volume 139

lecture notes in pure and applied mathematics

graphs, matrices, and designs

edited by
Rolf S. Rees

graphs, matrices, and designs

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Preface

It is customary in mathematics to show one's love and respect for a senior colleague by dedicating a research paper to him or her—birthdays and special anniversaries are two usual occasions. This is a book containing 21 such research papers in honor of the sixtieth birthday of Professor Norman J. Pullman on March 31, 1991.

Professor Pullman, who hails from New York, obtained his Ph.D. at Syracuse University in 1962. He taught for three years at McGill University before taking up a postdoctoral fellowship at the University of Alberta in 1965. Since then he has been on the faculty of Queen's University, Kingston, Ontario, where he was promoted to Professor in 1971. In addition to being his sixtieth birthday, 1991 also marks his 25th year of service at Queen's. In this time Professor Pullman has supervised 13 graduate students, three of whom are represented in this collection (D. de Caen, W. Jackson, and R. Rees). He has been an Invited Lecturer at six different professional meetings over the last 12 years, including annual meetings of the American Mathematical Society and the Australian Mathematical Society.

Professor Pullman has a long-standing association with Curtin University of Technology (formerly Western Australian Institute of Technology), Perth, Australia. He has been a Visiting Scholar there on several occasions in the last ten years.

Professor Pullman's research has spanned a wide range of topics in matrix theory, linear algebra, and graph theory. He has made significant contributions to the theory of tournaments and tournament matrices, the study of clique and biclique covering numbers and their relation to the problem of determining the boolean and real ranks of binary matrices, and the study of linear operators that preserve some prescribed property of a matrix (over some semiring). The 21 excellent chapters in this volume cover many aspects of his interests and constitute a representative sampler of current research in these areas. As such, we expect that this book will be of interest to anyone working in one or more of these areas.

To those who are familiar with Professor Pullman's work, the inclusion of design theory as one of his interests may at first seem to be something of a curiosity. Keeping in mind, however, that a pairwise balanced design is just an edge-clique partition of the complete graph, one of Professor Pullman's favorite problems (that of determining the clique partition number of the graph $K_v \backslash K_k$) is very closely related to a well-known extremal problem in design theory (that of determining the smallest number $g^{(k)}(v)$ of blocks required to construct a pairwise balanced design on v points in which the largest block has size k).

I would like to thank the referees, without whose invaluable assistance this volume would not have been possible. In this regard, special thanks go to D. Archdeacon, R. Brualdi, D. Hoffman, E. Kramer, E. Lamken, and C. H. Yang. I would also like to thank Professors Dominique de Caen and Walter D. Wallis for their encouragement and support in the early stages of this project. Finally, I wish to thank Professor Pullman for being my supervisor, mentor, and very dear friend.

I am certain that I speak for all the contributors in wishing our colleague and friend Norman J. Pullman a very happy birthday and continued success in all his endeavors.

Rolf S. Rees

Contents

Contributors

Jerzy K. Baksalary Department of Mathematics, Tadeusz Kotarbiński Pedagogical University, Zielona Góra, Poland, and Department of Mathematical Sciences, University of Tampere, Tampere, Finland

LeRoy B. Beasley Department of Mathematics, Utah State University, Logan, Utah

David C. Bigelow Department of Mathematics, Malaspina College, Nanaimo, British Columbia, Canada

John Wesley Brown Department of Mathematics, University of Illinois, Urbana, Illinois

L. Caccetta School of Mathematics and Statistics, Curtin University of Technology, Perth, Western Australia, Australia

Fred Cherry Brooklyn, New York

Lane Clark Department of Mathematics, University of New Mexico, Albuquerque, New Mexico

Charles J. Colbourn* Curtin University of Technology, Perth, Western Australia, Australia

L. J. Cummings Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario, Canada

*On leave from the University of Waterloo, Waterloo, Ontario, Canada

D. de Caen Department of Mathematics and Statistics, Queen's University, Kingston, Ontario, Canada

B. C. deGopi Utah State University, Logan, Utah, and Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario, Canada

J. H. Dinitz Department of Mathematics, University of Vermont, Burlington, Vermont

Roger Entringer Department of Mathematics, University of New Mexico, Albuquerque, New Mexico

Paul Erdös Mathematical Institute, Hungarian Academy of Science, Budapest, Hungary

Ivan Friš Department of Mathematics, Statistics and Computing Science, University of New England, Armidale, Australia

S. C. Furino Department of Mathematics, St. Jerome's College, Waterloo, Ontario, Canada

D. A. Gregory Department of Mathematics and Statistics, Queen's University, Kingston, Ontario, Canada

Alan Hartman IBM Israel-Science and Technology, Technion City, Haifa, Israel

Nora Hartsfield Department of Mathematics, Western Washington University, Bellingham, Washington

Kim A. S. Hefner Department of Mathematics, United States Naval Postgraduate School, Monterey, California

Katherine Heinrich Department of Mathematics, Simon Fraser University, Burnaby, British Columbia, Canada

Peter Horák Katedra matematiky, Bratislava, Czechoslovakia

Bill Jackson Department of Mathematical Studies, Goldsmiths' College, London, England

Christos Koukouvinos* Department of Mathematics, University of Thessaloniki, Thessaloniki, Greece

---

* *Present affiliation*: Department of Mathematics, University of Athens, Athens, Greece

