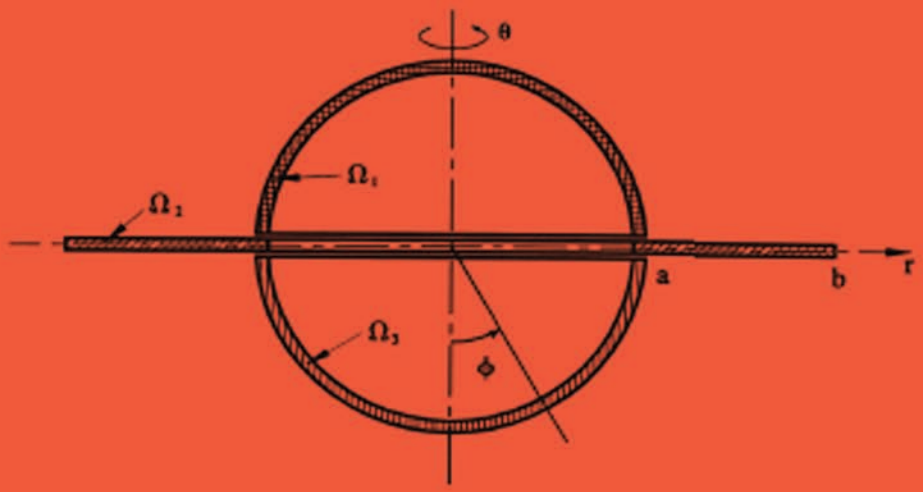
INFLUENCE FUNCTIONS AND MATRICES



YURI A. MELNIKOV

INFLUENCE FUNCTIONS AND MATRICES

MECHANICAL ENGINEERING

A Series of Textbooks and Reference Books

Editor

L. L. Faulkner

Columbus Division, Battelle Memorial Institute
and Department of Mechanical Engineering
The Ohio State University
Columbus, Ohio

1. *Spring Designer's Handbook*, Harold Carlson
2. *Computer-Aided Graphics and Design*, Daniel L. Ryan
3. *Lubrication Fundamentals*, J. George Wills
4. *Solar Engineering for Domestic Buildings*, William A. Himmelman
5. *Applied Engineering Mechanics: Statics and Dynamics*, G. Boothroyd and C. Poli
6. *Centrifugal Pump Clinic*, Igor J. Karassik
7. *Computer-Aided Kinetics for Machine Design*, Daniel L. Ryan
8. *Plastics Products Design Handbook, Part A: Materials and Components; Part B: Processes and Design for Processes*, edited by Edward Miller
9. *Turbomachinery: Basic Theory and Applications*, Earl Logan, Jr.
10. *Vibrations of Shells and Plates*, Werner Soedel
11. *Flat and Corrugated Diaphragm Design Handbook*, Mario Di Giovanni
12. *Practical Stress Analysis in Engineering Design*, Alexander Blake
13. *An Introduction to the Design and Behavior of Bolted Joints*, John H. Bickford
14. *Optimal Engineering Design: Principles and Applications*, James N. Siddall
15. *Spring Manufacturing Handbook*, Harold Carlson
16. *Industrial Noise Control: Fundamentals and Applications*, edited by Lewis H. Bell
17. *Gears and Their Vibration: A Basic Approach to Understanding Gear Noise*, J. Derek Smith
18. *Chains for Power Transmission and Material Handling: Design and Applications Handbook*, American Chain Association
19. *Corrosion and Corrosion Protection Handbook*, edited by Philip A. Schweitzer
20. *Gear Drive Systems: Design and Application*, Peter Lynwander
21. *Controlling In-Plant Airborne Contaminants: Systems Design and Calculations*, John D. Constance
22. *CAD/CAM Systems Planning and Implementation*, Charles S. Knox
23. *Probabilistic Engineering Design: Principles and Applications*, James N. Siddall
24. *Traction Drives: Selection and Application*, Frederick W. Heilich III and Eugene E. Shube
25. *Finite Element Methods: An Introduction*, Ronald L. Huston and Chris E. Passerello
26. *Mechanical Fastening of Plastics: An Engineering Handbook*, Brayton Lincoln, Kenneth J. Gomes, and James F. Braden
27. *Lubrication in Practice: Second Edition*, edited by W. S. Robertson
28. *Principles of Automated Drafting*, Daniel L. Ryan
29. *Practical Seal Design*, edited by Leonard J. Martini
30. *Engineering Documentation for CAD/CAM Applications*, Charles S. Knox
31. *Design Dimensioning with Computer Graphics Applications*, Jerome C. Lange
32. *Mechanism Analysis: Simplified Graphical and Analytical Techniques*, Lyndon O. Barton

33. *CAD/CAM Systems: Justification, Implementation, Productivity Measurement,* Edward J. Preston, George W. Crawford, and Mark E. Coticchia
34. *Steam Plant Calculations Manual,* V. Ganapathy
35. *Design Assurance for Engineers and Managers,* John A. Burgess
36. *Heat Transfer Fluids and Systems for Process and Energy Applications,* Jasbir Singh
37. *Potential Flows: Computer Graphic Solutions,* Robert H. Kirchhoff
38. *Computer-Aided Graphics and Design: Second Edition,* Daniel L. Ryan
39. *Electronically Controlled Proportional Valves: Selection and Application,* Michael J. Tonyan, edited by Tobi Goldoftas
40. *Pressure Gauge Handbook,* AMETEK, U.S. Gauge Division, edited by Philip W. Harland
41. *Fabric Filtration for Combustion Sources: Fundamentals and Basic Technology,* R. P. Donovan
42. *Design of Mechanical Joints,* Alexander Blake
43. *CAD/CAM Dictionary,* Edward J. Preston, George W. Crawford, and Mark E. Coticchia
44. *Machinery Adhesives for Locking, Retaining, and Sealing,* Girard S. Haviland
45. *Couplings and Joints: Design, Selection, and Application,* Jon R. Mancuso
46. *Shaft Alignment Handbook,* John Piotrowski
47. *BASIC Programs for Steam Plant Engineers: Boilers, Combustion, Fluid Flow, and Heat Transfer,* V. Ganapathy
48. *Solving Mechanical Design Problems with Computer Graphics,* Jerome C. Lange
49. *Plastics Gearing: Selection and Application,* Clifford E. Adams
50. *Clutches and Brakes: Design and Selection,* William C. Orthwein
51. *Transducers in Mechanical and Electronic Design,* Harry L. Trietley
52. *Metallurgical Applications of Shock-Wave and High-Strain-Rate Phenomena,* edited by Lawrence E. Murr, Karl P. Staudhammer, and Marc A. Meyers
53. *Magnesium Products Design,* Robert S. Busk
54. *How to Integrate CAD/CAM Systems: Management and Technology,* William D. Engelke
55. *Cam Design and Manufacture: Second Edition; with cam design software for the IBM PC and compatibles, disk included,* Preben W. Jensen
56. *Solid-State AC Motor Controls: Selection and Application,* Sylvester Campbell
57. *Fundamentals of Robotics,* David D. Ardayfio
58. *Belt Selection and Application for Engineers,* edited by Wallace D. Erickson
59. *Developing Three-Dimensional CAD Software with the IBM PC,* C. Stan Wei
60. *Organizing Data for CIM Applications,* Charles S. Knox, with contributions by Thomas C. Boos, Ross S. Culverhouse, and Paul F. Muchnicki
61. *Computer-Aided Simulation in Railway Dynamics,* by Rao V. Dukkipati and Joseph R. Amyot
62. *Fiber-Reinforced Composites: Materials, Manufacturing, and Design,* P. K. Mallick
63. *Photoelectric Sensors and Controls: Selection and Application,* Scott M. Juds
64. *Finite Element Analysis with Personal Computers,* Edward R. Champion, Jr., and J. Michael Ensminger
65. *Ultrasonics: Fundamentals, Technology, Applications: Second Edition, Revised and Expanded,* Dale Ensminger
66. *Applied Finite Element Modeling: Practical Problem Solving for Engineers,* Jeffrey M. Steele
67. *Measurement and Instrumentation in Engineering: Principles and Basic Laboratory Experiments,* Francis S. Tse and Ivan E. Morse
68. *Centrifugal Pump Clinic: Second Edition, Revised and Expanded,* Igor J. Karassik
69. *Practical Stress Analysis in Engineering Design: Second Edition, Revised and Expanded,* Alexander Blake
70. *An Introduction to the Design and Behavior of Bolted Joints: Second Edition, Revised and Expanded,* John H. Bickford
71. *High Vacuum Technology: A Practical Guide,* Marsbed H. Hablanian
72. *Pressure Sensors: Selection and Application,* Duane Tandeske

73. *Zinc Handbook: Properties, Processing, and Use in Design*, Frank Porter
74. *Thermal Fatigue of Metals*, Andrzej Weronski and Tadeusz Hejwowski
75. *Classical and Modern Mechanisms for Engineers and Inventors*, Preben W. Jensen
76. *Handbook of Electronic Package Design*, edited by Michael Pecht
77. *Shock-Wave and High-Strain-Rate Phenomena in Materials*, edited by Marc A. Meyers, Lawrence E. Murr, and Karl P. Staudhammer
78. *Industrial Refrigeration: Principles, Design and Applications*, P. C. Koelet
79. *Applied Combustion*, Eugene L. Keating
80. *Engine Oils and Automotive Lubrication*, edited by Wilfried J. Bartz
81. *Mechanism Analysis: Simplified and Graphical Techniques, Second Edition, Revised and Expanded*, Lyndon O. Barton
82. *Fundamental Fluid Mechanics for the Practicing Engineer*, James W. Murdock
83. *Fiber-Reinforced Composites: Materials, Manufacturing, and Design, Second Edition, Revised and Expanded*, P. K. Mallick
84. *Numerical Methods for Engineering Applications*, Edward R. Champion, Jr.
85. *Turbomachinery: Basic Theory and Applications, Second Edition, Revised and Expanded*, Earl Logan, Jr.
86. *Vibrations of Shells and Plates: Second Edition, Revised and Expanded*, Werner Soedel
87. *Steam Plant Calculations Manual: Second Edition, Revised and Expanded*, V. Ganapathy
88. *Industrial Noise Control: Fundamentals and Applications, Second Edition, Revised and Expanded*, Lewis H. Bell and Douglas H. Bell
89. *Finite Elements: Their Design and Performance*, Richard H. MacNeal
90. *Mechanical Properties of Polymers and Composites: Second Edition, Revised and Expanded*, Lawrence E. Nielsen and Robert F. Landel
91. *Mechanical Wear Prediction and Prevention*, Raymond G. Bayer
92. *Mechanical Power Transmission Components*, edited by David W. South and Jon R. Mancuso
93. *Handbook of Turbomachinery*, edited by Earl Logan, Jr.
94. *Engineering Documentation Control Practices and Procedures*, Ray E. Monahan
95. *Refractory Linings: Thermomechanical Design and Applications*, Charles A. Schacht
96. *Geometric Dimensioning and Tolerancing: Applications and Techniques for Use in Design, Manufacturing, and Inspection*, James D. Meadows
97. *An Introduction to the Design and Behavior of Bolted Joints: Third Edition, Revised and Expanded*, John H. Bickford
98. *Shaft Alignment Handbook: Second Edition, Revised and Expanded*, John Piotrowski
99. *Computer-Aided Design of Polymer-Matrix Composite Structures*, edited by S. V. Hoa
100. *Friction Science and Technology*, Peter J. Blau
101. *Introduction to Plastics and Composites: Mechanical Properties and Engineering Applications*, Edward Miller
102. *Practical Fracture Mechanics in Design*, Alexander Blake
103. *Pump Characteristics and Applications*, Michael W. Volk
104. *Optical Principles and Technology for Engineers*, James E. Stewart
105. *Optimizing the Shape of Mechanical Elements and Structures*, A. A. Seireg and Jorge Rodriguez
106. *Kinematics and Dynamics of Machinery*, Vladimír Stejskal and Michael Valášek
107. *Shaft Seals for Dynamic Applications*, Les Horve
108. *Reliability-Based Mechanical Design*, edited by Thomas A. Cruse
109. *Mechanical Fastening, Joining, and Assembly*, James A. Speck
110. *Turbomachinery Fluid Dynamics and Heat Transfer*, edited by Chunill Hah
111. *High-Vacuum Technology: A Practical Guide, Second Edition, Revised and Expanded*, Marsbed H. Hablanian

