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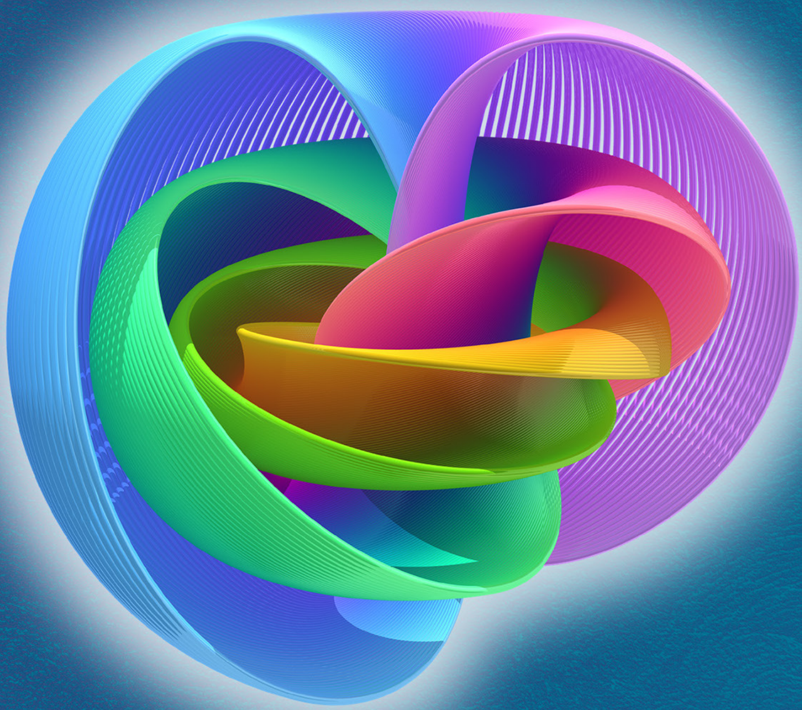

Edited by
Haynes Miller

CRC Press
Taylor & Francis Group
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Handbook of Homotopy Theory

CRC Press/Chapman and Hall Handbooks in Mathematics Series

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CRC Press
Taylor & Francis Group
Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group, an **informa** business
A CHAPMAN & HALL BOOK

CRC Press
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

Printed on acid-free paper

International Standard Book Number-13: 978-0-815-36970-7 (Hardback)

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Preface

This Handbook collects twenty-two essays by some thirty-one authors on a wide range of topics of current interest in homotopy theory. It is intended to present a snapshot of some key elements of contemporary work in this area, along with indications of potential avenues for future research. I hope it is useful to those interested in pursuing research in this field, as well as to the growing class of mathematicians in neighboring areas who find themselves employing the methods of homotopy theory in their own research.

This volume may be regarded as a successor to the "Handbook of Algebraic Topology," edited by Ioan James and published a quarter of a century ago. In calling it the "Handbook of Homotopy Theory," I am recognizing that the discipline has expanded and deepened, and traditional questions of topology, as classically understood, are now only one of many distinct mathematical disciplines in which it has had a profound impact and which serve as sources of motivation for research directions within homotopy theory proper.

The topics represented here range from axiomatic to applied, from applications in symplectic and algebraic geometry to the modern chromatic development of stable homotopy theory, from local methods in group theory to the theory and use of ∞-categories. It would nevertheless have been easy to come up with as many more topics in the subject that are active and promising today, and as many more authors engaged in trail-blazing research and ready to write about it. The chapters vary widely in style and in the demands placed on the reader, but taken together form a surprisingly highly connected narrative.

The strong response to my call for participation in this project is a reflection of the excitement and vitality of research today in homotopy theory. In fact, every author took the topic I proposed in unanticipated directions, often adding authors to the team, and in the aggregate providing a much broader and more vibrant coverage than I imagined possible at the outset. I am very grateful to them all. I also thank the many others who gave advice and assistance of all sorts during the assembly of this book. In particular I thank Ashley Fernandes for his unflagging assistance with the LaTeX, CRC editor Bob Ross for his assistance in bringing this together, Niles Johnson for allowing us to use his beautiful image of the Hopf fibration (a seminal example in our subject), and Steve Krantz, who proposed this project in the first place.

1

Goodwillie calculus

Gregory Arone and Michael Ching

Goodwillie calculus is a method for analyzing functors that arise in topology. One may think of this theory as a categorification of the classical differential calculus of Newton and Leibnitz, and it was introduced by Tom Goodwillie in a series of foundational papers [44, 45, 46].

The starting point for the theory is the concept of an *n-excisive* functor, which is a categorification of the notion of a polynomial function of degree n. One of Goodwillie's key results says that every homotopy functor F has a universal approximation by an n-excisive functor P_nF, which plays the role of the n-th Taylor approximation of F. Together, the functors P_nF fit into a tower of approximations of F: the *Taylor tower*

$$F \longrightarrow \cdots \longrightarrow P_nF \longrightarrow \cdots \longrightarrow P_1F \longrightarrow P_0F.$$

It turns out that 1-excisive functors are the ones that represent generalized homology theories (roughly speaking). For example, if $F = I$ is the identity functor on the category of based spaces, then P_1I is the functor $P_1I(X) \simeq \Omega^\infty\Sigma^\infty X$. This functor represents stable homotopy theory in the sense that $\pi_*(P_1I(X)) \cong \pi_*^s(X)$. Informally, this means that the best approximation to the homotopy groups by a generalized homology theory is given by the stable homotopy groups. The Taylor tower of the identity functor then provides a sequence of theories, satisfying higher versions of the excision axiom, that interpolate between stable and unstable homotopy.

The analogy between Goodwillie calculus and ordinary calculus reaches a surprising depth. To illustrate this, let D_nF be the homotopy fiber of the map $P_nF \to P_{n-1}F$. The functors D_nF are the homogeneous pieces of the Taylor tower. They are controlled by Taylor "coefficients" or derivatives of F. This means that for each n there is a spectrum with an action of Σ_n that we denote ∂_nF, and there is an equivalence of functors

$$D_nF(X) \simeq \Omega^\infty \left(\partial_nF \wedge X^{\wedge n}\right)_{h\Sigma_n}.$$

Here for concreteness F is a homotopy functor from the category of pointed spaces to itself; similar formulas apply for functors to and from other categories. The spectrum ∂_nF plays the role of the n-th derivative of F, and the spectra ∂_nF are relatively easy to calculate. There is an obvious similarity between the formula for D_nF and the classical formula for the n-th term of the Taylor series of a function.

Mathematics Subject Classification. 55P65, 55P48, 55P42, 19D10.

Key words and phrases. calculus of functors, identity functor, operads, algebraic K-theory.

Of course there are differences between the classical differential calculus and Goodwillie calculus, and they are just as interesting as the similarities. One place where the analogy breaks down is in the complex ways that the homogeneous pieces can be "added up" to create the full Taylor tower. Homogeneous layers only determine the Taylor tower up to extensions. Theorems of Randy McCarthy, Nick Kuhn and the present authors reveal situations in which these extension problems can be understood via interesting connections to Tate spectra. Considerable simplification occurs by passing to chromatic homotopy theory, a fact that forms the basis for recent work of Gijs Heuts on the classification of unstable v_n-periodic homotopy theory via spectral Lie algebras.[1]

Also unlike ordinary calculus, a crucial example is provided by the identity functor (for based spaces or, more generally, for any ∞-category of interest).[2] As hinted at above, the identity functor typically has an interesting and non-trivial Taylor tower controlled by its own sequence of Taylor coefficients. For based spaces, these derivatives were first calculated by Brenda Johnson. Mark Mahowald and the first author used a detailed description of these objects to get further information about the Taylor tower of the identity in unstable v_n-periodic homotopy. The derivatives of the identity also play an important theoretical role in the calculus; in particular, by a result of the second author they form an operad that encodes structure possessed by the derivatives of any functor to or from a given ∞-category. For topological spaces, this operad is a topological analogue of the Lie operad, explaining the role of Lie algebras in Quillen's work on rational homotopy theory, and in Heuts's work mentioned above. Structures related to the Lie operad also form the basis of a classification of functors up to Taylor tower equivalence given by the present authors.

The nature of Goodwillie calculus lends itself to both computational and conceptual applications. Goodwillie originally developed the subject in order to understand more systematically certain calculations in algebraic K-theory, and this area remains a compelling source of specific examples. However, the deeply universal nature of these constructions gives functor calculus a crucial role in the foundations of homotopy theory, especially given the expanding role therein of higher category theory.

Indeed it seems that the calculus has not yet found its most general form. The similarities to Goodwillie calculus borne by the manifold and orthogonal "calculi" of Michael Weiss suggest some deeper structure that is still to be properly worked out. There are also important but not fully understood connections to manifolds and factorization homology of E_n-algebras,[3] and a properly equivariant version of Goodwillie calculus has been hinted at by work of Dotto, but much remains to be explored.

Notwithstanding such future developments, the fundamental role of Goodwillie calculus in homotopy theory is as clear as that of ordinary calculus in other areas of mathematics: it provides a systematic interpolation between the linear (homotopy-theoretically, this usually means the stable) and nonlinear (or unstable) worlds, and thus brings our deep intuition of the nature of change to bear on our understanding of homotopy theory.

[1] See Gijs Heuts's chapter in this Handbook for an account of this development.

[2] See Moritz Groth's chapter in this Handbook for an introduction to the theory of ∞-categories.

[3] See the chapter in this Handbook by David Ayala and John Francis for a survey of factorization homology.

1.1 Polynomial Approximation and the Taylor Tower

Goodwillie's calculus of functors is modelled after ordinary differential calculus with the role of smooth maps between manifolds played by *homotopy functors* $F : \mathcal{C} \to \mathcal{D}$, i.e. those that preserve some notion of weak equivalence. In Goodwillie's original formulation [44], the categories $\mathcal{C}$ and $\mathcal{D}$ were each taken to be some category of topological spaces or spectra, but modern higher-category-theoretic technology allows the theory to be developed for functors F between any $(\infty,1)$-categories that are suitably well-behaved. The basic tenets of the theory are independent of any particular model for $(\infty,1)$-categories. In this paper, we will mostly use the language of ∞-categories (i.e. quasicategories) from [63], and the details of Goodwillie calculus have been developed in the greatest generality by Lurie in that context, see [64, Sec. 6]. Thus our typical assumption will be that $F : \mathcal{C} \to \mathcal{D}$ is a functor between ∞-categories. The reader can equally, however, view F as a functor between model categories that preserves weak equivalences. In that setting, many of the basic constructions are described by Kuhn in [58].

We will make considerable use of the notions of (homotopy) limit and colimit inside an ∞-category. We will refer to these simply as limits and colimits, though everywhere in this paper the appropriately homotopy-invariant concepts are intended. When working with a functor $F : \mathcal{C} \to \mathcal{D}$, we will usually require that $\mathcal{C}$ and $\mathcal{D}$ admit limits and colimits of particular shapes and that (especially in $\mathcal{D}$) certain limits and colimits commute. The relevant conditions will be made explicit when necessary.

Polynomial functors in the homotopy calculus

To some extent, the theory of the calculus of functors is completely determined by making a choice as to which functors $F : \mathcal{C} \to \mathcal{D}$ are to be considered the analogues of degree n polynomials. In Goodwillie's version, this choice is described in terms of cubical diagrams.

Definition 1.1.1. An *n-cube* in an ∞-category $\mathcal{C}$ is a functor $\mathcal{X} : \mathcal{P}(I) \to \mathcal{C}$, where $\mathcal{P}(I)$ is the poset of subsets of some finite set I of cardinality n. An n-cube $\mathcal{X}$ is *cartesian* if the canonical map

$$\mathcal{X}(\emptyset) \to \operatorname{holim}_{\emptyset \neq S \subseteq I} \mathcal{X}(S)$$

is an equivalence, and *cocartesian* if

$$\operatorname{hocolim}_{S \subsetneq I} \mathcal{X}(S) \to \mathcal{X}(I)$$

is an equivalence. When $n = 2$, these conditions reduce to the familiar notions of pullback and pushout, respectively. We also say that an n-cube $\mathcal{X}$ is *strongly cocartesian* if every 2-dimensional face is a pushout. Note that a strongly cocartesian n-cube is also cocartesian if $n \geq 2$.

We can now give Goodwillie's condition on a functor that plays the role of "polynomial of degree $\leq n$".

Definition 1.1.2. Let $\mathcal{C}$ be an ∞-category that admits pushouts. A functor $F : \mathcal{C} \to \mathcal{D}$ is *n-excisive* if it takes every strongly cocartesian $(n+1)$-cube in $\mathcal{C}$ to a cartesian $(n+1)$-cube in $\mathcal{D}$. We will say that F is *polynomial* if it is n-excisive for some integer n.

Let $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ be the ∞-category of functors from $\mathcal{C}$ to $\mathcal{D}$, and let $\mathrm{Exc}_n(\mathcal{C}, \mathcal{D})$ denote the full subcategory whose objects are the n-excisive functors.

Example 1.1.3. In the somewhat degenerate case $n = 0$, Definition 1.1.2 reduces to the statement that F is 0-excisive if and only if it is homotopically constant, i.e. F takes every morphism in $\mathcal{C}$ to an equivalence in $\mathcal{D}$. (In an even more degenerate case, F is (-1)-excisive if and only if $F(X)$ is a terminal object of $\mathcal{D}$ for all X in $\mathcal{C}$.)

Example 1.1.4. A functor $F : \mathcal{C} \to \mathcal{D}$ is 1-excisive if and only if it takes pushout squares in $\mathcal{C}$ to pullback squares in $\mathcal{D}$. The prototypical example of such a functor (when $\mathcal{C}$ and $\mathcal{D}$ are both the category $\mathcal{T}op_*$ of based topological spaces) is

$$X \mapsto \Omega^\infty(E \wedge X)$$

where E is some spectrum. In fact, these examples constitute a classification of those functors that are 1-excisive, *reduced* (i.e. preserve the null object) and *finitary* (i.e. preserve filtered colimits). This fact illustrates the key role played by stable homotopy theory in Goodwillie calculus.

Remark 1.1.5. It is notable that the identity functor $I : \mathcal{C} \to \mathcal{C}$ is typically *not* 1-excisive (or n-excisive for any n) unless $\mathcal{C}$ is a stable ∞-category. One might naturally think it would make more sense if a 1-excisive functor were defined to preserve either pushouts or pullbacks, rather than to mix the two notions. Indeed, one can make such a definition and explore its properties. However, the notion defined by Goodwillie has turned out to be much more useful. This is partly because of the close connection to stable homotopy theory hinted at in Example 1.1.4, but also because the fact that the identity functor is not 1-excisive makes the theory *more* useful rather than less, since, as we shall see, the Taylor tower of the identity functor provides an interesting decomposition of a space that we would not have if the identity were automatically linear.

Remark 1.1.6. The property of a functor F being n-excisive can also be described as a condition on sequences of $n+1$ morphisms in $\mathcal{C}$ with a common source, say $f_i : A \to X_i$ for $i = 0, \dots, n$. The condition relates the value of F on the pushouts, over A, of all possible subsets of the sequence $(f_0, \dots, f_n)$. This can be viewed as the analogue of a way to specify when a function $f : \mathbb{R} \to \mathbb{R}$ is a degree $\leq n$ polynomial by considering the values of f on sums of subsets of a set $(x_0, \dots, x_n)$ of real numbers.

Remark 1.1.7. Johnson and McCarthy [53] have given a different, slightly broader definition for a functor $F : \mathcal{C} \to \mathcal{D}$ to be *degree* $\leq n$: that the $(n+1)$-th cross-effect of F vanishes. This choice leads to a different version of the Taylor tower described in the next section, although the difference seems not to be important in most cases of interest. In particular, for "analytic" functors, the two towers agree within the 'radius of convergence'.

As one might expect, the conditions of being n-excisive for varying n are nested.

Lemma 1.1.8. *If $F : \mathcal{C} \to \mathcal{D}$ is n-excisive, then F is also $n+1$-excisive. We therefore have a sequence of inclusions of subcategories*

$$\mathrm{Exc}_0(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Exc}_1(\mathcal{C}, \mathcal{D}) \subseteq \cdots \subseteq \mathrm{Exc}_n(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Exc}_{n+1}(\mathcal{C}, \mathcal{D}) \subseteq \cdots .$$

Polynomial approximation in the homotopy calculus

The fundamental construction in ordinary differential calculus is that of polynomial approximation: given a smooth function $f : \mathbb{R} \to \mathbb{R}$ and a real number x, there is a unique "best" degree $\leq n$ polynomial that approximates f in a "neighbourhood" of x. To transfer this idea to the calculus of functors, we need to be able to compare the values of functors on objects in $\mathcal{C}$ that are related in some sense. In particular, we require a map between the objects in order to make this comparison. Thus the n-excisive approximation to a functor $F : \mathcal{C} \to \mathcal{D}$ in a neighbourhood of an object $X \in \mathcal{C}$ is only defined on objects Y that come equipped with a map $Y \to X$ in $\mathcal{C}$, that is, on Y in the slice ∞-category $\mathcal{C}_{/X}$.

Definition 1.1.9. We say that functors $\mathcal{C} \to \mathcal{D}$ *admit n-excisive approximations* at X in $\mathcal{C}$ if the composite

$$\mathrm{Exc}_n(\mathcal{C}_{/X}, \mathcal{D}) \hookrightarrow \mathrm{Fun}(\mathcal{C}_{/X}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

has a left adjoint, which, when it exists, we write P_n^X. Here (as everywhere in this article) we mean an adjunction in the $(\infty, 1)$-categorical sense: see, for example Lurie [63, 5.2.2.1] or Riehl-Verity [74, 1.1].

Remark 1.1.10. Biedermann, Chorny and Röndigs showed in [27] that the n-excisive approximation is a left Bousfield localization of a suitable category of functors, and it provides a best approximation *on the right* to a given functor $F : \mathcal{C} \to \mathcal{D}$ by one that is n-excisive. Explicitly, the n-excisive approximation to F at X, if it exists, consists of a natural transformation

$$F \to P_n^X F$$

of functors $\mathcal{C}_{/X} \to \mathcal{D}$ that is initial (up to homotopy) among natural transformations from F (restricted to $\mathcal{C}_{/X}$) to an n-excisive functor.

The first main theorem of Goodwillie calculus is that such n-excisive approximations exist under mild conditions on $\mathcal{C}$ and $\mathcal{D}$. The following result is stated by Lurie, but the proof is no different than that given originally by Goodwillie in the context of functors of topological spaces and spectra.

Theorem 1.1.11 (Goodwillie [46, 1.13], Lurie [64, 6.1.1.10])**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories, and suppose that $\mathcal{C}$ has pushouts, and that $\mathcal{D}$ has sequential colimits, and finite limits, which commute. Then functors $\mathcal{C} \to \mathcal{D}$ admit n-excisive approximations at any object $X \in \mathcal{C}$.*

Example 1.1.12. The 0-excisive approximation to F at X is, as you would expect, equivalent to the constant functor with value $F(X)$.

There is no loss of generality (and the notation is simpler) if we focus on the case where X is a terminal object of $\mathcal{C}$. In this setting, the n-excisive approximation to $F : \mathcal{C} \to \mathcal{D}$ is another functor from $\mathcal{C}$ to $\mathcal{D}$, which we simply denote by $P_n F$.

Example 1.1.13. Goodwillie gives an explicit construction of P_nF which is easiest to describe when $n = 1$ and where $\mathcal{C}$ and $\mathcal{D}$ are both pointed with $F : \mathcal{C} \to \mathcal{D}$ *reduced*, i.e. preserving the null object. In this case, $P_1F(Y)$ can be written as the colimit of the following sequence of maps

$$F(Y) \to \Omega F(\Sigma Y) \to \Omega^2 F(\Sigma^2 Y) \to \cdots$$

where Σ and Ω are the suspension and loop-space functors for $\mathcal{C}$ and $\mathcal{D}$ respectively. For $F : \mathcal{T}op_* \to \mathcal{T}op_*$, we have

$$P_1F(Y) \simeq \Omega^\infty \bar{F}(\Sigma^\infty Y)$$

where $\bar{F}$ is a reduced 1-excisive functor from spectra to spectra. For Y a finite CW-complex, such a functor can be written in the form of Example 1.1.4:

$$P_1F(Y) \simeq \Omega^\infty(\partial_1 F \wedge Y)$$

where $\partial_1 F$ is a spectrum which we refer to as the *(first) derivative of* F (at the one-point space $*$).

Example 1.1.14. The 1-excisive approximation to the identity functor on *based* spaces is simply the stable homotopy functor

$$P_1I(Y) \simeq \Omega^\infty \Sigma^\infty Y =: Q(Y)$$

or equivalently, $\partial_1 I \simeq S^0$, the sphere spectrum.

The unbased case is slightly more subtle: the 1-excisive approximation to the identity functor in that context can be written as

$$P_1I(Y) \simeq \operatorname{hofib}(Q(Y_+) \to Q(S^0)) =: \tilde{Q}(Y)$$

where the homotopy fibre is calculated over the point in $Q(S^0)$ corresponding to the *identity* map on the sphere spectrum (as opposed to the null map).

Taylor tower and convergence

The explicit description of P_nF is hard to make use of for $n > 1$, even in the case of the identity functor. The real power of the calculus of functors derives from the tower formed by the n-excisive approximations for varying n.

Definition 1.1.15. The *Taylor tower* (or *Goodwillie tower*) of $F : \mathcal{C} \to \mathcal{D}$ at $X \in \mathcal{C}$ is the sequence of natural transformations (of functors $\mathcal{C}_{/X} \to \mathcal{D}$):

$$F \to \cdots \to P^X_{n+1}F \to P^X_n F \to \cdots \to P^X_1 F \to P^X_0 F \simeq F(X)$$

where it follows from the universal property of $P^X_{n+1}F$, and Lemma 1.1.8, that each $F \to P^X_n F$ factors as shown.

For a given map $f : Y \to X$ in $\mathcal{C}$, the Taylor tower provides a sequence of factorizations of $F(f) : F(Y) \to F(X)$. Ideally, we would be able to recover the value $F(Y)$ from this sequence of approximations $P^X_n F(Y)$ in the following way.

Definition 1.1.16. The Taylor tower of $F : \mathcal{C} \to \mathcal{D}$ *converges* at $Y \in \mathcal{C}_{/X}$ if the induced map

$$F(Y) \to \operatorname{holim}_n P_n^X F(Y)$$

is an equivalence in $\mathcal{D}$.

Arguably the question of convergence of the Taylor tower is the most important step in actually applying the calculus of functors to a particular functor F. Very general approaches to proving convergence seem rare, but Goodwillie has developed an extensive set of tools, based on connectivity estimates, in the contexts of topological spaces and spectra.

These tools are based on measuring the failure of a functor F to be n-excisive via connectivity. This can be done by applying F to a strongly cocartesian $(n+1)$-cube, and examining the failure of the resulting cube to be cartesian, in terms of the connectivity of the map from the initial vertex to the homotopy limit of the rest of the diagram.

Roughly speaking, a functor F is *stably n-excisive* if this connectivity is controlled relative to the connectivities of the maps in the original cocartesian cube. Goodwillie then says that F is *ρ-analytic*, for some real number ρ, if it is stably n-excisive for all n where these connectivity estimates depend linearly on n with slope ρ. See [45] for complete details.

The upshot of these definitions is the following theorem.

Theorem 1.1.17 (Goodwillie [46, 1.13])**.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a ρ-analytic functor where $\mathcal{C}$ and $\mathcal{D}$ are each either spaces or spectra. Then the Taylor tower of F at $X \in \mathcal{C}$ converges on those objects Y in $\mathcal{C}_{/X}$ whose underlying map $Y \to X$ is ρ-connected.*

Examples 1.1.18. The identity functor $I : \mathcal{T}op \to \mathcal{T}op$ is 1-analytic [45, 4.3]. This depends on higher dimensional versions of the Blakers-Massey Theorem relating pushouts and pullbacks in $\mathcal{T}op$. Indeed, the usual Blakers-Massey Theorem implies that I is stably 1-excisive. Waldhausen's algebraic K-theory of spaces functor $A : \mathcal{T}op \to \mathcal{S}p$ is also 1-analytic [45, 4.6]. In particular, the Taylor towers at $*$ of both of these functors converge on simply connected spaces.

