SECOND EDITION

Crocheting Adventures with Hyperbolic Planes

Tactile Mathematics, Art and Craft for all to Explore

Daina Taimina

CRC Press
Taylor & Francis Group

Crocheting Adventures with Hyperbolic Planes

tactile mathematics, art and craft
for all to explore

Second Edition

Crocheting Adventures with Hyperbolic Planes

tactile mathematics, art and craft
for all to explore

Second Edition

Daina Taimiņa

CRC Press
Taylor & Francis Group
Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group, an **informa** business

CRC Press
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

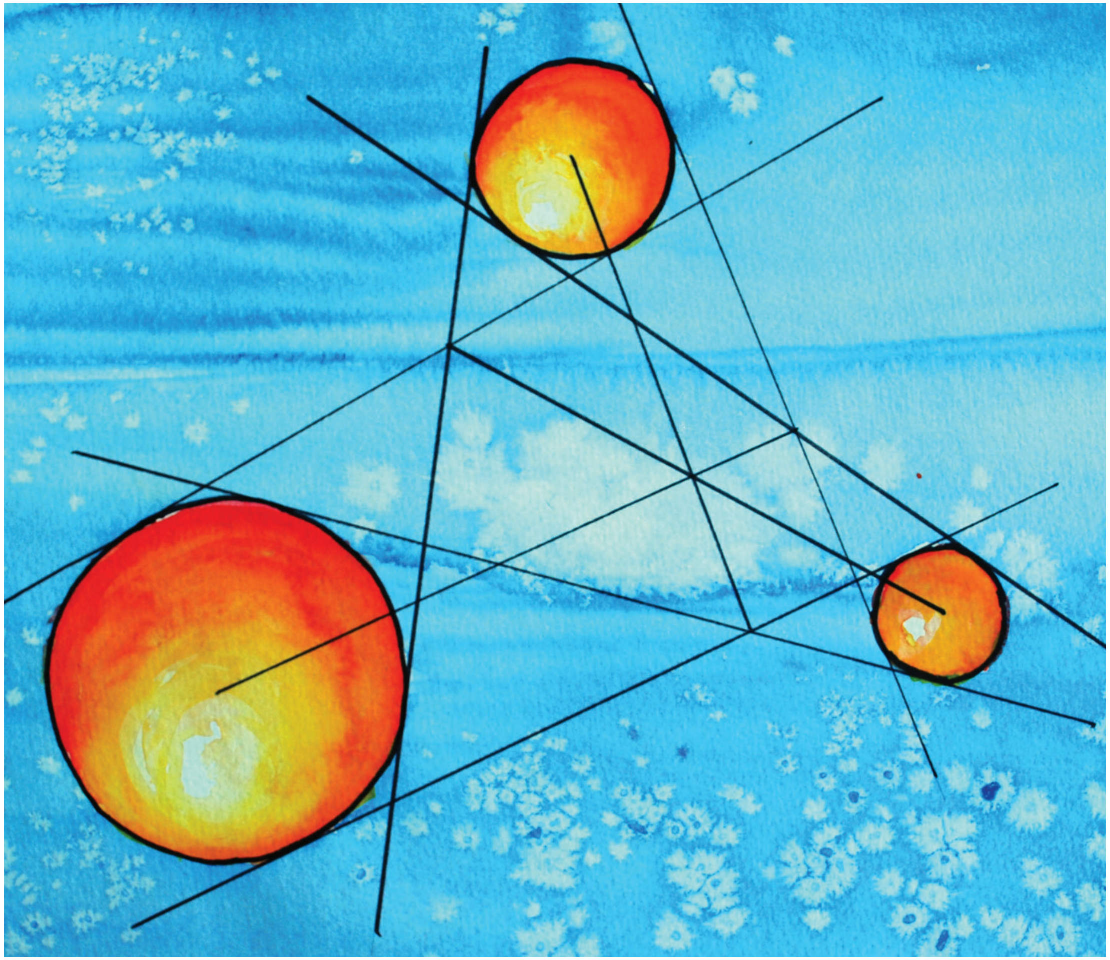
Printed on acid-free paper
Version Date: 20180119