112. *Geometric Dimensioning and Tolerancing: Workbook and Answerbook*, James D. Meadows
113. *Handbook of Materials Selection for Engineering Applications*, edited by G. T. Murray
114. *Handbook of Thermoplastic Piping System Design*, Thomas Sixsmith and Reinhard Hanselka
115. *Practical Guide to Finite Elements: A Solid Mechanics Approach*, Steven M. Lepi
116. *Applied Computational Fluid Dynamics*, edited by Vijay K. Garg
117. *Fluid Sealing Technology*, Heinz K. Muller and Bernard S. Nau
118. *Friction and Lubrication in Mechanical Design*, A. A. Seireg
119. *Influence Functions and Matrices*, Yuri A. Melnikov
120. *Machining of Ceramics and Composites*, edited by Said Jahanmir, M. Ramulu, and Philip Koshy

Additional Volumes in Preparation

Heat Exchange Design Handbook, T. Kuppan

Couplings and Joints: Second Edition, Revised and Expanded, Jon R. Mancuso

Mechanical Engineering Software

Spring Design with an IBM PC, Al Dietrich

Mechanical Design Failure Analysis: With Failure Analysis System Software for the IBM PC, David G. Ullman

INFLUENCE FUNCTIONS AND MATRICES

YURI A. MELNIKOV
Middle Tennessee State University
Murfreesboro, Tennessee

MARCEL DEKKER, INC. NEW YORK • BASEL

ISBN: 0-8247-1941-7

This book is printed on acid-free paper.

Headquarters
Marcel Dekker, Inc.
270 Madison Avenue, New York, NY 10016
tel: 212-696-9000; fax: 212-685-4540

Eastern Hemisphere Distribution
Marcel Dekker AG
Hutgasse 4, Postfach 812, CH-4001 Basel, Switzerland
tel: 44-61-261-8482; fax: 44-61-261-8896

World Wide Web
http://www.dekker.com

The publisher offers discounts on this book when ordered in bulk quantities. For more information, write to Special Sales/Professional Marketing at the headquarters address above.

Current printing (last digit):
10 9 8 7 6 5 4 3 2 1

PRINTED IN THE UNITED STATES OF AMERICA

To the memory of my parents,
Afanasy and Antonina,
my wisest teachers in this life

Preface

Notwithstanding their established significance in the qualitative theory of differential equations, Green's functions are rarely used in the realm of computational mechanics. As a working tool, the influence (Green's) function-based methods are virtually absent in texts related to numerical methods in mechanics. The cause of this discrepancy is likely to be found in the scarcity of implementable representations for Green's functions in the existing literature. This deficiency occurs because, even for the simplest boundary value problems, the construction of Green's functions is not a routine procedure. In addition, under-representation of methods based on Green's functions results from the widespread availability of algorithms based on standard approaches, such as the boundary element method.

Motivation for writing this book arose from the author's over twenty-five years experience of teaching the influence function-related courses to a variety of students at various institutions (Dnepropetrovsk State University, Brooklyn Polytechnic Institute, Northwestern University, Vanderbilt University, and Middle Tennessee State University). With this text, the author wished to popularize the influence function method among young researchers in the applied sciences and engineering.

Some classes of problems originating in applied mechanics form the basis of this book. Although the problems themselves are classical, their treatment in this context differs from that of more traditional texts in the field; the major difference being in the emphasis on influence (Green's) functions in the analysis of these problems.

A broad spectrum of problems in continuum mechanics, related to such fields as fluid flow, acoustics, electromagnetism, heat transfer, and elasticity, can be formulated as boundary value problems involving (ordinary or partial) differential equations. The Green's function is a key concept in the area of differential equations, resembling, in some sense, the inverse matrix in linear algebra. The Green's function formalism has been utilized as a powerful means of development and presentation in classical as well as contemporary treatments in the analysis of differential equations.

About two decades ago, in an attempt to make a contribution to this intriguing and challenging area, a technique was proposed [24, 42] for the construction of compact expressions of Green's functions and matrices for some elliptical PDEs that occur in mechanics. The technique is based on the classical method of eigenfunction expansion (see, for example, [63]). This productive approach is included in [45], a collection of elaborations on influence (Green's) functions methods, aimed at introducing advances in this field to the computational mechanics community.

Also included in [45] are computational implementations of influence functions with emphasis on shape complexity, on various types of nonlinearities, and on some inverse formulations, including optimal shape design. As such, [45] can serve as a source for those who are already familiar with the influence function concept and its strengths and weaknesses.

This volume is intended as a graduate text in influence (Green's) functions and is designed for students in mechanical engineering or in related fields in applied mathematics. The primary goal has been to create a supplement for existing texts in these fields, a book providing an alternative approach to boundary value problems of applied mechanics. In addition, the book is recommended to users of the boundary element method, where the use of influence functions can significantly increase the effectiveness of existing numerical procedures.

The level of presentation has been aimed at readers with standard background in calculus, linear algebra, ODEs, PDEs, and numerical methods. An elementary knowledge of integral equations is also assumed. We hope the reader will find use for the large collection of Green's functions and matrices for (systems of) ordinary and partial differential equations contained in this single volume. Many of these representations for Green's functions and matrices appears in book form for the first time.

It is with gratitude that the author acknowledges Drs. V.V. Loboda and N.V. Polyakov, Dnepropetrovsk State University and Dr. J.O. Powell, Middle Tennessee State University, for presently sharing his research interest in influence (Green's) functions and in the use of such functions for developing computational algorithms of the boundary integral equation method in engineering and science.

While writing this book, the author has received assistance from many of his colleagues in the Department of Mathematical Sciences at MTSU. And heartfelt thanks goes out to them all. Especially, the author wishes to acknowledge gratefully the many helpful suggestions about the exposition and organization, offered by Dr. Jan Zijlstra. Invaluable comments made by Drs. Leslie Aspinwall and Diane Miller during the work have substantially improved the quality of the text. The author also expresses his appreciation to Scott McDaniel, a junior faculty member, who worked through drafts of

some sections of the manuscript and made valuable comments from the point of view of the student.

The author is appreciative of the outstanding job done by the following students: Yugeny Bobylov (Dnepropetrovsk State University), Lynn Roubides, Miller Hall, and Ed Slowey (Middle Tennessee State University). Their knowledge in applied mathematics and mechanics, along with highly developed skills in computer techniques, helped with the preparation of some of the test examples and illustrative materials.

The author wishes to express his personal appreciation to each of the following reviewers, whose comments, constructive criticisms, and compliments have helped to make a much improved book: Dr. Richard Dippery (Kettering University, Flint, Michigan), Dr. J. Tinsley Oden (University of Texas at Austin), and Dr. Eduard Ventsel (The Pennsylvania State University, College Park).

It is also a great pleasure to thank B. J. Clark (acquisitions editor) and Brian Black (production editor) of Marcel Dekker, Inc., for their thoughtful guidance and useful prodding at critical junctures during the development and production of the book.

During the three years period of working on this project, the author has been supported by the Foundation of Middle Tennessee State University in the form of the Faculty Research Grant in 1995, and Summer Research Grants in 1996, 1997, and 1998. This support has significantly hastened the work and is very much appreciated.

Yuri A. Melnikov
Murfreesboro, Tennessee

Taylor & Francis
Taylor & Francis Group
http://taylorandfrancis.com

Contents

Introduction

There exists a remarkable analogy between two concepts from different areas of science, the *influence function* in mechanics and the *Green's function* in mathematics. This correspondence is of primary importance in understanding the basic idea which underlies this book. Essentially, the Green's function for a certain boundary value problem in mathematical physics (for ordinary or partial differential equation) can be identified with the influence function of that phenomenon in mechanics, for which the boundary value problem serves as a mathematical model.

To illustrate the Green's function/influence function analogy, we consider, as a first example, the differential equation

$$\frac{d}{dx}\left(m(x)\frac{dy(x)}{dx}\right) = f(x), \quad x \in (0, a) \tag{0.1}$$

subject to the boundary conditions

$$y(0) = 0, \quad y(a) = 0 \tag{0.2}$$

This boundary value problem can, for instance, be regarded as a model for the lateral displacement $y(x)$ of a string of length a, caused by a transverse distributed load proportional to $f(x)$ in eqn (0.1). The string has variable mass density $m(x)$, and according to the boundary conditions imposed by eqn (0.2), the string's edges $x=0$ and $x=a$ are fixed.

The Green's function $g(x, s)$ of the boundary value problem posed by the relations in eqn (0.2) for the homogeneous equation

$$\frac{d}{dx}\left(m(x)\frac{dy(x)}{dx}\right) = 0, \tag{0.3}$$

corresponding to that of eqn (0.1), can be identified as the displacement (or response) $y_s(x)$ of the string of length a, caused by a transverse unit force concentrated at s.

This is by no means the only possible influence function interpretation of the Green's function for the boundary value problem posed by eqns (0.2) and (0.3). The problem can for instance also be associated with steady-state heat conduction in a one dimensional rod of finite length a. In this case, $y(x)$ denotes the temperature at position x in a rod made of a nonhomogeneous material with heat conductivity $m(x)$. The boundary conditions in eqn (0.2) indicate that the temperature $y(x)$ is zero at all times at the end-points $x=0$ and $x=a$ of the rod. In this setting, the Green's function $g(x,s)$ can be identified as the influence function of a unit heat source acting permanently at position s on the rod.

These two examples illustrate the important fact that the Green's function – influence function correspondence is not necessarily a one-to-one relationship. In many instances in mathematical physics, a particular differential equation may be associated with different physical phenomena.

Another example on the Green's function – influence function correspondence stems from classical Kirchhoff's beam theory (e.g., [10, 18, 27, 30, 64, 67]) where the differential equation

$$\frac{d^2}{dx^2}\left(EI(x)\frac{d^2w(x)}{dx^2}\right) = q(x), \quad x \in (0,a) \tag{0.4}$$

subject to the boundary conditions

$$w(0) = \frac{dw(0)}{dx} = 0, \quad w(a) = \frac{d^2w(a)}{dx^2} = 0 \tag{0.5}$$

models the lateral deflection $w(x)$ of a beam of length a, caused by the transverse distributed load $q(x)$. The left end-point of the beam at $x=0$ is assumed to be clamped, while the beam is simply supported at its right edge, $x=a$. Other physically feasible boundary conditions can be imposed at the end-points of the beam. The quantity $EI(x)$ represents the so-called flexural rigidity of the beam.