Definition 1.1.19. Let $F : \mathcal{C} \to \mathcal{T}op_*$ be a homotopy functor with values in pointed spaces. The *Goodwillie spectral sequence* associated to F at $Y \in \mathcal{C}_{/X}$ is the homotopy spectral sequence of the tower of pointed spaces $(P_n^X F(Y))_{n \geq 1}$ [32, section IX.4]. This spectral sequence converges to

$$\pi_* P_\infty^X F(Y)$$

where $P_\infty^X F := \operatorname{holim}_n P_n^X F$, and has E^1-term given by the homotopy groups of the *layers* of the Taylor tower, i.e.

$$E^1_{s,t} \cong \pi_{t-s} D_s^X F(Y)$$

where $D_s^X F := \operatorname{hofib}(P_s^X F \to P_{s-1}^X F)$. If F is analytic (and $Y \to X$ is suitably connected), then this spectral sequence converges strongly [32, section IX.5].

Example 1.1.20. For the identity functor on based spaces, the Goodwillie spectral sequence at $Y \in \mathcal{T}op_*$ takes the form

$$E^1_{s,t} = \pi_{t-s} D_s I(Y)$$

and converges to the homotopy groups of Y when Y is simply-connected (or, more generally, when the Taylor tower of the identity converges for Y. This spectral sequence has been studied extensively by Behrens [25]. The spectral sequence also motivates the study of the layers $D_n^X F$ of the Taylor tower in general, and we turn to these now.

1.2 The Classification of Homogeneous Functors

Let $F : \mathcal{C} \to \mathcal{D}$ be a homotopy functor, where $\mathcal{C}$ and $\mathcal{D}$ are as in Theorem 1.1.11, and suppose further that $\mathcal{D}$ is a *pointed* ∞-category. Then we make the following definition, generalizing that given at the end of the previous section.

Definition 1.2.1. The n-th *layer* of the Taylor tower of F at X is the functor $D_n^X F : \mathcal{C}_{/X} \to \mathcal{D}$ given by

$$D_n^X F(Y) := \mathrm{hofib}(P_n^X F(Y) \to P_{n-1}^X F(Y)).$$

These layers play the role of *homogeneous* polynomials in the theory of calculus, and satisfy the following definition.

Definition 1.2.2. Let $F : \mathcal{C} \to \mathcal{D}$ be a homotopy functor that admits n-excisive approximations, and where $\mathcal{D}$ has a terminal object $*$. We say that F is *n-homogeneous* if F is n-excisive and $P_{n-1}F \simeq *$.

Another of Goodwillie's main theorems from [46] is the existence of a natural delooping, and consequently a classification, of homogeneous functors. To state this in the generality of ∞-categories, we first recall that any suitable pointed ∞-category $\mathcal{C}$ admits a *stabilization*, that is a stable ∞-category $\mathcal{S}p(\mathcal{C})$ together with an adjunction

$$\Sigma^\infty_{\mathcal{C}} : \mathcal{C} \rightleftarrows \mathcal{S}p(\mathcal{C}) : \Omega^\infty_{\mathcal{C}}$$

that generalizes the suspension spectrum / infinite-loop space adjunction, which we write simply as $(\Sigma^\infty, \Omega^\infty)$, for $\mathcal{C} = \mathcal{T}op_*$.

Theorem 1.2.3. *Let $F : \mathcal{C} \to \mathcal{D}$ be an n-homogeneous functor between pointed ∞-categories. Then there is a* symmetric multilinear *functor $H : \mathcal{S}p(\mathcal{C})^n \to \mathcal{S}p(\mathcal{D})$, and a natural equivalence*

$$F(X) \simeq \Omega^\infty_{\mathcal{D}} \left[H(\Sigma^\infty_{\mathcal{C}} X, \dots, \Sigma^\infty_{\mathcal{C}} X)_{h\Sigma_n} \right]$$

where we are taking the homotopy orbit construction with respect to the action of the symmetric group Σ_n that permutes the entries of H.

Example 1.2.4. For functors $\mathcal{T}op_* \to \mathcal{T}op_*$, a symmetric multilinear functor is uniquely determined (on finite CW-complexes at least) by a single spectrum with a symmetric group action. Applying this classification to the layers of the Taylor tower of $F : \mathcal{T}op_* \to \mathcal{T}op_*$, we get an equivalence

$$D_n F(X) \simeq \Omega^\infty(\partial_n F \wedge (\Sigma^\infty X)^{\wedge n})_{h\Sigma_n}$$

where $\partial_n F$ is a spectrum with a (naive) action of the symmetric group Σ_n, which we refer to as the *n-th derivative* of F (at $*$). Similar formulas apply when $\mathcal{C}$ and/or $\mathcal{D}$ is $\mathcal{S}p$ instead of $\mathcal{T}op_*$, and sense can be made of the object $\partial_n F$ for more general $\mathcal{C}$ and $\mathcal{D}$, though in such cases $\partial_n F$ is a diagram of spectra (indexed by generators for the stable ∞-categories $\mathcal{S}p(\mathcal{C})$ and $\mathcal{S}p(\mathcal{D})$) rather than a single spectrum, e.g. see [37, 1.1].

Example 1.2.5. The n-th derivative of the identity functor $I : \mathcal{T}op_* \to \mathcal{T}op_*$ was calculated by Brenda Johnson in [52]. The spectrum $\partial_n I$ is equivalent to a wedge of $(n-1)!$ copies of the $(1-n)$-sphere spectrum. In particular, $\partial_1 I \simeq S^0$ (as mentioned above) and $\partial_2 I \simeq S^{-1}$ (with trivial Σ_2-action), so

$$D_2 I(X) \simeq \Omega^\infty \Sigma^{-1} (\Sigma^\infty X)^{\wedge 2}_{h\Sigma_2}.$$

We can now attempt to calculate $P_2 I$ using the fibre sequence $D_2 I \to P_2 I \to P_1 I$. This takes the form

$$\Omega^\infty \Sigma^{-1} (\Sigma^\infty X)^{\wedge 2}_{h\Sigma_2} \to P_2 I(X) \to \Omega^\infty \Sigma^\infty X.$$

Goodwillie's results imply that this sequence deloops, so that $P_2 I(X)$ can be written as the fibre of a certain natural transformation.

Example 1.2.6. The n-th derivative of the functor $\Sigma^\infty \Omega^\infty : \mathcal{S}p \to \mathcal{S}p$ is equivalent to S^0 (with trivial Σ_n-action) [1, Corollary 1.3]. Therefore

$$D_n(\Sigma^\infty \Omega^\infty)(X) \simeq X^{\wedge n}_{h\Sigma_n}.$$

This tells us that, for a spectrum X, the spectrum $\Sigma^\infty \Omega^\infty X$ is given by piecing together the extended powers $X^{\wedge n}_{h\Sigma_n}$. When X is a 0-connected suspension spectrum, Snaith splitting provides an equivalence

$$\Sigma^\infty \Omega^\infty X \simeq \bigvee_{n \geq 1} X^{\wedge n}_{h\Sigma_n}$$

which can be interpreted as the splitting of the Taylor tower. For arbitrary X, the layers of the tower are pieced together in a less trivial way.

Splitting Results for the Taylor Tower

As illustrated in Example 1.2.6, the simplest situation holds when the Taylor tower is a product of its layers:

$$P_n F(X) \simeq \prod_{k=0}^{n} D_k F(X).$$

Example 1.2.7. Let K be a d-dimensional based finite CW-complex and consider the functor $\Sigma^\infty \operatorname{Hom}_*(K, -) : \mathcal{T}op_* \to \mathcal{S}p$ where $\operatorname{Hom}_*(-,-)$ denotes the space of basepoint-preserving maps between two based spaces. The first author, in [6], proved (1) that

$$\partial_n(\Sigma^\infty \operatorname{Hom}_*(K, -)) \simeq \operatorname{Map}(\Sigma^\infty K^{\wedge n}/\Delta^n K, S^0)$$

where $\operatorname{Map}(-,-)$ is the mapping spectrum construction, and $\Delta^n K$ denotes the "fat diagonal", i.e. the subspace of $K^{\wedge n}$ consisting of those points $(k_1, \ldots, k_n)$ where $k_i = k_j$ for some $i \neq j$, (2) that the Taylor tower converges on d-connected X, and (3) when K is a parallelizable d-dimensional manifold and X is a d-fold suspension, that the Taylor tower splits. Thus in the latter case we have:

$$\Sigma^\infty \operatorname{Hom}_*(K, X) \simeq \prod_{n=1}^{\infty} \operatorname{Map}(\Sigma^\infty K^{\wedge n}/\Delta^n K, \Sigma^\infty X^{\wedge n})_{h\Sigma_n}.$$

Kuhn has proved, in [56], the following splitting result which reveals some of the interesting interaction between Goodwillie calculus, Tate cohomology and chromatic homotopy theory.

Theorem 1.2.8 (Kuhn)**.** *Let $F : \mathcal{S}p \to \mathcal{S}p$ be a homotopy functor. Then the Taylor tower of F splits after $T(k)$-localization. (Here $T(k)$ denotes the spectrum given by the telescope of a v_k-self-map of a finite type-k complex.) In other words*

$$L_{T(k)} P_n F(X) \simeq \prod_{j=1}^{n} L_{T(k)} D_j F(X).$$

The proof of Theorem 1.2.8 relies on a number of interesting ingredients, in particular the vanishing of the $T(k)$-localization of the Tate construction associated to a finite group action on a spectrum. The specific part coming from functor calculus, however, is the following result of McCarthy [66].

Theorem 1.2.9 (McCarthy)**.** *For a functor $F : \mathcal{S}p \to \mathcal{S}p$ that preserves filtered colimits, there is a natural homotopy pullback square of the form*

$$\begin{tikzcd} P_n F(X) \arrow[r] \arrow[d] & (\partial_n F \wedge X^{\wedge n})^{h\Sigma_n} \arrow[d] \\ P_{n-1} F(X) \arrow[r] & (\partial_n F \wedge X^{\wedge n})^{t\Sigma_n}. \end{tikzcd}$$

Here Y^{tG} denotes the Tate construction of the action of a finite group G on a spectrum Y, that is the cofibre of the norm map $N : Y_{hG} \to Y^{hG}$, and the right-hand vertical map above is the canonical map from the homotopy fixed points to the Tate construction.

Note that the induced map between the homotopy fibres of the vertical maps in this diagram is the equivalence $D_n F(X) \xrightarrow{\sim} (\partial_n F \wedge X^{\wedge n})_{h\Sigma_n}$ that appears in the classification of n-homogeneous functors from spectra to spectra.

Kuhn's theorem, roughly speaking, follows from McCarthy's by taking $T(k)$-localization of the homotopy pullback square, and using the vanishing of the Tate construction.

The results quoted here begin to address the question of how the layers of the Taylor tower of a homotopy functor are pieced together to form the tower itself, and they start to illustrate the role of the Tate spectrum construction in that picture. We will return to this topic in Section 1.4.

1.3 The Taylor tower of the identity functor for based spaces

Let I be the identity functor from the category of based spaces to itself. From the perspective of functor calculus, this is a highly non-trivial object. As noted above, I is not an n-excisive functor for any n, and its derivatives have a lot of structure. Applying homotopy groups to the Taylor tower, we get a sequence

$$\pi_* X \to \cdots \to \pi_* P_n I(X) \to \pi_* P_{n-1} I(X) \to \cdots \to \pi_* P_1(X) = \pi_*^s X$$

interpolating between the unstable and stable homotopy groups of a space X.

The first step to understanding the Taylor tower of a functor is to calculate the derivatives (and hence the layers). For the identity functor, the derivatives were, as mentioned above, calculated by Johnson in [52]. Here we give the reformulation of her result produced in [14].

Definition 1.3.1. Let $\mathcal{P}_n$ be the poset of partitions of the set $\{1, \ldots, n\}$, ordered by refinement. Let $|\mathcal{P}_n|$ be the geometric realization of $\mathcal{P}_n$. Note that $\mathcal{P}_n$ has both an initial and a final object. It follows in particular that $|\mathcal{P}_n|$ is contractible. Let $\partial|\mathcal{P}_n|$ be the subcomplex of $|\mathcal{P}_n|$ spanned by simplices that do not contain both the initial and final element as vertices. Let $T_n = |\mathcal{P}_n|/\partial|\mathcal{P}_n|$.

It is easy to see that T_n is an $n-1$-dimensional complex with an action of the symmetric group Σ_n. It is well known that non-equivariantly T_n is equivalent to $\bigvee_{(n-1)!} S^{n-1}$. In fact, this equivalence holds already after restricting the action from Σ_n to Σ_{n-1}, where Σ_{n-1} is considered a subgroup of Σ_n in the standard way. This means that there is a Σ_{n-1}-equivariant equivalence $T_n \simeq \Sigma_{n-1+} \wedge S^{n-1}$ [8].

Theorem 1.3.2. [52, 14] *There is a Σ_n-equivariant equivalence of spectra*

$$\partial_n I \simeq \mathbb{D}(T_n)$$

between the n-th derivative of the identity functor and the Spanier-Whitehead dual of the complex T_n.

It turns out that the Taylor tower of the identity functor has some rather special properties when evaluated at a sphere. The results are cleanest to state for an odd-dimensional sphere, so we will mostly focus on this case.

We begin by noting that rationally the tower is constant.

Theorem 1.3.3. [14, Proposition 3.1] *Let X be an odd-dimensional sphere. The spectrum*

$$(\partial_n I \wedge X^{\wedge n})_{h\Sigma_n}$$

is rationally contractible for $n > 1$.

Proof sketch. Since $\partial_n I \simeq \mathbb{D}(T_n)$, it is enough to prove that $\Sigma^\infty X_{hG}^{\wedge n}$ has trivial rational homology when G is any isotropy group of T_n. This follows by an easy spectral sequence argument. □

This strengthens the following classical computation of Serre:

Corollary 1.3.4 (Serre, 1953)**.** *When X is an odd-dimensional sphere, the map $X \to \Omega^\infty\Sigma^\infty X$ is a rational homotopy equivalence.*

It follows from the theorem that for X an odd sphere, the homology of $(\partial_n I \wedge X^{\wedge n})_{h\Sigma_n}$ is torsion for all $n > 1$. The following theorem gives considerably more information about how the torsion is distributed among the layers.

Theorem 1.3.5. [14, 3] *Let X be an odd-dimensional sphere, and let p be a prime. The homology with mod p coefficients of the spectrum $(\partial_n I \wedge X^{\wedge n})_{h\Sigma_n}$ is non-trivial only if n is a power of p.*

Note that it follows that if n is not a prime power then the spectrum $(\partial_n I \wedge X^{\wedge n})_{h\Sigma_n}$ is contractible, since it is a connective spectrum of finite type whose homology is trivial with rational and mod p coefficients for all primes p. Theorem 1.3.5 implies that if one is willing to pick a prime p and localize all spaces at p, then the only non-trivial layers in the tower are the ones numbered by powers of p. Thus the tower converges exponentially faster than usual in this case.

Theorem 1.3.5 was first proved in [14], by a brute force calculation of the homology with mod p coefficients. A more conceptual proof was given in [3]. Furthermore, when $n = p^k$, it turns out that the spectrum $(\partial_n I \wedge X^{\wedge n})_{h\Sigma_n}$ is closely related to well-studied spectra in the literature (remember that X is an odd sphere throughout the discussion). In particular, this spectrum is equivalent to (the k-fold desuspension of) the direct summand of the spectrum $\Sigma^\infty X^{\wedge p^k}_{h(\mathbb{Z}/p)^k}$ split off by the Steinberg idempotent. Such wedge summands were studied by Mitchell, Kuhn, Priddy and others. In the case $X = S^1$, this spectrum is equivalent (up to a suspension) to the n-th subquotient in the filtration of $H\mathbb{Z}$ by symmetric powers of the sphere spectrum.

It turns out that Serre's theorem on rational homotopy groups of spheres (corollary 1.3.4) admits a rather dramatic generalization to v_k-periodic homotopy for all k (rational homotopy fits in as the case $k = 0$). We will now state the result, and then spend the rest of the section explaining what it says and outlining its proof.

Theorem 1.3.6. [14] *Let X be an odd-dimensional sphere, and work p-locally for a prime p. For $k \geq 0$, the map $X \mapsto P_{p^k} I(X)$ is a v_k-periodic equivalence.*

We will now take a detour to review some background on v_k-periodic homotopy. For an omnibus reference on this material we suggest Ravenel's orange book [73] (a revised version of which is available online).[4] Fix a prime p and let all spaces be implicitly localized at p. Homology and cohomology groups will be taken with mod p coefficients. Recall that for each integer $n \geq 0$ there is a generalized homology theory $K(n)$ called the n-th Morava K-theory. For $n = 0$, $K(0) = H\mathbb{Q}$, and for $n > 0$ the coefficient ring of $K(n)$ is $K(n)_* = \mathbb{F}_p[v_n, v_n^{-1}]$, where $|v_n| = 2p^n - 2$.

Definition 1.3.7. A finite complex V is said to be of *type* k if $K(n)_*V$ is trivial for $n < k$ and non-trivial for $n = k$.

[4] See also the chapters in this Handbook by Gijs Heuts and by Tobias Barthel and Agnès Beaudry.

By the Periodicity Theorem [51] [73, Theorem 1.5.4], for each $k \geq 0$ there exists a finite complex of type k. Furthermore, suppose $k \geq 1$, and V_k is a complex of type k. Then there exists a self-map $f\colon \Sigma^{d|v_k|+i} V_k \to \Sigma^i V_k$ for some $i \geq 0, d \geq 1$, whose effect on $K(n)_*$ is an isomorphism for $n = k$ and zero for $n > k$. A map with these properties is called a v_k-periodic map. By the uniqueness part of the Periodicity Theorem, any two v_k-periodic self-maps of V_k are equivalent after taking some suspensions and iterations.

Suppose X is either a pointed space or a spectrum, and V is a finite complex with a basepoint. If X is a space, let X^V denote the space of basepoint-preserving maps from V to X. If X is a spectrum, then let X^V denote the mapping spectrum from $\Sigma^\infty V$ to X. Clearly, if X is a spectrum then $\Omega^\infty(X^V) \cong (\Omega^\infty X)^V$. Let V_k be a complex of type k. By replacing V_k with some suspension thereof if necessary, we may assume that V_k has a self-map of the form $\Sigma^{d|v_k|} V_k \to V_k$. This map induces a map $X^{V_k} \to \Omega^{d|v_k|} X^{V_k}$, where X is still either a pointed space or a spectrum. We can form a mapping telescope as follows

$$v_k^{-1} X^{V_k} := \operatorname{hocolim}\left(X^{V_k} \to \Omega^{d|v_k|} X^{V_k} \to \Omega^{2d|v_k|} X^{V_k} \to \cdots \right).$$

The mapping telescope serves as the definition of $v_k^{-1} X^{V_k}$. If X is a space, then $v_k^{-1} X^{V_k}$ is, by definition, a space, that is easily seen to be an infinite loop space. If X is a spectrum then $v_k^{-1} X^{V_k}$ is a spectrum. Clearly, if X is a spectrum then

$$\Omega^\infty(v_k^{-1} X^{V_k}) \simeq v_k^{-1} (\Omega^\infty X)^{V_k}.$$

We define the v_k-periodic homotopy groups of X with coefficients in V_k as follows.

$$v_k^{-1}\pi_*(X; V_k) := \pi_*\left(v_k^{-1} X^{V_k}\right) \cong \operatorname{colim}(\pi_*(X^{V_k}) \to \pi_{*+d|v_k|}(X^{V_k}) \to \cdots).$$

As before, the groups $v_k^{-1}\pi_*(X; V_k)$ are defined if X is either a space or a spectrum. It is easy to see that if X is a spectrum then there is a canonical isomorphism

$$v_k^{-1}\pi_*(X; V_k) \cong v_k^{-1}\pi_*(\Omega^\infty X; V_k). \tag{1.3.1}$$

The groups $v_k^{-1}\pi_*(X; V_k)$ are periodic with period that divides $d|v_k|$. They depend on the choice of V_k, but by the uniqueness statement above they do not depend on the self-map of V_k (up to a dimension shift).

While the groups $v_k^{-1}\pi_*(X; V_k)$ depend on a choice of V_k, the following is a well-known consequence of the Thick Subcategory Theorem [51].

Proposition 1.3.8. *Let $f\colon X \to Y$ be a map of spaces. If there exists one complex V_k of type k for which f induces an isomorphism $v_k^{-1}\pi_*(X; V_k) \to v_k^{-1}\pi_*(Y; V_k)$ then f induces an isomorphism for every such V_k.*

We say that a map $X \to Y$ is a v_k-equivalence if it induces an isomorphism on v_k-periodic groups for some, and therefore every, choice of a complex V_k of type k. This explains the use of the term in Theorem 1.3.6.

A convenient subclass of complexes of type k are ones that are *strongly of type* k. They are defined by certain freeness properties of the Steenrod algebra action, and the requirement that the Atiyah-Hirzebruch spectral sequence for Morava K-theory collapses. For a precise definition see [73, Definition 6.2.3]. Proposition 1.3.9 summarizes the relevant facts about complexes that are strongly of type k. Recall that the Adams spectral sequence has the following form, where E and F are spectra

$$\mathrm{Ext}^{s,t}_{\mathcal{A}}(H^*(F), H^*(E)) \Rightarrow \pi_{t-s}(\mathrm{Map}(E,F)^\wedge_p).$$

Proposition 1.3.9. [73, Sections 6.2–6.4] *For every k there exists a complex strongly of type k. Let V_k be such a complex. After some suspension, V_k has a v_k-periodic self-map $\Sigma^{d|v_k|}V_k \to V_k$ whose stabilization is represented in the second page of the Adams spectral sequence by an element of $\mathrm{Ext}^{d,d|v_k|+d}_{\mathcal{A}}(H^*(V_k), H^*(V_k))$. In particular, the self-map has Adams filtration d.*

It is often convenient to bigrade the Adams spectral sequence by $(t-s, s)$, so that the horizontal axis corresponds to the topological degree and the vertical axis is the Adams filtration. Vanishing lines in the spectral sequence are calculated in these coordinates. Thus a vanishing line gives an upper bound on the possible Adams filtration of a non-zero element of the homotopy groups of a spectrum in terms of its topological dimension. If one uses the grading $(t-s, s)$ then Proposition 1.3.9 says the self-map of V_k is represented by multiplication by an element on a line of slope $\frac{1}{|v_k|}$ and intercept zero.

Let $\mathrm{End}(\Sigma^\infty V_k) \simeq \Sigma^\infty V_k \wedge D(\Sigma^\infty V_k)$ be the endomorphism spectrum of $\Sigma^\infty V_k$. The self-map $\Sigma^{d|v_k|}V_k \to V_k$ gives rise to an element of $\pi_{d|v_k|}(\mathrm{End}(\Sigma^\infty V_k))$. Proposition 1.3.9 says that this element has Adams filtration d. It follows that if E is any spectrum, then the induced map $E^{V_k} \to \Omega^{d|v_k|}E^{V_k}$ also has Adams filtration at least d. In fact, a slightly stronger statement is true: the self-map of E^{V_k} is represented in Adams spectral sequence by an operation that raises topological degree by $d|v_k|$ and raises Adams filtration by d. In other words, it moves elements in the Adams spectral sequence for E^{V_k} along a line of slope $\frac{1}{|v_k|}$. It follows that if the Adams spectral sequence for E^{V_k} has a vanishing line of slope smaller than $\frac{1}{|v_k|}$ then the action of the self-map of V_k on the homotopy groups of E^{V_k} is nilpotent.

It turns out that layers in the Goodwillie tower of the identity have good vanishing lines. So now it is time to get back to the Goodwillie tower.

We remind the reader that X is an odd sphere and everything is localised at a prime p. By Theorem 1.3.5, the only non-trivial layers of the Goowillie tower of the identity at X are the ones indexed by powers of p. Their underlying spectra have the form $(\partial_{p^k}I \wedge X^{\wedge p^k})_{h\Sigma_{p^k}}$. It turns out that these spectra have interesting properties in v_k-periodic homotopy. Our way to see it is via their cohomology with the Steenrod action. For an integer $k \geq 0$ let A_k be the subalgebra of the Steenrod algebra generated by $\{Sq^1, Sq^2, Sq^4, \ldots, Sq^{2^k}\}$ for $p=2$ and by $\{\beta, P^1, P^p, \ldots, P^{p^{k-1}}\}$ for $p>2$.