International Standard Book Number-13: 978-1-1383-0115-3 (Hardback)

Visit the Taylor & Francis Web site at
http://www.taylorandfrancis.com

and the CRC Press Web site at
http://www.crcpress.com

to those who walked me into the beautiful world of geometry:



Alfreds Grava

David Henderson

William Thurston


Contents

Foreword

Many people have an impression, based on years of schooling, that mathematics is an austere and formal subject concerned with complicated and ultimately confusing rules for the manipulation of numbers, symbols and equations: rather like the preparation of a complicated income tax return, where there are myriad unexplained steps, rules, exceptions, and gotchas.

Good mathematics is quite opposite to this. Mathematics is an art of human understanding. Billions of years of evolution have given us many extraordinary capabilities that we ordinarily take for granted—but we deny those capabilities at our peril. In the abstract, the mere act of walking through a room without bumping into other people or things is a far greater accomplishment than the most sophisticated formal computation ever done by mathematicians. Computers are far better than humans at formal computations, but humans far surpass current computers at informal and intuitive reasoning.

Our brains are complicated devices, with many specialized modules working behind the scenes to give us an integrated understanding of the world. Mathematical concepts are abstract, so it ends up that there are many different ways they can sit in our brains. A given mathematical concept might be primarily a symbolic equation, a picture, a rhythmic pattern, a short movie—or best of all, an integrated combination of several different representations. The non-symbolic mental models for mathematical concepts are extremely important, but unfortunately, many of them are hard to share.

Mathematics sings when we feel it in our whole brain. People are generally inhibited about even trying to share their personal mental models. People like music, but they are afraid to sing. You only learn to sing by singing.

How do you think of a cat? You probably have mental images, you probably can see in your mind how cats move, you can probably hear in your mind sounds that cats make—perhaps meowing, purring, squeaking, yowling. But can you draw a good picture of a cat, can you give a good description or animation of how a cat moves, can you describe or recreate the sounds a cat makes? It's a lot harder to communicate a cat than to see a cat, and it requires serious cultivation of special talents.

We learn about cats by watching cats; with most mathematical ideas, our culture has provided no comparable shortcuts. Without a good way for whole-brain communication, understanding is denatured.

Non-Euclidean or hyperbolic geometry was a topic of great mystery and confusion for many centuries, as Daina recounts in this book. Insights people may have developed were hard to document, so they crumbled away.

Why do Daina's crochet models have such a great resonance with so many people? It's because they break through the austere, formal stereotype of mathematics and present a path to a whole-brain understanding of a beautiful cluster of simple and significant but rarely understood ideas. The crochet models also break through the stereotype that mathematics is only relevant to traditionally male interests.

These models have a fascination far beyond their visual appearance. As illustrated in the book, there is actually negative curvature and hyperbolic geometry all around us, but people generally see it without seeing it. You will develop an entirely new understanding by actually following the simple instructions and crocheting! The models are deceptively interesting. Perhaps you will come up with your own variations and ideas.

In any case, I hope this book gives you pause for thought and changes your way of thinking about mathematics.

Bill Thurston (1946–2012)
Ithaca, NY

Introduction

> What I hear I forget,
> What I see I remember,
> What I touch I understand.
> —Confucius (555–479 BC)

Each day is like the stitch in the fabric of our lives. Over the years, days bring different colors and change patterns. In knitting or crocheting, there is always an option to repeat the pattern we enjoy or to unravel the piece we dislike and start it all over again—with different stitches, different colors, choosing a different pattern. For our own story, all we can do is to follow the thread to find where it begins.