The Green's function $g(x,s)$ of the boundary value problem which consists of the homogeneous equation

$$\frac{d^2}{dx^2}\left(EI(x)\frac{d^2w(x)}{dx^2}\right) = 0, \quad x \in (0,a)$$

subjected to the boundary conditions imposed by eqn (0.5), can be identified with the influence function of a unit force applied transversely to the beam at position s. Thus, $g(x,s)$ represents the deflection of the beam at point x, in response to this transverse unit force.

A last illustration of the analogy between Green's functions and influence functions comes from classical Poisson-Kirchhoff plate theory (e.g., [72]),

according to which the biharmonic equation

$$D\Delta\Delta w(x,y) = F(x,y), \quad (x,y)\in\Omega \tag{0.6}$$

posed on a simply connected region Ω bounded with a smooth contour Γ and subject to the following boundary conditions

$$w(x,y) = \frac{\partial w(x,y)}{\partial n} = 0, \quad (x,y)\in\Gamma \tag{0.7}$$

models the bending of a thin elastic clamped plate occupying the region Ω with boundary Γ. The load on the plate, $F(x,y)$, is transversely distributed and the constant D represents the flexural rigidity of the plate.

The Green's function $G(x,y;s,t)$ of the homogeneous problem corresponding to that stated with eqns (0.6) and (0.7) can be interpreted as the influence function of a transverse unit force concentrated at position (s,t) on the plate. Thus, $G(x,y;s,t)$ represents the deflection of the plate at the point (x,y), caused by this transverse unit force.

In these examples, dealing with boundary value problems from applied mechanics, one can adopt either Green's function or influence function terminology. Green's function terminology is the more suitable of the two for the discussion in Chapter 1. However, since most of the material in Chapters 2 through 8 is drawn from mechanics, influence function terminology is mainly used there.

The reader is probably aware of the important role that Green's functions play in the qualitative theory of differential equations. From the current literature on Green's (influence) functions, it becomes increasingly clear that these functions are also effective in obtaining numerical solutions, once they can be expressed explicitly in a compact manner.

There are two basic stages of the influence function method (IFM). The first stage, the focus of much of the early chapters of this book, deals with the construction of the influence function. In the second stage, a computational algorithm is developed using this influence function.

Chapter 1 is devoted to the basic concepts of the Green's function theory for linear boundary value problems for ordinary differential equations. In Section 1.1, the first of two traditional methods for the construction of the Green's function is discussed, a method based on its defining properties. Conditions for symmetry of the Green's function, in the sense that its variables are interchangeable, are considered in Section 1.2. The second traditional method for the construction of Green's functions, the method of variation of parameters, is described in Section 1.3.

Influence functions for single span Kirchhoff beams are constructed and utilized in Chapter 2. In Section 2.1 influence functions for beams of uniform flexural rigidity are presented, taking into account physically natural sets

of edge conditions. In Section 2.2 it is shown how the response of a beam to transverse loads, forces or moments can be expressed in terms of the influence function of a transversely concentrated unit force. Combinations of loads are treated by means of this approach. Beams resting on elastic foundations are discussed in Section 2.3. Analytic expressions for influence functions of such beams are given and the practical use of such expressions is discussed. Section 2.4 illustrates the practical solvability of problems for beams of variable flexural rigidity by means of the influence function method.

Chapter 3 illustrates some indirect applications of beam influence functions. It is shown how one can profit from the knowledge of the influence function of a transverse concentrated unit force for a beam in solving other beam problems. Frequencies and mode shapes of natural vibrations of a beam are determined in Section 3.1. In Section 3.2, influence functions of a transverse point force are used to find such 'critical' values of axial forces applied to a beam, that cause the loss of stability of the initial equilibrium state. The classical Euler formulation of buckling problems is considered. One simple nonlinear contact problem for a beam is discussed in Section 3.3 where a Kirchhoff's beam is spaced a small distance above a Winkler foundation.

Chapter 4 deals with some specific problems from applied mechanics, which reduce to the so-called multi-point posed boundary value problems for systems of linear ordinary differential equations. These are not, however, boundary value problems for systems of equations in the conventional sense, where several unknown functions are supposed to have a common domain for an independent variable, and at least one of the equations in the system involves more than one unknown function. Instead, each equation in the systems that are considered in Chapter 4 governs a single unknown function and is formulated over an individual domain. The system is actually formed by letting those single domains interact with each other at their end-points. End-points for the individual equations become contact points in the system at which appropriate conditions are formulated.

Section 4.1 introduces the notion of a *matrix of Green's type* appropriate for a particular type of multi-point posed boundary value problems stated for a sandwich type media. Based on that, in Section 4.2 we apply the notion of a matrix of Green's type to multi-spanned Kirchhoff beams (in this case, we call it just the *influence matrix*). Several particular examples are considered where we do not only construct influence matrices but also show how they can be used in computing components of stress-strain states for particular multi-spanned beams.

The applicability of the influence matrix formalism developed in Section 4.1 is limited to a sandwich type assembly in which the material is piecewise

homogeneous. To broaden its application range, the formalism is extended to a more general type of multi-point posed boundary value problems in Section 4.3. These problems, of a more complex type, represent a wider variety of situations in applied mechanics. The framework of graph theory is used for that purpose. Sets of linear ordinary differential equations are considered, formulated on finite weighted graphs in such a way that every equation in the set governs a single unknown function and is stated on a single edge of the graph. The individual equations in the set are put into system form by imposing contact and boundary conditions at the vertices and end-points of the graph, respectively. Based on this a setup, a new definition of the matrix of Green's type is introduced. Existence and uniqueness of such matrices are discussed and two methods for their construction are proposed and some particular examples are considered.

In Chapters 5 through 8, we turn to problems described by partial differential equations where the list of available Green's (influence) functions is very limited. In Chapter 5, a technique is described which was originally developed (see [24, 42, 44, 45, 53]) for the construction of Green's functions and matrices for elliptic equations in two dimensions. The technique is based on the so-called method of eigenfunction expansion [63] and has proven to be especially effective for a variety of problems in computational continuum mechanics (based on Laplace's equation, Klein-Gordon equation, biharmonic equation, as well as on Lame's system for the displacement formulation of the plane problem in the theory of elasticity).

In this technique, influence functions are first represented in terms of their Fourier expansions with respect to one of the independent variables. This consequently results in the construction of Green's functions for ordinary differential equations in the coefficients of the Fourier expansions (the first stage of the technique). This construction can be done by using either the method based on the defining properties of Green's functions (cf. Section 1.1), or by the method of variation of parameters, as described in Section 1.3. The influence function of interest is then constructed by complete or partial summation of the Fourier series (the second stage of the technique).

Potential fields and related fields are the subject matter of Chapter 5. A number of compact representations of influence functions for Laplace's and Klein-Gordon equations expressed in various coordinate systems are presented. A variety of region configurations and types of boundary conditions are considered. Some mathematical issues are addressed as they are closely related to our discussion. Among these are: (i) convergence of the series representing influence functions, (ii) splitting off singular components of influence functions, and (iii) expressing regular components in terms of uniformly convergent series.

In Chapter 6, the technique for the construction of influence functions and matrices is applied to a variety of boundary value problems occurring in solid mechanics. The material in this chapter is conceptually similar to that of Chapter 5, which may serve as a reference. Section 6.1 deals with traditional formulations from classical Poisson-Kirchhoff plate theory, in which the bending of thin plates is reduced to a boundary value problem involving the biharmonic equation. A number of influence functions are obtained for plates of various shapes and edge conditions. Section 6.2 extends the influence function formalism to thin plate problems formulated within the scope of Reissner theory (see [65]). Some of the displacement formulations for isotropic and orthotropic media from the plane problem in the theory of elasticity are discussed in Sections 6.3 and 6.4. In Section 6.5, it is shown that the elastic equilibrium of thin shells of revolution can also be treated with the suggested technique.

Chapter 7 is devoted to influence matrices for problems of continuum mechanics formulated in media whose properties are piecewise continuous functions of spatial variables. In Section 7.1, the notion of an influence matrix is introduced as that appropriate for problem classes in continuum mechanics of compound media in two dimensions. Section 7.2 considers specific problems arising in the theory of potential on thin-walled structures consisting of different plates and shells. Several closed form influence matrices are obtained for structures composed of circular and annular plates, cylindrical, spherical, and toroidal shells. Finally, in Section 7.3, it is shown how influence matrices can be constructed for plane problems which arise in the theory of elasticity for compound media.

The discussion in the final Chapter 8 focuses on the classical heat equation, a PDE of parabolic type. Explicit, easily computable expressions for influence functions of some initial-boundary value problems involving the heat equation can also be obtained by utilizing the ideas developed and widely used here. The potential of the present approach in the area of the heat equation is briefly discussed. Section 8.1 recalls some basics of the Laplace transform which this method requires. In Sections 8.2 and 8.3, the present approach is used to obtain some classical and new representations of influence functions in this area.

Each section of the text is supplied with examples illustrating how the material can be used in practice. A set of review exercises accompanies every chapter except for Chapter 3. Appendix A presents a catalogue of explicit representations of Green's functions and matrices available in the book. Appendix B contains answers and comments to most of the review exercises.

Chapter 1

Green's Function for ODE

Being a mathematical prototype of the influence function, the notion of a Green's function provides a theoretical background for comprehending the material of the present text. The development in this chapter touches upon ordinary differential equations, utilizing two classical approaches, which are commonly used in the current literature (see, for example, [7, 17, 30, 31, 68, 70]), for the construction of Green's functions for linear boundary value problems for ODEs. One of these approaches is associated with the proof of existence and uniqueness theorem for the Green's function (see Section 1.1). It appears that in the existing literature, this approach is more popular compared to the other one that is based on the Lagrange's method of variation of parameters.

In this book, the approach based on the variation of parameters method (see Section 1.3) is, however, more frequently employed because it is more universal in the sense that it does not deal with defining properties of Green's functions, which are individual for each type of an equation.

In addition, in Section 1.2 we will focus on the symmetry of Green's functions. Their symmetry is directly related to the so-called self-adjointness of the differential operator involved. This feature of a Green's function is of great theoretical and practical importance. It will be frequently discussed in the further discussion in the text.

1.1 Defining properties and construction

In this section, we introduce the definition of the Green's function for a linear boundary value problem for an ordinary differential equation of the n-th order with variable coefficients. We then give a detailed description of the traditional method for the construction of Green's functions, based on their defining properties. Many of the examples presented here are related to various problems of continuum mechanics.

The discussion that follows concerns a linear homogeneous boundary value problem for the ordinary differential equation

$$\begin{aligned} L[y(x)] \equiv\, & p_0(x)\, y^{(n)}(x) + p_1(x)\, y^{(n-1)}(x) + p_2(x)\, y^{(n-2)}(x) \\ & + \ldots + p_{n-1}(x)\, y'(x) + p_n(x)\, y(x) = 0, \quad x \in (a,b) \end{aligned} \tag{1.1}$$

subject to the boundary conditions written as

$$M_k(y;a,b) \equiv \sum_{j=0}^{n-1} [\alpha_j^k y^{(j)}(a) + \beta_j^k y^{(j)}(b)] = 0, \quad (k = \overline{1,n}) \tag{1.2}$$

In this formulation, the coefficients $p_j(x), (j = 0, 1, \ldots, n)$ of the governing equation are continuous functions on (a, b), where $p_0(x) \neq 0; M_k, (k = 1, \ldots, n)$ represent linearly independent forms with constant coefficients α_j^k and β_j^k.

