $\ln A = \ln P + rn$

3 Rewrite $R = r\sqrt{\dfrac{f+P}{f-P}}$ in log form.

4 Evaluate by calculator or by change of base where necessary (to 3 dp):

(a) $\log 5{\cdot}324$

(b) $\ln 0{\cdot}0023$

(c) $\log_4 1{\cdot}2$

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1 (a) $a^6 \times a^5 = a^{6+5} = a^{11}$

(b) $x^7 \div x^3 = x^{7-3} = x^4$

(c) $(w^2)^m \div (w^m)^3 = w^{2m} \div w^{3m} = w^{2m} \times w^{-3m} = w^{-m}$

(d) $s^3 \div t^{-4} \times (s^{-3}t^{-2})^3 = s^3 \times t^4 \times s^{-9}t^{-6} = s^{-6}t^{-2}$

(e) $\dfrac{8x^{-3} \times 3x^2}{6x^{-4}} = \dfrac{24x^{-1}}{6x^{-4}} = 4x^3$

(f)
$$\begin{aligned}
&(4a^3b^{-1}c)^2 \times (a^{-2}b^4c^{-2})^{\frac{1}{2}} \div \left[64(a^6b^4c^2)^{-\frac{1}{2}}\right]\\
&= (16a^6b^{-2}c^2) \times (a^{-1}b^2c^{-1}) \div \left[64(a^{-3}b^{-2}c^{-1})\right]\\
&= (16a^6b^{-2}c^2) \times (a^{-1}b^2c^{-1}) \times \left[64^{-1}(a^3b^2c^1)\right]\\
&= \frac{a^8b^2c^2}{4}
\end{aligned}$$

(g)
$$\begin{aligned}
\sqrt[3]{8a^3b^6} \div \sqrt{\frac{1}{25}a^4b^7} \times \left(16\sqrt{a^4b^6}\right)^{-\frac{1}{2}} &= (2ab^2) \div \frac{a^2b^{\frac{7}{2}}}{5} \times \left(4ab^{\frac{3}{2}}\right)^{-1}\\
&= (2ab^2) \times \frac{5}{a^2b^{\frac{7}{2}}} \times \frac{1}{4ab^{\frac{3}{2}}}\\
&= \frac{5ab^2}{2a^2b^{\frac{7}{2}}ab^{\frac{3}{2}}}\\
&= \frac{5}{2a^2b^3}
\end{aligned}$$

2 (a) $K = \dfrac{PV^{1\cdot 3}}{T}$

(b) $A = Pe^{rn}$

3 $\log R = \log r + \dfrac{1}{2}(\log(f+P) - \log(f-P))$

4 (a) 0·726

(b) −6·075

(c) 0·132

Now move to the next topic

Algebraic multiplication and division

39

Multiplication

Example 1

$$\begin{aligned}
(x+2)(x+3) &= x(x+3) + 2(x+3)\\
&= x^2 + 3x + 2x + 6\\
&= x^2 + 5x + 6
\end{aligned}$$

Now a slightly harder one

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Example 2

$(2x+5)(x^2+3x+4)$

Each term in the second expression is to be multiplied by $2x$ and then by 5 and the results added together, so we set it out thus:

$$\begin{array}{lrrrr}
 & x^2 & +3x & +4 & \\
 & 2x & +5 & & \\
\hline
\text{Multiply throughout by } 2x & 2x^3 & +\ 6x^2 & +\ 8x & \\
\text{Multiply by } 5 & & 5x^2 & +15x & +20 \\
\hline
\text{Add the two lines} & 2x^3 & +11x^2 & +23x & +20 \\
\hline
\text{So } (2x+5)(x^2+3x+4) = & 2x^3 & +11x^2 & +23x & +20
\end{array}$$

Be sure to keep the same powers of the variable in the same column.

Next frame

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Now look at this one.

Example 3

Determine $(2x+6)(4x^3-5x-7)$

You will notice that the second expression is a cubic (highest power x^3), but that there is no term in x^2. In this case, we insert $0x^2$ in the working to keep the columns complete, that is:

$$\begin{array}{l}
4x^3+0x^2-5x-7 \\
2x\ \ +6 \\
\hline
\end{array}$$

which gives ....