In early September, 1954, trees along the canals of Amsterdam started to get some fall colors in their leaves. The city was still busy healing World War II wounds, but some people were rushing in the streets of Amsterdam with suitcases and serious faces. The International Congress of Mathematicians (ICM) was once again held in Europe.[1]

Several prominent mathematicians were invited to give plenary talks in Amsterdam, but the most interest seemed to be centered on astoundingly creative John von Neumann (1903–1957), who was invited to give the lecture "Unsolved Problems in Mathematics." He was expected to summarize what had been accomplished since 1900, when David Hilbert (the most famous mathematician at that time) talked about unsolved problems in mathematics, and also what new problems were challenging mathematicians after half a century. Von Neumann mostly talked about problems in which he was interested, because as he said:

> The total subject of mathematics is clearly too broad for any of us. I do not think that any mathematician since Gauss has covered it uniformly and fully, even Hilbert did not, and all of us are of considerably lesser width (quite apart from the question of depth) than Hilbert.[2]

Geometry was of special interest in this congress because it had been 100 years since Georg Friedrich Bernhard Riemann (1826–1866) gave his famous inaugural lecture at the University of Göttingen, "On the Hypotheses which Lie in the Foundation of Geometry." One of the problems that later grew out of that 1854 talk was: Is there any surface in 3-space that has constant negative curvature? Such a surface is called a *hyperbolic plane*. Hilbert answered this question in 1901, by proving that one cannot have an equation describing such a surface. (More about it will be discussed in Chapter 1.) However, the crocheting construction of such surfaces (hyperbolic planes) is the subject of this book. I will talk about history of crochet in more detail in Chapter 4; but, in 1954, there were several other connections to the story of this book.

For example, at the congress, a young graduate student, who later became a well-known mathematician, Robert Osserman (1926–2011), presented a talk related to a problem about embedding a hyperbolic plane in 3-space.[3]

This problem was suggested to Osserman by his thesis advisor Lars Ahlfors (1907–1996), who was one of the first recipients of the Fields Medal. There is an interesting story about it that Robert Osserman told me.

In order to celebrate the 100th anniversary of the birth of the Riemann surface in Riemann's doctoral dissertation, a conference was held at Princeton in December 1951. Osserman was a graduate student at the time and had recently begun his doctoral studies at Harvard under the supervision of Ahlfors. Ahlfors was to give the opening address at the conference, and he suggested that his graduate student Osserman attend also. Prior to the opening talk, Lipman Bers (1914–1993) asked if he could present the assembled experts with two open problems concerning ordinary surfaces in Euclidean 3-space regarded as Riemannian surfaces. Lipman Bers, who became president of the American Mathematical Society (1975–1976), was born in Riga, Latvia (the same city where I was born in 1954). Osserman tried to crack one of these problems, but he got his breakthrough only in 1954 while in Paris continuing his studies. He wrote to me in 2008:

> Instead of trying to construct a surface in the form of a graph that I could prove had to be hyperbolic, I could try starting with a classical Riemann surface given as a branched covering of the plane that I knew to be hyperbolic, and then see if I could "unravel" it so that it could be embedded as a graph over the whole plane. *I wrote to Ahlfors to explain my "unraveling" process, but he wasn't able to understand my description, so I made a model* using a neighborhood of a simple branch point connecting two sheets; one could cut it along a ray from the branch point, fold it up into a fan shape consisting of 16 wedge-shaped sectors with a 45-degree angle each, and then glue the two sides of the cut back together, obtaining an isometric image of the original surface, with its 720-degree total angle at the branch point. *I mailed the model to Ahlfors, and finally got his affirmation that my (very geometric) argument was valid.* It turned out to be a bit harder than I had anticipated to find a branched surface known to be hyperbolic that could be unraveled in its entirety in this fashion and displayed as a graph over the whole plane, but I was able to do so, and that was the subject of my talk at Amsterdam whose abstract was published in the proceedings of the International Congress. [The emphasis is mine.]