The boundary conditions in eqn (1.2) are written in a general form, which implies that a certain physically natural formulation of the boundary conditions can be obtained from this form as a particular case. If, for example, the displacement formulation of a problem from the theory of elasticity is considered (beam, plate, shell problems, etc.), then the condition in eqn (1.2) may model either clamped, or simply supported, or free, or even elastically supported edge. Periodic boundary conditions can also follow from this form.

In the discussion that follows, we will be using conventional and customary notations (a, b), $[a, b]$, and $(a, b]$ or $[a, b)$ to specify open, closed, and half-open intervals, respectively.

To avoid possible confusion that may occur in regard with the term *homogeneous*, the reader must discern different meanings of this term in mathematics and mechanics. Indeed, in mathematics we usually say *homogeneous* boundary value problem, *homogeneous* equation, or *homogeneous* boundary condition, when the right-hand side in the corresponding equality is zero. In mechanics, on the other hand, when specifying properties of materials, we usually use the term *homogeneous* to indicate that an object under consideration is composed of a material whose properties do not vary with space coordinates. Within the present book, this term will be frequently used in both senses.

We now turn the reader's attention to one of the most important definitions in this study. Let us define the Green's function for the homogeneous boundary value problem that occurred in eqns (1.1) and (1.2).

Definition: The function $g(x, s)$ is said to be the *Green's function* for the boundary value problem in eqns (1.1) and (1.2), if as a function of its first variable x, it meets the following properties, for any $s \in (a, b)$:

1. On both of the intervals $[a, s)$ and $(s, b]$, $g(x, s)$ is a continuous function having continuous derivatives up to the n-th order included, and it satisfies the governing equation (1.1) on (a, s) and (s, b), i. e. :

$$L[g(x,s)] = 0, \quad x \in (a,s); \qquad L[g(x,s)] = 0, \quad x \in (s,b)$$

2. For $x = s$, $g(x, s)$ is continuous along with all its derivatives up to the $(n-2)$-nd order included

$$\frac{\partial^m g(s+0,s)}{\partial x^m} - \frac{\partial^m g(s-0,s)}{\partial x^m} = 0, \quad (m = \overline{0, n-2})$$

3. The $(n-1)$-st derivative of $g(x, s)$ is discontinuous when $x = s$, providing

$$\frac{\partial^{n-1} g(s+0,s)}{\partial x^{n-1}} - \frac{\partial^{n-1} g(s-0,s)}{\partial x^{n-1}} = -\frac{1}{p_0(s)}$$

where $p_0(s)$ represents the leading coefficient of eqn (1.1);

4. It satisfies the boundary conditions in eqn (1.2), i. e. :

$$M_k(g) = 0, \qquad (k = \overline{1, n})$$

The following theorem is valid specifying the conditions of existence and uniqueness for the Green's function.

Theorem 1.1 (of existence and uniqueness): If the homogeneous boundary value problem in eqns (1.1) and (1.2) has only the trivial (zero) solution, then there exists its unique Green's function $g(x, s)$.

The reader is insistently suggested to carefully go through this proof because it actually provides a straightforward algorithm for the practical construction of Green's functions. Throughout the present text, we will be frequently using this algorithm.

Proof. Let functions $y_i(x), (i = 1, \ldots, n)$ represent the fundamental solution set for eqn (1.1). That is, $y_i(x)$ are linearly independent on (a, b) particular solutions of eqn (1.1).

In numerous practical situations, one can find an analytic form for $y_i(x)$. This can, in particular, be easily done for equations with constant coefficients. If, however, the governing differential equation does not allow an analytical solution, then appropriate numerical procedures may be employed for obtaining approximate ones. Later in this book we will discuss this point in more detail.

In compliance with property 1 of the definition, for any arbitrarily fixed value of $s \in (a, b)$, the Green's function $g(x, s)$ must be a solution of eqn

(1.1) in (a, s) (on the left of s), as well as in (s, b) (on the right of s). Since any solution of eqn (1.1) can be expressed as a linear combination of the components $y_i(x)$ of the fundamental solution set, one may write $g(x, s)$ in the following form

$$g(x,s) = \sum_{i=1}^{n} \begin{cases} y_i(x)A_i(s), & \text{for } a \le x \le s \\ y_i(x)B_i(s), & \text{for } s \le x \le b \end{cases} \tag{1.3}$$

where $A_i(s)$ and $B_i(s)$ represent the functions to be determined. Clearly, the number of these functions is $2n$ and the number of the relations for them, which can be derived from properties 2, 3, and 4 of the definition, is also $2n$. Thus, the situation is promising so far. Indeed, we are going to derive a system of $2n$ equations in $2n$ unknowns ($(n-1)$ equations can be obtained from property 2, one equation comes from property 3, and n equations follow from property 4). Hence, the key issues to be highlighted in the remaining part of this proof are whether that system is going to be consistent and whether it has a unique solution.

By virtue of property 2, which stipulates the continuity of $g(x, s)$ itself and its partial derivatives with respect to x up to the $(n-2)$-nd order, as $x = s$, one derives the following system of $(n-1)$ linear algebraic equations

$$\sum_{i=1}^{n} C_i(s) y_i^{(j)}(s) = 0, \qquad (j = \overline{0, n-2}) \tag{1.4}$$

in n unknown functions

$$C_i(s) = B_i(s) - A_i(s), \qquad (i = \overline{1, n}) \tag{1.5}$$

The superscript j on $y_i(s)$ in eqn (1.4) specifies the differentiation order.

The system in eqn (1.4) is underdetermined, because the number of equations in it $(n-1)$ is fewer than the number of unknowns (n) involved. This drawback can be eluded, however, by applying property 3 to the expression in eqn (1.3). This yields one more linear algebraic equation

$$\sum_{i=1}^{n} C_i(s) y_i^{(n-1)}(s) = -\frac{1}{p_0(s)} \tag{1.6}$$

in the same set $C_i(s)$ of unknowns. Hence, the relations in eqn (1.4) together with those in eqn (1.6) form a system of n simultaneous linear algebraic equations in n unknowns. The determinant of the coefficient matrix of this system is not zero, because it represents the Wronskian for the fundamental solution set $y_i(s)$. Thus, this system has a unique solution. So, one can readily obtain the explicit expressions for $C_i(s)$ from eqns (1.4) and (1.6).

In order to obtain the values of $A_i(s)$ and $B_i(s)$, we take advantage of property 4. In doing so, let us first break down the forms $M_k(y)$ in the boundary conditions in eqn (1.2) into two parts as

$$M_k(y) = P_k(y) + Q_k(y), \qquad (k = \overline{1,n}) \tag{1.7}$$

with $P_k(y)$ and $Q_k(y)$ being defined as

$$P_k(y) = \sum_{j=0}^{n-1} \alpha_j^k y^{(j)}(a), \qquad Q_k(y) = \sum_{j=0}^{n-1} \beta_j^k y^{(j)}(b)$$

In compliance with property 4, we now substitute the expression for $g(x,s)$ from eqn (1.3) into eqn (1.2)

$$M_k(g) \equiv P_k(g) + Q_k(g) = 0, \qquad (k = \overline{1,n})$$

Since P_k in eqn (1.7) governs the values of $g(x,s)$ at the left-end point $x = a$ of the interval $[a,b]$, while Q_k governs the values of $g(x,s)$ at the right-end point $x = b$, the upper branch

$$\sum_{i=1}^{n} y_i(x) A_i(s)$$

of $g(x,s)$ from eqn (1.3) goes to $P_k(g)$, while the lower branch

$$\sum_{i=1}^{n} y_i(x) B_i(s)$$

must be substituted into $Q_k(g)$, resulting in

$$M_k(g) \equiv \sum_{i=1}^{n} [P_k(y_i) A_i(s) + Q_k(y_i) B_i(s)] = 0, \qquad (k = \overline{1,n})$$

Replacing the values of $A_i(s)$ in the above equation with $B_i(s) - C_i(s)$ in accordance with eqn (1.5), one rewrites it in the form

$$\sum_{i=1}^{n} [P_k(y_i)(B_i(s) - C_i(s)) + Q_k(y_i) B_i(s)] = 0, \qquad (k = \overline{1,n})$$

Combining then the terms with $B_i(s)$ and taking the term with $C_i(s)$ to the right-hand side, one obtains

$$\sum_{i=1}^{n} [P_k(y_i) + Q_k(y_i)] B_i(s) = \sum_{i=1}^{n} P_k(y_i) C_i(s), \qquad (k = \overline{1,n})$$

Upon recalling the partitioning from eqn (1.7), the above relations can finally be rewritten in the form

$$\sum_{i=1}^{n} M_k(y_i)B_i(s) = \sum_{i=1}^{n} P_k(y_i)C_i(s), \quad (k = \overline{1,n}) \tag{1.8}$$

These relations constitute a system of n linear algebraic equations in the n unknowns $B_i(s)$. The coefficient matrix of this system is not singular, since the forms M_k are linearly independent. The right-hand side vector in eqn (1.8) is defined in terms of the known values of $C_i(s)$. This system has, consequently, a unique solution for $B_i(s)$. Based on this, the values of $A_i(s)$ can readily be obtained from eqn (1.5). Hence, this final step completes the proof of Theorem 1.1, because upon substituting the values of $A_i(s)$ and $B_i(s)$ into eqn (1.3), we finally obtain an explicit expression for $g(x, s)$.

As we have already mentioned, the proof just completed suggests a consistent way to practically construct the Green's function. This point is illustrated below with a series of particular examples.

In each of the following examples, we present and analyze different peculiarities in statements of boundary value problems, which may occur while considering practical situations in computational mechanics.

EXAMPLE 1 Consider the following differential equation

$$\frac{d^2y(x)}{dx^2} = 0, \quad x \in (0, a) \tag{1.9}$$

subject to the boundary conditions written as

$$y(0) = y(a) = 0 \tag{1.10}$$

This boundary value problem can be associated with many phenomena in continuum mechanics (it represents a particular case of the problem in eqns (0.2) and (0.3) from the introduction to this text, as $m(x)$ is set to a constant).

The most elementary set of functions constituting a fundamental solution set for eqn (1.9) is represented by

$$y_1(x) \equiv 1, \quad y_2(x) \equiv x$$

Therefore, the general solution $y_g(x)$ for this equation can be written as

$$y_g(x) = D_1 + D_2 x$$

where D_1 and D_2 represent arbitrary constants.

A substitution of this function into the boundary conditions in eqn (1.10) yields the homogeneous system of linear algebraic equations in D_1 and D_2,

with a well-posed coefficient matrix. Hence, the problem in eqns (1.9) and (1.10) has only the trivial solution.

Thus, there exists a unique Green's function for this problem. According to the procedure described earlier, it can be sought in the form

$$g(x,s) = \begin{cases} A_1(s) + xA_2(s), & \text{for } 0 \le x \le s \\ B_1(s) + xB_2(s), & \text{for } s \le x \le a \end{cases} \tag{1.11}$$

Introducing then, as it is suggested in eqn (1.5), $C_1(s) = B_1(s) - A_1(s)$ and $C_2(s) = B_2(s) - A_2(s)$, we form a system of linear algebraic equations in these unknowns (see the system in eqns (1.4) and (1.6)) written as

$$\begin{cases} C_1(s) + sC_2(s) &= 0 \\ C_2(s) &= -1 \end{cases} \tag{1.12}$$

Its obvious solution is $C_1(s) = s$ and $C_2(s) = -1$.