Proposition 1.3.10. *The cohomology of $(\partial_{p^k}I \wedge X^{\wedge p^k})_{h\Sigma_{p^k}}$ is free over A_{k-1}.*

Again, this proposition was first proved in [14] by a brute force computation. The cohomology of the spectrum $(\partial_{p^k}I \wedge X^{\wedge p^k})_{h\Sigma_{p^k}}$ was calculated explicitly, and it was observed that in the case $X = S^1$ it is isomorphic (up to degree shift) to the cohomology of the quotient of symmetric product spectra

$$\mathrm{Sp}^{p^k}(S^0)/\mathrm{Sp}^{p^k-1}(S^0).$$

The cohomology of the latter is A_{k-1}-free by a theorem of Welcher [83], and Welcher's argument can be adapted to the case of a more general X.

It was pointed out to us by Nick Kuhn that there is a more direct way to deduce proposition 1.3.10 from Welcher's result. We know from [3] that the spectrum $(\partial_{p^k} I \wedge S^{p^k})_{h\Sigma_{p^k}}$ is homotopy equivalent (up to a suspension) to

$$\mathrm{Sp}^{p^k}(S^0)/\mathrm{Sp}^{p^k-1}(S^0),$$

and for a general odd sphere X, the spectrum $(\partial_{p^k} I \wedge X^{\wedge p^k})_{h\Sigma_{p^k}}$ is (roughly speaking) a Thom spectrum over the case $X = S^1$. Thus there is a Thom isomorphism between the cohomologies of the two spectra:

$$H^*\left(\left(\partial_{p^k} I \wedge S^{p^k}\right)_{h\Sigma_{p^k}}\right) \cong H^{*+2lp^k}\left(\left(\partial_{p^k} I \wedge S^{(2l+1)p^k}\right)_{h\Sigma_{p^k}}\right).$$

Furthermore, one can show that A_{k-1} acts trivially on the Thom class. This can be done by identifying the Thom class with a power of the top Dickson invariant, and using the formulas in [84]. From here it follows that in our case the Thom isomorphism respects the A_{k-1}-module structure.

Remark 1.3.11. It seems likely that Proposition 1.3.10 can also be proved by adapting the methods of Steve Mitchell [69].

Proposition 1.3.10 has consequences regarding the v_k-periodic homotopy thanks to the following theorem, due to Anderson-Davis [2] for $p = 2$ and Miller-Wilkerson [68] for $p > 2$.

Theorem 1.3.12. *Let M be a connected A-module that is free over A_k. Then $\mathrm{Ext}_A(M, \mathbb{F}_p)$ has a vanishing line of slope that is strictly smaller than $\frac{1}{|v_k|}$ and an intercept that is bounded above by a number that depends only on k. This bound is a slowly growing function of k.*

Corollary 1.3.13. *Suppose E is a spectrum for which $H^*(E)$ is free over A_{l-1}. Then E and $\Omega^\infty E$ have trivial v_k-periodic homotopy for $k < l$.*

Proof. It is enough to show that for a complex V_k that is strongly of type k, $v_k^{-1}\pi_*(E; V_k) = 0$. Since $H^*(E)$ is free over A_{l-1}, it follows that $H^*(E^{V_k})$ is free over A_{l-1}. By theorem 1.3.12, the Adams spectral sequence for $\pi_*(E^{V_k})$ has a vanishing line of slope smaller than $\frac{1}{|v_{l-1}|}$, and therefore smaller than $\frac{1}{|v_k|}$. By proposition 1.3.9 and subsequent comments, multiplication by a self-map of V_k is represented, on the level of second page of the Adams spectral sequence, by multiplication by an element on a line of slope $\frac{1}{|v_k|}$, which is larger than the slope of the vanishing line. Therefore, every element of $\pi_*(E^{V_k})$ is annihilated by some power of the self-map of V_k. It follows that inverting the self-map kills everything in the homotopy groups. □

Corollary 1.3.14. *If X is an odd sphere, then the only layers of Goodwillie tower of the identity evaluated at X that are non-trivial in v_k-periodic homotopy are $D_1 I(X), D_p I(X), D_{p^2} I(X), \ldots, D_{p^k} I(X)$.*

Proof. Everything is localized at p, and by theorem 1.3.5, $D_n I(X)$ is non-trivial at p only if n is a power of p. We need to show that if $l > k$ then $D_{p^l} I(X)$ is trivial in v_k-periodic homotopy. Recall that

$$D_{p^l} I(X) \simeq \Omega^\infty \left(\left(\partial_{p^l} I \wedge S^{p^l} \right)_{h\Sigma_{p^l}} \right).$$

By Proposition 1.3.10, $H^* \left(\left(\partial_{p^l} I \wedge S^{p^l} \right)_{h\Sigma_{p^l}} \right)$ is free over $\mathcal{A}_{l-1}$. The result follows by Corollary 1.3.13. $\square$

Corollary 1.3.14 suggests that the map $X \to P_{p^k} I(X)$ induces an isomorphism on v_k-periodic homotopy groups. This conclusion is not automatic, because inverting v_k does not always commute with inverse homotopy limits, but it turns out to be true in this case. So we can now sketch the proof of Theorem 1.3.6.

Proof of Theorem 1.3.6. Let k be fixed. Let $F_k(X)$ be the homotopy fibre of the map $X \to P_{p^k} I(X)$. For $n \geq k$ let us define $\overline{P_{p^n}}(X)$ to be the homotopy fibre of the map $P_{p^n} I(X) \to P_{p^k} I(X)$. Note that the homotopy fibre of the map $\overline{P_{p^n}}(X) \to \overline{P_{p^{n-1}}}(X)$ is $D_{p^n} I(X)$. Choose a complex V_k that is strongly of type k with a self-map $\nu_k \colon \Sigma^{d|v_k|} V_k \to V_k$ of Adams filtration d, as per proposition 1.3.9. There is a tower of the following form, where $F_k(X)^{V_k}$ is the homotopy inverse limit of the top row:

$$\begin{array}{ccccccc} F_k(X)^{V_k} \cdots \to & \overline{P_{p^n}}(X)^{V_k} & \to & \overline{P_{p^{n-1}}}(X)^{V_k} & \longrightarrow \cdots \longrightarrow & \overline{P_{p^{k+1}}}(X)^{V_k} \\ & \uparrow & & \uparrow & & \uparrow\sim \\ & D_{p^n} I(X)^{V_k} & & D_{p^{n-1}} I(X)^{V_k} & & D_{p^{k+1}} I(X)^{V_k}. \end{array}$$

We need to prove that $v_k^{-1}\pi_*(F_k; V_k)$ is zero. Let $\alpha \in \pi_*(F_k^{V_k})$. We need to show that some power of the self-map ν_k annihilates α. Let us say that α has height $\geq m$ if the image of α in $\pi_* \left(\overline{P_{p^n}}^{V_k} \right)$ is zero for $n < m$. Suppose α has height $\geq m$. Then the image of α in $\pi_* \left(\overline{P_{p^m}}^{V_k} \right)$ is in the image of some element of $\pi_* \left(D_{p^m} I(X)^{V_k} \right)$. We know that ν_k acts along a line of higher slope than the vanishing line of the Adams spectral sequence for the underlying spectrum of $D_{p^m} I(X)^{V_k}$. Because of this, some power ν_k^l of ν_k annihilates this element of $\pi_* \left(D_{p^m} I(X)^{V_k} \right)$. This means that $v_k^l(\alpha)$ has height $\geq m+1$. Repeating the process, one finds that for every $n > k$, there exists some l_n for which $v_k^{l_n}$ has height $\geq n$. Let l_n be the lowest such l. Using Theorem 1.3.12, it is not very difficult to write down an effective upper bound on the topological dimension of $\nu_k^{l_n}(\alpha)$ as a function of n. Crucially, it is not very difficult to show that this function grows slower than the connectivity of $D_{p^n} I(X)^{V_k}$ (the calculation is done in [14]). It follows that there exists an n for which $\nu_k^{l_n}(\alpha)$ has lower topological dimension than the connectivity of $D_{p^n} I(X)^{V_k}$. It follows that the image of $\nu_k^{l_n}(\alpha)$ in $\pi_* \left(\overline{P_{p^m}}^{V_k} \right)$ is zero for all m. It follows that $\nu_k^{l_n}(\alpha) = 0$. $\square$

Theorem 1.3.6 says that the unstable v_k-periodic homotopy type of an odd sphere can be resolved into $(k+1)$ stable homotopy types. The case $k = 0$ is essentially 1.3.4.

A Bousfield-Kuhn functor reformulation

Theorem 1.3.6 has a more modern reformulation in terms of the Bousfield-Kuhn functor. Let us recall what this functor is. Choose a v_k self-map $\Sigma^{d|v_k|}V_k \to V_k$. Use it to form a direct system of spectra

$$\Sigma^\infty V_k \to \Omega^{d|v_k|}\Sigma^\infty V_k \to \cdots.$$

Let $T(k)$ be the homotopy colimit of this system, and $L_{T(k)}$ be the Bousfield localization with respect to $T(k)$. It follows from the uniqueness part of the Periodicity Theorem that the functor $L_{T(k)}$ does not depend on the choice of V_k or the self-map.

The Bousfield-Kuhn functor Φ_k is a functor from pointed spaces to spectra, whose main property is that there is an equivalence $\Phi_k(\Omega^\infty E) \simeq L_{T(k)}(E)$. Here E is any spectrum, the equivalence is natural in E. The functor Φ_k is constructed as an inverse homotopy limit

$$\Phi_k(X) = \mathrm{holim}_\alpha\, v_k^{-1} X^{V_k^\alpha},$$

where $\{V_k^\alpha\}$ is a direct system of complexes of type k with certain properties. See [59] for more details. The following is now an immediate corollary of Theorem 1.3.6.

Theorem 1.3.15. *When X is an odd sphere, the map $X \to P_{p^k}I(X)$ becomes an equivalence after applying the Bousfield-Kuhn functor Φ_k.*

The case of even-dimensional spheres, and beyond

There is a version of Theorems 1.3.6 and 1.3.15 that holds for even-dimensional spheres:

Theorem 1.3.16. [14, Theorems 4.4 and 4.5] *Fix a prime p and localize everything at p. Let X be an even-dimensional sphere. Then $D_nI(X) \simeq *$ unless n is a power of p or twice a power of p. The map*

$$X \to P_{2p^k}I(X)$$

is a v_k-periodic equivalence.

The easiest way to prove this theorem that we know is to use the EHP sequence.

$$X \mapsto \Omega\Sigma X \to \Omega\Sigma(X \wedge X).$$

This is a fibration sequence if X is an odd-dimensional sphere, and one can show that it induces a fibration sequence of Taylor towers. Given this, it is easy to deduce Theorem 1.3.16 from Theorem 1.3.6.

Remark 1.3.17. The connection between the Goodwillie tower and the EHP sequence was investigated much more deeply by M. Behrens at [25].

Theorems 1.3.6 and 1.3.16 tell us that the behavior of the Taylor tower of the identity in v_k-periodic homotopy has two non-obvious properties when evaluated at spheres:

1. It is finite.

2. It converges.

It is therefore reasonable to ask if there exist other spaces for which the Taylor tower of the identity has these properties. Regarding the finiteness property, it seems clear that spheres (or at best homology spheres) are the only spaces for which the Taylor tower is finite in v_k-periodic homotopy. For example, it is easy to show that the only spaces for which the tower is rationally finite are rational homology spheres. On the other hand, there do exist other spaces for which the tower converges in v_k-periodic homotopy. Such results were obtained by Behrens-Rezk [26, section 8] and by Heuts [50]. For example, convergence holds for products of spheres and special unitary groups $SU(k)$. It would be interesting to characterize the spaces for which v_k-periodic convergence holds.

The case $X = S^1$. Theorems of Behrens and Kuhn

We noted above that there is a relationship between the layers of the Taylor tower of the identity evaluated at S^1, and the subquotients of the filtration of the Eilenberg-Mac Lane spectrum $H\mathbb{Z} = \mathrm{Sp}^\infty(S^0)$ by the symmetric powers of the sphere spectrum. More precisely, if n is not a power prime, then both layers are trivial, and if $n = p^k$ then there is an equivalence

$$D_{p^k} I(S^1) \simeq \Omega^{\infty+2k-1}\mathrm{Sp}^{p^k}(S^0)/\mathrm{Sp}^{p^k-1}(S^0).$$

In fact, there is a deeper connection between the two filtrations. Roughly speaking, one can recast each filtration as a "chain complex" of infinite loop spaces or spectra, and each chain complex can serve as a kind of contracting homotopy for the other one. This implies, in particular, that each filtration is trivial in the sense that its homotopy spectral sequence collapses at the second page. The triviality result for the symmetric powers filtration used to be known as the Whitehead conjecture. It was proved by Kuhn [55] at the prime 2 and by Kuhn and Priddy [61] at odd primes. The triviality result for the Goodwillie tower of the identity, and the connection between the two filtrations was proved by Behrens [24] at the prime 2 and in another way by Kuhn [60] at all primes.

1.4 Operads and Tate data: the Classification of Taylor towers

The layers of a Taylor tower (say, of a functor $F : \mathcal{T}op_* \to \mathcal{T}op_*$) are homogeneous functors classified by the spectra $\partial_* F = (\partial_n F)_{n \geq 1}$, along with the action of Σ_n on $\partial_n F$, i.e. a *symmetric sequence* of spectra. As we have seen, however, further information is needed to encode the full Taylor tower, and hence under convergence conditions, the functor F itself.

Roughly speaking, there are two approaches to understanding this extra information: (1) inductive techniques based on the delooped fibre sequences

$$P_n F \to P_{n-1} F \to \Omega^{-1} D_n F;$$

and (2) analysis of operad/module structures on the symmetric sequence $\partial_* F$ in its entirety, based on a chain rule philosophy for the calculus of functors.

We saw approach (1) applied to $P_2 I$ in Example 1.2.5 and, for functors from spectra to spectra, in McCarthy's Theorem (1.2.9). Here we focus on (2).

Chain Rules in Functor Calculus

The first version of a "Chain Rule" for the calculus of functors was proved by Klein and Rognes in [54]. The simplest version of this states the following: for reduced functors $F, G : \mathcal{T}op_* \to \mathcal{T}op_*$, we have

$$\partial_1(FG) \simeq \partial_1 F \wedge \partial_1 G.$$

More generally, Klein and Rognes provided a formula for the first derivative of FG at an arbitrary space X, in terms of the first derivatives of G (at X) and F (at $G(X)$).

For higher derivatives, the Chain Rule for functors of spectra is much simpler than that for spaces. Suppose first that $F, G : \mathcal{S}p \to \mathcal{S}p$ are given by the formulas

$$F(X) \simeq \bigvee_{k=1}^{\infty} (\partial_k F \wedge X^{\wedge k})_{h\Sigma_k}, \quad G(X) \simeq \bigvee_{l=1}^{\infty} (\partial_l G \wedge X^{\wedge l})_{h\Sigma_l}.$$

A simple calculation then shows that

$$\partial_n(FG) \simeq \bigvee_{\text{partitions of } \{1,...,n\}} \partial_k F \wedge \partial_{n_1} G \wedge \cdots \wedge \partial_{n_k} G$$

where $n_1, \ldots, n_k \geq 1$ are the sizes of the terms in a given partition. This is also the formula for the composition product of symmetric sequences, and, more succinctly, we can write

$$\partial_*(FG) \simeq \partial_* F \circ \partial_* G. \tag{1.4.1}$$

The second author proved in [35] that (1.4.1) holds for any reduced functors $F, G : \mathcal{S}p \to \mathcal{S}p$ where F preserves filtered colimits.

For functors of based spaces (or more general ∞-categories), the formula (1.4.1) does not hold, but there is nonetheless a natural map

$$l : \partial_* F \circ \partial_* G \to \partial_*(FG)$$

making ∂_*, at least up to homotopy, into a lax monoidal functor from a suitable ∞-category of functors to the ∞-category of symmetric sequences. Specializing the map l to the case when one or both of F, G is the identity, we obtain a number of important consequences. The following results were proved in [9] for functors between based spaces and spectra, and (with the exception of the final statement of the chain rule, though a version of this appears in [64, 6.3]) in [37] for arbitrary ∞-categories.

Theorem 1.4.1. *The derivatives $\partial_* I_{\mathcal{C}}$ of the identity functor on an ∞-category $\mathcal{C}$ have a canonical operad structure, and for an arbitrary functor $F : \mathcal{C} \to \mathcal{D}$, the derivatives $\partial_* F$ form a $(\partial_* I_{\mathcal{D}}, \partial_* I_{\mathcal{C}})$-bimodule. Moreover, if F preserves filtered colimits and $G : \mathcal{B} \to \mathcal{C}$ is reduced, then there is an equivalence (of bimodules):*

$$\partial_*(FG) \simeq \partial_* F \circ_{\partial_* I_{\mathcal{C}}} \partial_* G$$

where the right-hand side involves a (derived) relative composition product of bimodules over the operad $\partial_* I_{\mathcal{C}}$.

Remark 1.4.2. When $\mathcal{C} = \mathcal{T}op_*$, the ∞-category of based spaces, the operad $\partial_* I_{\mathcal{T}op_*}$ from Theorem 1.4.1 can be viewed as the analogue in stable homotopy theory of the operad encoding the structure of a Lie algebra. This perspective can be justified in a number of ways. Firstly, it follows from Johnson's calculation that taking homology groups of the spectra $\partial_* I_{\mathcal{T}op_*}$ recovers precisely the ordinary Lie operad (up to a shift in degree). A deeper connection is given by viewing $\partial_* I_{\mathcal{T}op_*}$ as an example of bar-cobar (or Koszul) duality for operads of spectra. Ginzburg and Kapranov developed the theory of bar-cobar duality for differential-graded operads in [43], and identified the Lie operad as the dual of the commutative cooperad. The following result was proved in [34] (or in the given form in [9]) and justifies viewing $\partial_* I_{\mathcal{T}op_*}$ as a version of the Lie operad.

Theorem 1.4.3. *The operad $\partial_* I_{\mathcal{T}op_*}$ is equivalent to the cobar construction, or (derived) Koszul dual, of the cooperad $\partial_*(\Sigma^\infty\Omega^\infty)$, which itself can be identified with the commutative cooperad of spectra.*

Remark 1.4.4. Theorem 1.4.3 can actually be understood quite easily from the point of view of calculus. For simply-connected X, the adjunction $(\Sigma^\infty, \Omega^\infty)$ determines an equivalence

$$X \xrightarrow{\simeq} \mathrm{Tot}(\Omega^\infty(\Sigma^\infty\Omega^\infty)^\bullet\Sigma^\infty X)$$

which connects the identity functor to a cobar construction on the comonad $\Sigma^\infty\Omega^\infty$. Taking derivatives of each side, and applying the chain rule for spectra from (1.4.1) we recover 1.4.3 with a little work. Similar arguments were used in [9] to understand the bimodule structures on the derivatives of functors to/from based spaces.

Remark 1.4.5. A version of Theorem 1.4.3 seems likely to be valid for the identity functor on an arbitrary $\mathcal{C}$. Lurie constructs in [64, 6.3.0.14] a cooperad that represents the derivatives $\partial_*(\Sigma^\infty_{\mathcal{C}}\Omega^\infty_{\mathcal{C}})$. (In Lurie's language, this object is actually a "stable corepresentable ∞-operad", but the connection with cooperads is explained in [64, 6.3.0.12].) A similar proof to that of 1.4.3 should imply that the cobar construction on this cooperad recovers the operad $\partial_* I_{\mathcal{C}}$.

Heuts's theorem on spectral Lie algebras in chromatic homotopy theory

The calculations of Arone and Mahowald [14] described above already show that the Taylor tower of the identity functor on based spaces has interesting structure when viewed through the lens of v_k-periodic homotopy theory. Recent work of Heuts [50] has further developed this connection, taking the Lie operad structure on $\partial_* I_{\mathcal{T}op_*}$ into account.

First recall that one of Quillen's models for rational homotopy theory is in terms of Lie algebras. Specifically, he constructs in [72] a Quillen equivalence between simply connected rational spaces and 0-connected differential-graded rational Lie algebras. Heuts's work extends Quillen's to higher chromatic height in the following way.

Theorem 1.4.6 (Heuts)**.** *Fix a prime p and positive integer k. Let $\mathcal{M}_k^f$ denote the ∞-category obtained from that of p-local based spaces by inverting those maps that induce an equivalence on v_k-periodic homotopy groups. Then there is an equivalence of ∞-categories*

$$\mathcal{M}_k^f \simeq \mathcal{S}p_{T(k)}(\partial_* I_{\mathcal{T}op_*})$$

between $\mathcal{M}_k^f$ and the category of $T(k)$-local spectra with an algebra structure over the operad $\partial_* I_{\mathcal{T}op_*}$. Moreover, to a space X this equivalence assigns a $\partial_* I_{\mathcal{T}op_*}$-algebra whose underlying spectrum is given by applying the Bousfield-Kuhn functor Φ_k to X.

Theorem 1.4.6 should be compared to Kuhn's Theorem 1.2.8 which also described a simplification to Goodwillie calculus that appears in the presence of chromatic localization. Both results illustrate the general principle that analysis in Goodwillie calculus typically comprises two pieces: (1) an operadic part related to the derivatives of the identity functor, and (2) something related to the Tate spectrum construction, which vanishes chromatically. Notice that in Kuhn's result, the operadic part is absent because the derivatives of the identity on $\mathcal{S}p$ form the trivial operad.

Tate data and the classification of Taylor towers

We have already seen in McCarthy's Theorem 1.2.9 that the extra information for reconstructing the Taylor tower of a functor $F : \mathcal{S}p \to \mathcal{S}p$ can be described in terms of Tate spectrum constructions for the Σ_n-action on $\partial_n F$. For functors $F : \mathcal{T}op_* \to \mathcal{T}op_*$, we instead think of the Taylor tower information being given by a combination of such "Tate data" with the $\partial_* I$-bimodule structure of Theorem 1.4.1.

To understand this perspective, we use the fact the functor ∂_* (when viewed with values in $\partial_* I$-bimodules) admits a right adjoint which we denote Ψ, i.e. there is an adjunction

$$\partial_* : \mathrm{Exc}_n^*(\mathcal{C}, \mathcal{D}) \rightleftarrows \mathrm{Bimod}_{\leq n}(\partial_* I_{\mathcal{D}}, \partial_* I_{\mathcal{C}}) : \Psi \tag{1.4.2}$$

between the ∞-category of reduced n-excisive functors $\mathcal{C} \to \mathcal{D}$, and the ∞-category of n-truncated bimodules over the derivatives of the identity on $\mathcal{C}$ and $\mathcal{D}$. (The right adjoint was denoted Φ in [10] but we use Ψ here to avoid confusion with the Bousfield-Kuhn functor.) In [10], we proved the following:

Theorem 1.4.7. *The adjunction (∂_*, Ψ) of (1.4.2) is comonadic. In particular, an n-excisive functor $F : \mathcal{C} \to \mathcal{D}$ is classified by the bimodule $\partial_* F$ together with an action of the comonad $\mathbf{C} := \partial_* \Psi$.*

Understanding the full structure on the derivatives of a functor $F : \mathcal{C} \to \mathcal{D}$ thus involves a calculation of the comonad $\mathbf{C}$ of Theorem 1.4.7. This is most easily described in the case of functors $F : \mathcal{T}op_* \to \mathcal{S}p$ where we obtain the following consequence.

Theorem 1.4.8. *The Taylor tower of $F : \mathcal{T}op_* \to \mathcal{S}p$ is determined by the right $\partial_* I$-module structure on $\partial_* F$ together with (suitably compatible) lifts $\psi_{n_1,\dots,n_k}$ of the form*

$$\begin{tikzcd} & \mathrm{Map}(\partial_{n_1} I \wedge \cdots \wedge \partial_{n_k} I, \partial_n F)_{h\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}} \arrow[d, "N"] \\ \partial_k F \arrow[r, "\phi_{n_1,\dots,n_k}"'] \arrow[ur, dotted, "\psi_{n_1,\dots,n_k}"] & \mathrm{Map}(\partial_{n_1} I \wedge \cdots \wedge \partial_{n_k} I, \partial_n F)^{h\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}} \end{tikzcd}$$

where N is the norm map from homotopy orbits to homotopy fixed points, and $\phi_{n_1,\dots,n_k}$ is the map associated to the right $\partial_* I$-module structure.