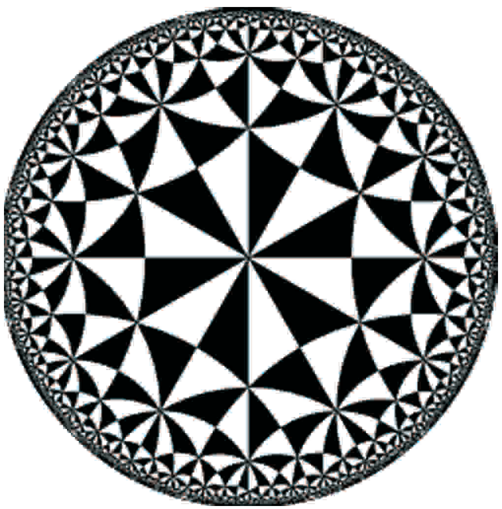
Osserman's result is connected to hyperbolic planes because, in 1871, Henri Poincaré mapped the hyperbolic plane to a disk without changing angles—this mapping is called the *Poincaré model*.

The Poincaré model of the hyperbolic plane was used by some mathematicians but after this particular ICM it became extremely popular because of a Dutch artist, M. C. Escher (1898–1972). As a part of the 1954 ICM cultural program, there was a special exhibition organized in the Stedelijk Museum, in which the shy Dutch artist showed many fascinating prints and carved wooden balls, as well as some hand-drawn and colored drawings of periodic tessellations of reptiles, birds, and other interlocked creatures.

M. C. Escher, *Reptiles*, Lithograph, 1943.

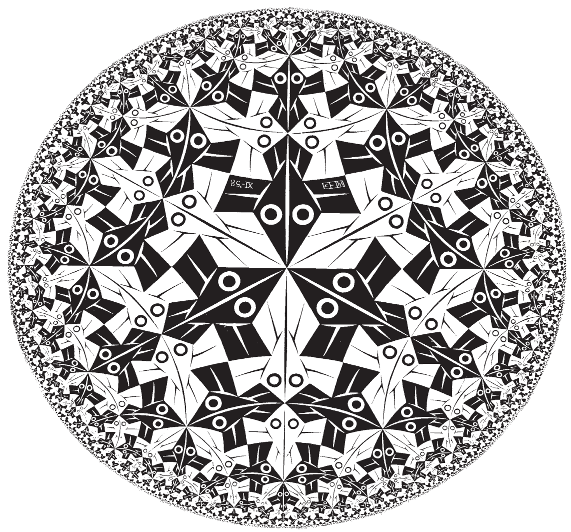
Although Escher did not have any mathematical training, he had an amazing understanding of mathematics, both visually and intuitively. The mathematical influence emerged in his works after 1936, especially after his visit to the Alhambra (discussed in Chapter 3), but Escher was very nervous about having an exhibit at a Congress of Mathematicians. Dutch mathematician N. G. de Bruijn wrote in the catalog,

> In view of the fact that Mr. Escher's work may be said to be a point of contact between art and mathematics, the Organizing Committee … took the initiative to inaugurating this exhibition. Probably mathematicians will not only be interested in the geometric motifs; the same playfulness which constantly appears in mathematics in general and which, to great many mathematicians is the peculiar charm of their subject, will be a more important element.[4]



H. S. M. Coxeter's tiling of Poincaré disk.

Escher's work was extremely well received, and this exhibit was the beginning of a long-term friendship between the famous geometer H. S. M. Coxeter and the artist. Coxeter asked Escher's permission to use some of his work as illustrations in his mathematical papers—a tradition he continued for years, promoting the popularity of Escher's work among the mathematical community. Later Escher in his turn asked Coxeter a question that puzzled him from the artistic point of view: Was there any way he could map the whole plane in a circle? Coxeter knew the mathematical answer to this question and introduced Escher to the Poincaré disk model of the hyperbolic plane, and for many years Escher's famous woodcuts *Circle Limit I*, *II*, *III*, and *IV* became iconic representations of the hyperbolic plane.