D. L. Kreher Department of Mathematical Sciences, Michigan Technological University, Houghton, Michigan

J. Richard Lundgren Department of Mathematics, University of Colorado at Denver, Denver, Colorado

John S. Maybee Department of Mathematics, University of Colorado at Boulder, Boulder, Colorado

Mirka Miller* Department of Mathematics, Statistics and Computing Science, University of New England, Armidale, Australia

Lee Most San Francisco, California

Mel Most† New York, New York

E. T. Parker† Department of Mathematics, University of Illinois, Urbana, Illinois

D. Pritikin Department of Mathematics and Statistics, Miami University, Oxford, Ohio

Purwanto School of Mathematics and Statistics, Curtin University of Technology, Perth, Western Australia, Australia

Rolf S. Rees Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada

R. Bruce Richter Department of Mathematics and Statistics, Carleton University, Ottawa, Ontario, Canada

Alexander Rosa Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, Canada

Jennifer Seberry Department of Computer Science, University College, University of New South Wales, Australian Defence Force Academy, Canberra, Australia

W. F. Smyth Department of Computer Science and Systems, McMaster University, Hamilton, Ontario, Canada

---

**Present affiliation*: Department of Computer Science, University of Newcastle, Newcastle, Australia

† Deceased

D. R. Stinson Department of Computer Science and Engineering, University of Nebraska at Lincoln, Lincoln, Nebraska

George P. H. Styan Department of Mathematics and Statistics, McGill University, Montreal, Quebec, Canada

Huicheng Sun Department of Mathematics, Nanjing University, Nanjing, People's Republic of China

László Székely* Department of Mathematics, University of New Mexico, Albuquerque, New Mexico

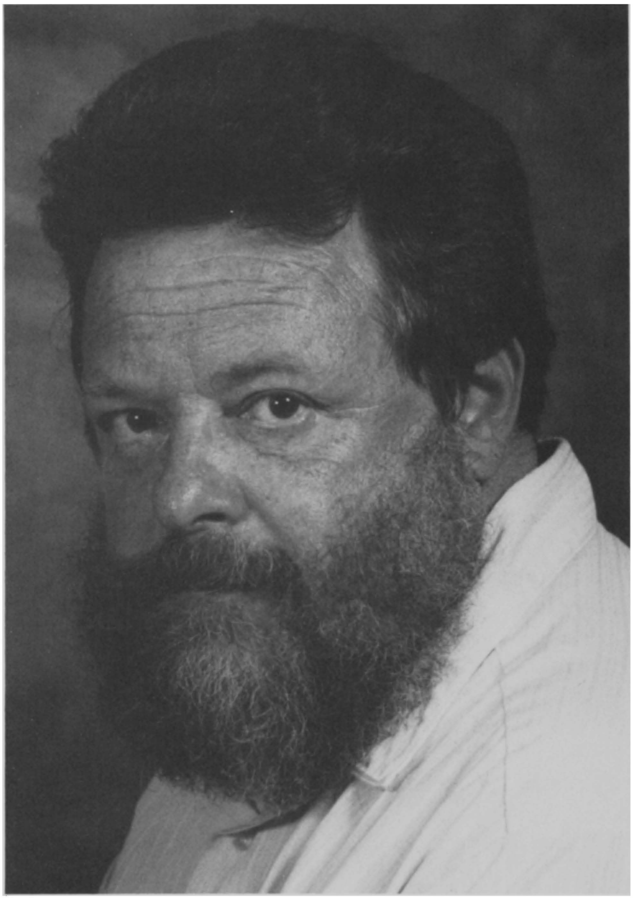
S. A. Vanstone Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Ontario, Canada

W. D. Wallis Department of Mathematics, Southern Illinois University at Carbondale, Carbondale, Illinois

Nicholas C. Wormald Department of Mathematics, University of Melbourne, Parkville, Victoria, Australia

Guo-Hui Zhang Department of Mathematics, Sonoma State University, Rohnert Park, California

* On leave from Eötvös University, Budapest, Hungary



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graphs, matrices, and designs

1

Around a Formula for the Rank of a Matrix Product with Some Statistical Applications

Jerzy K. BAKSALARY

Dept. of Mathematics, Tadeusz Kotarbiński Pedagogical University, PL 65-069 Zielona Góra, Poland, and Dept. of Mathematical Sciences, University of Tampere, SF-33101 Tampere, Finland

George P. H. STYAN

Dept. of Mathematics and Statistics, McGill University, 805 ouest, rue Sherbrooke, Montreal, Quebec, Canada H3A 2K6

Dedicated to Norman J. Pullman on the Occasion of his 60th Birthday.

ABSTRACT

We consider the rank formula $r(\mathbf{A}^*\mathbf{B}) = r(\mathbf{A}) + r(\mathbf{B}) - r(\mathbf{A} : \mathbf{B}) + r(\mathbf{A}^*\mathbf{Q}_\mathbf{B}\mathbf{Q}_\mathbf{A}\mathbf{B})$, where $\mathbf{Q}_\mathbf{A}$ and $\mathbf{Q}_\mathbf{B}$ are the orthogonal projectors on the orthocomplements of the ranges, respectively, of **A** and **B**. We offer a short proof, and show how this formula leads to a strengthened version of Sylvester's Law of Nullity; we also obtain several extensions. These results are then applied to some problems in mathematical statistics concerned with the concepts of connectedness and orthogonality in the theory of experimental designs in the two-way and three-way layouts.