The first boundary condition $y(0) = 0$ in eqn (1.10), being satisfied with the upper branch of $g(x,s)$, results in $A_1(s) = 0$. The upper branch is chosen because $x = 0$ belongs to its domain $0 \le x \le s$. Since $B_1(s) = C_1(s) + A_1(s)$, it follows that $B_1(s) = s$.

The second condition $y(a) = 0$ in eqn (1.10), being treated with the lower branch of $g(x,s)$, yields $B_1(s) + aB_2(s) = 0$. Hence, $B_2(s) = -s/a$, and finally, since $A_2(s) = B_2(s) - C_2(s)$, it follows that $A_2(s) = 1 - s/a$. Substituting these into eqn (1.11), we ultimately obtain the Green's function that we are looking for in the form

$$g(x,s) = \begin{cases} a^{-1}x(a-s), & \text{for } 0 \le x \le s \\ a^{-1}s(a-x), & \text{for } s \le x \le a \end{cases} \tag{1.13}$$

EXAMPLE 2 We now formulate another boundary value problem

$$\frac{dy(0)}{dx} = 0, \quad \frac{dy(a)}{dx} = 0$$

for eqn (1.9) over the interval $(0,a)$.

This problem is not uniquely solvable. Indeed, any constant function represents the solution to it. Hence, the condition of existence and uniqueness for Green's function does not hold for the above statement. Therefore, a Green's function cannot be constructed in this case, because it does not exist.

EXAMPLE 3 Consider one more boundary value problem

$$\frac{dy(0)}{dx} = 0, \quad \frac{dy(a)}{dx} + my(a) = 0 \tag{1.14}$$

for equation (1.9) over $(0, a)$, where m is thought to be a nonzero constant.

It can easily be shown (see exercise 1.1(a) of this chapter) that the problem in eqns (1.9) and (1.14) has only the trivial solution. Consequently, there exists a unique Green's function for this problem.

The first part of the construction procedure precisely resembles that from the problem stated with eqns (1.9) and (1.10). The Green's function is again expressed by eqn (1.11), the coefficients $C_1(s)$ and $C_2(s)$ again satisfy the system in eqn (1.12), resulting in $C_1(s) = s$ and $C_2(s) = -1$.

The first boundary condition in eqn (1.14), being treated by the upper branch in eqn (1.11), yields $A_2(s)=0$. This immediately results in $B_2(s)=-1$. The second condition in (1.14), being treated by the lower branch in eqn (1.11), yields the following equation

$$B_2(s) + m[B_1(s) + aB_2(s)] = 0$$

in $B_1(s)$ and $B_2(s)$. Based on the known value of $B_2(s)$, one obtains $B_1(s) = (1+ma)/m$. This in turn yields $A_1(s) = [1 + m(a-s)]/m$.

Substituting the values of $A_i(s)$ and $B_i(s)$ just found, into eqn (1.11), we finally obtain the Green's function to the boundary value problem posed by eqns (1.9) and (1.14) in the form

$$g(x,s) = \begin{cases} (a-s) + m^{-1}, & \text{for } 0 \le x \le s \\ (a-x) + m^{-1}, & \text{for } s \le x \le a \end{cases} \tag{1.15}$$

Notice that as m is taken to infinity, the second term m^{-1} in (1.15) vanishes yielding the Green's function

$$g(x,s) = \begin{cases} a-s, & \text{for } 0 \le x \le s \\ a-x, & \text{for } s \le x \le a \end{cases}$$

for equation (1.9) subject to the following boundary conditions

$$\frac{dy(0)}{dx} = 0, \quad y(a) = 0$$

In applied mechanics, one frequently is required to work out research projects for phenomena occurring in infinite media. The influence (Green's) function formalism can successfully be applied to the associated boundary value problems formulated over infinite intervals. As our next example, we construct the Green's function for such a problem.

EXAMPLE 4 Consider the following differential equation

$$\frac{d^2y(x)}{dx^2} - k^2y(x) = 0, \quad x \in (0, \infty) \tag{1.16}$$

subject to the boundary conditions imposed as

$$y(0) = 0, \quad |y(\infty)| < \infty \tag{1.17}$$

It can be shown (see exercise 1.1(b)) that the conditions of existence and uniqueness for the Green's function are met in this case assuring a unique Green's function of the above formulation.

Since the following two functions

$$y_1(x) \equiv \exp(kx), \quad y_2(x) \equiv \exp(-kx)$$

represent the fundamental solution set for eqn (1.16), one can express the Green's function for the boundary value problem in eqns (1.16) and (1.17) in the following form

$$g(x,s) = \begin{cases} A_1(s)\exp(kx) + A_2(s)\exp(-kx), & \text{for } x \le s \\ B_1(s)\exp(kx) + B_2(s)\exp(-kx), & \text{for } s \le x \end{cases} \tag{1.18}$$

Denoting $C_i(s) = B_i(s) - A_i(s), (i = 1, 2)$, one obtains the following system of linear algebraic equations

$$\begin{cases} \exp(ks)C_1(s) + \exp(-ks)C_2(s) & = & 0 \\ k\exp(ks)C_1(s) - k\exp(-ks)C_2(s) & = & -1 \end{cases}$$

in $C_1(s)$ and $C_2(s)$. Its solution is expressed as

$$C_1(s) = -\frac{1}{2k}\exp(-ks), \quad C_2(s) = \frac{1}{2k}\exp(ks) \tag{1.19}$$

The first condition in eqn (1.17) implies

$$A_1(s) + A_2(s) = 0 \tag{1.20}$$

while the second condition results in $B_1(s) = 0$, because the exponential function $\exp(kx)$ is unbounded as x approaches infinity. And the only way to satisfy the second condition in eqn (1.17) is to set $B_1(s)$ to zero. This immediately yields

$$A_1(s) = \frac{1}{2k}\exp(-ks)$$

and the relation in eqn (1.20) consequently provides

$$A_2(s) = -\frac{1}{2k}\exp(-ks)$$

Hence, based on the known values of $C_2(s)$ and $A_2(s)$, one obtains

$$B_2(s) = \frac{1}{2k}[\exp(ks) - \exp(-ks)]$$

Upon substituting the values of the coefficients $A_i(s)$ and $B_i(s)$ just found into eqn (1.18), one finally obtains the Green's function to the problem posed by eqns (1.16) and (1.17) in the form

$$g(x,s) = \frac{1}{2k}\begin{cases} \exp(k(x-s)) - \exp(-k(x+s)), & \text{for } x \le s \\ \exp(k(s-x)) - \exp(-k(s+x)), & \text{for } s \le x \end{cases} \tag{1.21}$$

EXAMPLE 5 Consider a boundary value problem for the same equation as in the previous example but formulated over a different domain. Let

$$\frac{d^2y(x)}{dx^2} - k^2y(x) = 0, \quad x \in (0,a) \tag{1.22}$$

be subjected to the boundary conditions written as

$$y(0) = y(a), \quad \frac{dy(0)}{dx} = \frac{dy(a)}{dx} \tag{1.23}$$

This boundary value problem represents one more important type of formulations in applied mechanics. The relations in eqn (1.23) specify conditions of the a-periodicity of the solution.

It can be shown (see exercise 1.1(c) of this chapter) that this boundary value problem has only the trivial solution, providing existence of the unique Green's function for it.

Since the formulation in eqns (1.22) and (1.23) again entails the same differential equation which was considered in *EXAMPLE 4*, the beginning of the construction procedure for the Green's function resembles that from the previous problem. We again express the Green's function by eqn (1.18), and the coefficients $C_1(s)$ and $C_2(s)$ are again given with eqn (1.19).

Satisfying the first condition in eqn (1.23), we utilize the upper branch in eqn (1.18) in order to compute the value of $y(0)$, while its lower branch is used for computing the value of $y(a)$. This results in

$$A_1(s) + A_2(s) = B_1(s)\exp(ka) + B_2(s)\exp(-ka) \tag{1.24}$$

Satisfying the second condition in eqn (1.23), we compute the derivative of $y(x)$ at $x = 0$ by using the upper branch in eqn (1.18), while the value of the derivative of $y(x)$ at $x = a$ is computed by using the lower branch of eqn (1.18). This yields

$$A_1(s) - A_2(s) = B_1(s)\exp(ka) - B_2(s)\exp(-ka) \tag{1.25}$$

So the relations in eqns (1.24) and (1.25) along with those in eqn (1.19) form a system of four linear algebraic equations in $A_1(s)$, $A_2(s)$, $B_1(s)$, and $B_2(s)$. To find the values of $A_1(s)$ and $B_1(s)$, we add eqns (1.24) and (1.25) to each other. This provides

$$A_1(s) - B_1(s)\exp(ka) = 0 \tag{1.26}$$

The first relation in eqn (1.19) can be rewritten in the form

$$-A_1(s) + B_1(s) = -\frac{1}{2k}\exp(-ks) \tag{1.27}$$

Solving eqns (1.26) and (1.27) simultaneously, one obtains

$$A_1(s) = \frac{\exp(k(a-s))}{2k[\exp(ka)-1]}, \quad B_1(s) = \frac{\exp(-ks)}{2k[\exp(ka)-1]}$$

To find the values of $A_2(s)$ and $B_2(s)$, we subtract eqn (1.25) from eqn (1.24) resulting in

$$A_2(s) - B_2(s)\exp(-ka) = 0 \tag{1.28}$$

Rewriting then the second relation from eqn (1.19) in the form

$$-A_2(s) + B_2(s) = \frac{1}{2k}\exp(ks) \tag{1.29}$$

we solve eqns (1.28) and (1.29) simultaneously. This yields

$$A_2(s) = \frac{\exp(ks)}{2k[\exp(ka)-1]}, \quad B_2(s) = \frac{\exp(k(s+a))}{2k[\exp(ka)-1]}$$

Substituting the values of $A_1(s), A_2(s), B_1(s)$, and $B_2(s)$ just found into eqn (1.18), we finally obtain

$$g(x,s) = K_0 \begin{cases} \exp(k(x-s+a)) + \exp(k(s-x)), & \text{for } x \le s \\ \exp(k(s-x+a)) + \exp(k(x-s)), & \text{for } s \le x \end{cases} \tag{1.30}$$

where $K_0 = \{2k[\exp(ka)-1]\}^{-1}$.

In all of the examples considered so far, we have dealt with ordinary differential equations having constant coefficients. Clearly, variable coefficients do not bring any limitations to the algorithm described, if the fundamental solution set of the equation under consideration is obtainable in terms of elementary functions. In other words, if the governing differential equation allows exact solution, one can readily construct a Green's function by means of this algorithm.

EXAMPLE 6 To address the last issue, consider the equation

$$\frac{d}{dx}\left((mx+p)\frac{dy}{dx}\right) = 0, \qquad x \in (0,a) \tag{1.31}$$

with the boundary conditions imposed as

$$\frac{dy(0)}{dx} = 0, \quad y(a) = 0 \tag{1.32}$$

where we assume $m > 0$ and $p > 0$, which means $mx + p \neq 0$ on $x \in [0, a]$.