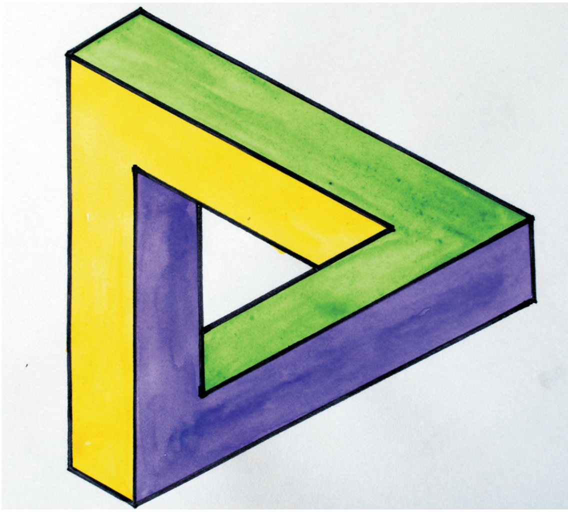
M. C. Escher, *Circle Limit I*, 1958, woodcut.

In addition, the young Dutch mathematician Nicolaas Kuiper (1920–1994) was apparently thinking in 1954 about the problem of a hyperbolic surface because in mid-1955 he published a proof that there must exist a hyperbolic surface (not defined by equations), but he was not able to produce a concrete construction. (This history will be discussed more in Chapter 1.)

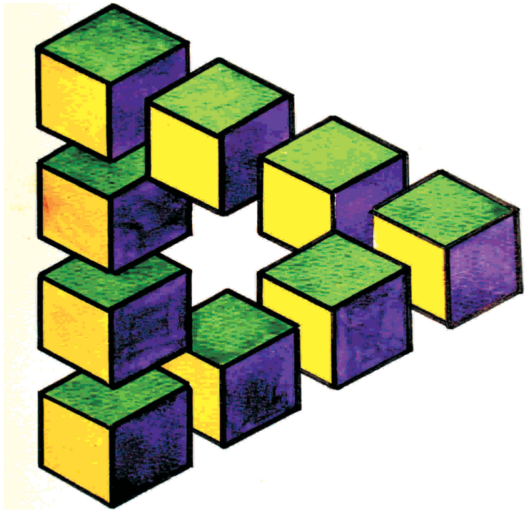
In 1954 Sir Roger Penrose was only a second-year graduate student when he attended the ICM. Somebody suggested that he go and see Escher's exhibit. Escher's work inspired Penrose to try to draw something impossible himself, and he came up with the discovery of an impossible triangle. In 1958, a friend showed the Swedish artist Oscar Reutersvärd the now classical article on impossible figures by Lionel Penrose and Roger Penrose, which featured the Penroses' versions of the impossible triangle and impossible staircase.[5] Reutersvärd then recollected that he already drew a similar object 20 years earlier while doodling in a Latin grammar notebook during class. The Penroses' article prompted Reutersvärd to see the impossible figures as new and worthy as a serious subject to explore, and he became well known as the "father of impossible figures."[6]

Aesthetics have always been important to Penrose. Later, in his famous book *The Emperor's New Mind*, he wrote:

> My impression is that the strong conviction of the validity of a flash of inspiration (not 100 percent reliable; I should add, but at least far more reliable than just a chance, is very closely bound up with its aesthetic qualities. A beautiful idea has a much greater chance of being a correct idea than an ugly one. At least, that has been my own experience.[7]



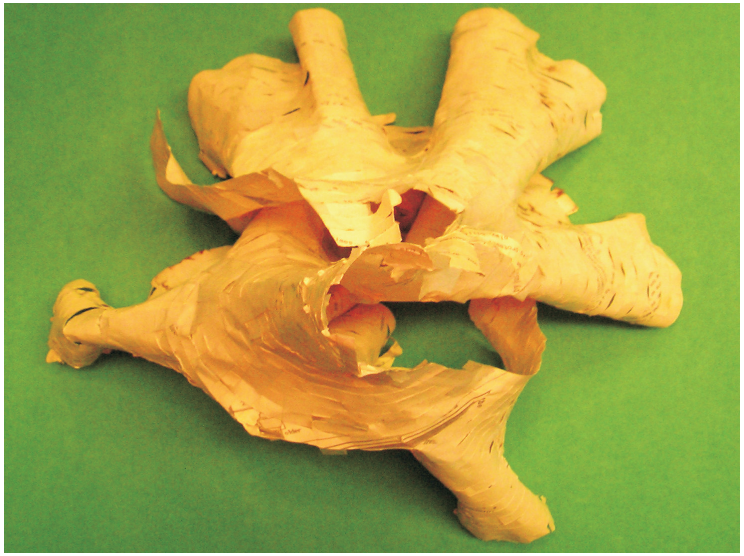
Penrose's impossible triangle.