1. INTRODUCTION

Let $\mathbb{C}_{m,n}$ denote the set of $m \times n$ complex matrices. For a given matrix $\mathbf{L} \in \mathbb{C}_{m,n}$ the symbols $\mathbf{L}^*$, $\mathcal{R}(\mathbf{L})$, and $\mathrm{r}(\mathbf{L})$ are used for the conjugate transpose, range (column space), and rank, respectively, of $\mathbf{L}$. In addition, $\mathbf{L}^+$ is the Moore-Penrose inverse of $\mathbf{L}$, that is, the unique $n \times m$ matrix satisfying the equations $\mathbf{LL}^+\mathbf{L} = \mathbf{L}$, $\mathbf{L}^+\mathbf{LL}^+ = \mathbf{L}^+$, $\mathbf{LL}^+ = (\mathbf{LL}^+)^*$, and $\mathbf{L}^+\mathbf{L} = (\mathbf{L}^+\mathbf{L})^*$. Then $\mathbf{P}_\mathbf{L} = \mathbf{LL}^+$ and $\mathbf{Q}_\mathbf{L} = \mathbf{I}_m - \mathbf{LL}^+$ are the orthogonal projectors, respectively, on $\mathcal{R}(\mathbf{L})$ and on its orthocomplement $\mathcal{R}^\perp(\mathbf{L})$; here $\mathbf{I}_m$ is the $m \times m$ identity matrix. Moreover, for given $\mathbf{L} \in \mathbb{C}_{m,n}$ and $\mathbf{M} \in \mathbb{C}_{m,p}$ the symbol $(\mathbf{L} : \mathbf{M})$ denotes the $m \times (n + p)$ partitioned matrix with $\mathbf{M}$ placed next to $\mathbf{L}$.

Our purpose in this paper is to discuss several aspects of the following formula (1) for the rank of a matrix product.

LEMMA 1. *For any $\mathbf{A} \in \mathbb{C}_{m,n}$ and $\mathbf{B} \in \mathbb{C}_{m,p}$ the rank of the product $\mathbf{A}^*\mathbf{B}$ admits the representation*

$$\mathrm{r}(\mathbf{A}^*\mathbf{B}) = \mathrm{r}(\mathbf{A}) + \mathrm{r}(\mathbf{B}) - \mathrm{r}(\mathbf{A} : \mathbf{B}) + \mathrm{r}(\mathbf{A}^*\mathbf{Q}_\mathbf{B}\mathbf{Q}_\mathbf{A}\mathbf{B}), \tag{1}$$

where $\mathbf{Q}_\mathbf{A}$ and $\mathbf{Q}_\mathbf{B}$ are the orthogonal projectors on the orthocomplements of the ranges, respectively, of $\mathbf{A}$ and $\mathbf{B}$.

We believe that the formula (1) in the form above may be new, but it follows easily from results on canonical correlations obtained by Latour and Styan (1985) and Styan (1985, Section 2). In Section 2 we provide a short direct (and new) proof of Lemma 1; we note that in the formula (1) we may replace $\mathrm{r}(\mathbf{A}) + \mathrm{r}(\mathbf{B}) - \mathrm{r}(\mathbf{A} : \mathbf{B})$ by the dimension $\dim[\mathcal{R}(\mathbf{A}) \cap \mathcal{R}(\mathbf{B})]$ of the intersection of the ranges $\mathcal{R}(\mathbf{A})$ and $\mathcal{R}(\mathbf{B})$ and the rank $\mathrm{r}(\mathbf{A}^*\mathbf{Q}_\mathbf{B}\mathbf{Q}_\mathbf{A}\mathbf{B})$ by either $\mathrm{r}(\mathbf{Q}_\mathbf{B}\mathbf{Q}_\mathbf{A}\mathbf{B})$ or $\mathrm{r}(\mathbf{Q}_\mathbf{B}\mathbf{P}_\mathbf{A}\mathbf{B})$, where $\mathbf{P}_\mathbf{A} = \mathbf{I}_m - \mathbf{Q}_\mathbf{A}$; cf. (2) below. We also show how Lemma 1 leads to the well-known Sylvester's Law of Nullity, which has recently [Cohn, 1989] received renewed attention. In addition, we present several extensions, including a result concerning the coincidence of certain eigenvalues of the products $\mathbf{P}_\mathbf{A}\mathbf{P}_\mathbf{B}$ and $\mathbf{P}_{\mathbf{Q}_\mathbf{B}\mathbf{A}}\mathbf{P}_{\mathbf{Q}_\mathbf{A}\mathbf{B}}$ in Theorem 1, a formula for $\mathrm{r}(\mathbf{A}^*\mathbf{Q}_\mathbf{C}\mathbf{B})$ in Theorem 2, and an orthogonal decomposition of a linear subspace in Theorem 3.

In Section 3 we show how our algebraic results may be applied to some problems in mathematical statistics concerned with the concepts of connectedness and orthogonality in the theory of experimental designs in the two-way and three-way layouts.

2. ALGEBRAIC RESULTS

We begin by proving Lemma 1, and then show how it leads to a strengthened version of Sylvester's Law of Nullity.