The fundamental solution set

$$y_1(x) \equiv 1, \qquad y_2(x) \equiv \ln(mx + p)$$

required for the construction of the Green's function for the problem in eqns (1.31) and (1.32) can be obtained by two successive integrations of eqn (1.31).

In view of exercise 1.1(d), the problem in eqns (1.31) and (1.32) has only the trivial solution. Hence, there exists a unique Green's function which can be presented in the form

$$g(x,s) = \begin{cases} A_1(s) + \ln(mx+p)A_2(s), & \text{for } 0 \le x \le s \\ B_1(s) + \ln(mx+p)B_2(s), & \text{for } s \le x \le a \end{cases} \tag{1.33}$$

Following then our customary procedure, one obtains the system of linear algebraic equations

$$\begin{cases} C_1(s) + \ln(ms+p)C_2(s) &= 0 \\ m(ms+p)^{-1}C_2(s) &= -(ms+p)^{-1} \end{cases}$$

in $C_i(s) = B_i(s) - A_i(s), (i = 1, 2)$. Its solution is

$$C_1(s) = \frac{1}{m}\ln(ms+p), \quad C_2(s) = -\frac{1}{m} \tag{1.34}$$

The first boundary condition in eqn (1.32) yields $A_2(s) = 0$. Consequently, $B_2(s) = -1/m$. The second condition in eqn (1.32) gives

$$B_1(s) + \ln(ma+p)B_2(s) = 0$$

resulting in $B_1(s) = [\ln(ma+p)]/m$, which provides

$$A_1(s) = \frac{1}{m}\ln\frac{ma+p}{ms+p}$$

Substituting the values of $A_i(s)$ and $B_i(s)$ just found into eqn (1.33), one obtains the Green's function that we are looking for in the form

$$g(x,s) = \frac{1}{m}\begin{cases} \ln[(ma+p)(ms+p)^{-1}], & \text{for } 0 \le x \le s \\ \ln[(ma+p)(mx+p)^{-1}], & \text{for } s \le x \le a \end{cases} \tag{1.35}$$

Sometimes in applied mechanics, we consider boundary value problems formulated over finite intervals, where one of the end-points is a singular point for the governing differential equation. The algorithm described in this section can also be used to construct Green's functions for such problems.

EXAMPLE 7 As an illustrative example on this issue, we consider a boundary value problem for the following differential equation

$$\frac{d}{dx}\left(x\frac{dy(x)}{dx}\right) = 0, \quad x \in (0,a) \tag{1.36}$$

subject to the boundary conditions written as

$$|y(0)| < \infty, \qquad \frac{dy(a)}{dx} + hy(a) = 0 \tag{1.37}$$

Clearly, the left end-point $x = 0$ of the domain is a point of singularity for eqn (1.36). Therefore, instead of formulating a traditional boundary condition at this point, we require in eqn (1.37) for $y(0)$ to be bounded.

Integrating eqn (1.36) successively two times, one obtains its fundamental solution set that can be written as

$$y_1(x) \equiv 1, \quad y_2(x) \equiv \ln x \tag{1.38}$$

The problem in eqns (1.36) and (1.37) has only the trivial solution (see exercise 1.1(e)), allowing a unique Green's function in the form

$$g(x,s) = \begin{cases} A_1(s) + \ln x A_2(s), & \text{for } 0 \le x \le s \\ B_1(s) + \ln x B_2(s), & \text{for } s \le x \le a \end{cases} \tag{1.39}$$

In compliance with our customary procedure, we form a system of linear algebraic equations

$$\begin{cases} C_1(s) + \ln s\, C_2(s) &= 0 \\ s^{-1}\, C_2(s) &= -s^{-1} \end{cases}$$

whose solution is $C_1(s) = \ln s$ and $C_2(s) = -1$.

The boundedness of the Green's function at $x = 0$ implies $A_2(s) = 0$. Consequently, $B_2(s) = -1$. The second condition in eqn (1.37) yields

$$B_2(s)/a + h[B_1(s) + \ln a B_2(s)] = 0$$

Hence, $B_1(s) = 1/ah + \ln a$, and ultimately, $A_1(s) = 1/ah - \ln s/a$. Thus, we finally obtain

$$g(x,s) = \begin{cases} (ah)^{-1} - \ln[(a)^{-1}s], & \text{for } 0 \le x \le s \\ (ah)^{-1} - \ln[(a)^{-1}x], & \text{for } s \le x \le a \end{cases} \tag{1.40}$$

Notice that as the value of h is taken to infinity, the first term $(ah)^{-1}$ in eqn (1.40) vanishes, yielding the Green's function

$$g(x,s) = \begin{cases} -\ln[(a)^{-1}s], & \text{for } 0 \le x \le s \\ -\ln[(a)^{-1}x], & \text{for } s \le x \le a \end{cases}$$

for eqn (1.36) subject to the boundary conditions $|y(0)| < \infty$ and $y(a) = 0$.

EXAMPLE 8 For the next example, we formulate a boundary value problem for the equation of the fourth order

$$\frac{d^4y(x)}{dx^4} = 0, \quad x \in (0,1) \tag{1.41}$$

with boundary conditions written as

$$y(0) = \frac{dy(0)}{dx} = 0, \quad y(1) = \frac{d^2y(1)}{dx^2} = 0 \tag{1.42}$$

As is known, this formulation relates to the bending phenomenon of a beam of unit length, if its left edge is clamped while the right edge is simply supported. In Chapters 2 and 3, we consider a number of other problems from the beam theory.

The following set of functions

$$y_1(x) \equiv 1, \; y_2(x) \equiv x, \; y_3(x) \equiv x^2, \; y_4(x) \equiv x^3 \tag{1.43}$$

constitutes the simplest fundamental solution set for eqn (1.41). Hence, its general solution is

$$y_g(x) = D_1 + D_2x + D_3x^2 + D_4x^3$$

Applying the boundary conditions from eqn (1.42), one derives a homogeneous system of linear algebraic equations in D_i. The coefficient matrix

of that system is not singular, providing only the trivial solution for the system. Consequently, there exists a unique Green's function for the problem posed by eqns (1.41) and (1.42).

Based on the fundamental solution set presented in eqn (1.43), the Green's function can be written in the form

$$g(x,s) = \begin{cases} A_1(s) + A_2(s)x + A_3(s)x^2 + A_4(s)x^3, & \text{for } x \le s \\ B_1(s) + B_2(s)x + B_3(s)x^2 + B_4(s)x^3, & \text{for } s \le x \end{cases} \tag{1.44}$$

From properties 2 and 3 of the definition of the Green's function, one derives the following system of linear equations in $C_i(s) = B_i(s) - A_i(s)$, written in a matrix form

$$\begin{pmatrix} 1 & s & s^2 & s^3 \\ 0 & 1 & 2s & 3s^2 \\ 0 & 0 & 2 & 6s \\ 0 & 0 & 0 & 6 \end{pmatrix} \times \begin{pmatrix} C_1(s) \\ C_2(s) \\ C_3(s) \\ C_4(s) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

whose solution

$$C_1 = \frac{1}{6}s^3, \quad C_2 = -\frac{1}{2}s^2, \quad C_3 = \frac{1}{2}s, \quad C_4 = -\frac{1}{6} \tag{1.45}$$

is easily obtained because of the triangular form of its coefficient matrix.

By virtue of property 4 in the definition, the boundary conditions in eqn (1.42) provide

$$A_1 = 0, \quad A_2 = 0, \quad B_1 + B_2 + B_3 + B_4 = 0, \quad 2B_3 + 6B_4 = 0$$

while the rest of the coefficients for $g(x,s)$

$$A_3 = -\frac{1}{4}s^3 + \frac{3}{4}s^2 - \frac{1}{2}s, \quad A_4 = \frac{1}{12}s^3 - \frac{1}{4}s^2 + \frac{1}{6}$$

$$B_1 = \frac{1}{6}s^3, \quad B_2 = -\frac{1}{2}s^2, \quad B_3 = -\frac{1}{4}s^3 + \frac{3}{4}s^2, \quad B_4 = \frac{1}{12}s^3 - \frac{1}{4}s^2$$

are computed through the values of $C_i(s)$ presented in eqn (1.45).

Substituting all the coefficients $A_i(s)$ and $B_i(s)$ just obtained into eqn (1.44), we obtain the Green's function $g(x,s)$ for the boundary value problem posed by eqns (1.41) and (1.42). For $x \le s$, it is found in the form

$$g(x,s) = -\left(\frac{1}{4}s^3 - \frac{3}{4}s^2 + \frac{1}{2}s\right)x^2 + \left(\frac{1}{12}s^3 - \frac{1}{4}s^2 + \frac{1}{6}\right)x^3 \tag{1.46}$$

while for $x \geq s$, its expression is

$$g(x,s) = -\left(\frac{1}{4}x^3 - \frac{3}{4}x^2 + \frac{1}{2}x\right)s^2 + \left(\frac{1}{12}x^3 - \frac{1}{4}x^2 + \frac{1}{6}\right)s^3$$

This example shows that even for equations of higher order, the procedure for the construction of Green's functions utilized here results in a reasonable amount of computation.

Analyzing the form of all of the Green's functions constructed so far in this section, one may notice their common property (they are symmetric). That is, $g(x,s) = g(s,x)$. Indeed, the interchange of x with s in the expression valid for $x \leq s$ yields that valid for $x \geq s$ and vice versa. In the next section, we will discuss this issue in more detail. The conditions will be found under which the symmetry takes place.

EXAMPLE 9 We close the discussion in this section with a problem whose Green's function, contrary to all previous ones, appears to be in a nonsymmetrical form. Namely, consider the following equation

$$\frac{d^2y(x)}{dx^2} + \frac{dy(x)}{dx} - 2y(x) = 0, \qquad x \in (0,\infty) \tag{1.47}$$

subject to the boundary conditions

$$y(0) = 0, \qquad |y(\infty)| < \infty \tag{1.48}$$

Clearly, this problem has only the trivial solution (see exercise 1.1(f)), allowing, subsequently, a unique Green's function. Since $y_1(x) = \exp(x)$ and $y_2(x) = \exp(-2x)$ represent the fundamental solution set to eqn (1.47), one can express the Green's function to this problem in the form

$$g(x,s) = \begin{cases} A_1(s)\exp(x) + A_2(s)\exp(-2x), & \text{for } x \leq s \\ B_1(s)\exp(x) + B_2(s)\exp(-2x), & \text{for } s \leq x \end{cases} \tag{1.49}$$

This results in the system of linear equations in $C_i(s) = B_i(s) - A_i(s)$

$$\begin{pmatrix} \exp(s) & \exp(-2s) \\ \exp(s) & -2\exp(-2s) \end{pmatrix} \times \begin{pmatrix} C_1(s) \\ C_2(s) \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

whose solution is found as

$$C_1(s) = -\frac{1}{3}\exp(-s), \quad C_2(s) = \frac{1}{3}\exp(2s)$$

The first condition in eqn (1.48) provides $A_1(s)+A_2(s)=0$, while the second condition implies $B_1(s)=0$. Therefore $A_1(s)=[\exp(-s)]/3$, resulting in $A_2(s)=-[\exp(-s)]/3$, and, finally, $B_2(s)=[\exp(2s)-\exp(-s)]/3$. Substituting these values into eqn (1.49), one obtains

$$g(x,s)=\frac{1}{3}\begin{cases}\exp(-s)\,[\exp(x)-\exp(-2x)], & \text{for } x\le s\\ \exp(-2x)\,[\exp(2s)-\exp(-s)], & \text{for } s\le x\end{cases} \tag{1.50}$$

It is clearly seen that this Green's function fails to be symmetric. Why? What makes the statement of this last problem different from all the ones considered earlier in this section? The reader will find the reasoning for this occurrence in the next section.