Such lifts make $\partial_* F$ into what we call a *divided power right $\partial_* I$-module.*

The data of the lifts $\psi_{n_1,\dots,n_k}$ in Theorem 1.4.8 can be reframed as choices of nullhomotopies for maps

$$\partial_k F \to \mathrm{Map}(\partial_{n_1} I \wedge \cdots \wedge \partial_{n_k} I, \partial_n F)^{t\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}}$$

into the corresponding Tate construction. We think of this as the *Tate data* corresponding to the Taylor tower of the functor F.

Problem 1.4.9. *It is still unclear how to describe the structure on the derivatives of a functor $F : \mathcal{T}op_* \to \mathcal{T}op_*$ as explicitly as that in Theorem 1.4.8. The comonad guaranteed by Theorem 1.4.7 is hard to understand in this case. In [10] we gave a concrete description of this structure for 3-excisive functors, but a more general picture was too elusive.*

Vanishing Tate data and applications

Another way to see how the Tate spectrum construction comes up is via the unit map $\eta : P_n F \to \Psi \partial_{\leq n} F$ of the adjunction (1.4.2). The right adjoint Ψ can be written in terms of mapping spaces for the ∞-category of bimodules. This leads to the existence of the following diagram for, for example, a functor $F : \mathcal{T}op_* \to \mathcal{T}op_*$:

$$\begin{array}{ccc} \Omega^\infty(\partial_n F \wedge X^{\wedge n})_{h\Sigma_n} & \xrightarrow{N} & \Omega^\infty(\partial_n F \wedge X^{\wedge n})^{h\Sigma_n} \\ \downarrow & & \downarrow \\ P_n F(X) & \xrightarrow{\eta} & \mathrm{Bimod}_{\partial_* I}(\partial_*(\mathrm{Hom}(X,-)), \partial_{\leq n} F) \\ \downarrow & & \downarrow \\ P_{n-1} F(X) & \xrightarrow{\eta} & \mathrm{Bimod}_{\partial_* I}(\partial_*(\mathrm{Hom}(X,-)), \partial_{\leq n-1} F) \end{array} \tag{1.4.3}$$

where the columns are fibration sequences, and $\mathrm{Bimod}_{\partial_* I}(-,-)$ denotes the space of maps of $\partial_* I$-bimodules between the given derivatives. Here $\partial_{\leq n} F$ denotes the *n-truncation* of a $\partial_* I$-bimodule, given by setting all terms in degree larger than n to be the trivial spectrum.

We think of (1.4.3) as a generalization of McCarthy's Theorem 1.2.9, and there is a similar picture for functors from spectra to spectra that reduces to McCarthy's result. For functors $F : \mathcal{T}op_* \to \mathcal{S}p$, we can similarly deduce the following version, which is from [10, 4.17].

Theorem 1.4.10. *Let $F : \mathcal{T}op_* \to \mathcal{S}p$ be a reduced functor that preserved filtered colimits. Then there is a pullback square of the form*

$$\begin{array}{ccc} P_n F(X) & \longrightarrow & (\partial_n F \wedge X^{\wedge n}/\Delta^n X)_{h\Sigma_n} \\ \downarrow & & \downarrow \\ P_{n-1} F(X) & \longrightarrow & (\partial_n F \wedge \Sigma \Delta^n X)_{h\Sigma_n} \end{array}$$

where $\Delta^n X$ is the fat diagonal inside $X^{\wedge n}$.

All the results mentioned here have similar consequences to McCarthy's Theorem in cases where the Tate spectra vanish. In such situations, the Taylor tower of a functor is completely determined by the relevant module or bimodule structure.

Proposition 1.4.11. *Suppose $F : \mathcal{T}op_* \to \mathcal{T}op_*$ preserves filtered colimits and X has the property that*

$$(\partial_n F \wedge X^{\wedge n})^{t\Sigma_n} \simeq *$$

for $2 \leq n \leq N$. Then

$$P_N F(X) \simeq \mathrm{Bimod}_{\partial_* I}(\partial_* \operatorname{Hom}(X, -), \partial_{\leq N} F).$$

For $F : \mathcal{T}op_* \to \mathcal{S}p$ satisfying the same assumptions

$$P_N F(X) \simeq \mathrm{RMod}_{\partial_* I}(\partial_* \Sigma^\infty \operatorname{Hom}(X, -), \partial_{\leq N} F)$$

where $\mathrm{RMod}_{\partial_* I}(-,-)$ is the mapping spectrum for the ∞-category of right $\partial_* I$-modules.

Since the Tate construction vanishes for any rational spectrum, we see that the hypothesis of Proposition 1.4.11 holds if either:

- $\partial_n F$ is a rational spectrum for $2 \leq n \leq N$; or
- X is a rational (simply-connected) topological space.

In particular, if F is a functor either to or from the category of rational based spaces, then the Taylor tower of F is given by the formula in Proposition 1.4.11.

Proposition 1.4.11 allows us to extend Kuhn's Theorem 1.2.8 to functors from $\mathcal{T}op_*$ to $\mathcal{S}p$.

Proposition 1.4.12. *Let F be a functor from $\mathcal{T}op_*$ to $T(n)$-local spectra. Then there is an equivalence*

$$P_N F(X) \simeq \mathrm{RMod}_{\partial_* I}(\mathbb{D}(X^{\wedge *}/\Delta^* X), \partial_{\leq N} F).$$

Similarly, if F is a functor from $\mathcal{T}op_*$ to $\mathcal{T}op_*$, and V_k is a finite complex of type k, then there is an equivalence

$$P_N v_k^{-1} F(X)^{V_k} \simeq \mathrm{RMod}_{\partial_* I}(\mathbb{D}(X^{\wedge *}/\Delta^* X), v_k^{-1} \partial_{\leq N} F^{V_k}).$$

The spectra $\mathbb{D}X^{\wedge *}/\Delta^* X$ form a right $\partial_* I$-module, by identifying them as the derivatives of the functor $\Sigma^\infty \operatorname{Hom}(X, -) : \mathcal{T}op_* \to \mathcal{S}p$.

It seems less straightforward to obtain an analogous result for functors from spaces to spaces than in the spectrum-valued case, yet Heuts's Theorem 1.4.6 simplifies this to the case of functors between spectral Lie algebras.

Functors from spaces to spectra and modules over the little disc operads

There is a close connection between the spectral Lie operad $\partial_* I$ and the little disc operads. Write E_n for the operad of spectra formed by taking suspension spectra of the terms in the little n-discs operad of May [65], and write KE_n for the (derived) Koszul dual of E_n in the sense of [36]. Then $\partial_* I$ can be expressed as the inverse limit of the sequence of Koszul duals:

$$\partial_* I \to \cdots \to KE_n \to KE_{n-1} \to \cdots \to KE_1.$$

It is conjectured that KE_n is equivalent as an operad to a desuspension of E_n itself. This is proved in the context of chain complexes by Fresse [42].

The connection between $\partial_* I$ and the sequence KE_n has powerful consequences. Any divided power module over KE_n determines, by pulling back along the map $\partial_* I \to KE_n$, a divided power module over $\partial_* I$. Moreover, the terms in the operad KE_n have free symmetric group actions, so the norm map for the Σ_k-action on $\mathrm{Map}(KE_n(k), X^{\wedge k})$ is an equivalence. It follows that any right KE_n-module has a unique divided power structure and hence determines a divided power $\partial_* I$-module.

This allows us to ask the following question: for a given functor $F : \mathcal{T}op_* \to \mathcal{S}p$, is the Taylor tower of F determined by a right KE_n-module structure on $\partial_* F$?

The following result of [11] gives a criterion for this to be the case.

Theorem 1.4.13. *A polynomial functor $F : \mathcal{T}op_* \to \mathcal{S}p$ is determined by a KE_n-module structure on $\partial_* F$ if and only if F is the left Kan extension of a functor $f\mathcal{M}an_n \to \mathcal{S}p$ along the inclusion $f\mathcal{M}an_n \to \mathcal{T}op_*$, where $f\mathcal{M}an_n$ is the subcategory of $\mathcal{T}op_*$ consisting of certain "pointed framed n-dimensional manifolds" (that is, one-point compactifications of framed n-manifolds) and "pointed framed embeddings".*

In the next section we will see an application of 1.4.13 to the Taylor tower of algebraic K-theory of spaces.

Remark 1.4.14. The category $f\mathcal{M}an_n$ in Theorem 1.4.13 is related to the "zero-pointed manifolds" of Ayala and Francis [18], and this result suggests deeper connections between Goodwillie calculus and factorization homology that are yet to be explored.[5]

1.5 Applications and calculations in algebraic K-theory

Much of Goodwillie's initial motivation for developing the calculus of functors came from algebraic K-theory, and aside from the identity functor, most of the calculations and applications of calculus have been in this area, largely by Randy McCarthy and coauthors. We review here what is known about the Taylor tower of algebraic K-theory both in the context of spaces and ring spectra.

Algebraic K-theory of spaces

Let $A : \mathcal{T}op \to \mathcal{S}p$ denote Waldhausen's functor calculating the algebraic K-theory of a topological space X, i.e. of the category of spaces over and under X; see [79]. For calculus to be at all relevant to the study of the functor A, we first have to know that the Taylor tower converges for some spaces X.

[5] See the chapter in this Handbook by David Ayala and John Francis for information about factorization homology.

Theorem 1.5.1 (Goodwillie [45]). *The functor $A : \mathcal{T}op \to \mathcal{S}p$ is* 1*-analytic. Thus the Taylor tower of A converges on simply connected spaces.*

Goodwillie's initial application of this result, however, was not to the convergence of the Taylor tower, but to the cyclotomic trace map τ from $A(X)$ to the *topological cyclic homology* $TC(X)$ of Bökstedt, Hsiang and Madsen [31].

Theorem 1.5.2 (Böksted, Carlsson, Cohen, Goodwillie, Hsiang, Madsen [30]). *For a simply connected finite CW-complex X, there is a pullback square*

$$\begin{array}{ccc} A(X) & \xrightarrow{\tau} & TC(X) \\ \downarrow & & \downarrow \\ A(*) & \xrightarrow{\tau} & TC(*). \end{array}$$

The main step in the proof of Theorem 1.5.2 is to show that the cyclotomic trace τ induces equivalences between the first derivatives of A and TC (at an arbitrary space X). The general theory of calculus then implies that the fibre of τ is "locally constant", i.e. takes a suitably connected map of spaces to an equivalence of spectra.

The first derivative of A at the space X can be thought of as a parametrized spectrum over the space X with fibre over $x \in X$ given by

$$\partial_1 A(X)_x \simeq \Sigma^\infty (\Omega_x X)_+.$$

In [46], Goodwillie generalizes this formula to higher derivatives. The n-th derivative is a spectrum parametrized over X^n and for simplicity, we will only describe the case $X = *$, where

$$\partial_n A \simeq \Sigma^\infty (\Sigma_n / C_n)_+ \tag{1.5.1}$$

and where C_n is an order n cyclic subgroup of the symmetric group Σ_n.

What can we now say about the Taylor tower of A? Firstly, a choice of basepoint for a space Y determines a splitting

$$A(Y) \simeq A(*) \times \tilde{A}(Y)$$

where $\tilde{A} : \mathcal{T}op_* \to \mathcal{S}p$ is the corresponding reduced functor. Next, Waldhausen's splitting result [79] implies that the 1-excisive approximation to A splits off too, so we have

$$A(Y) \simeq A(*) \times \Sigma^\infty Y \times \tilde{\mathrm{Wh}}(Y)$$

where $\tilde{\mathrm{Wh}}(Y)$ is the "reduced Whitehead spectrum" of Y, which contains all of the higher degree information from the Taylor tower of A.

The formula (1.5.1) implies the following simple calculation of the layers of the Taylor tower for A: for $n \geq 2$ we have (using the chosen basepoint for Y):

$$D_n A(Y) \simeq (\Sigma^\infty Y)^{\wedge n}_{hC_n}.$$

How are these layers attached to each other to form $\tilde{\mathrm{Wh}}(Y)$?

Recall that Theorem 1.4.13 gave us conditions for the Taylor tower of a functor F: $\mathcal{T}op_* \to \mathcal{S}p$ to be determined by a KE_n-module structure on $\partial_* F$. The following result of [11] applies this to the functor A.

Theorem 1.5.3. *The divided power module structure on $\partial_* A$, and hence the Taylor tower of A, is determined by a certain KE_3-module structure on $\partial_* A$.*

Note that the KE_3-module structure on $\partial_* A$ pulls back to a KE_n structure for any $n > 3$, but we do not know if in fact it comes from a KE_2-, or KE_1-module.

Problem 1.5.4. *Extract from the proof of Theorem 1.5.3 an explicit description of the KE_3-module structure on $\partial_* A$, and a corresponding construction of the Taylor tower of A.*

Algebraic K-theory of rings

For an A_∞-ring spectrum R, the K-theory spectrum $K(R)$ is defined as the algebraic K-theory of the subcategory of compact objects in the ∞-category of R-modules [41, VI.3.2]. Our goal in this section is to describe what is known about the Taylor towers of the functor K defined in this way.

It is easiest to describe results in the pointed case, i.e. for augmented algebras. Let us fix an A_∞-ring spectrum R, and let $\mathcal{A}lg_R^{\mathrm{aug}}$ denote the ∞-category of augmented R-algebras. We are interested in the Taylor tower (at the terminal object R of $\mathcal{A}lg_R^{\mathrm{aug}}$) of the functor

$$\tilde{K}_R : \mathcal{A}lg_R^{\mathrm{aug}} \to \mathcal{S}p$$

given by $\tilde{K}_R(A) := \mathrm{hofib}(K(A) \to K(R))$.

The first calculation of part of the Taylor tower of $\tilde{K}_R$ was made by Dundas and McCarthy in [40].

Theorem 1.5.5 (Dundas-McCarthy)**.** *Let R be the Eilenberg-Mac Lane spectrum of a discrete ring. Then*

$$\partial_1 \tilde{K}_R \simeq \Sigma\, \mathrm{THH}(R)$$

the topological Hochschild homology of R of Bökstedt.

To recover the first layer $D_1 \tilde{K}_R$ of the Taylor tower from Theorem 1.5.5, we need to know the stabilization of the category of augmented R-algebras. By work of Basterra and Mandell [23], this is equivalent to the category of R-bimodules. The suspension spectrum construction takes an augmented R-algebra A to its *topological André-Quillen homology* $\mathrm{taq}_R(A)$, a derived version of I/I^2 where I is the augmentation ideal of A. We then have

$$D_1 \tilde{K}_R(A) \simeq \Sigma\, \mathrm{THH}(R; \mathrm{taq}_R(A))$$

where $\mathrm{THH}(R; M) := R \wedge_{R \wedge R^{op}} M$ is the topological Hochschild homology with coefficients.

Lindenstrauss and McCarthy [62] have extended Theorem 1.5.5 to higher layers in the following way. The generalization to all connective ring spectra is due to Pancia [70].

Theorem 1.5.6 (Lindenstrauss-McCarthy, Pancia)**.** *Let R be a connective ring spectrum. Then for an R-bimodule A*

$$D_n \tilde{K}_R(A) \simeq \Sigma U^n(R; \mathrm{taq}_R(A))_{hC_n}$$

where $U^n(R;M)$ is a generalization of $\mathrm{THH}(R;M)$ *given by the cyclic tensoring of n copies of a bimodule M over R, with action of the cyclic group C_n given by permutation.*

The proof of Theorem 1.5.6 is a consequence of Lindenstrauss and McCarthy's calculation of the complete Taylor tower of the functor

$$\tilde{K}_R(T_R(-)) : \mathrm{Bimod}_R \to \mathcal{S}p$$

where $T_R : \mathrm{Bimod}_R \to \mathcal{A}lg_R^{\mathrm{aug}}$ is the free tensor R-algebra functor given by

$$T_R(M) := \bigoplus_{n \geq 0} M^{\otimes_R}.$$

It is shown in [62] that

$$P_n(\tilde{K}_R T_R)(M) \simeq \mathrm{holim}_{k \leq n} \Sigma U^k(R;M)^{C_k}$$

where the homotopy limit is formed over a diagram of restriction maps

$$U^k(R;M)^{C_k} \to U^l(R;M)^{C_l}$$

for positive integers l dividing k. It is reasonable to expect that more information about the Taylor tower of $\tilde{K}_R$ itself, beyond just the layers, can be extracted from this description, but this has not been done.

One way to approach this question is via the general theory of Section 1.4. According to Theorem 1.4.1, one would expect the derivatives of $\tilde{K}_R$ to be a right module over the derivatives of the identity functor on $\mathcal{A}lg_R^{\mathrm{aug}}$. Theorem 1.4.7 then tells us that the Taylor tower of $\tilde{K}_R$ is determined by the action of a certain comonad on that right module.

Interestingly, and unlike the case for topological spaces, the Taylor tower for the identity functor on $\mathcal{A}lg_R^{\mathrm{aug}}$ is rather easy to describe:

$$P_n I_{\mathcal{A}lg_R^{\mathrm{aug}}}(A) \simeq I/I^{n+1} \tag{1.5.2}$$

where the right-hand side is a derived version of the quotient of the augmentation ideal I by its $(n+1)$-th power, generalizing the construction of $\mathrm{taq}_R(A) \simeq I/I^2$. This formula seems to have been written down first by Kuhn [57], and a more formal version is developed by Pereira [71]. With this calculation it should now be possible to make more progress with the calculation of the Taylor tower of $\tilde{K}_R$.

Remark 1.5.7. One final remark on the connection between algebraic K-theory and Goodwillie calculus is worth making here. Barwick [22] identifies the process of forming algebraic K-theory itself as the first layer of a Taylor tower. He defines an ∞-category $\mathcal{W}ald_\infty$ of

Waldhausen ∞-categories that are ∞-category-theoretic analogues of Waldhausen's categories with cofibrations. He then identifies the algebraic K-theory functor as

$$K \simeq P_1(\iota)$$

where $\iota : \mathcal{W}ald_\infty \to \mathcal{T}op$ is the functor that sends a Waldhausen ∞-category to its underlying ∞-groupoid of objects. The 1-excisive property for K is a version of Waldhausen's Additivity Theorem, and this result identifies K-theory as the universal example of an additive theory on $\mathcal{W}ald_\infty$ together with a natural transformation from ι.

1.6 Taylor towers of infinity-categories

Recent work of Heuts has taken Goodwillie calculus in a new direction. In [49] he constructs, for each pointed compactly-generated ∞-category $\mathcal{C}$, a *Taylor tower of ∞-categories* for $\mathcal{C}$. This is a sequence of adjunctions

$$\mathcal{C} \rightleftarrows \cdots \rightleftarrows \mathcal{P}_n\mathcal{C} \rightleftarrows \mathcal{P}_{n-1}\mathcal{C} \rightleftarrows \cdots \rightleftarrows \mathcal{P}_1\mathcal{C} \tag{1.6.1}$$

where $\mathcal{P}_n\mathcal{C}$ is a universal approximation to $\mathcal{C}$ by an *n-excisive ∞-category*.

This approximation has the property that the identity functor on $\mathcal{P}_n\mathcal{C}$ is n-excisive, and both the unit and counit of the adjunction $\mathcal{C} \rightleftarrows \mathcal{P}_n\mathcal{C}$ are P_n-equivalences. In the case $n = 1$, we have $\mathcal{P}_1\mathcal{C} \simeq \mathcal{S}p(\mathcal{C})$, the stabilization of $\mathcal{C}$. Thus, the sequence above can be viewed as interpolating between $\mathcal{C}$ and its stabilization via ∞-categories that better and better approximate the potentially unstable ∞-category $\mathcal{C}$.

Heuts's main results concern the classification of n-excisive ∞-categories such as $\mathcal{P}_n\mathcal{C}$. Just as in the classification of Taylor towers of functors, this process is broken down into two parts: one *operadic* and the other related to the *Tate* construction.

The first part associates to the ∞-category $\mathcal{C}$ the *cooperad* constructed by Lurie that represents the derivatives of the functor $\Sigma^\infty_{\mathcal{C}}\Omega^\infty_{\mathcal{C}}$, as described in Remark 1.4.5. (Recall that this cooperad is actually an ∞-*operad* in Lurie's terminology.) The notation for this cooperad in [49] is $\mathcal{S}p(\mathcal{C})^{\otimes}$, but we will denote it by $\partial_*(\Sigma^\infty_{\mathcal{C}}\Omega^\infty_{\mathcal{C}})$.

Proposition 1.6.1. *Let $\mathcal{C}$ be a pointed compactly generated ∞-category. Then the suspension spectrum construction for $\mathcal{C}$ can be made into a functor*

$$\Sigma^\infty_{\mathcal{C}} : \mathcal{C} \to \mathrm{Coalg}(\partial_*(\Sigma^\infty_{\mathcal{C}}\Omega^\infty_{\mathcal{C}}))$$

where the right-hand side is the ∞-category of non-unital coalgebras over the cooperad $\partial_*(\Sigma^\infty_{\mathcal{C}}\Omega^\infty_{\mathcal{C}})$.

Example 1.6.2. For $\mathcal{C} = \mathcal{T}op_*$, the cooperad $\partial_*(\Sigma^\infty_{\mathcal{C}}\Omega^\infty_{\mathcal{C}})$ is the commutative cooperad in $\mathcal{S}p$. In this case the functor in Proposition 1.6.1 associates to a based space X the commutative coalgebra structure on $\Sigma^\infty X$ given by the diagonal on X.

The category of coalgebras described in Proposition 1.6.1 can be thought of as the best approximation to the ∞-category $\mathcal{C}$ based on the information in the cooperad $\partial_*(\Sigma^\infty_{\mathcal{C}}\Omega^\infty_{\mathcal{C}})$.

The main focus of [49] is an understanding of the additional information needed to recover $\mathcal{C}$ itself (or at least its Taylor tower) from this category of coalgebras.

Heuts identifies $\mathcal{P}_n\mathcal{C}$ with an ∞-category of *n-truncated Tate coalgebras* over the cooperad $\partial_*(\Sigma^\infty_{\mathcal{C}}\Omega^\infty_{\mathcal{C}})$. We do not have space here to provide a precise definition of Tate coalgebra, but we can give the general idea. Full details are in [49].

An n-truncated Tate coalgebra over a cooperad $\mathcal{Q}$ consists first of an ordinary (truncated) $\mathcal{Q}$-coalgebra structure on an object E in $\mathcal{P}_1\mathcal{C} \simeq \mathcal{S}p(\mathcal{C})$. That is, for $k \leq n$, we have structure maps of the form

$$c_k : E \to [\mathcal{Q}(k) \wedge E^{\wedge k}]^{h\Sigma_k}.$$

Each map c_k is then required to be compatible (via the canonical map from fixed points to the Tate construction) with a certain natural transformation

$$t_k : E \to [\mathcal{Q}(k) \wedge E^{\wedge k}]^{t\Sigma_k},$$

that Heuts calls the *Tate diagonal* for $\mathcal{C}$.

Crucially, the natural transformation t_k is defined only when E is already a $(k-1)$-truncated Tate $\mathcal{Q}$-coalgebra, and t_k must be compatible, via the cooperad structure on $\mathcal{Q}$, with the maps $c_1, \dots, c_{k-1}$. Also note that the Tate diagonal t_k depends on the ∞-category $\mathcal{C}$ and not just the cooperad $\mathcal{Q}$. It is the choice of these maps that carries the extra information needed to reconstruct the Taylor tower of $\mathcal{C}$.

We can now state Heuts's main result from [49].

Theorem 1.6.3. *Let $\mathcal{C}$ be a pointed compactly generated ∞-category with $\mathcal{Q} := \partial_*(\Sigma^\infty_{\mathcal{C}}\Omega^\infty_{\mathcal{C}})$, the corresponding cooperad in the stable ∞-category $\mathcal{S}p(\mathcal{C})$. Then there is a unique sequence of Tate diagonals, i.e. natural transformations*

$$t_k : E \to [\mathcal{Q}(k) \wedge E^{\wedge k}]^{t\Sigma_k}$$

where t_k is defined for $E \in \mathcal{P}_{k-1}\mathcal{C}$, such that $\mathcal{P}_k\mathcal{C}$ is equivalent to the ind-completion of the ∞-category of k-truncated Tate $\mathcal{Q}$-coalgebras whose underlying object of $\mathcal{S}p(\mathcal{C})$ is compact.

Note that this is intrinsically an inductive result: the construction of $\mathcal{P}_{k-1}\mathcal{C}$ needs to be made in order to understand the Tate diagonal t_k that allows for the definition of $\mathcal{P}_k\mathcal{C}$. Here is how this process plays out for based spaces.