Proof of Lemma 1. In view of (2.10) and (2.35) in Marsaglia and Styan (1974), we have

$$r(\mathbf{A}^*\mathbf{Q_B}\mathbf{Q_A}\mathbf{B}) = r(\mathbf{A}^*\mathbf{Q_B}\mathbf{P_A}\mathbf{B}) = r(\mathbf{Q_B}\mathbf{P_A}\mathbf{B}) = r(\mathbf{B} : \mathbf{P_A}\mathbf{B}) - r(\mathbf{B}). \tag{2}$$

Since elementary column operations do not change rank and $\mathbf{B} - \mathbf{P_A}\mathbf{B} = \mathbf{Q_A}\mathbf{B}$, it follows that

$$r(\mathbf{B} : \mathbf{P_A}\mathbf{B}) = r(\mathbf{Q_A}\mathbf{B} : \mathbf{P_A}\mathbf{B}) = r(\mathbf{Q_A}\mathbf{B}) + r(\mathbf{P_A}\mathbf{B}) = r(\mathbf{A} : \mathbf{B}) - r(\mathbf{A}) + r(\mathbf{A}^*\mathbf{B}). \tag{3}$$

Combining (3) with (2) yields (1) . ❐

COROLLARY 1. *For any* $\mathbf{A} \in \mathbb{C}_{m,n}$ *and* $\mathbf{B} \in \mathbb{C}_{m,p}$ *the rank of the product* $\mathbf{A}^*\mathbf{B}$ *satisfies the inequality string*

$$r(\mathbf{A}^*\mathbf{B}) \geq r(\mathbf{A}) + r(\mathbf{B}) - r(\mathbf{A} : \mathbf{B}) \geq r(\mathbf{A}) + r(\mathbf{B}) - m. \tag{4}$$

Equality holds on the left of (4) *if and only if the projectors* $\mathbf{P_A}$ *and* $\mathbf{P_B}$ *commute, i.e.,* $\mathbf{P_A}\mathbf{P_B} = \mathbf{P_B}\mathbf{P_A}$, *and on the right if and only if the range* $\mathcal{R}(\mathbf{A} : \mathbf{B}) = \mathbb{C}_{m,1}$, *i.e., the partitioned matrix* $(\mathbf{A} : \mathbf{B})$ *has full row rank. Equality holds throughout* (4) *if and only if* $\mathbf{P_A} + \mathbf{P_B} - \mathbf{P_A}\mathbf{P_B} = \mathbf{I}_m$.

Proof. The first inequality in (4) follows directly from (1) and the second since clearly $r(\mathbf{A} : \mathbf{B}) \leq m$. Equality holds on the left of (4) if and only if $\mathbf{A}^*\mathbf{Q_B}\mathbf{Q_A}\mathbf{B} = \mathbf{0}$, and hence, using (2), $\mathbf{P_A}\mathbf{P_B} = \mathbf{P_B}\mathbf{P_A}\mathbf{P_B}$, which is Hermitian and so $\mathbf{P_A}\mathbf{P_B} = \mathbf{P_B}\mathbf{P_A}$. The condition for equality on the right of (4) is obvious. Since $r(\mathbf{A} : \mathbf{B})$ having full row rank equal to m is equivalent to $r(\mathbf{A}^*\mathbf{Q_B}) = r(\mathbf{Q_B})$ as well as to $r(\mathbf{Q_A}\mathbf{B}) = r(\mathbf{Q_A})$, it follows that equality holds throughout (4) if and only if $\mathbf{Q_B}\mathbf{Q_A} = \mathbf{0}$, or alternatively $\mathbf{P_A} + \mathbf{P_B} - \mathbf{P_A}\mathbf{P_B} = \mathbf{I}_m$. ❐

A necessary and sufficient condition for equality holding throughout (4) may also be obtained by combining $\mathbf{P}_{(\mathbf{A}:\mathbf{B})} = \mathbf{I}_m$ with the equivalence between the representation $\mathbf{P}_{(\mathbf{A}:\mathbf{B})} = \mathbf{P}_\mathbf{A} + \mathbf{P}_\mathbf{B} - \mathbf{P}_\mathbf{A}\mathbf{P}_\mathbf{B}$ and the commutativity property $\mathbf{P}_\mathbf{A}\mathbf{P}_\mathbf{B} = \mathbf{P}_\mathbf{B}\mathbf{P}_\mathbf{A}$, established by Rao and Yanai (1979). For a rather complete list of algebraic characterizations of this commutativity condition, however, we refer the reader to Baksalary (1987), where some statistical implications are also presented.

Removing the middle part of the inequality string (4) yields

$$\mathrm{r}(\mathbf{A}^*\mathbf{B}) \geq \mathrm{r}(\mathbf{A}) + \mathrm{r}(\mathbf{B}) - m, \tag{5}$$

which is Sylvester's Law of Nullity, recently discussed by Cohn (1989). Moreover,

$$\mathrm{r}(\mathbf{A}^*\mathbf{B}) = \mathrm{r}(\mathbf{A}) + \mathrm{r}(\mathbf{B}) + \mathrm{r}(\mathbf{Q}_\mathbf{B}\mathbf{Q}_\mathbf{A}) - m, \tag{6}$$

which is due to Marsaglia and Styan (1974, Theorem 6). From (6) it follows at once that equality holds in (5), or equivalently throughout (4), if and only if $\mathbf{Q}_\mathbf{B}\mathbf{Q}_\mathbf{A} = \mathbf{0}$ or, alternatively, $\mathbf{P}_\mathbf{A} + \mathbf{P}_\mathbf{B} - \mathbf{P}_\mathbf{A}\mathbf{P}_\mathbf{B} = \mathbf{I}_m$.