1.2 Symmetry of Green's functions

In order to address the basic issue of this section, a certain preparatory work has to be carried out. Let us write down the linear homogeneous differential equation of the n-th order

$$\begin{aligned}L[y(x)] \;\equiv\; & p_0(x)\,y^{(n)}(x)+p_1(x)\,y^{(n-1)}(x)+p_2(x)\,y^{(n-2)}(x)\\ & +\ldots+p_{n-1}(x)\,y'(x)+p_n(x)\,y(x)=0\end{aligned}$$

From the general theory of linear ODEs (see, for example, [31, 68]), it is known that the equation

$$\begin{aligned}L_a[y(x)] \;\equiv\; & (-1)^n[p_0(x)\,y(x)]^{(n)}+(-1)^{n-1}[p_1(x)\,y(x)]^{(n-1)}\\ & +\ldots-[p_{n-1}(x)\,y(x)]'+p_n(x)\,y(x)=0\end{aligned}$$

is said to be *adjoint* to $L[y(x)]=0$. The operator L_a is called *adjoint* to L, and if $L\equiv L_a$, then L is said to be a *self-adjoint* operator and the equation $L[y(x)]=0$ is said to be a *self-adjoint* equation.

For the sake of simplicity, the discussion herein is limited to equations of the second order

$$L[y(x)]\equiv p_0(x)\frac{d^2y(x)}{dx^2}+p_1(x)\frac{dy(x)}{dx}+p_2(x)y(x)=0 \tag{1.51}$$

The leading coefficient $p_0(x)$ is not supposed to equal zero at any single point in (a,b) except, maybe, for one of its end-points. In addition, in the discussion that follows, we require the coefficient $p_0(x)$ to be two times differentiable and $p_1(x)$ to be just differentiable on (a,b).

According to what we recently recalled, the following equation

$$L_a[y(x)]\equiv\frac{d^2}{dx^2}[p_0(x)y(x)]-\frac{d}{dx}[p_1(x)y(x)]+p_2(x)y(x)=0 \tag{1.52}$$

is adjoint to that in eqn (1.51).

We will review here a brief discussion on the self-adjointness of differential equations and relevant issues, which are required just to analyze the symmetry of Green's functions. The more detailed discussion on this subject can be found, for example, in [17, 30, 31, 35, 63, 66, 68].

Using the product rule for the differentiation in eqn (1.52), the operator L_a can be rewritten in the form

$$L_a[y(x)] \equiv \frac{d}{dx}\left(y\frac{dp_0}{dx} + p_0\frac{dy}{dx}\right) - \left(y\frac{dp_1}{dx} + p_1\frac{dy}{dx}\right) + p_2 y$$

Differentiating and combining the like terms, one obtains

$$L_a[y(x)] \equiv p_0\frac{d^2y}{dx^2} + \left(2\frac{dp_0}{dx} - p_1\right)\frac{dy}{dx} + \left(\frac{d^2p_0}{dx^2} - \frac{dp_1}{dx} + p_2\right)y \qquad (1.53)$$

Suppose eqn (1.51) is self-adjoint, that is $L[y(x)] \equiv L_a[y(x)]$. If so, then upon comparing the coefficients of y' in $L[y(x)]$ and $L_a[y(x)]$, one obtains the following relation for the coefficients $p_0(x)$ and $p_1(x)$

$$2\frac{dp_0(x)}{dx} - p_1(x) = p_1(x)$$

which must hold for the self-adjointness of eqn (1.51). This implies

$$p_1(x) = \frac{dp_0(x)}{dx} \qquad (1.54)$$

Recall then the coefficient $p_0''(x) - p_1'(x) + p_2(x)$ of $y(x)$ in eqn (1.53). Since the sum of the first two terms equals zero (to realize this, one needs to differentiate the relation in eqn (1.54)), it follows that the self-adjointness puts no additional constraints on the coefficient $p_2(x)$ in eqn (1.51). Hence, if eqn (1.51) is self-adjoint, it can be written in the form

$$p_0(x)\frac{d^2y(x)}{dx^2} + \frac{dp_0(x)}{dx}\frac{dy(x)}{dx} + p_2(x)y(x) = 0$$

The first two terms in this equation can be combined, providing the following compact form

$$\frac{d}{dx}\left(p_0(x)\frac{dy(x)}{dx}\right) + p_2(x)y(x) = 0 \qquad (1.55)$$

This form is usually referred to as the standard form of a self-adjoint equation of the second order.

Hence, if the coefficients $p_0(x)$ and $p_1(x)$ satisfy the relation in eqn (1.54), then eqn (1.51) is in a self-adjoint form, regardless of the form of the coefficient $p_2(x)$. This prompts an idea of how eqn (1.51) can be reduced to a self-adjoint form. Indeed, multiplying eqn (1.51) through by a certain nonzero function (the integrating factor) and applying then the relation in eqn (1.54) to the coefficients of $y''(x)$ and $y'(x)$ of the resultant equation, one can readily formulate a simple relation from which the integrating factor can afterwards be found. We leave the completion of this procedure for one of the exercises of this section.

Assume now that L represents a self-adjoint operator of the second order

$$L \equiv \frac{d}{dx}\left(p_0(x)\frac{d}{dx}\right) + p_2(x)$$

Consider two functions $u(x)$ and $v(x)$ both being two times continuously differentiable on (a, b), and form the following bilinear combination of them

$$u(x)\, L[v(x)] - v(x)\, L[u(x)]$$

which can be rewritten explicitly

$$u\left(\frac{d}{dx}\left(p_0(x)\frac{dv}{dx}\right) + p_2(x)\, v\right) - v\left(\frac{d}{dx}\left(p_0(x)\frac{du}{dx}\right) + p_2(x)\, u\right)$$

Removing the outer parentheses in both the components above and cancelling the terms $p_0(x)uv$, we have

$$uL(v) - vL(u) = u\frac{d}{dx}\left(p_0(x)\frac{dv}{dx}\right) - v\frac{d}{dx}\left(p_0(x)\frac{du}{dx}\right)$$

Applying the product rule to the exterior differentiation for both the terms above, one obtains

$$u\frac{d}{dx}\left(p_0(x)\frac{dv}{dx}\right) - v\frac{d}{dx}\left(p_0(x)\frac{du}{dx}\right) = \frac{d}{dx}\left(p_0(x)\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\right)$$

Hence, this bilinear combination is finally reduced to

$$u\, L(v) - v\, L(u) = \frac{d}{dx}\left(p_0(x)\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\right) \tag{1.56}$$

Integrating both sides of eqn (1.56) throughout the interval $[a, b]$, one obtains the following relation

$$\int_a^b [u\, L(v) - v\, L(u)]\, dx = p_0(x)\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Bigg|_a^b \tag{1.57}$$

which is usually referred to as *Green's formula* for a self-adjoint operator.

If in addition to being two times continuously differentiable on (a, b), $u(x)$ and $v(x)$ are functions for which the right-hand side in eqn (1.57) vanishes. That is, when

$$p_0(x)\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Bigg|_a^b = 0 \tag{1.58}$$

then Green's formula reduces to the very compact form

$$\int_a^b [u\,L(v) - v\,L(u)]\,dx = 0 \tag{1.59}$$

So, the Green's formula in eqn (1.59) is valid for a self-adjoint operator L, with $u(x)$ and $v(x)$ being two times continuously differentiable on (a, b) satisfying the relation in eqn (1.58). This relation is, however, implicit in nature, which makes it too cumbersome to deal with over and over again in actual computation. Therefore, it is important to find some of its explicit equivalents which are more convenient to use in practice.

In doing so, we rewrite the relation in eqn (1.58) in the extended form

$$p_0(b)\,[u(b)v'(b) - v(b)u'(b)] - p_0(a)\,[u(a)v'(a) - v(a)u'(a)] = 0 \tag{1.60}$$

Since this relation contains the values of $u(x)$, $v(x)$, and their derivatives $u'(x)$ and $v'(x)$ at the end-points of the interval $[a, b]$, it should be directly seen that the equality in eqn (1.60) holds, if $u(x)$ and $v(x)$ both satisfy either of the following sets of boundary conditions at $x = a$ and $x = b$:

- $y(a) = 0, \quad y(b) = 0$
- $y(a) = 0, \quad y'(b) = 0$
- $y'(a) = 0, \quad y'(b) = 0$

It should also be directly seen that the condition in eqn (1.60) is valid in the so-called *singular* case, when the leading coefficient $p_0(x)$ in eqn (1.55) equals zero at one of the end-points of $[a, b]$. In such a case we usually require $y(x)$ to be bounded at that end-point, with a value of either $y(x)$ or $y'(x)$ being prescribed at the other end-point, that is:

- $|y(a)| < \infty, \quad y(b) = 0$
- $|y(a)| < \infty, \quad y'(b) = 0$

In addition, from exercises 1.5(a)–1.5(e), it follows that the condition in eqn (1.60) holds also for both $u(x)$ and $v(x)$ satisfying one of the following sets of boundary conditions:

- $y(a) = 0, \quad y'(b) + hy(b) = 0$
- $y'(a) = 0, \quad y'(b) + hy(b) = 0$
- $y'(a) + h_1 y(a) = 0, \quad y'(b) + h_2 y(b) = 0$
- $y(a) = y(b), \quad p_0(a)\, y'(a) = p_0(b)\, y'(b)$
- $|y(a)| < \infty, \quad y'(b) + hy(b) = 0$

The last conditions set presumes that the leading coefficient $p_0(x)$ of eqn (1.55) equals zero at $x = a$.

Boundary value problems formulated for eqn (1.55) subject to either one of the sets of boundary conditions listed above, belong to the important class of the so-called *self-adjoint boundary value problems.*

To provide the definition of the self-adjointness for a boundary value problem, we consider an equation in the self-adjoint form

$$\frac{d}{dx}\left(p_0(x)\frac{dy(x)}{dx}\right) + p_2(x)y(x) = 0, \quad x \in (a, b) \tag{1.61}$$

subject to the boundary conditions

$$B_1(y; a, b) = 0, \quad B_2(y; a, b) = 0 \tag{1.62}$$

The boundary conditions in eqn (1.62) are supposed to match the relation in eqn (1.58) in the sense that for any allowable functions $u(x)$ and $v(x)$, each satisfying the conditions in eqn (1.62), the relation in eqn (1.58) is also satisfied.

With these assumptions in mind, we say that the formulation in eqns (1.61) and (1.62) represents a *self-adjoint boundary value problem*, if Green's formula in eqn (1.59) is valid for any two times continuously differentiable functions $u(x)$ and $v(x)$ satisfying the boundary conditions in eqn (1.62).

We now turn the reader's attention to the basic question in this section, that is, the symmetry of a Green's function. Based on the self-adjointness of a boundary value problem, we formulate the condition for a Green's function to be symmetric by the following theorem.

Theorem 1.2: If the boundary value problem in eqns (1.61) and (1.62) is self-adjoint and has only the trivial solution, then its Green's function $g(x, s)$ is symmetric, provided that $g(x, s) = g(s, x)$.