Example 1.6.4. We have $\mathcal{P}_1\mathcal{T}op_* \simeq \mathcal{S}p$ and $\partial_*(\Sigma^\infty\Omega^\infty) \simeq \mathcal{C}om$, the commutative cooperad of spectra with $\mathcal{C}om(n) \simeq S^0$ for all n. The first Tate diagonal has no compatibility requirements and is simply a natural transformation

$$t_2 : Y \to (Y \wedge Y)^{t\Sigma_2}$$

defined for all $Y \in \mathcal{S}p$.

Since both the source and target are 1-excisive functors, such a natural transformation is determined by its value on the sphere spectrum, where it takes the form of a map

$$t_2 : S^0 \to (S^0)^{t\Sigma_2} \simeq (S^0)^\wedge_2$$

whose target is the 2-complete sphere by the Segal Conjecture [33]. The specific Tate diagonal that corresponds to the 2-excisive ∞-category $\mathcal{P}_2\mathcal{T}op_*$ is then the ordinary 2-completion map $S^0 \to (S^0)^\wedge_2$.

According to Theorem 1.6.3, a (compact) object of $\mathcal{P}_2\mathcal{T}op_*$ consists of a finite spectrum Y with map

$$c_2 : Y \to (Y \wedge Y)^{h\Sigma_2}$$

and a homotopy between t_2 and the composite

$$Y \to (Y \wedge Y)^{h\Sigma_2} \to (Y \wedge Y)^{t\Sigma_2}.$$

Now consider a Tate diagonal for $\mathcal{P}_2\mathcal{T}op_*$. This should be a natural transformation

$$t_3 : Y \to (Y^{\wedge 3})^{t\Sigma_3}$$

that is compatible with the map

$$Y \to \left(\prod_3 Y \wedge (Y \wedge Y) \right)^{t\Sigma_3}$$

that is built from iterating c_2, where the three copies of $Y \wedge (Y \wedge Y)$ are indexed by the three binary trees with leaves labelled $\{1, 2, 3\}$, and composing with the map from fixed points to the Tate spectrum. As in the $n = 2$ case, there is a specific such natural transformation t_3 that corresponds to $\mathcal{P}_3\mathcal{T}op_*$ though it doesn't seem to be as easy to describe explicitly for general $Y \in \mathcal{P}_2\mathcal{T}op_*$. (For objects Y that are of the form $\Sigma^\infty X$ for a space X, t_3 is induced by the diagonal on X.)

An object of $\mathcal{P}_3\mathcal{T}op_*$ then consists of an object of $\mathcal{P}_2\mathcal{T}op_*$ together with a map

$$c_3 : Y \to (Y^{\wedge 3})^{h\Sigma_3}$$

that lifts t_3 and is compatible with c_2.

As in the case of Taylor towers of functors, a substantial simplification occurs when the Tate data vanishes, in which case the Taylor tower of the ∞-category $\mathcal{C}$ is completely determined by the cooperad $\partial_*(\Sigma^\infty_{\mathcal{C}}\Omega^\infty_{\mathcal{C}})$.

Corollary 1.6.5. [49, 6.14] *Let $\mathcal{C}$ be an ∞-category such that for any object $X \in \mathcal{S}p(\mathcal{C})$ with Σ_k-action, the Tate construction $X^{t\Sigma_k}$ is trivial, for all $k \geq 2$. Then $\mathcal{P}_n\mathcal{C}$ is equivalent to the ind-completion of the category of n-truncated $\partial_*(\Sigma^\infty_{\mathcal{C}}\Omega^\infty_{\mathcal{C}})$-coalgebras whose underlying object is compact in $\mathcal{S}p(\mathcal{C})$.*

Remark 1.6.6. The Tate diagonals for an ∞-category $\mathcal{C}$ bear a close relationship to the Taylor tower of the functor $\Sigma^\infty_{\mathcal{C}}\Omega^\infty_{\mathcal{C}} : \mathcal{S}p(\mathcal{C}) \to \mathcal{S}p(\mathcal{C})$. In particular t_n can be written as a composite

$$E \to P_{n-1}(\Sigma^\infty_{\mathcal{C}}\Omega^\infty_{\mathcal{C}})(E) \xrightarrow{t'_n} (\partial_n(\Sigma^\infty_{\mathcal{C}}\Omega^\infty_{\mathcal{C}}) \wedge E^{\wedge n})^{t\Sigma_n}$$

where the second map here is the bottom horizontal map in McCarthy's square (1.2.9), and the first map contains the structure of an $(n-1)$-truncated Tate coalgebra on E. The map on derivatives induced by t'_n can furthermore be identified with the coalgebra structure from Theorem 1.4.7 that classifies the Taylor tower of the functor $\Sigma^\infty_{\mathcal{C}}\Omega^\infty_{\mathcal{C}}$.

1.7 The manifold and orthogonal calculi

While this article has been focused on Goodwillie's calculus of homotopy functors, there are two other theories of "calculus" developed by Michael Weiss, that are inspired by, and related to, Goodwillie calculus to varying degrees. They are called *manifold calculus* (initially known as embedding calculus) and *orthogonal calculus*. In this section we give a review of these theories and some applications they have had.

Manifold calculus

Manifold calculus was initially developed in [81, 48], see also [29]. It concerns *contravariant* functors on manifolds. Of all the brands of calculus, this one is closest to classical sheaf theory, and therefore may look the most familiar.

Suppose M is an m-manifold. Let F be a presheaf of spaces on M. In other words, F is a contravariant functor $F\colon \mathcal{O}(M) \to \mathcal{T}op$, where $\mathcal{O}(M)$ is the poset of open subsets of M.[6] We assume that F takes isotopy equivalences to homotopy equivalences and filtered colimits to inverse limits. The motivating example is the presheaf $U \mapsto \mathrm{Emb}(U, N)$, where N is a fixed manifold, and $\mathrm{Emb}(-,-)$ denotes the space of smooth embeddings.

The notion of excisive functor in manifold calculus is analogous to that in Goodwillie's homotopy calculus. We say that a presheaf is *n-excisive* if it takes strongly cocartesian $(n+1)$-cubes in $\mathcal{O}(M)^{op}$ to cartesian cubes in $\mathcal{T}op$. For example, F is 1-excisive if for any open sets $U, V \subset M$, the following diagram is a (homotopy) pull-back square

$$\begin{tikzcd} F(U\cup V) \arrow[r] \arrow[d] & F(V) \arrow[d] \\ F(U) \arrow[r] & F(U\cap V). \end{tikzcd}$$

Thus we see that 1-excisive presheaves are, essentially, sheaves, except that the sheaf condition has to be interpreted in a homotopy invariant way. Sometimes 1-excisive functors are called homotopy sheaves.

Similarly, n-excisive functors can be interpreted as homotopy sheaves with respect to a different Grothendieck topology, where one says that $V_1, \ldots, V_k$ cover U if $U^n = V_1^n \cup \cdots \cup V_k^n$.

Just as in Goodwillie calculus, the inclusion of the category of n-excisive presheaves into the category of all presheaves has a left adjoint. The adjoint is constructed by a process of sheafification, rather than stabilization. There is another, slightly different procedure for constructing the approximation that is often useful. We will describe it now.

Definition 1.7.1. For an m-manifold U, let $\mathcal{O}_n(U) \subset \mathcal{O}(U)$ be the poset of open subsets of U that are diffeomorphic to a disjoint union of at most n copies of $\mathbb{R}^m$. Given a presheaf F on M, define a new presheaf T_nF by the formula

$$T_nF(U) = \mathrm{holim}_{V \in \mathcal{O}_n(U)} F(V).$$

[6]For concreteness we focus on presheaves in $\mathcal{T}op$, but it seems likely that the theory can be extended without difficulty to presheaves in a general $(\infty, 1)$-category.

Theorem 1.7.2 (Weiss [81])**.** *Assume that F takes isotopy equivalences to homotopy equivalences and filtered colimits to inverse homotopy limits. Then T_nF is n-excisive, and the natural transformation $F \to T_nF$ is initial among all natural transformations from F to an n-excisive functor.*

Thus T_nF is the universal n-excisive approximation of F. As in homotopy calculus, there are natural transformations $T_nF \to T_{n-1}F$. So the approximations fit into a "tower" of functors under F

$$F \to \cdots \to T_nF \to T_{n-1}F \to \cdots$$

sometimes called the *embedding tower* when F is $\mathrm{Emb}(-, N)$ for a manifold N.

Continuing the analogy with homotopy calculus, there is a classification theorem for homogeneous functors that can be used to describe the layers in the tower of approximations. Unlike in homotopy calculus, a homogeneous functor is not classified by a spectrum, but by a fibration over a space of configurations of points in M. We will give below a description of the homotopy fibre of the map $T_nF \to T_{n-1}F$ (for space-valued F).

Let $\binom{M}{n}$ be the space of unordered n-tuples of pairwise distinct points of M. Given a point $\mathbf{x} = [x_1, \ldots, x_n]$ of $\binom{M}{n}$, let $U_{\mathbf{x}}$ be a tubular neighborhood of $\{x_1, \ldots, x_n\}$ in M. In particular, $U_{\mathbf{x}} = \coprod_{i=1}^n U_i$ is diffeomorphic to a disjoint union of n copies of $\mathbb{R}^n$. We have an n-dimensional cubical diagram, indexed by subsets of $\{1, \ldots, n\}$ and opposites of inclusions, that sends a subset $S \subset \{1, \ldots, n\}$ to $F(\coprod_{i \in S} U_i)$. Let $\widehat{F}(U_{\mathbf{x}})$ be the total homotopy fibre of this diagram. One can naturally construct a fibred space over $\binom{M}{n}$ where the fibre at $\mathbf{x}$ is equivalent to $\widehat{F}(U_{\mathbf{x}})$.

Theorem 1.7.3 (Weiss [81])**.** *A choice of point in $T_{n-1}F(M)$ gives a germ of a section of this fibration near the fat diagonal. The homotopy fibre of $T_nF(M) \to T_{n-1}F(M)$ is equivalent to the space of sections of the fibration, that agree near the fat diagonal with the local section prescribed by the choice of point in $T_{n-1}F(M)$.*

Remark 1.7.4. Suppose $f \colon \mathbb{R} \to \mathbb{R}$ is a function. For each $n \geq 0$, let t_nf be the unique polynomial of degree n for which $f(i) = t_nf(i)$ for $i = 0, 1, \ldots, n$. There is a well known formula for t_nf. Namely $t_nf(m) = \sum_{i=0}^n \hat{f}(i) \cdot \binom{m}{i}$. Here $\hat{f}(i) = \sum_{j=0}^i (-1)^j f(i-j)$ is the i-th cross-effect of f. In particular, the n-th term of $t_nf(m)$ is $\hat{f}(n) \cdot \binom{m}{n}$. Note the formal similarity of this formula with the formula for the n-th layer in the manifold calculus tower in Theorem 1.7.3. This suggests that the construction T_nF of manifold calculus is analogous to the interpolating polynomial t_nf rather than to the Taylor polynomial of F. In fact, it is true that the map $F \to T_nF$ is characterized by the property that it is an equivalence on objects of $\mathcal{O}_n(M)$. This confirms the intuition that the approximations in manifold calculus are analogous to interpolation polynomials.

Just as in Goodwillie calculus, the question of convergence is important, and there is a general theory of analytic functors, for which the tower of approximations converges strongly. The main result on the subject is the following deep theorem of Goodwillie and Klein. Let Emb denote the space of smooth embeddings.

Theorem 1.7.5. [47] *Let M^m, N^d be smooth manifolds. Consider the presheaf on M given by the formula $U \mapsto \mathrm{Emb}(U, N)$. The map $\mathrm{Emb}(M, N) \to T_n\,\mathrm{Emb}(M, N)$ is $(d-m-2)n - m + 1$-connected.*

The theorem says in particular that if $d-m \geq 3$, then the connectivity of the map goes to infinity linearly in n, and the tower converges strongly. Even in situations where there is no strong convergence, the tower may be useful for constructing invariants of embeddings and obstructions to existence of embeddings. For example, in the case of $m = 1$, $d = 3$, there is a close connection between the manifold tower and finite type knot invariants [78].

It is often possible to express the tower T_kF in terms of mapping spaces between modules over the little discs operad. This idea probably first appeared in [76] in the context of spaces of knots, and then was developed in [12, 15, 77], and finally and definitively in [29]. Thanks to this connection, facts about the homotopy of the little disc operads (most notably Kontsevich's formality theorem) have consequences regarding the homotopy type of spaces of embeddings [12, 5, 16].

Recently manifold calculus was used by Michael Weiss in the course of proving some striking results about Pontryagin classes of topological bundles [82].

Orthogonal calculus

Let $\mathcal{J}$ be the category of finite-dimensional Euclidean spaces and linear isometric inclusions. Orthogonal calculus [80] concerns continuous functors from $\mathcal{J}$ to the category of topological spaces. It seems likely that topological spaces can be replaced in a routine way with a more general topologically enriched category. The paper [20] develops orthogonal calculus using the perspective of Quillen model categories. Formally, orthogonal calculus is more similar to homotopy calculus than manifold calculus is, and the paper [19] explores this similarity in some depth. Orthogonal calculus has probably received the least attention of the three main brands, and we believe it is ripe for exploration.

Let V denote a generic Euclidean space. Here are some examples of functors to which one might profitably apply orthogonal calculus: $V \mapsto BO(V)$, $V \mapsto G(V)$ (the space of homotopy self equivalences of the unit sphere of V), $V \mapsto BTop(V)$, $V \mapsto \mathrm{Emb}(M, N \times V)$, etc.

Such functors tend to be easier to understand when evaluated at high dimensional vector spaces. The rough idea of orthogonal calculus is to analyze the behavior of a functor on high-dimensional vector spaces and then to extrapolate to low-dimensions. As a result one obtains a kind of Taylor expansion at infinity. Thus to a functor $F \colon \mathcal{J} \to \mathcal{T}op$ one associates a tower of functors under F

$$F \to \cdots \to P_nF \to P_{n-1}F \to \cdots \to P_0F$$

where P_nF plays the role of the Taylor tower of F at infinity. In particular, the constant term P_0F is equivalent to $\mathrm{colim}_{n\to\infty} F(\mathbb{R}^n)$. The higher Taylor approximations are usually difficult to describe explicitly, but the layers in the tower can be described in terms of certain spectra that play the role of derivatives. This is similar to homotopy calculus, but unlike in homotopy calculus, the sequence of derivatives of a functor in orthogonal calculus is an *orthogonal* sequence of spectra, rather than a symmetric one. This means that the n-th derivative is a spectrum with an action of $O(n)$. The following theorem summarizes some results from [80] about homogeneous functors and layers in orthogonal calculus. Compare with Theorem 1.2.3 and Example 1.2.4.

Theorem 1.7.6. *To a continuous functor $F \colon \mathcal{J} \to \mathcal{T}op$ one can associate an orthogonal sequence of spectra, $\partial_1 F, \ldots, \partial_n F, \ldots$. The homotopy fibre of the map $P_nF(V) \to P_{n-1}F(V)$*

is naturally equivalent to the functor that sends V to $\Omega^\infty\left(\partial_n F \wedge S^{nV}\right)_{hO(n)}$. Here S^{nV} is the one-point compactification of the vector space nV.

Examples 1.7.7. Recall that $\mathcal{J}$ is the category of Euclidean spaces and linear isometric inclusions. Suppose V_0, V are Euclidean spaces. Let $\mathrm{Hom}_{\mathcal{J}}(V_0, V)$ denote the space of linear isometric inclusions from V_0 to V. Now fix V_0 and consider the functor $\Omega^\infty\Sigma^\infty \mathrm{Hom}_{\mathcal{J}}(V_0, -)_+$. The Taylor tower of this functor was described in [7]. This is a rare example of a functor whose Taylor tower can be described rather explicitly. When $\dim(V_0) \leq \dim(V)$, the Taylor tower splits as a product of its layers, and in fact this splitting is equivalent to a classical stable splitting of Stiefel manifolds discovered by Haynes Miller [67, 4].

For another example, consider the functor $V \mapsto BO(V)$. The n-th derivative of this functor can be described in terms of the complex of direct-sum decompositions of $\mathbb{R}^n$. A (proper) direct-sum decomposition is a collection (at least two) of pairwise-orthogonal non-zero subspaces of $\mathbb{R}^n$ whose direct sum is $\mathbb{R}^n$. We can partially order such decompositions by refinement and obtain a topological poset whose realization we denote by L_n. Let $L_n^\diamond$ be the unreduced suspension of that realization. The space $L_n^\diamond$ is analogous to the complex of partitions T_n from definition 1.3.1. The n-th derivative of the functor $BO(V)$ is equivalent to $\mathbb{D}(L_n^\diamond) \wedge S^{ad_n}$, where S^{ad_n} is the one-point compactification of the adjoint representation of $O(n)$. Compare with Theorem 1.3.2 about the Goodwillie derivatives of the identity functor. The space $L_n^\diamond$, and even more so its complex analogue, has many striking properties. It appeared in several recent works in an interesting way. For example, it was used in the study of the stable rank filtration of complex K-theory [13, 17], and in the study of the Balmer spectrum of the equivariant stable homotopy category [21]. [7]

It would be especially interesting to understand the derivatives of the functor $V \mapsto \mathrm{Emb}(M, N \times V)$. We speculate that the n-th derivative of this functor can be expressed in terms of moduli space of connected graphs with points marked in M, for which the quotient space of the graph by the subset of marked points is homotopy equivalent to a wedge of n circles.

1.8 Further directions

The basic concepts of Goodwillie calculus are very general and can be applied to a wide variety of homotopy-theoretic settings. Nonetheless, not many calculations have been done outside of representable functors, the identity functor and algebraic K-theory. We have seen that the Taylor tower of the identity functor plays a key role in the theory, so a calculation of that Taylor tower, or at least its layers, would be valuable in other contexts. Obvious candidates for further exploration include:

- unpointed and parametrized homotopy theory: that is, a more detailed understanding of Taylor towers at base objects other than the one-point space;
- unstable equivariant homotopy theory: presumably the derivatives of the identity here are some equivariant version of the spectral Lie operad;

[7] See Paul Balmer's chapter in this Handbook for more on the spectrum of a tensor-triangulated category.

- unstable motivic homotopy theory: as far as we know, no work has been done in this direction despite the wealth of structure the Taylor tower of the identity should possess in this case.

In each of these cases calculations of the derivatives of the identity, their operad structure and Heuts's Tate diagonals, could reveal something deep about how unstable information is built from stable.

The equivariant setting is potentially very interesting. Dotto [39] has generalized Goodwillie calculus to a G-equivariant context, for a finite group G, in which the Taylor tower of a functor is replaced by a diagram of approximations indexed by the finite G-sets. This is based on an idea of Blumberg [28] for a G-equivariant version of excision.

Dotto's approach seems to be better able to make use of the power of modern equivariant stable homotopy theory, something rather neglected by Goodwillie's original version. In particular, the derivatives of a functor in this setting are genuine G-spectra. Dotto calculates the derivatives of the identity functor on pointed G-spaces as equivariant Spanier-Whitehead duals of the partition poset complexes of [14]. It seems reasonable to expect that much of the theory described in this article could be extended the equivariant setting, but little has been done yet.

Manifold and orthogonal calculus have been applied to problems in geometric topology, but there surely is much that remains to be done. In particular orthogonal calculus should have a lot of potential that has barely began to be tapped. We mentioned above that it would be interesting to apply orthogonal calculus to the study of embedding spaces by considering the functor $V \mapsto \mathrm{Emb}(M, N \times V)$. An even more interesting and challenging example is given by the functor $V \mapsto \mathrm{BTop}(V)$ and the closely related functor $V \mapsto B^{V+1}\mathrm{Diff}_c(V)$. Here B^{V+1} stands for $V+1$-fold delooping, and $\mathrm{Diff}_c(V)$ stands for the space of diffeomorphisms of V that equal the identity outside a compact set. We speculate that the n-th derivative of these functors is related to the moduli space of graphs homotopy equivalent to a wedge of n circles (without marked points).

The close analogies that connect Goodwillie calculus with the manifold and orthogonal versions suggests the existence of a more encompassing framework that could also provide new instances of "calculus" for other kinds of functors. One such framework is the theory of tangent categories of Rosický [75] and Cockett and Cruttwell [38], developed to axiomatize the structure of the category of manifolds and smooth maps. Lurie's notion of tangent bundle for an ∞-category [64, 7.3.1] fits Goodwillie calculus into this story, and opens up a way to make precise ideas of Goodwillie on applying differential-geometric ideas such as connections and curvature directly to homotopy theory. There is a helpful intuition that stable ∞-categories such as spectra are "flat", whereas unstable worlds are "curved". However, manifold and orthogonal calculus do not appear to fit into this same picture, so perhaps an even more general theory is still waiting to be discovered.

Acknowledgements

The first author was partially supported by the Swedish Research Council (grant number 2016-05440), and the second author was partially supported by the National Science Foundation (grant DMS-1709032) and the Isaac Newton Institute (EPSRC grant number

EP/R014604/1 and the HHH programme), during the writing of this article. Useful feedback and corrections to an earlier draft were provided by Gijs Heuts and Haynes Miller.

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Department of Mathematics, Stockholm University, Stockholm 106 91, Sweden

E-mail address: gregory.arone@math.su.se

Department of Mathematics and Statistics, Amherst College, Amherst, MA 01002, U.S.A.

E-mail address: mching@amherst.edu

2

A factorization homology primer

David Ayala and John Francis

2.1 Introduction

This article is an introduction to factorization homology—or factorization algebras—in the topological setting; see also Lurie [43]. For introductions in the closely related algebro-geometric or Riemannian settings, see Beilinson–Drinfeld [9], Francis–Gaitsgory [22], and Costello–Gwilliam [17].

Factorization homology takes:

- a geometric input: an n-manifold M;
- an algebraic input: an n-disk algebra A, or a stack X over n-disk algebras, in a symmetric monoidal ∞-category $\mathcal{V}$;[1]

the resulting factorization homology

$$\int_M A$$

is an object of $\mathcal{V}$. It can be thought of as the integral of the algebra A over the manifold M, in the same sense that ordinary homology $\mathsf{H}_*(M, A)$ is given by integrating an abelian group A over a space M.[2] The functor $\int A$ is covariant with respect to open embeddings in the manifold variable: for each fixed M, the functor on the poset of opens $\int A : \mathsf{Opens}(M) \to \mathcal{V}$, sending $U \subset M$ to $\int_U A$, defines a *factorization algebra* on M, both in the definitions of Beilinson–Drinfeld [9] and Costello–Gwilliam [17].

Factorization homology has three essential features making it technically advantageous:

1. **Local-to-global principle**: $\otimes$-excision, generalizing the Eilenberg–Steenrod axioms;
2. **Filtration**: a generalization of the Goodwillie–Weiss embedding calculus;
3. **Duality**: Poincaré/Koszul duality.

2010 *Mathematics Subject Classification.* primary: 55N40, secondary: 57R56, 57N35.

Key words and phrases. factorization homology, topological quantum field theory, derived algebraic geometry, Koszul duality, manifold calculus, Hochschild homology, stratified spaces, factorization algebras.

[1] $\mathcal{V}$ is usually $\mathsf{Mod}_{\Bbbk}$, $\mathsf{Spectra}$, or Spaces—respectively, chain complexes over a commutative ring $\Bbbk$, spectra, or spaces. See §2.1.2 of [43] for a thorough development of *symmetric monoidal ∞-categories.*

[2] This is the alpha version of the theory; there is a beta version, developed in [6], which takes more general geometric and algebraic inputs.

The algebraic input A can arise from a variety of sources: as an n-fold loop space in classical algebraic topology; or as a formal deformation of an n-Poisson algebra. This latter class of examples arises in higher quantization in mathematical physics, a major source of contemporary interest in factorization methods.

The following is a table of values for factorization homology, as the algebra input varies, which we will explain through this article.