It is clear that the rank formula (1) may also be written as

$$\mathrm{r}(\mathbf{P}_\mathbf{A}\mathbf{P}_\mathbf{B}) = \mathrm{r}(\mathbf{A}) + \mathrm{r}(\mathbf{B}) - \mathrm{r}(\mathbf{A} : \mathbf{B}) + \mathrm{r}(\mathbf{P}_{\mathbf{Q}_\mathbf{B}\mathbf{A}}\mathbf{P}_{\mathbf{Q}_\mathbf{A}\mathbf{B}})$$

$$= \mathrm{r}(\mathbf{P}_\mathbf{A}) + \mathrm{r}(\mathbf{P}_\mathbf{B}) - \mathrm{r}(\mathbf{P}_\mathbf{A} + \mathbf{P}_\mathbf{B}) + \mathrm{r}(\mathbf{P}_{\mathbf{Q}_\mathbf{B}\mathbf{A}}\mathbf{P}_{\mathbf{Q}_\mathbf{A}\mathbf{B}}); \tag{7}$$

this version is related to the following result (Theorem 1) concerning the eigenvalues of the products $\mathbf{P}_\mathbf{A}\mathbf{P}_\mathbf{B}$ and $\mathbf{P}_{\mathbf{Q}_\mathbf{B}\mathbf{A}}\mathbf{P}_{\mathbf{Q}_\mathbf{A}\mathbf{B}}$, which is a direct consequence of Theorems 2.1 and 2.3 in Styan (1985); cf. Latour and Styan (1985, Theorem 3). See also Anderson, Harner, and Trapp (1985) for relationships between the eigenvalues of the product and difference of two orthogonal projectors.

THEOREM 1. *For any* $\mathbf{A} \in \mathbb{C}_{m,n}$ *and* $\mathbf{B} \in \mathbb{C}_{m,p}$ *the nonzero eigenvalues of the product* $\mathbf{P}_\mathbf{A}\mathbf{P}_\mathbf{B}$ *of two orthogonal projectors are all nonnegative and less than or equal to* 1, *and the multiplicity of the unit eigenvalue is equal to* $\mathrm{r}(\mathbf{A}) + \mathrm{r}(\mathbf{B}) - \mathrm{r}(\mathbf{A} : \mathbf{B})$. *The nonzero eigenvalues of the product* $\mathbf{P}_{\mathbf{Q}_\mathbf{B}\mathbf{A}}\mathbf{P}_{\mathbf{Q}_\mathbf{A}\mathbf{B}}$ *are all nonnegative and strictly less than* 1 *and coincide with the nonzero eigenvalues of* $\mathbf{P}_\mathbf{A}\mathbf{P}_\mathbf{B}$ *that are not equal to* 1.

Our next two results show further applications of Lemma 1 — to deriving a more general rank formula and to establishing an orthogonal decomposition of a linear subspace.

THEOREM 2. *For any* $\mathbf{A} \in \mathbb{C}_{m,n}$, $\mathbf{B} \in \mathbb{C}_{m,p}$ *and* $\mathbf{C} \in \mathbb{C}_{m,q}$ *the rank of the product* $\mathbf{A}^*\mathbf{Q_C}\mathbf{B}$ *admits the representation*

$$\mathrm{r}(\mathbf{A}^*\mathbf{Q_C}\mathbf{B}) = \mathrm{r}(\mathbf{Q_C}\mathbf{A}) + \mathrm{r}(\mathbf{Q_C}\mathbf{B}) - \mathrm{r}(\mathbf{Q_C}\mathbf{A} : \mathbf{Q_C}\mathbf{B}) + \mathrm{r}(\mathbf{A}^*\mathbf{Q}_{(\mathbf{B}:\mathbf{C})}\mathbf{Q}_{(\mathbf{A}:\mathbf{C})}\mathbf{B}). \tag{8}$$

Proof. Replacing $\mathbf{A}$ and $\mathbf{B}$ in (1) with $\mathbf{Q_C}\mathbf{A}$ and $\mathbf{Q_C}\mathbf{B}$, respectively, yields (8) but with $\mathrm{r}(\mathbf{A}^*\mathbf{Q_C}\mathbf{Q}_{\mathbf{Q_C}\mathbf{B}}\mathbf{Q}_{\mathbf{Q_C}\mathbf{A}}\mathbf{Q_C}\mathbf{B})$ instead of $\mathrm{r}(\mathbf{A}^*\mathbf{Q}_{(\mathbf{B}:\mathbf{C})}\mathbf{Q}_{(\mathbf{A}:\mathbf{C})}\mathbf{B})$. Moreover $\mathcal{R}(\mathbf{A} : \mathbf{C}) = \mathcal{R}(\mathbf{C}) \oplus^{\perp} \mathcal{R}(\mathbf{Q_C}\mathbf{A})$, where $\oplus^{\perp}$ denotes the orthogonal direct sum, and hence $\mathbf{P}_{(\mathbf{A}:\mathbf{C})} = \mathbf{P_C} + \mathbf{P}_{\mathbf{Q_C}\mathbf{A}}$. Consequently,

$$\mathbf{Q}_{\mathbf{Q_C}\mathbf{A}}\mathbf{Q_C} = \mathbf{Q_C} - \mathbf{P}_{\mathbf{Q_C}\mathbf{A}} = (\mathbf{P_C} + \mathbf{Q_C}) - \mathbf{P}_{(\mathbf{A}:\mathbf{C})} = \mathbf{Q}_{(\mathbf{A}:\mathbf{C})} \tag{9}$$

and similarly $\mathbf{Q_C}\mathbf{Q}_{\mathbf{Q_C}\mathbf{B}} = \mathbf{Q}_{(\mathbf{B}:\mathbf{C})}$ completing the proof. ❐

The following corollary will prove to be useful in the statistical applications discussed in Section 3.