Reutersvärd's impossible figure.

In September 1954 I was just a few-weeks-old baby in Riga, Latvia, and had no idea about all these connections or the fact that I would become a mathematician, that I would work on the other side of the world, and that the hyperbolic plane would introduce me not only to many extraordinary mathematicians but also to the fiber arts and many wonderful people I have met because of these crocheted hyperbolic planes.

For me, adventures with the hyperbolic plane were unpredictable. I first learned about a paper model of the hyperbolic plane in June 1997. Professor David Henderson, now my husband, was leading a workshop about teaching geometry to university professors and was showing the model that you can see in a picture below. He told me that he learned how to make this model from Bill Thurston in the late 1970s.



From this paper model I learned about the hyperbolic plane.

That fall I was scheduled to teach a geometry class at Cornell and therefore was seriously thinking about how to better explain hyperbolic geometry to my students. We were spending summer vacation on a tree farm in Pennsylvania, and while watching my two little girls learning to swim, I was crocheting my classroom set of models of the hyperbolic plane. It turned out to be a success. Students really liked the tactile way of exploring hyperbolic geometry. They told me that after playing with these models, the acquired experiences helped them to move on. Later this set of models was used in several classes and many workshops (and still is used today and is in good shape!).

Crocheted hyperbolic plane models are often used in our workshops for school teachers (Washington, DC, summer 2006).

Crocheted models featured in *The Mathematical Intelligencer* 2001 (left) and 2004 (right).

The two "cover girls" finally met in 2008.

David and I decided to publish a pattern of how to crochet a model of the hyperbolic plane and also to tell about our own discovery of a formula for hyperbolic area—which arose after we played with these models ourselves. We convinced the editor of *The Mathematical Intelligencer* that the idea of using crochet to visualize mathematical concepts was worth publishing and that somebody else might use it too.[8]

Actually, we were right: a few years later there was another crochet model on a cover of *The Mathematical Intelligencer*—this time two mathematicians Hinke Osinga and Bernd Krauskopf (then from Bristol University, UK now University of Auckland, NZ), used crochet technique to show what the Lorenz manifold (from the area of mathematics called dynamical systems) looks like. They wrote that our paper inspired them to use crochet as a medium.[9]

The idea to write this book grew from countless questions I have received after my lectures, during workshops and exhibits, and over electronic mail. In this book I am trying to explain to crocheters the mathematics that can be experienced through these models. With weathered mathematicians, I am sharing a tactile way of exploring mathematical ideas. For all others, I hope this book will convey a different way to view mathematics.

Chapter 1

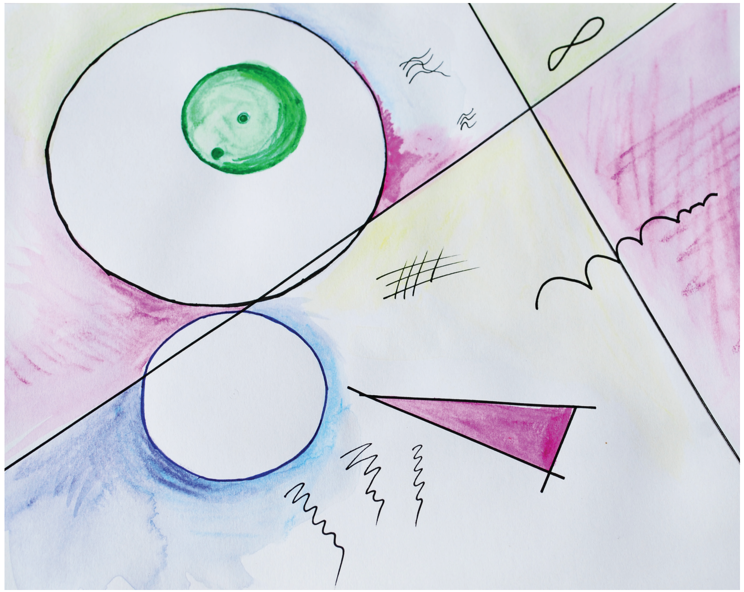
What Is the Hyperbolic Plane? Can We Crochet It?