Manifold input	**Algebra input**	**Factorization homology output**
$M = S^1$ the circle	A an associative algebra	$\int_{S^1} A \simeq \mathsf{HH}_\bullet(A)$ Hochschild homology
M an n-manifold	A an abelian group	$\int_M A \simeq \mathsf{H}_*(M, A)$ ordinary homology
M an n-manifold	A a spectrum	$\int_M A \simeq \Sigma^\infty_* M \wedge A$ generalized homology
M an n-manifold	A a commutative algebra (e.g., $\mathsf{Sym}(V)$)	$\int_M A \simeq M \boxtimes A$ categorical tensoring (e.g., $\int_M \mathsf{Sym}(V) \simeq \mathsf{Sym}(\mathsf{C}_*(M) \otimes V)$)
M an n-manifold	$A = \mathbb{F}_n V$ a free n-disk algebra	$\int_M \mathbb{F}_n V \simeq \bigoplus_{k \geq 0} \mathsf{C}_*(\mathsf{Conf}^{\mathsf{fr}}_k(M)) \underset{\Sigma_k \wr \mathsf{O}(n)}{\otimes} V^{\otimes k}$
M an n-manifold	$A = \Omega^n K$ the n-fold loop space of an n-connective space	$\int_M \Omega^n K \simeq \mathsf{Map}_{\mathsf{c}}(M, K)$ space of compactly-supported maps
M an n-manifold	$A = \mathsf{U}_n(\mathfrak{g})$ an n-disk enveloping algebra of a Lie algebra	$\int_M \mathsf{U}_n(\mathfrak{g}) \simeq \mathsf{C}^{\mathsf{Lie}}_*\big(\mathsf{C}^*_{\mathsf{c}}(M, \mathfrak{g})\big)$ Lie algebra homology
M an n-manifold	$A = \mathsf{Obs}(\mathbb{R}^n)$ the local observables in a TQFT	$\int_M \mathsf{Obs}(\mathbb{R}^n) \longrightarrow \mathsf{Obs}(M)$ $\otimes$-excisive left-approximation to global observables

The last row is not a genuine equivalence in general: the result of alpha factorization homology is not always the genuine global observables, but only those observables that are determined by the point-local ones. Additionally, the alpha version of factorization homology does not compute the state space of a field theory. This motivates a beta version of factorization homology, developed in [6], where the algebraic input is a more general object, an (∞, n)-category, and which satisfies a more general, and nuanced, form of excision.[3] As this beta version of factorization homology is more involved, it will not be discussed in this article.

[3]See Julia Bergner's chapter in this Handbook for a discussion of (∞, n)-categories.

While the subject of factorization homology is relatively new in name, it has important roots and antecedents. It derives foremost from the algebro-geometric theory of chiral and factorization algebras developed by Beilinson and Drinfeld in conformal field theory in [9]. Secondly, it has an antecedent in the labeled configuration space models of mapping spaces dating to the 1970s; it is closest to the models of Salvatore [51] and Segal [56], but see also [31], [14], [46], [45], and [53]. Aspects appear implicitly in other works, particularly Bott–Segal [15], in realizing the Gelfand–Fuks cohomology of vector fields as the homology of a section space, and in foliation theory [47]. Factorization homology thus arises from the broad nexus of Segal's ideas on conformal field theory [55], on mapping spaces [53] and [56], on foliations [54], and on Lie algebra homology [15].[4]

Outline of contents

We briefly outline the contents of this article. Section 2 concerns n-manifolds and certain topological structures thereon. In §2.2-2.5 we discuss a classification of sheaves on an ∞-category $\mathcal{M}\mathsf{fld}_n$ of n-manifolds and embeddings among them: *sheaves on* $\mathcal{M}\mathsf{fld}_n$ *are* n*-dimensional tangential structures*. In §2.5-2.7 we undergo a similar examination concerning *Weiss sheaves*, thereby motivating and introducing an ∞-category $\mathcal{D}\mathsf{isk}_n$, which plays a key role in factorization homology. In §2.8 we extend these developments to manifolds with boundary; in §2.9 we record a technical result concerning localizing with respect to isotopies, which plays a key role in essentially all of the technical results concerning factorization homology.

Section 3 concerns homology theories for manifolds, and introduces factorization homology. In §3.2 we relate factorization homology to factorization algebras. The remainder of this section is devoted to establishing a pushforward formula for factorization homology, and characterizing homology theories in terms of factorization homology. Section 4 applies this characterization to prove nonabelian Poincaré duality. Section 5 is devoted to a few formal calculations of factorization homology.

Section 6 concerns filtrations and cofiltrations of factorization homology, whose layers are explicit in terms of configuration spaces. These (co)filtrations offer access to identifying and controlling factorization homology. Section 7 demonstrates this with a statement for how factorization homology intertwines Poincaré duality and Koszul duality. This *Poincaré/Koszul duality* (Theorem 2.7.8) is the deepest result discussed in this article and supplies a physical interpretation to factorization homology as observables of perturbative sigma-models.

Section 8 is a synopsis of an adaptation of factorization homology for singular manifolds with coefficients in *disk*-algebras and module data among such.

Implementation of ∞-categories

In this work, we use Joyal's *quasi-category* model of ∞-category theory [30]. Boardman and Vogt first introduced these simplicial sets in [13], as weak Kan complexes, and their and Joyal's theory has been developed in great depth by Lurie in [42] and [43], our primary references; see the first article of [42] for an introduction. We use this model, rather than

[4] More recent works include [1], [2], [10], [11], [18], [25], [26], [27], [29], [34], [36], [35], among many others.

model categories or simplicial categories, because of the great technical advantages for constructions involving categories of functors, which are ubiquitous in this work.[5]

More specifically, we work inside of the quasi-category associated to this model category of Joyal's. In particular, each map between quasi-categories is understood to be an iso- and inner-fibration; (co)limits among quasi-categories are equivalent to homotopy (co)limits with respect to Joyal's model structure. As we work in this way, we refer the reader to these sources for ∞-categorical versions of numerous familiar results and constructions among ordinary categories. In particular, we will make repeated use of the ∞-categorical adjoint functor theorem (Corollary 5.5.2.9 of [42]); the straightening-unstraightening equivalence between Cartesian fibrations over an ∞-category $\mathcal{C}$ and Cat_∞-valued contravariant functors from $\mathcal{C}$ (Theorem 3.2.0.1 of [42]), and likewise between right fibrations over $\mathcal{C}$ and space-valued presheaves on $\mathcal{C}$ (Theorem 2.2.1.2 of [42]); the ∞-categorical version of the Yoneda functor $\mathcal{C} \to \mathsf{PShv}(\mathcal{C}) \simeq \mathsf{RFib}_{\mathcal{C}}$ as it evaluates on objects as $c \mapsto \mathcal{C}_{/c}$ (see §5.1 of [42]).

We will also make use of topological categories, such as $\mathcal{M}\mathsf{fld}_n$ of n-manifolds and embeddings among them. By a functor $\mathcal{S} \to \mathcal{C}$ from a topological category to an ∞-category $\mathcal{C}$ we will always mean a functor $\mathsf{N}\,\mathsf{Sing}\,\mathcal{S} \to \mathcal{C}$ from the simplicial nerve of the Kan-enriched category obtained by applying the product preserving functor Sing to the morphism topological spaces.

The reader uncomfortable with this language can substitute the words "topological category" for "∞-category" wherever they occur in this paper to obtain the correct sense of the results, but they should then bear in mind the proviso that technical difficulties may then abound in making the statements literally true. The reader only concerned with algebras in chain complexes, rather than spectra, can likewise substitute "pre-triangulated differential graded category" for "stable ∞-category" wherever those words appear, with the same proviso.

2.2 Manifolds with tangential structure

In this section, we first introduce a topological category of n-manifolds and embeddings among them. We then define the *tangent classifier* functor. Using this tangent classifier, we then introduce tangential structures, and define an ∞-category of such structured n-manifolds.

2.2.1 Manifolds and embeddings

Definition 2.2.1. A smooth manifold M is *finitary* if it admits a *good* cover, which is to say it admits a finite open cover $\mathcal{U} := \{U \subset M\}$ with the property that, for each finite subset $S \subset \mathcal{U}$, the intersection $\bigcap_{U \in S} U$ is either empty or diffeomorphic to Euclidean space.

Observation 2.2.2. A smooth manifold M is finitary if and only if it is the interior of a compact smooth manifold with (possibly empty) boundary.

Terminology 2.2.3. In this article, by "manifold" we will mean "finitary smooth manifold", unless otherwise stated.

[5] See Moritz Groth's chapter in this Handbook for a development of the theory of quasi-categories.

Remark 2.2.4. The size restriction of Terminology 2.2.3 is not an essential requirement. Indeed, each non-compact manifold is built as sequential colimit of finitary manifolds. Correspondingly, this smallness condition could be removed and one could instead add to Definition 2.3.28 the requirement that a homology theory preserves sequential colimits. With this modification to that definition, the main results are still valid.

Definition 2.2.5. The symmetric monoidal topological category $\mathcal{M}\mathsf{fld}_n$ has as objects smooth n-manifolds. The space of morphisms in $\mathcal{M}\mathsf{fld}_n$ from M to N is

$$\mathsf{Map}_{\mathcal{M}\mathsf{fld}_n}(M, N) := \mathsf{Emb}(M, N)$$

the set of embeddings of M into N equipped with the compact-open C^∞ topology. The symmetric monoidal structure is disjoint union.

Note that disjoint union is not the coproduct in $\mathcal{M}\mathsf{fld}_n$; in fact, there are almost no nontrivial colimits in $\mathcal{M}\mathsf{fld}_n$.

We will make ongoing use of the following result.

Proposition 2.2.6. *The continuous homomorphism of topological monoids* $\mathsf{O}(n) \to \mathsf{Emb}(\mathbb{R}^n, \mathbb{R}^n)$ *is a homotopy equivalence.*

Proof. This continuous homomorphism factors as a composite of continuous homomorphisms:

$$\mathsf{O}(n) \longrightarrow \mathsf{GL}(n) \longrightarrow \mathsf{Emb}_0(\mathbb{R}^n, \mathbb{R}^n) \longrightarrow \mathsf{Emb}(\mathbb{R}^n, \mathbb{R}^n);$$

where $\mathsf{GL}(n)$ is the topological group of linear automorphisms of $\mathbb{R}^n$, and $\mathsf{Emb}_0(\mathbb{R}^n, \mathbb{R}^n)$ is the submonoid of $\mathsf{Emb}(\mathbb{R}^n, \mathbb{R}^n)$ consisting of those smooth embeddings that preserve the origin. The result is established upon showing each of these continuous homomoprhisms is a section of a deformation retraction. Well, the Gram–Schmidt process demonstrates a deformation retraction to the inclusion $\mathsf{O}(n) \to \mathsf{GL}(n)$. Translation $(\mathbb{R}^n \xrightarrow{x \mapsto f(x)} \mathbb{R}^n) \mapsto (\mathbb{R}^n \xrightarrow{x \mapsto f(x) - tf(0)} \mathbb{R}^n)$ demonstrates a deformation retraction to the inclusion $\mathsf{Emb}_0(\mathbb{R}^n, \mathbb{R}^n) \to \mathsf{Emb}(\mathbb{R}^n, \mathbb{R}^n)$. The expression

$$f \mapsto \begin{cases} \frac{f(tx)}{t} & \text{for } t > 0 \\ D_0 f(x) & \text{for } t = 0 \end{cases}$$

demonstrates a deformation retraction to the inclusion $\mathsf{GL}(n) \to \mathsf{Emb}_0(\mathbb{R}^n, \mathbb{R}^n)$. □

Now, temporarily consider the full ∞-subcategory

$$\mathcal{E}\mathsf{uc}_n \subset \mathcal{M}\mathsf{fld}_n$$

consisting of the object $\mathbb{R}^n$; this ∞-category is a delooping of the topological monoid $\mathsf{Emb}(\mathbb{R}^n, \mathbb{R}^n)$ We draw an immediate consequence of Proposition 2.2.6. First, note that there exists a natural functor

$$\mathsf{BO}(n) \longrightarrow \mathcal{E}\mathsf{uc}_n, \qquad * \mapsto \mathbb{R}^n,$$

from the classifying space of the orthogonal group, defined by the inclusion $\mathsf{O}(n) \hookrightarrow \mathsf{Emb}(\mathbb{R}^n, \mathbb{R}^n)$.

Corollary 2.2.7. *The functor*

$$\mathsf{BO}(n) \longrightarrow \mathcal{E}\mathsf{uc}_n,$$

is an equivalence between ∞-categories. In particular, there is a canonical equivalence between the ∞-category of presheaves

$$\mathsf{PShv}(\mathcal{E}\mathsf{uc}_n) \ \simeq \ \mathsf{Spaces}_{/\,\mathsf{BO}(n)}$$

and spaces over $\mathsf{BO}(n)$*.*

2.2.2 Sheaves on n-manifolds

Proposition 2.2.6 offers a classification of sheaves on the ∞-category of B-framed n-manifolds, with respect to a standard Grothendieck topology.

Definition 2.2.8. The symmetric monoidal category Mfld_n has objects n-manifolds; its morphisms are smooth embeddings; the symmetric monoidal structure is disjoint union. The full subcategory $\mathsf{Euc}_n \subset \mathsf{Mfld}_n$ consists of the object $\mathbb{R}^n$. In the *standard* Grothendieck topology on Mfld_n, a sieve $\mathcal{U} \subset \mathsf{Mfld}_{n/M}$ is a covering sieve exactly if, for each element $x \in M$, there is an object $(U \xrightarrow{e} M) \in \mathcal{U}$ for which $\{x\} \subset e(U)$. The ∞-category of *sheaves* on Mfld_n is the full ∞-subcategory

$$\mathsf{Shv}(\mathsf{Mfld}_n) \ \subset \ \mathsf{PShv}(\mathsf{Mfld}_n)$$

consisting of those presheaves $\mathcal{F}\colon \mathsf{Mfld}_n^{\mathsf{op}} \to \mathsf{Spaces}$ for which, for each standard covering sieve $\mathcal{U} \subset \mathsf{Mfld}_{n/M}$, the canonical functor

$$(\mathcal{U}^{\mathsf{op}})^{\triangleleft} \ \simeq \ (\mathcal{U}^{\triangleright})^{\mathsf{op}} \longrightarrow (\mathsf{Mfld}_{n/M})^{\mathsf{op}} \longrightarrow \mathsf{Mfld}_n^{\mathsf{op}} \xrightarrow{\mathcal{F}} \mathsf{Spaces}$$

is a limit diagram, where

$$\mathcal{U}^{\triangleright} := \mathcal{U} \times [1] \coprod_{\mathcal{U}\times\{1\}} *$$

is the right-cone of $\mathcal{U}$ and likewise $\mathcal{U}^{\triangleleft}$ is the left-cone.

Note the evident functor between ∞-categories

$$\mathsf{Mfld}_n \longrightarrow \mathcal{M}\mathsf{fld}_n$$

defined by the natural identity mapping from sets of embeddings, with the discrete topology, to the same sets of embeddings endowed the compact-open C^∞ topology.

Definition 2.2.9. The ∞-category of *sheaves* on $\mathcal{M}\mathsf{fld}_n$ is the pullback among ∞-categories:

$$\begin{tikzcd}
\mathsf{Shv}(\mathcal{M}\mathsf{fld}_n) \arrow[r] \arrow[d] & \mathsf{PShv}(\mathcal{M}\mathsf{fld}_n) \arrow[d, "\text{restriction}"] \\
\mathsf{Shv}(\mathsf{Mfld}_n) \arrow[r] & \mathsf{PShv}(\mathsf{Mfld}_n).
\end{tikzcd}$$

The next result serves to contextualize the recurring role of spaces over $\mathsf{BO}(n)$. This result is not essential for the overall logic of this paper, so we do not supply a proof. The result and its proof is, however, analogous to (and simpler than) Proposition 2.2.22 and its proof.

Proposition 2.2.10. [7] *Restriction along the functor* $\mathsf{BO}(n) \to \mathcal{M}\mathsf{fld}_n$ *defines an equivalence between* ∞*-categories*

$$\mathsf{Shv}(\mathcal{M}\mathsf{fld}_n) \xrightarrow{\simeq} \mathsf{PShv}\big(\mathsf{BO}(n)\big) \ \simeq\ \mathsf{Spaces}_{/\,\mathsf{BO}(n)}\,.$$

Remark 2.2.11. Proposition 2.2.10 is notable in that the∞-topos $\mathsf{Shv}(\mathcal{M}\mathsf{fld}_n)$ is *free* on its completely compact objects , which is a connected ∞-groupoid: $\mathsf{BO}(n)$. Consequently, a sheaf on the ∞-category $\mathcal{M}\mathsf{fld}_n$ is uniquely determined by its value on $\mathbb{R}^n$ as an $\mathsf{O}(n)$-space, or equivalently a space over $\mathsf{BO}(n)$, without reference to a sheaf condition.

Remark 2.2.12. Proposition 2.2.10 asserts that a sheaf on $\mathcal{M}\mathsf{fld}_n$ is precisely the datum of a space B equipped with a map $B \to \mathsf{BO}(n)$, which is equivalent to a rank-n vector bundle $E \to B$ over B. In this equivalence, the value of the sheaf corresponding to $B \to \mathsf{BO}(n)$ on an object $M \in \mathcal{M}\mathsf{fld}_n$ is the space $\mathsf{Map}_{/\,\mathsf{BO}(n)}\big(M, B\big)$ of lifts:

$$\begin{tikzcd}
& B \arrow[d] \\
\mathsf{Emb}(\mathbb{R}^n, M)_{\mathsf{O}(n)} \arrow[r] \arrow[ur, dashed, "\varphi"] & \mathsf{BO}(n),
\end{tikzcd}$$

where the bottom left space is the $\mathsf{O}(n)$-coinvariants of the space of morphisms in $\mathcal{M}\mathsf{fld}_n$ from $\mathbb{R}^n$ to M. Corollary 2.2.31 will identify the bottom horizontal arrow as the tangent classifier, $M \xrightarrow{\tau_M} \mathsf{BO}(n)$. Through that identification, such a lift is equivalent data to a map $\varphi\colon M \to B$ together with an isomorphism $TM \cong \varphi^*E$ between vector bundles over M.

Remark 2.2.13. The canonical functor $\mathsf{Mfld}_n \to \mathcal{M}\mathsf{fld}_n$ carries morphisms $U \hookrightarrow V$ that are isotopy equivalences to equivalences in the ∞-category $\mathcal{M}\mathsf{fld}_n$. It follows that the restriction functor factors

$$\mathsf{Shv}(\mathcal{M}\mathsf{fld}_n) \longrightarrow \mathsf{Shv}^{\mathsf{l.c.}}(\mathsf{Mfld}_n) \ \subset\ \mathsf{Shv}(\mathsf{Mfld}_n)$$

through the ∞-subcategory of locally constant sheaves. We warn the reader that this first functor is *not* an equivalence. Indeed, it follows from a result of Segal ([54]) that the shape of the∞-topos $\mathsf{Shv}(\mathsf{Mfld}_n)$ is the classifying space of the groupoid completion $\mathsf{BEmb}^{\delta}(\mathbb{R}^n, \mathbb{R}^n)$

of the discrete monoid of smooth self-embeddings of $\mathbb{R}^n$. As such, through Proposition 2.2.10, this functor can be identified as

$$\mathsf{PShv}\big(\mathsf{BO}(n)\big) \longrightarrow \mathsf{PShv}\big(\mathsf{BEmb}^{\delta}(\mathbb{R}^n,\mathbb{R}^n)\big),$$

which is restriction along the canonical functor $\mathsf{BEmb}^{\delta}(\mathbb{R}^n,\mathbb{R}^n) \to \mathsf{BEmb}(\mathbb{R}^n,\mathbb{R}^n) \simeq \mathsf{BO}(n)$. The work [28] establishes that this map is not an equivalence.

The following result, proved in [7], contrasts with Remark 2.2.13. Indeed, the shape of the∞-topos $\mathsf{Shv}(\mathsf{Mfld}_{n/M})$ is the underlying space of M, while $\mathsf{Shv}(\mathcal{M}\mathsf{fld}_{n/M})$ is identified, as a consequence of Corollary 2.2.31, as the ∞-category $\mathsf{Spaces}_{/M}$ of local systems on M.

Proposition 2.2.14. *For each n-manifold M, the restriction functor* $\mathsf{Shv}(\mathcal{M}\mathsf{fld}_{n/M}) \to \mathsf{Shv}(\mathsf{Mfld}_{n/M})$ *factors through an equivalence between ∞-categories*

$$\mathsf{Shv}(\mathcal{M}\mathsf{fld}_{n/M}) \xrightarrow{\simeq} \mathsf{Shv}^{\mathsf{l.c.}}(\mathsf{Mfld}_{n/M}).$$

2.2.3 Tangent classifier

We will be particularly interested in n-manifolds that are equipped with the additional structure of a section of a given sheaf on $\mathcal{M}\mathsf{fld}_n$. Through Proposition 2.2.10, such a section is a continuous system of linear structures on each tangent space, such as an orientation, spin structure, or a framing. Tangential structure of this sort can be swiftly accommodated by way of the *tangent classifier*: each n-manifold M has a tangent bundle, and it is classified by a map $\tau_M\colon M \to \mathsf{BO}(n)$ to the classifying space of the topological group $\mathsf{O}(n)$ of linear isometries of $\mathbb{R}^n$. For $B \to \mathsf{BO}(n)$ a map between spaces, a B-framing on M is a homotopy commutative diagram among spaces

$$\begin{tikzcd} & B \arrow[d] \\ M \arrow[ur, "\varphi"] \arrow[r, "\tau_M"'] & \mathsf{BO}(n). \end{tikzcd}$$

Example 2.2.15. Consider the surjective homomorphism

$$\mathsf{O}(n) \xrightarrow{\text{determinant}} \mathsf{O}(1).$$

The kernel of this homomorphism is $\mathsf{SO}(n) \subset \mathsf{O}(n)$, and a $\mathsf{BSO}(n)$-framing on a topological n-manifold is precisely an orientation.

Toward defining an ∞-category $\mathcal{M}\mathsf{fld}_n^B$ of B-framed n-manifolds, we next explain how to make the tangent classifier continuously functorial among open embeddings. Namely, through Corollary 2.2.7, we define the tangent classifier as the restricted Yoneda functor:

$$\tau\colon\ \mathcal{M}\mathsf{fld}_n \xrightarrow{\text{Yoneda}} \mathsf{PShv}(\mathcal{M}\mathsf{fld}_n) \xrightarrow{\text{restriction}} \mathsf{PShv}(\mathcal{E}\mathsf{uc}_n) \underset{\text{Cor } 2.2.7}{\simeq} \mathsf{Spaces}_{/\,\mathsf{BO}(n)}\,. \qquad (2.2.1)$$

We postpone to Corollary 2.2.31 justification that the value τ_M is indeed the familiar tangent classifier, namely that the functor τ sends a manifold M to the map of spaces $M \to \mathsf{BO}(n)$, homotopy-coherently among embeddings in the manifold variable.

Observation 2.2.16. Because $\mathbb{R}^n$ is connected, this tangent classifier $\tau\colon \mathcal{M}\mathsf{fld}_n \to \mathsf{Spaces}_{/\,\mathsf{BO}(n)}$ is symmetric monoidal with respect to coproducts in the codomain. In other words, τ carries finite disjoint unions to finite coproducts over $\mathsf{BO}(n)$.

2.2.4 B-framed manifolds

In this section, we fix a space B as well as a map $B \to \mathsf{BO}(n) \simeq \mathsf{BGL}(n)$, which is equivalent to a rank-n vector bundle $E \to B$ over B. Through Proposition 2.2.10, such data defines a sheaf on $\mathcal{M}\mathsf{fld}_n$. We now consider an ∞-category $\mathcal{M}\mathsf{fld}_n^B$, of n-manifolds equipped with sections of this sheaf.

Definition 2.2.17. The symmetric monoidal ∞-category $\mathcal{M}\mathsf{fld}_n^B$ of B-framed n-manifolds is the limit in the following diagram:

$$\begin{tikzcd}
\mathcal{M}\mathsf{fld}_n^B \arrow[r] \arrow[d] & \mathsf{Spaces}_{/B} \arrow[d] \\
\mathcal{M}\mathsf{fld}_n \arrow[r, "\tau"] & \mathsf{Spaces}_{/\,\mathsf{BO}(n)}\,.
\end{tikzcd}$$

With coproduct as the symmetric monoidal structure on $\mathsf{Spaces}_{/B}$ and $\mathsf{Spaces}_{/\,\mathsf{BO}(n)}$, the right vertical functor is canonically symmetric monoidal. Observation 2.2.16 grants that the bottom horizontal functor τ in the diagram in Definition 2.2.17 is canonically symmetric monoidal. Formally, the forgetful functor $\mathsf{Cat}_\infty^\otimes \to \mathsf{Cat}_\infty$ from symmetric monoidal ∞-categories to ∞-categories creates and preserves limits. We conclude that Definition 2.2.17 indeed defines an ∞-category equipped with a symmetric monoidal structure, as claimed.