Finish it

42

$$\boxed{8x^4+24x^3-10x^2-44x-42}$$

Here it is set out:

$$\begin{array}{rrrrr}
4x^3 & +0x^2 & -5x & -7 & \\
2x & +6 & & & \\
\hline
8x^4 & +\ 0x^3 & -10x^2 & -14x & \\
 & 24x^3 & +\ 0x^2 & -30x & -42 \\
\hline
8x^4 & +24x^3 & -10x^2 & -44x & -42 \\
\hline
\end{array}$$

They are all done in the same way, so here is one more for practice.

Example 4

Determine the product $(3x-5)(2x^3-4x^2+8)$

You can do that without any trouble.

The product is ....

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$$\boxed{6x^4 - 22x^3 + 20x^2 + 24x - 40}$$

All very straightforward:

$$\begin{array}{l}
2x^3 - 4x^2 + 0x + 8 \\
\underline{3x - 5} \\
6x^4 - 12x^3 + 0x^2 + 24x \\
\underline{ - 10x^3 + 20x^2 + 0x - 40} \\
\underline{6x^4 - 22x^3 + 20x^2 + 24x - 40}
\end{array}$$

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Division

Let us consider $(12x^3 - 2x^2 - 3x + 28) \div (3x + 4)$. The result of this division is called the *quotient* of the two expressions and we find the quotient by setting out the division in the same way as we do for the long division of numbers:

$$\begin{array}{r|l}
3x + 4 & \overline{12x^3 \ - \ 2x^2 \ - 3x \ + 28}
\end{array}$$

To make $12x^3$, $3x$ must be multiplied by $4x^2$, so we insert this as the first term in the quotient, multiply the divisor $(3x + 4)$ by $4x^2$, and subtract this from the first two terms:

$$\begin{array}{r|l}
 & 4x^2 \\
3x + 4 & \overline{12x^3 \ - \ 2x^2 \ - 3x \ + 28} \\
 & \underline{12x^3 \ + 16x^2} \\
 & - 18x^2 \ - 3x
\end{array}$$

Bring down the next term $(-3x)$ and repeat the process

To make $-18x^2$, $3x$ must be multiplied by $-6x$, so do this and subtract as before, not forgetting to enter the $-6x$ in the quotient.

Do this and we get . . . .

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$$\boxed{\begin{array}{r|l}
 & 4x^2 \ - \ 6x \\
3x + 4 & \overline{12x^3 \ - \ 2x^2 \ - \ 3x + 28} \\
 & \underline{12x^3 \ + 16x^2} \\
 & - 18x^2 \ - \ 3x \\
 & \underline{- 18x^2 \ - 24x} \\
 & 21x
\end{array}}$$

Now bring down the next term and continue in the same way and finish it off.

So $(12x^3 - 2x^2 - 3x + 28) \div (3x + 4) =$ . . . .

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$$\boxed{4x^2 - 6x + 7}$$

As before, if an expression has a power missing, insert the power with zero coefficient. Now you can determine $(4x^3 + 13x + 33) \div (2x + 3)$

Set it out as before and check the result with the next frame

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$$\boxed{2x^2 - 3x + 11}$$

Here it is:

$$\begin{array}{rllll}
 & 2x^2 & -3x & +11 & \\
\cline{2-5}
\multicolumn{1}{r|}{2x+3} & 4x^3 & -0x^2 & +13x & +33 \\
 & 4x^3 & +6x^2 & & \\
\cline{2-3}
 & & -6x^2 & +13x & \\
 & & -6x^2 & -\ 9x & \\
\cline{3-4}
 & & & 22x & +33 \\
 & & & 22x & +33 \\
\cline{4-5}
 & & & \bullet & \bullet \\
\cline{4-5}
\end{array}$$

$$\text{So } (4x^3 + 13x + 33) \div (2x + 3) = 2x^2 - 3x + 11$$

And one more.

Determine $(6x^3 - 7x^2 + 1) \div (3x + 1)$

Setting out as before, the quotient is . . . .

48

$$\boxed{2x^2 - 3x + 1}$$

After inserting the x term with zero coefficient, the rest is straightforward.

At this point let us pause and summarize the main facts so far for multiplication and division of algebraic expressions

49

Review summary

1 *Multiplication of algebraic expressions* [39]

Two algebraic expressions are multiplied together by successively multiplying the second expression by each term of the first expression.

2 *Division of algebraic expressions* [44]

Two algebraic expressions are divided by setting out the division in the same way as we do for the long division of numbers.