Proof. This proof is based on a slight modification of that procedure which has been used in the proof of Theorem 1.1 in Section 1.1. Here we also consider two linearly independent particular solutions $y_1(x)$ and $y_2(x)$

of the governing equation in eqn (1.61). But contrary to Theorem 1.1, we put some restrictions on them choosing these solutions in a special way.

First, let $y_1(x)$ satisfy the first boundary condition in eqn (1.62), and let $y_2(x)$ satisfy the second condition in eqn (1.62). Clearly, $y_1(x)$ cannot satisfy the second boundary condition in eqn (1.62) and $y_2(x)$ cannot satisfy the first of those conditions, because according to what we have stated earlier, the trivial solution is the only solution of eqn (1.61) satisfying both of the boundary conditions in eqn (1.62).

Second, based on $y_1(x)$ and $y_2(x)$, let us form the bilinear combination

$$y_1(x)\, L[y_2(x)] - y_2(x)\, L[y_1(x)]$$

which identically equals zero on (a, b), since $L[y_1(x)] \equiv 0$ and $L[y_2(x)] \equiv 0$ for $x \in (a, b)$.

Recalling the relation in eqn (1.56), derived earlier in this section, and rewriting it in terms of $y_1(x)$ and $y_2(x)$ yields

$$y_1\, L(y_2) - y_2\, L(y_1) = \frac{d}{dx}\left(p_0(x)\left(y_1\frac{dy_2}{dx} - y_2\frac{dy_1}{dx}\right)\right)$$

Hence, in the case of a self-adjoint boundary value problem, $y_1(x)$ and $y_2(x)$ must satisfy the following relation

$$\frac{d}{dx}\left(p_0(x)\left(y_1\frac{dy_2}{dx} - y_2\frac{dy_1}{dx}\right)\right) = 0$$

which implies

$$p_0(x)\left(y_1\frac{dy_2}{dx} - y_2\frac{dy_1}{dx}\right) = C \tag{1.63}$$

where C is a constant.

Notice that $y_1(x)$ and $y_2(x)$ are determined up to a constant multiple. Indeed, if $y_1(x)$, for example, satisfies both the governing equation in eqn (1.61) and the first boundary condition in eqn (1.62), then, for any nonzero constant α, the product $\alpha\, y_1(x)$ also satisfies both of these relations. This is equally true for $y_2(x)$. Hence, we can rewrite the relation in eqn (1.63) in the form

$$p_0(x)\left(y_1\frac{dy_2}{dx} - y_2\frac{dy_1}{dx}\right) = 1 \tag{1.64}$$

Thus, without losing generality, we can assume that $y_1(x)$ and $y_2(x)$ meet the condition in eqn (1.64) throughout (a, b). We will return to this point later in this section.

Fix now an arbitrary point $s \in (a, b)$ and express the Green's function $g(x, s)$ to the problem in eqns (1.61) and (1.62) in the form

$$g(x,s) = \begin{cases} c_1(s)\, y_1(x), & \text{for } a \le x \le s \\ c_2(s)\, y_2(x), & \text{for } s \le x \le b \end{cases} \tag{1.65}$$

This function satisfies the boundary conditions in eqn (1.62) regardless of the values of $c_1(s)$ and $c_2(s)$. This occurs because $y_1(x)$ and $y_2(x)$ satisfy the first and second of those boundary conditions, respectively. Hence, $g(x, s)$ in the form of eqn (1.65) already meets properties 1 and 4 of the definition of Green's function.

By virtue of properties 2 and 3 of the definition, we obtain the following system of linear algebraic equations

$$\begin{pmatrix} y_2(s) & -y_1(s) \\ y_2'(s) & -y_1'(s) \end{pmatrix} \times \begin{pmatrix} c_2(s) \\ c_1(s) \end{pmatrix} = \begin{pmatrix} 0 \\ -p_0^{-1}(s) \end{pmatrix}$$

in $c_1(s)$ and $c_2(s)$. The coefficient matrix of this system is not singular, because its determinant $y_1(s)y_2'(s) - y_2(s)y_1'(s)$ is the Wronskian of the two linearly independent functions $y_1(s)$ and $y_2(s)$. Hence, this system has a unique solution which can be written in the form

$$c_1(s) = -\frac{y_2(s)}{p_0(s)\, W(s)}, \quad c_2(s) = -\frac{y_1(s)}{p_0(s)\, W(s)}$$

Upon substituting these values of $c_1(s)$ and $c_2(s)$ into eqn (1.65), one obtains, for the upper branch of the Green's function

$$g(x,s) = -\frac{y_1(x)\, y_2(s)}{p_0(s)\, W(s)}, \quad x \le s \tag{1.66}$$

while for the lower branch, we have

$$g(x,s) = -\frac{y_2(x)\, y_1(s)}{p_0(s)\, W(s)}, \quad s \le x \tag{1.67}$$

According to the relation in eqn (1.64), the denominator in eqns (1.66) and (1.67) meets the condition

$$p_0(s)\, W(s) \equiv p_0(s) \left(y_1(s) \frac{dy_2(s)}{dx} - y_2(s) \frac{dy_1(s)}{dx} \right) \equiv 1$$

In view of this fact, we can finally write the Green's function $g(x,s)$ for the self-adjoint boundary value problem posed by eqns (1.61) and (1.62) in the following symmetric compact form

$$g(x,s) = \begin{cases} -y_2(s)\,y_1(x), & \text{for } a \le x \le s \\ -y_1(s)\,y_2(x), & \text{for } s \le x \le b \end{cases} \tag{1.68}$$

From this representation, it follows that $g(x,s)$ is invariant to the interchange of x with s. In other words, the Green's function is symmetric in the case of a self-adjoint boundary value problem.

In the next section, we will return to the basic issue of this chapter, which is the construction of Green's functions. One more procedure available for this construction in the current literature will be discussed in detail.

1.3 Method of variation of parameters

So far in this text, we have discussed boundary value problems stated for homogeneous equations with homogeneous boundary conditions imposed.

In this section, we will recall [17, 30, 31, 63, 66, 68, 70] an important theorem that establishes a theoretical background for the utilization of Green's functions in solving boundary value problems for nonhomogeneous equations and boundary conditions. Then we will present the procedure for construction of Green's functions, which is based on that theorem and Lagrange's method of variation of parameters. As it is known, Lagrange's method is traditionally used to analytically solve nonhomogeneous linear differential equations if the fundamental solution set is available for the corresponding homogeneous equation.

Consider a boundary value problem for the nonhomogeneous equation

$$L(y) \equiv p_0(x)y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_{n-1}(x)y' + p_n(x)y = -f(x) \tag{1.69}$$

on (a,b), subject to the homogeneous boundary conditions written as

$$M_k(y;a,b) \equiv \sum_{j=0}^{n-1} \left(\alpha_j^k y^{(j)}(a) + \beta_j^k y^{(j)}(b)\right) = 0, \quad (k = \overline{1,n}) \tag{1.70}$$

where the coefficients $p_i(x)$ are continuous functions on (a,b), with $p_0(x) \neq 0$ and $M_k(y;a,b)$ represent linearly independent forms with constant coefficients.

Suppose the problem posed by eqns (1.69) and (1.70) has a unique solution. This consequently implies that the corresponding homogeneous boundary value problem has only the trivial solution.

The following theorem supports a direct method for solving boundary value problems that are formulated for nonhomogeneous equations subject to homogeneous boundary conditions.

Theorem 1.3: If $g(x,s)$ represents the Green's function of the homogeneous boundary value problem corresponding to that posed by eqns (1.69) and (1.70), then the unique solution of the problem in eqns (1.69) and (1.70) itself can be expressed by the integral

$$y(x) = \int_a^b g(x,s)\, f(s)\, ds \tag{1.71}$$

Proof. It is clear that two independent facts need to be proven. First, that the integral in eqn (1.71) satisfies the equation (1.69), and second, that it satisfies the boundary conditions in eqn (1.70).

Since the Green's function $g(x,s)$ is defined in pieces, we break down the integral in eqn (1.71) into two integrals as shown

$$y(x) = \int_a^x g(x,s)\, f(s)\, ds + \int_x^b g(x,s)\, f(s)\, ds \tag{1.72}$$

In order to differentiate $y(x)$, we should take into account its occurrence. Indeed, it is defined in terms of integrals containing a parameter and having variable limits. Therefore, one has to recall from calculus [68] that if

$$I(x) = \int_{a(x)}^{b(x)} F(x,s)ds$$

then the derivative of this function is written as

$$\frac{dI(x)}{dx} = F(x,b(x))b'(x) - F(x,a(x))a'(x) + \int_{a(x)}^{b(x)} F'_x(x,s)ds \tag{1.73}$$

Hence, since both of the integrals in eqn (1.72) contain x as a parameter and their limits depend on x, we obtain

$$y'(x) = \int_a^x g'_x(x,s)f(s)ds + g(x,x-0)f(x)$$

$$+ \int_x^b g'_x(x,s)f(s)ds - g(x,x+0)f(x)$$

The above integrals can be combined and nonintegral terms are eliminated due to the continuity of the Green's function as $x=s$. This yields

$$y'(x) = \int_a^b g'_x(x,s)f(s)ds$$

Recalling the continuity of the derivatives of the Green's function up to the $(n-2)$-nd order included as $x = s$ (see property 2 of the definition), the higher order derivatives of the integral in eqn (1.72) up to the $(n-1)$-st order included can be computed analogously to the first derivative

$$y^{(j)}(x) = \int_a^b g_x^{(j)}(x,s)f(s)ds, \quad (j = \overline{2, n-1})$$

Thus, the boundary conditions in eqn (1.70) are satisfied with $y(x)$ expressed by eqn (1.71), since all the differentiations in $M_k(y; a, b)$ can be carried out under the integration sign.

Additionally, in order to substitute $y(x)$ into eqn (1.69), we compute its n-th order derivative

$$y^{(n)}(x) = \int_a^b g_x^{(n)}(x,s)f(s)ds + [g_x^{(n-1)}(x, x-0) - g_x^{(n-1)}(x, x+0)]\, f(x)$$

which, in accordance with property 3 of the definition of Green's function, yields

$$y^{(n)}(x) = \int_a^b g_x^{(n)}(x,s)f(s)ds - f(x)p_0^{-1}(x)$$

Upon substituting $y(x)$ and its derivatives found above into eqn (1.69) and grouping all the integral terms together, one finally obtains

$$\int_a^b L[g(x,s)]\, f(s)\, ds - f(x) = -f(x)$$

The above equality is an identity, since $L[g(x,s)] = 0$ on (a, b). Thus, the theorem is proved.

Based on this theorem, we describe below one more approach which can be used for the construction of Green's functions. The idea behind this approach is to employ Lagrange's method of variation of parameters which is traditionally used to solve nonhomogeneous linear differential equations. For the sake of simplicity, we again consider a boundary value problem for the equation of the second order

$$L[y(x)] \equiv p_0(x)\frac{d^2y(x)}{dx^2} + p_1(x)\frac{dy(x)}{dx} + p_2(x)y(x) = -f(x) \tag{1.74}$$

subject to the simplest set of boundary conditions

$$y(a) = 0, \quad y(b) = 0 \tag{1.75}$$

Assume the above boundary value problem has a unique solution or, in other words, the corresponding homogeneous problem has only the trivial solution. Let $y_1(x)$ and $y_2(x)$ represent two linearly independent particular