Positive and Negative Curvature

When we talk about numbers, we can divide them into positive numbers, negative numbers, and zero.

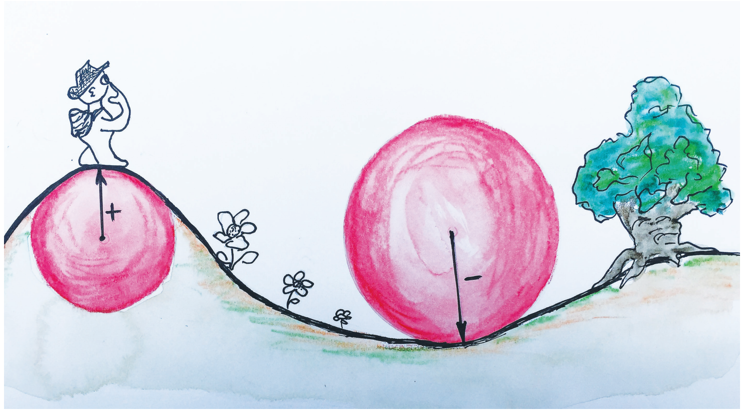
Elevator buttons in France: the ground floor is numbered 0

When we scribble different lines with a pencil on a piece of paper, we can get different shapes in the plane. The ones that are most useful in geometry are the ones that are "the same everywhere." In the plane, such shapes are straight lines and circles. A straight line is called straight because it is not curved—in other words, we can say it has *zero curvature*. Circles are curved everywhere in the same way—so they have *constant curvature*.



Straight lines and circles are different from other scribbles because they are the same everywhere.

The larger the radius of a circle, the less curved it is. For example, if you look at part of a circle with a very large radius, the line will seem straight. Mathematicians express this property by saying that circles with radius R have curvature $1/R$. The notion of the curvature is used to study curves that are different from circles and straight lines. We can say that the curvature of a curve measures the failure of a curve to be a straight line. We know from school geometry that two circles are the same if they have the same radius. Using the notion of the curvature we can now say that two circles are the same if they have the same curvature.

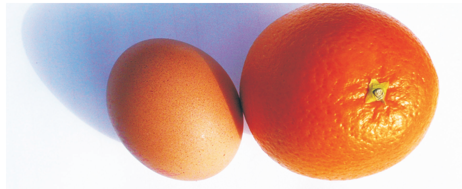


For curves in a plane, the curvature can be negative (in a valley) or positive (on the top of a hill).

Imagine that we are sketching a landscape as children first learn to do, just with different curves; there can be hills and there can be valleys in this landscape. In our example, we are avoiding cliffs and sharp peaks or valleys. The skyline of the landscape is not uniformly curved everywhere, but at each point we can measure the curvature by the circle that would "best fit" in the curve at that point, as you can see in the picture. To distinguish numerically where the hills are and where the valleys are, we use a notion of one-dimensional positive and negative curvature. We can decide that hills have positive curvature where the circle would be inside the hill, below the ground; for valleys we say that the curvature is negative where the circle is above the ground. At some places positive curvature will turn into negative curvature, and at that place the curvature will be zero. It will be zero also everywhere that the curve is straight.