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Review exercise

1 Perform the following multiplications and simplify your results:

(a) $(8x - 4)(4x^2 - 3x + 2)$

(b) $(2x + 3)(5x^3 + 3x - 4)$

▶

2 Perform the following divisions:

(a) $(x^2+5x-6)\div(x-1)$

(b) $(x^2-x-2)\div(x+1)$

(c) $(12x^3-11x^2-25)\div(3x-5)$

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1 (a) $$\begin{aligned}(8x-4)(4x^2-3x+2) &= 8x(4x^2-3x+2)-4(4x^2-3x+2)\\ &= 32x^3-24x^2+16x-16x^2+12x-8\\ &= 32x^3-40x^2+28x-8\end{aligned}$$

(b) $$\begin{aligned}(2x+3)(5x^3+3x-4) &= 2x(5x^3+3x-4)+3(5x^3+3x-4)\\ &= 10x^4+6x^2-8x+15x^3+9x-12\\ &= 10x^4+15x^3+6x^2+x-12\end{aligned}$$

2 (a)

$$(x^2+5x-6)\div(x-1) = \begin{array}{rr} & x+6 \\ \cline{2-2} x-1 & \big)\, x^2+5x-6 \\ & x^2-x \\ \cline{2-2} & 6x-6 \\ & 6x-6 \\ \cline{2-2} & \bullet\quad\bullet \\ \cline{2-2} \end{array}$$

(b)

$$(x^2-x-2)\div(x+1) = \begin{array}{rr} & x-2 \\ \cline{2-2} x+1 & \big)\, x^2-x-2 \\ & x^2+x \\ \cline{2-2} & -2x-2 \\ & -2x-2 \\ \cline{2-2} & \bullet\quad\bullet \\ \cline{2-2} \end{array}$$

(c)

$$(12x^3-11x^2-25)\div(3x-5) = \begin{array}{rrrrr} & & 4x^2 & +\;3x & +\;5 \\ \cline{2-5} 3x-5 & \big)\,12x^3 & -11x^2 & +\;0x & -25 \\ & 12x^3 & -20x^2 & & \\ \cline{2-4} & & 9x^2 & +\;0x & \\ & & 9x^2 & -15x & \\ \cline{3-5} & & & 15x & -25 \\ & & & 15x & -25 \\ \cline{4-5} & & & \bullet & \bullet \\ \cline{4-5} \end{array}$$

Algebraic fractions

52

A numerical fraction is represented by one integer divided by another. Division of symbols follows the same rules to create *algebraic fractions*. For example:

$5\div 3$ can be written as the fraction $\dfrac{5}{3}$ so

$a\div b$ can be written as $\dfrac{a}{b}$

▶

Addition and subtraction

The addition and subtraction of algebraic fractions follow the same rules as the addition and subtraction of numerical fractions – the operations can only be performed when the denominators are the same. For example, just as:

$$\begin{aligned}\frac{4}{5}+\frac{3}{7}&=\frac{4\times 7}{5\times 7}+\frac{3\times 5}{7\times 5}\\&=\frac{4\times 7+3\times 5}{5\times 7}\\&=\frac{43}{35}\qquad\text{(where 35 is the LCM of 5 and 7)}\end{aligned}$$

so:

$$\begin{aligned}\frac{a}{b}+\frac{c}{d}&=\frac{a\times d}{b\times d}+\frac{c\times b}{d\times b}\\&=\frac{ad+cb}{bd}\text{ provided } b\neq 0 \text{ and } d\neq 0\qquad\text{(where } bd \text{ is the LCM of } b \text{ and } d\text{)}\end{aligned}$$

So that:

$$\frac{a}{b}-\frac{c}{d^2}+\frac{d}{a}=\dots\dots\dots\dots$$

Answer in the next frame

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$$\boxed{\frac{a^2d^2-abc+bd^3}{abd^2}}$$

Because

$$\begin{aligned}\frac{a}{b}-\frac{c}{d^2}+\frac{d}{a}&=\frac{aad^2}{bad^2}-\frac{cab}{d^2ab}+\frac{dd^2b}{ad^2b}\\&=\frac{a^2d^2-abc+bd^3}{abd^2}\end{aligned}$$

where abd^2 is the LCM of a, b and d^2.