Remark 2.2.18. We describe the objects and spaces of morphism spaces in the ∞-category $\mathcal{M}\mathsf{fld}_n^B$.

- An object is a *B-framed n-manifold*, which is the data of an n-manifold M and a lift

  $$\begin{tikzcd}
  & B \arrow[d] \\
  M \arrow[ur, "\varphi"] \arrow[r, "\tau_M"'] & \mathsf{BO}(n)
  \end{tikzcd}$$

  of its tangent classifier. Here, we understand that this is a commutative diagram in the ∞-category Spaces, i.e. a homotopy coherently commutative diagram of spaces. Equivalently, for $E \to B$ the rank-n vector bundle over B classified by the given map $B \to \mathsf{BO}(n)$, such a commutative diagram is the data of a map $M \xrightarrow{\varphi} B$ between spaces together with an isomorphism $TM \cong \varphi^* E$ between vector bundles over M.

- Let (M, φ) and (N, ψ) be B-framed n-manifolds, as above. The space of morphisms in $\mathcal{M}\mathsf{fld}_n^B$ from (M, φ) to (N, ψ) is the space of *B-framed embeddings*, which is the limit space:

  $$\begin{tikzcd}
  \mathsf{Emb}^B(M,N) \arrow[r] \arrow[d] & \mathsf{Map}_{/B}(M,N) \arrow[d] \\
  \mathsf{Emb}(M,N) \arrow[r] & \mathsf{Map}_{/\,\mathsf{BO}(n)}(M,N).
  \end{tikzcd} \tag{2.2.2}$$

Here, for $M \xrightarrow{\varphi} X \xleftarrow{\psi} N$ a diagram in Spaces, then $\mathsf{Map}_{/X}(M,N)$ is the space of maps from M to N over X: it is the limit space:

$$\begin{tikzcd}
\mathsf{Map}_{/X}(M,N) \arrow[r] \arrow[d] & \mathsf{Map}(M,N) \arrow[d, "-\circ\psi"] \\
\ast \arrow[r, "\{\varphi\}"] & \mathsf{Map}(M,X).
\end{tikzcd}$$

That is, a point in $\mathsf{Map}_{/X}(M,N)$ is represented by a map $M \to N$ and a homotopy between the two resulting maps from M to X

Proposition 2.2.6 yields the following.

Observation 2.2.19. The space of B-framings on $\mathbb{R}^n$ is the fiber of the given map $B \to \mathsf{BO}(n)$ over the point $\ast \xrightarrow{\{\mathbb{R}^n\}} \mathsf{BO}(n)$ selecting the vector space $\mathbb{R}^n$. For φ a B-framing on $\mathbb{R}^n$, there is a canonical equivalence between monoid-objects in Spaces:

$$\Omega_\varphi B \ \simeq \ \mathsf{Emb}^B(\mathbb{R}^n, \mathbb{R}^n);$$

here, $\Omega_\varphi B$ is the monoid in Spaces, via concatenation, of loops in B based at the point $\ast \xrightarrow{\varphi} \mathsf{fiber}\big(B \to \mathsf{BO}(n)\big) \to B$.

2.2.5 Examples and discussion of B-framings

Framings. In the case that $\big(B \to \mathsf{BO}(n)\big) := \big(\ast \xrightarrow{\{\mathbb{R}^n\}} \mathsf{BO}(n)\big)$, then a B-framing on an n-manifold M is a *framing* of M, i.e., a trivialization of its tangent bundle. Such a trivialization is a sequence of n linearly independent vector fields $(X_1, \dots, X_n)$ on M.

Now, let (M,φ) and (N,ψ) be framed n-manifolds, with corresponding vector fields $(X_i)_{1\leq i\leq n}$ and $(Y_i)_{1\leq i\leq n}$. A naive definition of the space of *framed embeddings* might be the subspace,

$$\Big\{M \overset{e}{\hookrightarrow} N \mid De(X_i) = Y_i \text{ for each } 1 \leq i \leq n\Big\} \ \subset \ \mathsf{Emb}(M,N),$$

of those smooth embeddings that strictly respect the framings. However, for the purposes of defining manifold invariants from algebraic input (e.g., factorization homology), this naive definition of framed embeddings is deficient:

1. for (N,ψ) a framed n-manifold with N compact, it receives no framed embeddings from $\mathbb{R}^n$ (with its standard framing) since it receives no isometric embeddings from $\mathbb{R}^n$;

2. for each $i \geq 0$, this space of framed embeddings from $(\mathbb{R}^n)^{\sqcup i}$ to $\mathbb{R}^n$ (each with its standard framing) is *not* equivalent to the space $\mathcal{E}_n(i)$ of i-ary operations of the little n-disks operad.

One might try to fix the above naive definition of framed embeddings by weakening the condition $De(X_i) = Y_i$ to the *data* of a sequence of smooth functions $M \xrightarrow{\lambda_i} \mathbb{R}_{>0}$ subject to the condition of an equality $De(X_i) = \lambda_i Y_i$. But this, too, has similar shortcomings.

Practically, the essential feature of the 'correct' space of framed embeddings is so that there is a homotopy equivalence

$$\mathsf{Emb}^{\mathsf{fr}}\big((\mathbb{R}^n)^{\sqcup i},(N,\psi)\big) \;\simeq\; \mathsf{Conf}_i(N)$$

with the *configuration space* of injections from $\{1,\dots,i\}$ into N. Lemma 2.2.30, to come, grants that this is the case for the definition (2.2.2) of framed embeddings (in this case that $B=*$) as a homotopy pullback.

Linear structures. More generally, for G a Lie group and $\rho\colon G\to\mathsf{GL}(n)$ a smooth homomorphism, consider the composite map $\mathsf{B}G\to\mathsf{BGL}(n)\simeq\mathsf{BO}(n)$. A $\mathsf{B}G$-framing on an n-manifold M is a compatible system of lifts $D(\alpha^{-1}\beta)\colon U\cap V\to G$ along ρ of the derivatives of the transition maps of a smooth atlas for M. So a $\mathsf{BSO}(n)$-framing on M is a smooth structure on M together with an orientation on M; a $\mathsf{B}*$-framing on M is a framing, as discussed above; a $\mathsf{BSpin}(n)$-framing on M is a spin structure on M.

General structures. In general, for B a space, the space of maps $B\to\mathsf{BO}(n)$ is a moduli space of rank-n vector bundles on B. So, given such a vector bundle $E\to B$, a B-framing on an n-manifold M is a continuous map $f\colon M\to B$ together with an isomorphism $TM\cong f^*E$ between vector bundles over M. In the case that $\big(B\to\mathsf{BO}(n)\big)\simeq\big(X\times\mathsf{BO}(n)\xrightarrow{\mathsf{pr}}\mathsf{BO}(n)\big)$, then a B-framing on an n-manifold M is simply a map $M\to X$ to X from the underlying space of M.

2.2.6 Weiss sheaves on n-manifolds

Definition 2.2.20. [59] In the *Weiss* Grothendieck topology on the category Mfld_n, a sieve $\mathcal{U}\subset\mathsf{Mfld}_{n/M}$ is a covering sieve if, for each finite subset $S\subset M$, there is an object $(U\hookrightarrow M)\in\mathcal{U}$ for which $S\subset e(U)$. The ∞-category of *Weiss* sheaves on Mfld_n is the full ∞-subcategory

$$\mathsf{Shv}^{\mathsf{Weiss}}(\mathsf{Mfld}_n)\;\subset\;\mathsf{PShv}(\mathsf{Mfld}_n)$$

consisting of those presheaves $\mathcal{F}\colon\ \mathsf{Mfld}_n^{\mathsf{op}}\to\mathsf{Spaces}$ for which, for each Weiss covering sieve $\mathcal{U}\subset\mathsf{Mfld}_{n/M}$, the canonical functor

$$(\mathcal{U}^{\mathsf{op}})^{\triangleleft}\;\simeq\;(\mathcal{U}^{\triangleright})^{\mathsf{op}}\longrightarrow(\mathsf{Mfld}_{n/M})^{\mathsf{op}}\longrightarrow\mathsf{Mfld}_n^{\mathsf{op}}\xrightarrow{\mathcal{F}}\mathsf{Spaces}$$

is a limit diagram. The ∞-category of *Weiss* sheaves on $\mathcal{M}\mathsf{fld}_n$ is the pullback among ∞-categories:

$$\begin{array}{ccc}\mathsf{Shv}^{\mathsf{Weiss}}(\mathcal{M}\mathsf{fld}_n) & \longrightarrow & \mathsf{PShv}(\mathcal{M}\mathsf{fld}_n)\\ \downarrow & & \downarrow{\scriptstyle \text{restriction}}\\ \mathsf{Shv}^{\mathsf{Weiss}}(\mathsf{Mfld}_n) & \longrightarrow & \mathsf{PShv}(\mathsf{Mfld}_n).\end{array}$$

We next exhibit generators the for the ∞-category of Weiss sheaves.

Definition 2.2.21. The symmetric monoidal ∞-categories

$$\mathsf{Disk}_n\subset\mathsf{Mfld}_n\qquad\text{and}\qquad\mathcal{D}\mathsf{isk}_n\subset\mathcal{M}\mathsf{fld}_n$$

are the full ∞-subcategories consisting of disjoint unions of n-dimensional Euclidean spaces.

Notice the evident symmetric monoidal functors

$$\mathsf{Disk}_n \longrightarrow \mathcal{D}\mathsf{isk}_n \qquad \text{and} \qquad \mathsf{Mfld}_n \longrightarrow \mathcal{M}\mathsf{fld}_n\,.$$

We insert the next result to contextualize the fundamental role of disks, though this result is not essential for the overall logic of this paper. Its proof makes use of results established later on, which of course do not logically depend on it.

Proposition 2.2.22. *Restriction along $\mathcal{D}\mathsf{isk}_n \hookrightarrow \mathcal{M}\mathsf{fld}_n$ defines an equivalence between ∞-categories:*

$$\mathsf{Shv}^{\mathsf{Weiss}}(\mathcal{M}\mathsf{fld}_n) \xrightarrow{\simeq} \mathsf{PShv}(\mathcal{D}\mathsf{isk}_n).$$

Proof. Denote the fully-faithful inclusion $\iota\colon \mathcal{D}\mathsf{isk}_n \hookrightarrow \mathcal{M}\mathsf{fld}_n$. This functor determines an adjunction

$$\iota^*\colon\ \mathsf{PShv}(\mathcal{M}\mathsf{fld}_n) \rightleftarrows \mathsf{PShv}(\mathcal{D}\mathsf{isk}_n)\colon \iota_*$$

in which the left adjoint is restriction along ι and the right adjoint is right Kan extension along ι. It is enough to show that this adjunction restricts as an equivalence:

$$\iota^*\colon\ \mathsf{Shv}^{\mathsf{Weiss}}(\mathcal{M}\mathsf{fld}_n) \rightleftarrows \mathsf{PShv}(\mathcal{D}\mathsf{isk}_n)\colon \iota_*.$$

This amounts to verifying two assertions.

1. For each $\mathcal{F} \in \mathsf{Shv}^{\mathsf{Weiss}}(\mathcal{M}\mathsf{fld}_n)$, and each $M \in \mathcal{M}\mathsf{fld}_n$, the unit map

$$\mathcal{F}(M) \xrightarrow{\text{unit}} \lim\Big((\mathcal{D}\mathsf{isk}_{n/M})^{\mathsf{op}} \to \mathcal{D}\mathsf{isk}_n^{\mathsf{op}} \xrightarrow{\iota} \mathcal{M}\mathsf{fld}_n^{\mathsf{op}} \xrightarrow{\mathcal{F}} \mathsf{Spaces}\Big)$$

   is an equivalence between spaces.

2. For each $\mathcal{G} \in \mathsf{PShv}(\mathcal{D}\mathsf{isk}_n)$, and for each Weiss covering sieve $\mathcal{U} \subset \mathsf{Mfld}_{n/M}$, the canonical map

$$\iota_*\mathcal{G}(M) \longrightarrow \lim\Big(\mathcal{U}^{\mathsf{op}} \to \mathsf{Mfld}_n^{\mathsf{op}} \to \mathcal{M}\mathsf{fld}_n^{\mathsf{op}} \xrightarrow{\iota_*\mathcal{G}} \mathsf{Spaces}\Big)$$

   is an equivalence between spaces.

We first establish (1). The canonical functor $\mathsf{Disk}_{n/M} \to \mathcal{D}\mathsf{isk}_{n/M}$ determines the sequence of maps among spaces

$$\begin{tikzcd}
\mathcal{F}(M) \arrow[rr, "\text{unit}"] \arrow[drr] & & \lim\big((\mathcal{D}\mathsf{isk}_{n/M})^{\mathsf{op}} \to \mathcal{D}\mathsf{isk}_n^{\mathsf{op}} \xrightarrow{\iota} \mathcal{M}\mathsf{fld}_n^{\mathsf{op}} \xrightarrow{\mathcal{F}} \mathsf{Spaces}\big) \arrow[d] \\
& & \lim\big((\mathsf{Disk}_{n/M})^{\mathsf{op}} \to \mathsf{Disk}_n^{\mathsf{op}} \to \mathcal{D}\mathsf{isk}_n^{\mathsf{op}} \xrightarrow{\iota} \mathcal{M}\mathsf{fld}_n^{\mathsf{op}} \xrightarrow{\mathcal{F}} \mathsf{Spaces}\big).
\end{tikzcd}$$

Because smooth open embeddings from disjoint unions of n-dimensional Euclidean spaces to each smooth n-manifold is a basis for the Weiss Grothendieck topology thereon, the full subcategory $\mathsf{Disk}_n \subset \mathsf{Mfld}_n$ is a basis for the Weiss Grothendieck topology. It follows that the diagonal map in the above diagram is an equivalence between spaces. Furthermore, Proposition 2.2.38 implies the functor $(\mathsf{Disk}_{n/M})^{\mathsf{op}} \to (\mathcal{D}\mathsf{isk}_{n/M})^{\mathsf{op}}$ is initial. It follows that the downward map in the above diagram is an equivalence. We conclude that the horizontal map in the above diagram is an equivalence, as desired.

We now establish (2). The Weiss covering sieve $\mathcal{U} \subset \mathsf{Mfld}_{n/M}$ determines a functor between ∞-categories

$$\mathsf{colim}\Big(\mathcal{U} \to \mathsf{Mfld}_n \xrightarrow{\mathcal{D}\mathsf{isk}_{n/-}} \mathsf{Cat}_\infty\Big) \longrightarrow \mathcal{D}\mathsf{isk}_{n/M} \tag{2.2.3}$$

from the colimit indexed by $\mathcal{U}$. Through the standard formula computing values of right Kan extension as limits indexed by ∞-undercategories, the map in (2) is canonically identified as the map between spaces

$$\lim\Big((\mathcal{D}\mathsf{isk}_{n/M})^{\mathsf{op}} \to \mathcal{D}\mathsf{isk}_n^{\mathsf{op}} \xrightarrow{\mathcal{G}} \mathsf{Spaces}\Big) \longrightarrow$$

$$\lim\Big(\mathsf{colim}\big(\mathcal{U} \to \mathsf{Mfld}_n \xrightarrow{\mathcal{D}\mathsf{isk}_{n/-}} \mathsf{Cat}_\infty\big)^{\mathsf{op}} \to (\mathcal{D}\mathsf{isk}_{n/M})^{\mathsf{op}} \to \mathcal{D}\mathsf{isk}_n^{\mathsf{op}} \xrightarrow{\mathcal{G}} \mathsf{Spaces}\Big).$$

It is therefore enough to show that the functor (2.2.3) is final. Observe that, for each n-manifold W, the ∞-category $\mathcal{D}\mathsf{isk}_{n/W}$ is canonically a right fibration over $\mathcal{D}\mathsf{isk}_n$. To assess finality of (2.2.3), it is enough to compute the relevant colimit in the ∞-category $\mathsf{RFib}_{\mathcal{D}\mathsf{isk}_n}$ of right fibrations over $\mathcal{D}\mathsf{isk}_n$, and show that the morphism between right fibrations over $\mathcal{D}\mathsf{isk}_n$

$$\mathsf{colim}\Big(\mathcal{U} \to \mathsf{Mfld}_n \xrightarrow{\mathcal{D}\mathsf{isk}_{n/-}} \mathsf{RFib}_{\mathcal{D}\mathsf{isk}_n}\Big) \longrightarrow \mathcal{D}\mathsf{isk}_{n/M}$$

is an equivalence. A morphism between right fibrations over an ∞-category is an equivalence if and only if it is so when base changed over the maximal ∞-subgroupoid of the base. Through Lemma 2.2.30, the maximal ∞-subgroupoid of $\mathcal{D}\mathsf{isk}_n$ is the coproduct $\coprod_{i\geq 0} \mathsf{B}\big(\Sigma_i \wr \mathsf{O}(n)\big)$. For each n-manifold W, Lemma 2.2.30 identifies the base change of $\mathcal{D}\mathsf{isk}_{n/M}$ over the i-cofactor as $\mathsf{Conf}_i(W)_{\Sigma_i}$, the unordered configuration space. So let $i \geq 0$. So we are to show that the canonical map between spaces

$$\mathsf{colim}\Big(\mathcal{U} \to \mathsf{Mfld}_n \xrightarrow{\mathsf{Conf}_i(-)_{\Sigma_i}} \mathsf{Spaces}_{/\mathsf{B}(\Sigma_i \wr \mathsf{O}(n))}\Big) \longrightarrow \mathsf{Conf}_i(M)_{\Sigma_i} \tag{2.2.4}$$

is an equivalence between spaces over $\mathsf{B}\big(\Sigma_i \wr \mathsf{O}(n)\big)$. This is simply to show that this map is an equivalence between spaces, ignorant to the structure maps to $\mathsf{B}\big(\Sigma_i \wr \mathsf{O}(n)\big)$. Consider the smallest sieve $\mathcal{U}_i \subset \mathsf{Mfld}_{ni/\,\mathsf{Conf}_i(M)_{\Sigma_i}}$ containing, for each $U \in \mathcal{U}$, the object $\big(\mathsf{Conf}_i(U)_{\Sigma_i} \hookrightarrow \mathsf{Conf}_i(M)_{\Sigma_i}\big) \in \mathsf{Mfld}_{ni/\,\mathsf{Conf}_i(M)_{\Sigma_i}}$. Precisely because $\mathcal{U}$ is a Weiss covering sieve, the sieve $\mathcal{U}_i$ is a standard covering sieve. Using that the underlying topological space of $\mathsf{Conf}_i(M)_{\Sigma_i}$ is paracompact and Hausdorff, Theorem A.3.1 [43] can be applied to $\mathcal{U}_i$ with the result that the above map (2.2.4) between ∞-groupoids is an equivalence. This completes the proof. □

Definition 2.2.23. Let $\mathcal{C}$ be a presentable ∞-category. The ∞-category of *$\mathcal{C}$-valued Weiss cosheaves* on Mfld_n is the full ∞-subcategory

$$\mathsf{cShv}_{\mathcal{C}}^{\mathsf{Weiss}}(\mathsf{Mfld}_n) \;\subset\; \mathsf{Fun}(\mathsf{Mfld}_n, \mathcal{C})$$

consisting of those functors $\mathcal{A}\colon \mathsf{Mfld}_n \to \mathcal{C}$ for which, for each Weiss covering sieve $\mathcal{U} \subset \mathsf{Mfld}_{n/M}$, the canonical functor

$$\mathcal{U}^{\triangleright} \longrightarrow \mathsf{Mfld}_{n/M} \longrightarrow \mathsf{Mfld}_n \xrightarrow{\mathcal{A}} \mathcal{C}$$

is a colimit diagram. The ∞-category of *$\mathcal{C}$-valued Weiss cosheaves* on $\mathcal{M}\mathsf{fld}_n$ is the pullback among ∞-categories:

$$\begin{tikzcd}
\mathsf{cShv}^{\mathsf{Weiss}}_{\mathcal{C}}(\mathcal{M}\mathsf{fld}_n) \arrow[r] \arrow[d] & \mathsf{Fun}(\mathcal{M}\mathsf{fld}_n, \mathcal{C}) \arrow[d, "\text{restriction}"] \\
\mathsf{cShv}^{\mathsf{Weiss}}_{\mathcal{C}}(\mathsf{Mfld}_n) \arrow[r] & \mathsf{Fun}(\mathsf{Mfld}_n, \mathcal{C}).
\end{tikzcd}$$

The next result is routine consequence of Proposition 2.2.22.

Corollary 2.2.24. *Let $\mathcal{C}$ be a presentable ∞-category. Restriction along $\mathcal{D}\mathsf{isk}_n \to \mathcal{M}\mathsf{fld}_n$ defines an equivalence between ∞-categories:*

$$\mathsf{cShv}^{\mathsf{Weiss}}_{\mathcal{C}}(\mathcal{M}\mathsf{fld}_n) \xrightarrow{\simeq} \mathsf{Fun}(\mathcal{D}\mathsf{isk}_n, \mathcal{C}).$$

Remark 2.2.25. Corollary 2.2.24 says that a $\mathcal{C}$-valued Weiss cosheaf on $\mathcal{M}\mathsf{fld}_n$ is simply a functor $\mathcal{D}\mathsf{isk}_n \to \mathcal{C}$, without regard to a cosheaf condition.

Remark 2.2.26. We follow up on Remark 2.2.13 and Remark 2.2.14. Namely, restriction along $\mathsf{Mfld}_n \to \mathcal{M}\mathsf{fld}_n$ factors

$$\mathsf{cShv}^{\mathsf{Weiss}}_{\mathcal{C}}(\mathcal{M}\mathsf{fld}_n) \longrightarrow \mathsf{cShv}^{\mathsf{Weiss,l.c.}}_{\mathcal{C}}(\mathsf{Mfld}_n) \ \subset \ \mathsf{cShv}^{\mathsf{Weiss}}_{\mathcal{C}}(\mathsf{Mfld}_n)$$

through the ∞-subcategory of locally constant Weiss cosheaves. This first functor is not an equivalence. However, for each n-manifold M, the restriction functor factors as an equivalence between ∞-categories,

$$\mathsf{cShv}^{\mathsf{Weiss}}_{\mathcal{C}}(\mathcal{M}\mathsf{fld}_{n/M}) \xrightarrow{\simeq} \mathsf{cShv}^{\mathsf{Weiss,l.c}}_{\mathcal{C}}(\mathsf{Mfld}_{n/M}) \subset \mathsf{cShv}^{\mathsf{Weiss}}_{\mathcal{C}}(\mathsf{Mfld}_{n/M}),$$

involving the locally constant Weiss cosheaves on M.

2.2.7 Disks

Prompted by the results in §2.2.6, we now turn to consider B-framed n-disks, as they organize as a symmetric monoidal ∞-category. We identify the maximal ∞-subgroupoid of the ∞-category $\mathcal{D}\mathsf{isk}^B_{n/M}$ in terms of configuration spaces in a B-framed n-manifold M.

Definition 2.2.27. [3] The symmetric monoidal ∞-category $\mathcal{D}\mathsf{isk}^B_n \subset \mathcal{M}\mathsf{fld}^B_n$ is the full ∞-subcategory of $\mathcal{M}\mathsf{fld}^B_n$ whose objects are disjoint unions of B-framed n-dimensional Euclidean spaces.

Remark 2.2.28. Consider the framing structure, which is the map $* \to \mathsf{BO}(n)$ selecting the basepoint of $\mathsf{BO}(n)$. As discussed in §2.2.5, a $*$-framing on an n-manifold M is a trivialization of the tangent bundle of M. We denote the associated ∞-category of framed n-disks as $\mathcal{D}\mathsf{isk}_n^{\mathsf{fr}}$. This symmetric monoidal ∞-category $\mathcal{D}\mathsf{isk}_n^{\mathsf{fr}}$ is equivalent to the PROP associated to the operad $\mathcal{E}_n$, of Boardman-Vogt [13]. This is the case because the inclusion of rectilinear embeddings as framed embeddings determines an equivalence $\mathcal{E}_n(i) \xrightarrow{\simeq} \mathsf{Emb}^{\mathsf{fr}}\big((\mathbb{R}^n)^{\sqcup i}, \mathbb{R}^n\big)$ from the space of i-ary operations of the operad $\mathcal{E}_n$; see, e.g., [3] for a presentation of this equivalence.