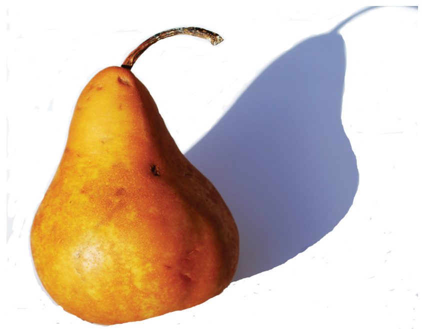
Can we make a similar distinction with surfaces?

Curvature is a mathematical notion widely used in differential geometry. What does it mean in simple words? If you look, for example, on the surface of your desk or on the floor in your room, you will notice that it is flat—there is no curvature, or we say that the curvature is *zero*. If you look at an orange or an egg, you will see that it is curved "outward"—we say that the curvature is *positive*. The egg is curved more at its tips and curved less in between and thus does not have constant positive curvature, but many oranges have almost constant positive curvature.



Two surfaces with positive curvature:
on an egg it is varying, but on an orange, it is (almost) constant.

Now look at the surface of the pear. Most of its surface has positive curvature like an orange or an egg, but there are some points were the curvature is different. How can we describe this diff



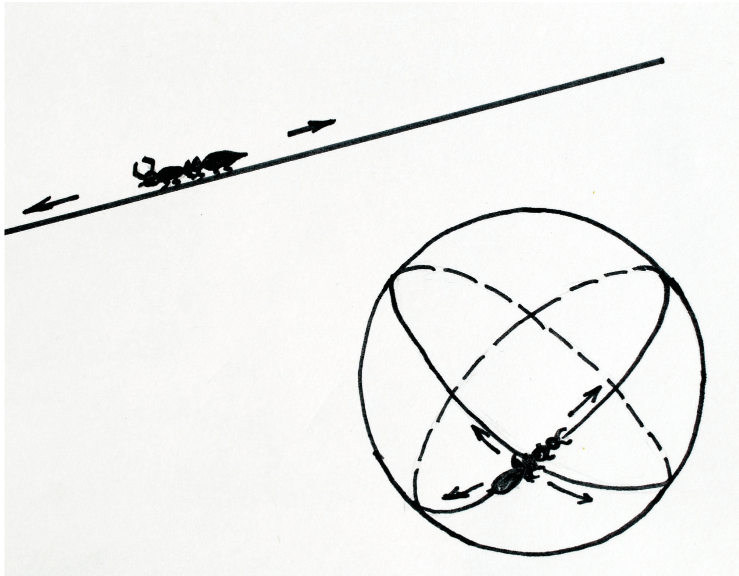
Not all points on the pear have positive curvature.

Many leaves have interesting surfaces like in the picture with kale leaves.

Kale leaves have negative curvature.

In the early nineteenth century one of the greatest mathematicians of all times, Carl Friedrich Gauss (1777–1855), explored the idea that surfaces can be distinguished by their *curvature* at different points, which can be positive, negative, or zero. In the 1820s Gauss was a professional surveyor. This work inspired him to study the intrinsic geometry of surfaces. His concern was how one can determine the curvature of an arbitrary surface without knowing anything about how this surface might be embedded in space—in other words, asking whether and how one could determine the curvature of a surface through measurements made only along this surface, without knowing anything about the shape of this surface.

Consider an ant crawling on a big sphere, where the ant cannot see that it is on a sphere; how could this ant distinguish whether it is on a flat surface or on a curved surface? If the ant crawls on a straight line or on any other one-dimensional curve, it can move only in two directions: forward or backward. If the ant is on a two-dimensional surface, then it can choose to go backward or forward but also right or left. Still, the ant cannot see the surface it is on from the outside; it can explore the surface only intrinsically, not leaving this surface.



An ant on a curve can go in only two directions, but on a surface there are more choices.

Another great mathematician, Leonard Euler (1707–1783), had already introduced a notion of surface curvature, which was used in eighteenth-century calculus. But to use Euler's method for surfaces, you had to know how the surface is embedded in space. This means that you had to be able to see the surface from another dimension. Gauss was able to prove that it is possible to find a way to determine the curvature of a surface that depends only on intrinsic properties of the surface.[1]