On now to the next frame

54

In the same way

$$\begin{aligned}\frac{2}{x+1}+\frac{4}{x+2}&=\frac{2(x+2)+4(x+1)}{(x+1)(x+2)}\\&=\frac{2x+4+4x+4}{(x+1)(x+2)}\\&=\frac{6x+8}{x^2+3x+2}\end{aligned}$$

55

Multiplication and division

Fractions are multiplied by multiplying their numerators and denominators separately. For example, just as:

$$\frac{5}{4}\times\frac{3}{7}=\frac{5\times 3}{4\times 7}=\frac{15}{28}$$

so:

$$\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$$

▶

The *reciprocal* of a number is unity divided by the number. For example, the reciprocal of a is $1/a$ and the reciprocal of $\dfrac{a}{b}$ is:

$$\frac{1}{a/b} = 1 \div \frac{a}{b} = 1 \times \frac{b}{a} = \frac{b}{a}$$ *the numerator and denominator in the divisor are interchanged*

To divide by an algebraic fraction we multiply by its reciprocal. For example:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$$

So that $\dfrac{2a}{3b} \div \dfrac{a^2b}{6} = \ldots\ldots\ldots\ldots$

Check with the next frame

56

$$\boxed{\frac{4}{ab^2}}$$

Because

$$\frac{2a}{3b} \div \frac{a^2b}{6} = \frac{2a}{3b} \times \frac{6}{a^2b} = \frac{4}{ab^2}$$

Try another one:

$$\frac{2a}{3b} \div \frac{a^2b}{6} \times \frac{ab}{2} = \ldots\ldots\ldots\ldots$$

The answer is in the next frame

57

$$\boxed{\frac{2}{b}}$$

Because

$$\frac{2a}{3b} \div \frac{a^2b}{6} \times \frac{ab}{2} = \frac{2a}{3b} \times \frac{6}{a^2b} \times \frac{ab}{2} = \frac{4}{ab^2} \times \frac{ab}{2} = \frac{2}{b}$$

Remember that by the rules of precedence we work through the expression from the left to the right so we perform the division before we multiply. If we were to multiply before dividing in the above expression we should obtain:

$$\begin{aligned}\frac{2a}{3b} \div \frac{a^2b}{6} \times \frac{ab}{2} &= \frac{2a}{3b} \div \frac{a^3b^2}{12}\\ &= \frac{2a}{3b} \times \frac{12}{a^3b^2}\\ &= \frac{24a}{3a^3b^3} = \frac{8}{a^2b^3}\end{aligned}$$

and this would be wrong.

58 Review summary

1 The manipulation of algebraic fractions follows identical principles as those for arithmetic fractions. [52]

2 Only fractions with identical denominators can be immediately added or subtracted. [52]

3 Two fractions are multiplied by multiplying their respective numerators and denominators separately. [55]

4 Two fractions are divided by multiplying the numerator fraction by the reciprocal of the divisor fraction. [55]

59 Review exercise

1 Perform the following multiplications and simplify your results:

(a) $(2a+4b)(a-3b)$ (b) $(9s^2+3)(s^2-4)$

2 Simplify each of the following into a single algebraic fraction:

(a) $\dfrac{ab}{c}+\dfrac{cb}{a}$ (b) $\dfrac{ab}{c}-1$ (c) $\left(\dfrac{ab}{c}+\dfrac{ac}{b}\right)+\dfrac{bc}{a}$

3 Perform the following division:

$$\frac{a^3+8b^3}{a+2b}$$

60

1 (a) $(2a+4b)(a-3b)=2a(a-3b)+4b(a-3b)=2a^2-2ab-12b^2$

(b) $(9s^2+3)(s^2-4)=9s^2(s^2-4)+3(s^2-4)$

$$=9s^4-36s^2+3s^2-12$$

$$=9s^4-33s^2-12$$

2 (a) $\dfrac{ab}{c}+\dfrac{cb}{a}=\dfrac{aab}{ac}+\dfrac{cbc}{ac}=\dfrac{b(a^2+c^2)}{ac}$

(b) $\dfrac{ab}{c}-1=\dfrac{ab}{c}-\dfrac{c}{c}=\dfrac{ab-c}{c}$

(c) $\left(\dfrac{ab}{c}+\dfrac{ac}{b}\right)+\dfrac{bc}{a}=\dfrac{ab}{c}+\dfrac{ac}{b}+\dfrac{bc}{a}=\dfrac{a^2b^2+a^2c^2+b^2c^2}{abc}$

3

$$\frac{a^3+8b^3}{a+2b}=\begin{array}{r|lllll} & & a^2 & -2ab & +4b^2 \\ \cline{2-5} a+2b & a^3 & & & +8b^3 \\ & a^3 & +2a^2b & & \\ \cline{2-3} & & -2a^2b & & \\ & & -2a^2b & -4ab^2 & \\ \cline{3-5} & & & 4ab^2 & +8b^3 \\ & & & 4ab^2 & +8b^3 \\ \cline{4-5} & & & \bullet & \bullet \\ \cline{4-5} \end{array}$$

Factorization of algebraic expressions

61

An algebraic fraction can often be simplified by writing the numerator and denominator in terms of their factors and cancelling where possible. For example:

$$\begin{aligned}\frac{25ab^2 - 15a^2b}{40ab^2 - 24a^2b} &= \frac{5ab(5b - 3a)}{8ab(5b - 3a)}\\ &= \frac{5}{8}\end{aligned}$$

This is an obvious example, but there are many uses for factorization of algebraic expressions in advanced processes.