COROLLARY 2. *Let* $\mathbf{A} \in \mathbb{C}_{m,n}$, $\mathbf{B} \in \mathbb{C}_{m,p}$ *and* $\mathbf{C} \in \mathbb{C}_{m,q}$ *be such that the range* $\mathcal{R}(\mathbf{C}) \subseteq \mathcal{R}(\mathbf{A}) \cap \mathcal{R}(\mathbf{B})$. *Then*

$$\mathrm{r}(\mathbf{A}^*\mathbf{Q_C}\mathbf{B}) = \mathrm{r}(\mathbf{A}^*\mathbf{B}) - \mathrm{r}(\mathbf{C}). \tag{10}$$

Proof. From (2.35) of Marsaglia and Styan (1974), if $\mathcal{R}(\mathbf{C}) \subseteq \mathcal{R}(\mathbf{A}) \cap \mathcal{R}(\mathbf{B})$ then $\mathrm{r}(\mathbf{Q_C}\mathbf{A}) = \mathrm{r}(\mathbf{A}) - \mathrm{r}(\mathbf{C})$, $\mathrm{r}(\mathbf{Q_C}\mathbf{B}) = \mathrm{r}(\mathbf{B}) - \mathrm{r}(\mathbf{C})$, and $\mathrm{r}(\mathbf{Q_C}\mathbf{A} : \mathbf{Q_C}\mathbf{B}) = \mathrm{r}(\mathbf{A} : \mathbf{B}) - \mathrm{r}(\mathbf{C})$. Moreover, $\mathbf{Q}_{(\mathbf{A}:\mathbf{C})} = \mathbf{Q_A}$ and $\mathbf{Q}_{(\mathbf{B}:\mathbf{C})} = \mathbf{Q_B}$. Hence (8) becomes

$$\mathrm{r}(\mathbf{A}^*\mathbf{Q_C}\mathbf{B}) = \mathrm{r}(\mathbf{A}) + \mathrm{r}(\mathbf{B}) - \mathrm{r}(\mathbf{A} : \mathbf{B}) - \mathrm{r}(\mathbf{C}) + \mathrm{r}(\mathbf{A}^*\mathbf{Q_B}\mathbf{Q_A}\mathbf{B}), \tag{11}$$

and combining (1) with (11) yields (10). ❐

We note that if $\mathcal{R}(\mathbf{C}) \subseteq \mathcal{R}(\mathbf{A}) \cap \mathcal{R}(\mathbf{B})$ then $r(\mathbf{A}^*\mathbf{P_C B}) = r(\mathbf{P_A P_C P_B}) = r(\mathbf{P_C}) = r(\mathbf{C})$, and therefore (10) may be rewritten $r(\mathbf{A}^*\mathbf{B} - \mathbf{A}^*\mathbf{P_C B}) = r(\mathbf{A}^*\mathbf{B}) - r(\mathbf{A}^*\mathbf{P_C B})$, and we see that rank is subtractive. As originally established by Hartwig (1980), the rank subtractivity property determines a partial ordering of matrices, which has been extensively studied in the literature; cf. e.g., Baksalary, Pukelsheim, and Styan (1989) and the references therein.

THEOREM 3. *For any* $\mathbf{A} \in \mathbb{C}_{m,n}$ *and* $\mathbf{B} \in \mathbb{C}_{m,p}$ *the orthogonal sum*

$$[\mathcal{R}(\mathbf{A}) \cap \mathcal{R}(\mathbf{B})] \oplus [\mathcal{R}(\mathbf{A}) \cap \mathcal{R}^{\perp}(\mathbf{B})] = \mathcal{R}(\mathbf{A}) \cap \mathcal{R}^{\perp}(\mathbf{Q_B Q_A B}). \tag{12}$$

Proof. Using (3.13) and (4.6) in Marsaglia and Styan (1974), we see that

$$r(\mathbf{A}^*\mathbf{B}) = r(\mathbf{A}) - \dim[\mathcal{R}(\mathbf{A}) \cap \mathcal{R}^{\perp}(\mathbf{B})] \tag{13}$$

and

$$r(\mathbf{A} : \mathbf{B}) = r(\mathbf{A}) + r(\mathbf{B}) - \dim[\mathcal{R}(\mathbf{A}) \cap \mathcal{R}(\mathbf{B})]. \tag{14}$$

Hence (1) is equivalent to the dimension identity

$$\dim[\mathcal{R}(\mathbf{A}) \cap \mathcal{R}(\mathbf{B})] + \dim[\mathcal{R}(\mathbf{A}) \cap \mathcal{R}^{\perp}(\mathbf{B})] = \dim[\mathcal{R}(\mathbf{A}) \cap \mathcal{R}^{\perp}(\mathbf{Q_B Q_A B})]. \tag{15}$$

Since $\mathcal{R}(\mathbf{Q_B Q_A B}) \subseteq \mathcal{R}^{\perp}(\mathbf{B})$, it follows that

$$\mathcal{R}(\mathbf{A}) \cap \mathcal{R}(\mathbf{B}) \subseteq \mathcal{R}(\mathbf{A}) \cap \mathcal{R}^{\perp}(\mathbf{Q_B Q_A B}). \tag{16}$$