Common factors

The simplest form of factorization is the extraction of highest common factors (HCF) from an expression. For example, $(10x + 8)$ can clearly be written $2(5x + 4)$.

Similarly with $(35x^2y^2 - 10xy^3)$:

the HCF of the coefficients 35 and 10 is 5
the HCF of the powers of x is x
the HCF of the powers of y is y^2
So $(35x^2y^2 - 10xy^3) = 5xy^2(7x - 2y)$

In the same way: (a) $8x^4y^3 + 6x^3y^2 = \dots\dots\dots\dots$
and (b) $15a^3b - 9a^2b^2 = \dots\dots\dots\dots$

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(a) $2x^3y^2(4xy + 3)$
(b) $3a^2b(5a - 3b)$

Common factors by grouping

Four-termed expressions can sometimes be factorized by grouping into two binomial expressions and extracting common factors from each.

For example: $2ac + 6bc + ad + 3bd$

$$\begin{aligned}&= (2ac + 6bc) + (ad + 3bd) = 2c(a + 3b) + d(a + 3b)\\ &= (a + 3b)(2c + d)\end{aligned}$$

Similarly:

$x^3 - 4x^2y + xy^2 - 4y^3 = \dots\dots\dots\dots$

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$(x - 4y)(x^2 + y^2)$

Because

$$\begin{aligned}x^3 - 4x^2y + xy^2 - 4y^3 &= (x^3 - 4x^2y) + (xy^2 - 4y^3)\\ &= x^2(x - 4y) + y^2(x - 4y) = (x - 4y)(x^2 + y^2)\end{aligned}$$

▶

In some cases it might be necessary to rearrange the order of the original four terms. For example:

$$\begin{aligned}12x^2 - y^2 + 3x - 4xy^2 &= 12x^2 + 3x - y^2 - 4xy^2\\ &= (12x^2 + 3x) - (y^2 + 4xy^2) = 3x(4x+1) - y^2(1+4x)\\ &= (4x+1)(3x - y^2)\end{aligned}$$

Likewise, $20x^2 - 3y^2 + 4xy^2 - 15x = \ldots\ldots\ldots\ldots$

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$$\boxed{(4x-3)(5x+y^2)}$$

Rearranging terms:

$$\begin{aligned}(20x^2 - 15x) + (4xy^2 - 3y^2) &= 5x(4x-3) + y^2(4x-3)\\ &= (4x-3)(5x+y^2)\end{aligned}$$

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Useful products of two simple factors

A number of standard results are well-worth remembering for the products of simple factors of the form $(a+b)$ and $(a-b)$. These are:

(a) $(a+b)^2 = (a+b)(a+b) = a^2 + ab + ba + b^2$

i.e. $(a+b)^2 = a^2 + 2ab + b^2$

(b) $(a-b)^2 = (a-b)(a-b) = a^2 - ab - ba + b^2$

i.e. $(a-b)^2 = a^2 - 2ab + b^2$

(c) $(a-b)(a+b) = a^2 + ab - ba - b^2$

i.e. $(a-b)(a+b) = a^2 - b^2$ *the difference of two squares*

For our immediate purpose, these results can be used in reverse:

$$\begin{aligned}a^2 + 2ab + b^2 &= (a+b)^2\\ a^2 - 2ab + b^2 &= (a-b)^2\\ a^2 - b^2 &= (a-b)(a+b)\end{aligned}$$

If an expression can be seen to be one of these forms, its factors can be obtained at once.

These expressions that involve the variables raised to the power 2 are examples of what are called *quadratic* expressions. If a quadratic expression can be seen to be one of these forms, its factors can be obtained at once.