Moreover, the vector $\mathbf{k} \in \mathcal{R}(\mathbf{A}) \cap \mathcal{R}^{\perp}(\mathbf{Q_B Q_A B})$ if and only if $\mathbf{k} = \mathbf{Am}$, say, where $\mathbf{m}^*\mathbf{A}^*\mathbf{Q_B Q_A B} = \mathbf{0}$, or equivalently, $\mathbf{m}^*\mathbf{A}^*\mathbf{P_B Q_A B} = \mathbf{0}$. Hence

$$\mathcal{R}(\mathbf{A}) \cap \mathcal{R}^{\perp}(\mathbf{Q_B Q_A B}) = \mathcal{R}(\mathbf{A}) \cap \mathcal{R}^{\perp}(\mathbf{P_B Q_A B}). \tag{17}$$

Since $\mathcal{R}(\mathbf{P_B Q_A B}) \subseteq \mathcal{R}(\mathbf{B})$, it is now clear that

$$\mathcal{R}(\mathbf{A}) \cap \mathcal{R}^{\perp}(\mathbf{B}) \subseteq \mathcal{R}(\mathbf{A}) \cap \mathcal{R}^{\perp}(\mathbf{Q_B Q_A B}). \tag{18}$$

Combining (16) and (18) with the dimension identity (15) establishes (12). ❒

We note that, in addition to (17), another alternative expression for the subspace on the right-hand side of (12) is $\mathcal{R}(\mathbf{A}) \cap \mathcal{R}^{\perp}(\mathbf{Q_B P_A B})$, since $\mathbf{Q_B Q_A B} = -\mathbf{Q_B P_A B}$.

3. STATISTICAL APPLICATIONS

3.1. The Two-way Layout—Block Design

Let us consider the statistical experiment corresponding to a two-way layout with a treatments allocated to n experimental units blocked in b sets according to an $a \times b$ (with a rows corresponding to treatments and b columns corresponding to sets) incidence matrix $\mathbf{N} = \{n_{ij}\}$, $i = 1, ..., a, j = 1, ..., b$, $\sum_{ij} n_{ij} = n$. If the researcher is primarily interested in comparing the effects of the treatments, while the effects due to the sets are considered to be "nuisance" and are assumed not to interact with the treatment effects, then the two-way layout is usually called a block design (with sets representing blocks). If y_{ijk} denotes the response observed on the kth experimental unit in the (i, j)th cell—where the ith treatment has been applied in the jth block—then the parametric model adopted for the analysis of data from such a block design consisting of the n observations from the nonempty cells (i.e., with $n_{ij} \geq 1$), may be expressed in the form:

$$y_{ijk} = \mu + \gamma_i + \delta_j + e_{ijk};\ i = 1, ..., a, j = 1, ..., b, k = 1, ..., n_{ij}. \tag{19}$$

In (19) the quantities μ, γ_i, and δ_j are unknown parameters denoting, respectively, the overall grand mean, and the ith treatment and jth block effects; the observational errors e_{ijk} are random variables, which we will assume to have means zero and to be uncorrelated and homoscedastic with unknown common variance σ^2 (white noise). We will use the terminology introduced above also in the sequel, although our considerations are valid as well for two-way layouts with two groups of parameters being of equal interest.

We may represent the model (19) in matrix notation, with $\mathbf{1}_n$ denoting the $n \times 1$ vector of ones, as

$$\mathrm{M}_1 = \{\mathbf{y}, \mathbf{1}_n\mu + \mathbf{W}\boldsymbol{\gamma} + \mathbf{Z}\boldsymbol{\delta}, \sigma^2\mathbf{I}_n\}, \tag{20}$$

which indicates that the $n \times 1$ vector $\mathbf{y} = \{y_{ijk}\}$ has expectation $\mathcal{E}(\mathbf{y}) = \mathbf{1}_n\mu + \mathbf{W}\boldsymbol{\gamma} + \mathbf{Z}\boldsymbol{\delta}$ and dispersion (variance-covariance) matrix $\mathcal{D}(\mathbf{y}) = \sigma^2\mathbf{I}_n$. The binary (each element equal to 0 or 1) "design" matrices $\mathbf{W}$, $n \times a$, and $\mathbf{Z}$, $n \times b$, describe the correspondence between components of $\mathbf{y}$ and, respectively, the treatments and blocks of the two-way layout, and the vectors $\boldsymbol{\gamma} = \{\gamma_i\}$, $a \times 1$, and $\boldsymbol{\delta} = \{\delta_j\}$, $b \times 1$, comprise, respectively, the treatment and block effects.

The design matrices $\mathbf{W}$ and $\mathbf{Z}$ are related to the $a \times b$ treatment-block incidence matrix $\mathbf{N} = \{n_{ij}\}$ through $\mathbf{N} = \mathbf{W'Z}$, where the superscript ' denotes transpose—in this Section 3 all our matrices are real. Since each experimental unit receives precisely one treatment and is in precisely one block it follows that $\mathbf{W1}_a = \mathbf{Z1}_b = \mathbf{1}_n$, and so

$$u = \dim[\mathcal{R}(\mathbf{W}) \cap \mathcal{R}(\mathbf{Z})] = \mathrm{r}(\mathbf{W}) + \mathrm{r}(\mathbf{Z}) - \mathrm{r}(\mathbf{W} : \mathbf{Z}) = \mathrm{r}(\mathbf{W}) - \mathrm{r}(\mathbf{Q_Z W}) \geq 1, \quad (21)$$

recalling that $\mathbf{Q_Z}$ is the orthogonal projector on the orthocomplement $\mathcal{R}^{\perp}(\mathbf{Z})$. Since at least one experimental unit receives each treatment and occurs in each block, the two matrices $\mathbf{W}$ and $\mathbf{Z}$ each have full column rank—the rank of the $a \times b$ incidence matrix $\mathbf{N}$, however, may vary from 1 to $\min(a, b)$. It then follows that u is equal to the (column) nullity of $\mathbf{Q_Z W}$ and hence also to the nullity of the so-called C-matrix $\mathbf{W'Q_Z W}$, which—from an algebraic point of view—is a suitably defined Schur complement, cf. e.g., Latour and Styan (1985, pp. 227-228).