On to the next frame

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Remember

$$\begin{aligned}a^2 + 2ab + b^2 &= (a+b)^2\\ a^2 - 2ab + b^2 &= (a-b)^2\\ a^2 - b^2 &= (a-b)(a+b)\end{aligned}$$

Example 1

$$\begin{aligned}x^2 + 10x + 25 &= (x)^2 + 2(x)(5) + (5)^2, \text{ like } a^2 + 2ab + b^2\\ &= (x+5)^2\end{aligned}$$

So $x^2 + 10x + 25 = (x+5)^2$

▶

Example 2

$$4a^2 - 12a + 9 = (2a)^2 - 2(2a)(3) + (3)^2, \text{ like } a^2 - 2ab + b^2$$
$$= (2a - 3)^2$$

So $4a^2 - 12a + 9 = (2a - 3)^2$

Example 3

$$25x^2 - 16y^2 = (5x)^2 - (4y)^2$$
$$= (5x - 4y)(5x + 4y)$$

So $25x^2 - 16y^2 = (5x - 4y)(5x + 4y)$

Now can you factorize the following:

(a) $16x^2 + 40xy + 25y^2 =$ . . . .
(b) $9x^2 - 12xy + 4y^2 =$ . . . .
(c) $(2x + 3y)^2 - (x - 4y)^2 =$ . . . .

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(a) $(4x + 5y)^2$
(b) $(3x - 2y)^2$
(c) $(x + 7y)(3x - y)$

Quadratic expressions as the product of two factors

1 $(x + a)(x + b) = x^2 + (a + b)x + ab$

2 $(x - a)(x - b) = x^2 - (a + b)x + ab$

3 $(x + a)(x - b) = x^2 + (a - b)x - ab$

where both a and b are natural numbers.

Example 1

$x^2 + 5x = 6 = x^2 + (a + b)x + ab = (x + a)(x + b)$.
Here $a + b = 5$ so $(a, b) = (0, 5)$, $(1, 4)$ or $(2, 3)$ and since $ab = 6$ so then $(a, b) = (2, 3)$.
Therefore $x^2 + 5x + 6 = (x + 2)(x + 3)$

Example 2

$x^2 - 9x + 20 = x^2 - (a + b)x + ab = (x - a)(x - b)$.
Here $a + b = 9$ so $(a, b) = (0, 9)$, $(1, 8)$, $(2, 7)$, $(3, 6)$ or $(4, 5)$ and since $ab = 20$ then $(a, b) = (4, 5)$.
Therefore $x^2 - 9x + 20 = (x - 4)(x - 5)$

Example 3

$x^2 - 2x - 24 = x^2 + (a - b)x - ab = (x + a)(x - b)$.
Here $a - b = -2$ and $ab = 24$ so $(a, b) = (4, 6)$.
Therefore $x^2 - 2x - 24 = (x + 4)(x - 6)$

▶

Now here is a short exercise for practice. Factorize each of the following into two linear factors:

(a) $x^2 + 7x + 12$
(b) $x^2 - 11x + 28$
(c) $x^2 - 3x - 18$
(d) $x^2 + 2x - 24$
(e) $x^2 - 2x - 35$
(f) $x^2 - 10x + 16$

Finish all six and then check with the next frame

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(a) $(x + 3)(x + 4)$
(b) $(x - 4)(x - 7)$
(c) $(x - 6)(x + 3)$
(d) $(x + 6)(x - 4)$
(e) $(x - 7)(x + 5)$
(f) $(x - 2)(x - 8)$

At this point let us pause and summarize the main facts so far on factorization of algebraic expressions

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Review summary

Factorization of algebraic expressions

(a) Common factors of binomial expressions.
(b) Common factors of expressions by grouping.

Useful standard factors:

$$a^2 + 2ab + b^2 = (a + b)^2$$
$$a^2 - 2ab + b^2 = (a - b)^2$$
$$a^2 - b^2 = (a - b)(a + b) \quad \text{(Difference of two squares)}$$

Quadratic expressions as the product of two factors

1 $(x + a)(x + b) = x^2 + (a + b)x + ab$

2 $(x - a)(x - b) = x^2 - (a + b)x + ab$

3 $(x + a)(x - b) = x^2 + (a - b)x - ab$ where both a and b are natural numbers.

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Review exercise

1 Factorize the following:

(a) $18xy^3 - 8x^3y$
(b) $x^3 - 6x^2y - 2xy + 12y^2$
(c) $16x^2 - 24xy - 18x + 27y$
(d) $(x - 2y)^2 - (2x - y)^2$
(e) $x^2 + 7x - 30$
(f) $4x^2 - 36$
(g) $x^2 + 10x + 25$
(h) $3x^2 - 11x - 4$