The treatments and blocks are (and the two-way layout is) said to be *connected* whenever all the treatment effect contrasts $\mathbf{c'}\boldsymbol{\gamma}$, for any $a \times 1$ vector $\mathbf{c}$ satisfying $\mathbf{c'1}_a = 0$ (and, equivalently, all the block effect contrasts) are unbiasedly estimable in the model M_1 defined in (20). An alternate combinatorial definition, cf. e.g., Raghavarao (1971, page 49), says that the layout is connected whenever, given any two treatments numbered i and i', it is possible to construct a chain of treatments numbered $i = i_0, i_1, \ldots, i_p = i'$, say, such that every consecutive pair of treatments in the chain occurs together in at least one block. Both the statistical and combinatorial formulations of the concept of connectedness are due to Bose (1947). It is well known, cf. e.g., Raghavarao (1971, page 50), that connectedness is equivalent to $u = 1$, since u is the nullity of the C-matrix $\mathbf{W'Q_Z W}$.

The *canonical correlations* between the vectors

$$\mathbf{y}_a = \{\Sigma_{jk} y_{ijk}\} = \mathbf{W'y} \text{ and } \mathbf{y}_b = \{\Sigma_{ik} y_{ijk}\} = \mathbf{Z'y} \quad (22)$$

of, respectively, *treatment totals* and *block totals*, are the positive square roots of the nonzero eigenvalues of the product $\mathbf{P_W P_Z}$ of the orthogonal projectors on $\mathcal{R}(\mathbf{W})$ and $\mathcal{R}(\mathbf{Z})$. Similarly, the canonical correlations between the vectors

$$\mathbf{z}_a = \mathbf{W'Q_Z y} = \mathbf{y}_a - \mathbf{N}(\mathbf{Z'Z})^{-1}\mathbf{y}_b \text{ and } \mathbf{z}_b = \mathbf{Z'Q_W y} = \mathbf{y}_b - \mathbf{N'}(\mathbf{W'W})^{-1}\mathbf{y}_a \tag{23}$$

of, respectively, *treatment totals adjusted for blocks* and *block totals adjusted for treatments* are the positive square roots of the nonzero eigenvalues of the product $\mathbf{P}_{\mathbf{Q_Z W}}\mathbf{P}_{\mathbf{Q_W Z}}$. It then follows from Theorem 1 that, cf. Latour and Styan (1985, Theorem 2), the canonical correlations between $\mathbf{z}_a$ and $\mathbf{z}_b$ are all less than 1 and coincide with the canonical correlations between $\mathbf{y}_a$ and $\mathbf{y}_b$ that are less than 1, the multiplicity of the unit canonical correlation between $\mathbf{y}_a$ and $\mathbf{y}_b$ being equal to u as defined in (21).

The concept of *orthogonality* in experimental designs, whose origins date back to Yates (1933), is not uniquely defined—so aptly reflected by the phrase "terminological muddle" in the title of the survey paper by Preece (1977); see also Styan (1986, page 434). We will consider two versions of orthogonality—in the first, cf. Chakrabarti (1962, page 19), the treatments and blocks (and the two-way layout or block design) are said to be orthogonal whenever the vectors $\mathbf{z}_a$ and $\mathbf{z}_b$, respectively, of treatment totals adjusted for blocks and block totals adjusted for treatments, as defined in (23), are uncorrelated, i.e.,

$$\mathbf{W'Q_Z Q_W Z} = \mathbf{0}. \tag{24}$$

In the second version, cf. Scheffé (1959, page 119) and Bradu (1965, page 95), it is required that the vectors $\mathbf{y}_a$ and $\mathbf{y}_b$ of treatment and block totals, each centered (i.e., adjusted only for the grand mean), be uncorrelated, i.e.,

$$\mathbf{W'Q}_{\mathbf{1}_n}\mathbf{Z} = \mathbf{0}, \tag{25}$$

where the $n \times n$ centering matrix $\mathbf{Q}_{\mathbf{1}_n} = \mathbf{I}_n - \mathbf{1}_n\mathbf{1}_n'/n$. Following the adaptation by Styan (1986, page 434) of the terminology proposed by Khatri and Shah (1986, page 219) for m-way layouts with $m \geq 3$, we will say that (24) defines *weak orthogonality* and that (25) defines *strict orthogonality* of the treatments and blocks (and of the underlying two-way layout or block design).

The weak-orthogonality criterion (24), which is equivalent, cf. Baksalary (1987, Theorem 2), to the commutativity property $\mathbf{P_W P_Z} = \mathbf{P_Z P_W}$, is a necessary and sufficient condition, cf. Baksalary (1984, Theorem 2.3), for all the treatment contrasts