Example 2.2.29. Consider the structure $\big(B \to \mathsf{BO}(n)\big) := \big(\mathsf{BO}(n) \xrightarrow{\mathsf{id}} \mathsf{BO}(n)\big)$. The symmetric monoidal ∞-category $\mathcal{D}\mathsf{isk}_n = \mathcal{D}\mathsf{isk}_n^{\mathsf{BO}(n)}$ is equivalent to the PROP associated to the unoriented version of the ribbon, or "framed," $\mathcal{E}_n$-operad; see [52] for a treatment of this operad.[6] This equivalence follows from Proposition 2.2.6.

Given a topological space X and a finite cardinality i, the *configuration space* of i (ordered) points in X is the subspace

$$\mathsf{Conf}_i(X) := \Big\{\{1,\dots,i\} \xrightarrow{c} X \mid c \text{ is injective}\Big\} \ \subset\ X^{\times i}.$$

This configuration space has an evident free action of the symmetric group Σ_i, as given by precomposition. The *unordered* configuration space is the Σ_i-coinvariants: $\mathsf{Conf}_i(X)_{\Sigma_i}$.

In the next result, for M a B-framed n-manifold, we consider the ∞-overcategory

$$\mathcal{D}\mathsf{isk}^B_{n/M} \ := \ \mathcal{D}\mathsf{isk}^B_n \underset{\mathcal{M}\mathsf{fld}^B_n}{\times} \mathcal{M}\mathsf{fld}^B_{n/M}\,.$$

An object in this ∞-category is given by a B-framed embedding $(\mathbb{R}^n)^{\sqcup i} \hookrightarrow M$ for some i.

Lemma 2.2.30. [3] *The maximal ∞-subgroupoid of $\mathcal{D}\mathsf{isk}^B_n$ is canonically identified as the space*

$$\coprod_{i\geq 0} B^{\times i}{}_{\Sigma_i} \ \simeq\ \big(\mathcal{D}\mathsf{isk}^B_n\big)^{\sim}$$

where the coproduct is indexed by finite cardinalities and each cofactor is the Σ_i-homotopy coinvariants of the i-fold product of the space B. In particular, the symmetric monoidal functor $[-]\colon \mathcal{D}\mathsf{isk}^B_n \to \mathsf{Fin}$, given by taking sets of connected components of underlying manifolds, is conservative.

For M a B-framed n-manifold, the maximal ∞-subgroupoid of $\mathcal{D}\mathsf{isk}^B_{n/M}$ is canonically identified as the space

$$\coprod_{i\geq 0} \mathsf{Conf}_i(M)_{\Sigma_i} \ \simeq\ \big(\mathcal{D}\mathsf{isk}^B_{n/M}\big)^{\sim}$$

where the coproduct is indexed by finite cardinalities, and each cofactor is an unordered configuration space.

We conclude this section by justifying the term *tangent classifier* for the functor $\mathcal{M}\mathsf{fld}_n \xrightarrow{\tau} \mathsf{Spaces}_{/\,\mathsf{BO}(n)}$ from (2.2.1).

Corollary 2.2.31. *The value of the tangent classifier (2.2.1) on an n-manifold M is the map between spaces $M \xrightarrow{\tau_M} \mathsf{BO}(n)$ classifying its tangent bundle.*

[6]The historical use of "framed" here is potentially misleading, since in the "framed" $\mathcal{E}_n$ operad the embeddings do not preserve the framing, while in the usual $\mathcal{E}_n$ operad the embeddings do preserve the framing (up to scale). It might lead to less confusion to replace the term "framed $\mathcal{E}_n$ operad" with "unoriented $\mathcal{E}_n$ operad."

Proof. First, recognize $\mathcal{E}\mathsf{uc}_n \subset \mathcal{D}\mathsf{isk}_n$ as the full ∞-subcategory consisting of the connected n-manifolds. Next, specialize the second statement of Lemma 2.2.30 at $i = 1$ to obtain an identification

$$M \underset{\text{Lem } 2.2.30}{\simeq} \mathcal{E}\mathsf{uc}_{n/M} \underset{\text{Prop } 2.2.6}{\simeq} \mathsf{Emb}(\mathbb{R}^n, M)_{\mathsf{O}(n)} \longrightarrow \mathsf{BO}(n)$$

involving the homotopy $\mathsf{O}(n)$-coinvariants. Unwinding the equivalences above recognizes this map in terms of the frame bundle for M, as

$$M \ \simeq \ \mathsf{Fr}(M)_{\mathsf{O}(n)} \longrightarrow \mathsf{BO}(n).$$

Evidently, this map classifies the tangent bundle of M. $\square$

2.2.8 Manifolds with boundary

We will also employ the category of n-manifolds with boundary.

Definition 2.2.32. A smooth manifold M with boundary is *finitary* if it admits a *good* cover, which is to say it admits a finite open cover $\mathcal{U} := \{U \subset M\}$ with the property that, for each finite subset $S \subset \mathcal{U}$, the intersection $\bigcap_{U\in S} U$ is either empty, diffeomorphic to Euclidean space, or diffeomorphic to Euclidean half-space, $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$.

Terminology 2.2.33. In this article, by "manifold with boundary" we mean "finitary smooth manifold with boundary," unless otherwise stated.

Definition 2.2.34. $\mathcal{M}\mathsf{fld}_n^\partial$ is the symmetric monoidal topological category of n-manifolds, possibly with boundary. The topological space of morphisms between two is the set of smooth open embeddings equipped with the compact-open C^∞ topology. The symmetric monoidal structure is disjoint union. The full symmetric monoidal topological category $\mathcal{D}\mathsf{isk}_n^\partial \subset \mathcal{M}\mathsf{fld}_n^\partial$ is that consisting of finite disjoint unions of $\mathbb{R}^n$ and $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$.

Remark 2.2.35. The category $\mathcal{D}\mathsf{isk}_n^\partial$ is minimal with respect to the condition that any finite subset of an n-manifold with boundary has an open neighborhood diffeomorphic to an object of $\mathcal{D}\mathsf{isk}_n^\partial$. In particular, the closed n-disk $\mathbb{D}^n$ is not an object of $\mathcal{D}\mathsf{isk}_n^\partial$.

The following result is an application of the Alexander trick.

Proposition 2.2.36. [3] *The symmetric monoidal functor*

$$\mathbb{R}_{\geq 0} \times - : \mathcal{M}\mathsf{fld}_{n-1} \longrightarrow \mathcal{M}\mathsf{fld}_n^\partial$$

is fully faithful. Namely, for each pair of $(n-1)$-manifolds M and N, the map

$$\mathsf{Emb}(M, N) \longrightarrow \mathsf{Emb}\big(\mathbb{R}_{\geq 0} \times M, \mathbb{R}_{\geq 0} \times N\big)$$

is an equivalence between spaces.

Remark 2.2.37. Together with Proposition 2.2.6, the previous proposition implies that the continuous homomorphism between topological monoids

$$\mathsf{O}(n-1) \hookrightarrow \mathsf{Emb}(\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}, \mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1})$$

is a homotopy equivalence. Consequently, $\mathcal{D}\mathsf{isk}_n^{\partial}$ is an unoriented variant of the Swiss cheese operad of Voronov [58]. Specifically, the framed variant $\mathcal{D}\mathsf{isk}_n^{\partial,\mathsf{fr}}$ is equivalent to the PROP associated to the Swiss cheese operad.

2.2.9 Localizing with respect to isotopy equivalences

Here we show that the ∞-category $\mathcal{D}\mathsf{isk}^B_{n/M}$ is a localization of its more discrete version $\mathsf{Disk}^B_{n/M}$ on the collection of those embeddings that are isotopic to isomorphisms. This comparison plays a fundamental role in recognizing certain colimit expressions in this theory, for instance those that support the pushforward formula of §2.3.5.

Recall Definition 2.2.21, and the symmetric monoidal functors:

$$\mathsf{Disk}_n \longrightarrow \mathcal{D}\mathsf{isk}_n \qquad \text{and} \qquad \mathsf{Mfld}_n \longrightarrow \mathcal{M}\mathsf{fld}_n.$$

We denote the pullback symmetric monoidal ∞-categories:

$$\begin{tikzcd} \mathsf{Disk}^B_n \arrow[r] \arrow[d] & \mathcal{D}\mathsf{isk}^B_n \arrow[d] \\ \mathsf{Disk}_n \arrow[r] & \mathcal{D}\mathsf{isk}_n \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \mathsf{Mfld}^B_n \arrow[r] \arrow[d] & \mathcal{M}\mathsf{fld}^B_n \arrow[d] \\ \mathsf{Mfld}_n \arrow[r] & \mathcal{M}\mathsf{fld}_n. \end{tikzcd}$$

For each n-manifold M, we denote the ∞-subcategory

$$\mathcal{I}_M \ \subset \ \mathsf{Disk}^B_{n/M} := \mathsf{Disk}^B_n \underset{\mathsf{Mfld}^B_n}{\times} \mathsf{Mfld}^B_{n/M} \tag{2.2.5}$$

that consists of the same objects but only those morphisms $(U \hookrightarrow M) \hookrightarrow (V \hookrightarrow M)$ whose image in $\mathcal{D}\mathsf{isk}^B_{n/M}$ is an equivalence.

Proposition 2.2.38. [3] *The functor* $\mathsf{Disk}^B_{n/M} \longrightarrow \mathcal{D}\mathsf{isk}^B_{n/M}$ *witnesses a localization between* ∞*-categories:*

$$\big(\mathsf{Disk}^B_{n/M}\big)[\mathcal{I}_M^{-1}] \ \simeq \ \mathcal{D}\mathsf{isk}^B_{n/M}\,.$$

Let M be a B-framed n-manifold. Each of the ∞-categories $\mathsf{Disk}^B_{n/M}$ and $\mathcal{D}\mathsf{isk}^B_{n/M}$ is naturally the active ∞-subcategory of an ∞-operad,[7] each of which we again denote as $\mathsf{Disk}^B_{n/M}$ and $\mathcal{D}\mathsf{isk}^B_{n/M}$, respectively.

- The ∞-operad structure on $\mathsf{Disk}^B_{n/M}$ is such that the ∞-category of colors is the poset in which an object is an open subset $U \subset M$ that is abstractly diffeomorphic to Euclidean space. There is a unique i-ary morphism from an i-fold collection $(U_k)_{1\leq k\leq i}$ of such to

[7] Recall, §2 of [43], that a morphism of finite based sets $I_* \xrightarrow{f} J_*$ is *active* if $f^{-1}\{*\} = \{*\}$ and is *inert* if the restriction $f_| : f^{-1}J \to J$ is injective. A morphism in an ∞-operad $\mathcal{O} \to \mathsf{Fin}_*$ is active or inert if its image in Fin_* is.

another V precisely if the U_k are pairwise disjoint and their union $\bigcup_{1\leq k\leq i} U_k \subset V$ is contained in V.

- The ∞-operad structure on $\mathcal{D}\mathsf{isk}^B_{n/M}$ is such that the ∞-category of colors is the ∞-groupoid $\mathsf{Emb}(\mathbb{R}^n, M)_{\mathsf{O}(n)} \simeq M$ of an open disk in M. The space of i-ary morphisms from an i-fold collection $(U_j)_{1\leq j\leq i}$ of such to another V is the fiber of the composite map
$$\mathsf{Emb}\Big(\bigsqcup_{j=1}^{i} U_j, V\Big) \longrightarrow \prod_{j=1}^{i} \mathsf{Emb}(U_j, V) \longrightarrow \prod_{j=1}^{i} \mathsf{Emb}(U_j, M)$$
over the point selecting the given sequence of embeddings $(U_j)_{1\leq j\leq i}$.

The next result also appears in [43] as Theorem 5.4.5.9.

Corollary 2.2.39. *Let $\mathcal{V}$ be a symmetric monoidal ∞-category. For each B-framed n-manifold M, restriction along the morphism $\mathsf{Disk}^B_{n/M} \to \mathcal{D}\mathsf{isk}^B_{n/M}$ defines a fully faithful functor*
$$\mathsf{Alg}_{\mathcal{D}\mathsf{isk}^B_{n/M}}(\mathcal{V}) \xrightarrow{\text{f.f.}} \mathsf{Alg}_{\mathsf{Disk}^B_{n/M}}(\mathcal{V}).$$

In the case $M = \mathbb{R}^n$, Corollary 2.2.39 can be used to prove that locally constant factorization algebras on $\mathbb{R}^n$ are equivalent to $\mathcal{E}_n$-algebras—see §2.3.2 for a discussion of factorization algebras.

The next construction is made possible using Proposition 2.2.38.

Construction 2.2.40. Let M be a B-framed n-manifold, and let N be a B'-framed k-manifold possibly with boundary. Let $f\colon M \to N$ be a continuous map that satisfies the following regularity condition:

> Each of the restrictions
> $$f_|\colon f^{-1}(N \smallsetminus \partial N) \to N \smallsetminus \partial N \qquad \text{and} \qquad f_|\colon f^{-1}(\partial N) \to \partial N$$
> is a smooth fiber bundle.

We produce a composite morphism between ∞-operads
$$f^{-1}\colon\ \mathsf{Disk}^{\partial,B'}_{k/N} \longrightarrow \mathsf{Mfld}^B_{n/M} \longrightarrow \mathcal{M}\mathsf{fld}^B_{n/M},$$
in which the second morphism is the standard one. We now describe the first functor. For formal reasons, we can assume the maps $B \to \mathsf{BO}(n)$ and $B' \to \mathsf{BO}(k)$ are equivalences. For this case, the first functor is given by $(U \hookrightarrow N) \mapsto (U \times_N M \hookrightarrow M)$, which is evidently a multi-functor. By inspection, this functor f^{-1} carries isotopy equivalences to equivalences. Through Proposition 2.2.38, there results a multi-functor
$$f^{-1}\colon\ \mathcal{D}\mathsf{isk}^{B'}_{k/N} \longrightarrow \mathcal{M}\mathsf{fld}^B_{n/M}, \tag{2.2.6}$$
as desired.

2.3 Homology theories for manifolds

The value of factorization homology over an n-manifold M of an n-disk algebra A is a sort of average, indexed by n-disks embedded in M, of the value of A on such n-disks. We make this definition precise, as well as observe a property of this definition for which it is universal, by defining factorization homology as left Kan extension of an n-disk algebra $A\colon \mathcal{D}\mathsf{isk}_n \to \mathcal{V}$ along the inclusion $\mathcal{D}\mathsf{isk}_n \hookrightarrow \mathcal{M}\mathsf{fld}_n$.

Throughout this entire section, we fix a space $B \to \mathsf{BO}(n)$ over $\mathsf{BO}(n)$ as well as a symmetric monoidal ∞-category $\mathcal{V}$ that is $\otimes$-*presentable* in the following sense.

Definition 2.3.1. A symmetric monoidal ∞-category $\mathcal{V}$ is $\otimes$-*presentable* if it satisfies both of the following conditions.

- The underlying ∞-category of $\mathcal{V}$ is presentable: $\mathcal{V}$ admits colimits and every object is a filtered colimit of compact objects.[8]
- The symmetric monoidal structure of $\mathcal{V}$ distributes over colimits: for each object $V \in \mathcal{V}$, the functor $V \otimes -\colon \mathcal{V} \to \mathcal{V}$ carries colimit diagrams to colimit diagrams.

Example 2.3.2. Let $\mathcal{S}$ be a presentable ∞-category. Consider the Cartesian symmetric monoidal ∞-category $(\mathcal{S}, \times)$, in which the symmetric monoidal structure is categorical product. Provided $\mathcal{S}$ is Cartesian closed, then this symmetric monoidal ∞-category is $\otimes$-presentable. In particular, $(\mathsf{Spaces}, \times)$ is $\otimes$-presentable; for $\mathcal{X}$ any ∞-topos, then $(\mathcal{X}, \times)$ is $\otimes$-presentable; also, $(\mathsf{Cat}_\infty, \times)$ is $\otimes$-presentable.

Example 2.3.3. Let R be a commutative ring. The symmetric monoidal ∞-category $\big(\mathsf{Mod}_R, \otimes_R\big)$ of R-modules with tensor product over R is $\otimes$-presentable. Note, however, that its opposite $\big(\mathsf{Mod}_R^{\mathsf{op}}, \otimes_R\big)$ is not $\otimes$-presentable.

Remark 2.3.4. The results in §2.3 (in particular, the Eilenberg–Steenrod axioms for factorization homology) only require that the symmetric monoidal structure of $\mathcal{V}$ distributes over sifted colimits. (This generality is established as a special case of §2 of [8].) However, the calculations of §2.5 onwards require the symmetric monoidal structure to distribute over all colimits, so for simplicity of exposition we enforce this stronger hypothesis throughout.

2.3.1 Disk algebras

Definition 2.3.5. The ∞-category of $\mathcal{D}\mathsf{isk}_n^B$-*algebras* in $\mathcal{V}$ is that of symmetric monoidal functors from $\mathcal{D}\mathsf{isk}_n^B$ to $\mathcal{V}$:

$$\mathsf{Alg}_{\mathcal{D}\mathsf{isk}_n^B}(\mathcal{V}) \ := \ \mathsf{Fun}^{\otimes}(\mathcal{D}\mathsf{isk}_n^B, \mathcal{V}).$$

Remark 2.3.6. Let $G \to \mathsf{O}(n)$ be a morphism between group-objects in the ∞-category Spaces (equivalently, a map of loop spaces). Through this representation, change-of-framing

[8] This is with respect to an understood fixed uncountable cardinal κ, i.e., $\mathcal{V}$ admits κ-small colimits and every object is a κ-filtered colimit of κ-compact objects.

defines an action of G on the symmetric monoidal ∞-category $\mathcal{D}\mathsf{isk}_n^{\mathsf{fr}}$. As such, the symmetric monoidal forgetful functor

$$\mathcal{D}\mathsf{isk}_n^{\mathsf{fr}} \longrightarrow \mathcal{D}\mathsf{isk}_n^{\mathsf{B}G}$$

witnesses the G-coinvariants: $\big(\mathcal{D}\mathsf{isk}_n^{\mathsf{fr}}\big)_G \xrightarrow{\simeq} \mathcal{D}\mathsf{isk}_n^{\mathsf{B}G}$. Consequently, for each symmetric monoidal ∞-category $\mathcal{V}$, the restriction functor canonically factors through the G-invariants,

$$\mathsf{Alg}_{\mathcal{D}\mathsf{isk}_n^{\mathsf{B}G}}(\mathcal{V}) \xrightarrow{\simeq} \big(\mathsf{Alg}_{\mathcal{E}_n}(\mathcal{V})\big)^G \longrightarrow \mathsf{Alg}_{\mathcal{E}_n}(\mathcal{V})$$

as an equivalence. In particular, there is a canonical equivalence from the ∞-category of $\mathcal{D}\mathsf{isk}_n$-algebras

$$\mathsf{Alg}_{\mathcal{D}\mathsf{isk}_n}(\mathcal{V}) \xrightarrow{\simeq} \big(\mathsf{Alg}_{\mathcal{E}_n}(\mathcal{V})\big)^{\mathsf{O}(n)}$$

and that of $\mathsf{O}(n)$-invariant cE_n-algebras.

We denote the restricted Yoneda functor

$$\mathbb{E}\colon \mathcal{M}\mathsf{fld}_n^B \xrightarrow{\text{Yoneda}} \mathsf{PShv}(\mathcal{M}\mathsf{fld}_n^B) \xrightarrow{\text{restriction}} \mathsf{PShv}(\mathcal{D}\mathsf{isk}_n^B).$$

Definition 2.3.7. Let M be a B-framed n-manifold. Let A be a $\mathcal{D}\mathsf{isk}_n^B$-algebra in $\mathcal{V}$. The *factorization homology (of M with coefficients in A)* is the object in $\mathcal{V}$ given either as the colimit (provided it exists), or the coend (provided it exists):

$$\begin{aligned}\int_M A \quad &:= \quad \mathsf{colim}\big(\mathcal{D}\mathsf{isk}_{n/M}^B \to \mathcal{D}\mathsf{isk}_n^B \xrightarrow{A} \mathcal{V}\big)\\ &\simeq \quad \mathbb{E}_M \bigotimes_{\mathcal{D}\mathsf{isk}_M^B} A.\end{aligned}$$

Remark 2.3.8. We comment on the two equivalent expressions defining factorization homology: as a colimit indexed by an overcategory and as a coend. This is analogous to the familiar fact that, for $X_\bullet\colon \Delta^{\mathsf{op}} \to \mathsf{Spaces}$ a simplicial space, its geometric realization can be expressed

$$|X_\bullet| \ := \ \mathsf{colim}\big(\Delta_{/X_\bullet} \to \Delta \xrightarrow{\Delta^\bullet} \mathsf{Spaces}\big) \ \simeq \ X_\bullet \bigotimes_{\Delta} \Delta^\bullet$$

as a colimit of topological simplices indexed by the category of simplices in $X_\bullet$, or as the standard coend expression which is a quotient of $\coprod_{p\geq 0} X_p \times \Delta^p$.

The fully faithful symmetric monoidal functor $\iota\colon \mathcal{D}\mathsf{isk}_n^B \hookrightarrow \mathcal{M}\mathsf{fld}_n^B$ gives the restriction functor

$$\mathsf{Alg}_{\mathcal{D}\mathsf{isk}_n^B}(\mathcal{V}) := \mathsf{Fun}^\otimes(\mathcal{D}\mathsf{isk}_n^B, \mathcal{V}) \ \longleftarrow \ \mathsf{Fun}^\otimes\big(\mathcal{M}\mathsf{fld}_n^B, \mathcal{V}\big) : \iota^*.$$

The next result, proved in [3], identifies factorization homology as the values of a left adjoint to this restriction functor, provided $\mathcal{V}$ is $\otimes$-presentable.

Proposition 2.3.9. [3] *Let $\mathcal{V}$ be $\otimes$-presentable ∞-category. The restriction functor ι^* admits a left adjoint,*

$$\iota_!\colon \mathsf{Alg}_{\mathcal{D}\mathsf{isk}_n^B}(\mathcal{V}) \ \rightleftarrows \ \mathsf{Fun}^\otimes\big(\mathcal{M}\mathsf{fld}_n^B, \mathcal{V}\big) : \iota^*,$$

over a left adjoint $\iota_!\colon \mathsf{Fun}(\mathcal{D}\mathsf{isk}_n^B, \mathcal{V}) \rightleftarrows \mathsf{Fun}(\mathcal{M}\mathsf{fld}_n^B, \mathcal{V})\colon \iota^*$*. Furthermore, this left adjoint evaluates on a* $\mathcal{D}\mathsf{isk}_n^B$*-algebra* A *as factorization homology:*

$$\iota_!(A)\colon M \mapsto \int_M A.$$

Remark 2.3.10. Proposition 2.3.9 implies factorization homology can be expressed as a symmetric monoidal left Kan extension, at least when $\mathcal{V}$ is $\otimes$-presentable. Consequently, the definition of factorization homology given above is equivalent to an operadic left Kan extension (after parsing Definitions 3.1.1.2 and 3.1.2.2 of [43]), which is the definition of factorization homology, or topological chiral homology, given by Lurie [43] (Definition 5.5.2.6).

The next result explains how factorization homology transforms under change of tangential structure.

Proposition 2.3.11. *Let* $B \xrightarrow{\alpha} B'$ *be a map between spaces over* $\mathsf{BO}(n)$*. Let* $M = (M, \varphi)$ *be a* B*-framed* n*-manifold. Consider the* B'*-framed* n*-manifold* $\alpha M := (M, \alpha\varphi)$*. Let* A *be a* $\mathcal{D}\mathsf{isk}_n^{B'}$*-algebra* $A\colon \mathcal{D}\mathsf{isk}_n^{B'} \to \mathcal{V}$*. Consider the* $\mathcal{D}\mathsf{isk}_n^B$*-algebra* $\alpha A\colon \mathcal{D}\mathsf{isk}_n^B \to \mathcal{D}\mathsf{isk}_n^{B'} \xrightarrow{A} \mathcal{V}$*. The canonical morphism in* $\mathcal{V}$

$$\int_M \alpha A \xrightarrow{\simeq} \int_{\alpha M} A$$

is an equivalence.

Proof. Note the canonical commutative diagram among ∞-categories:

$$\begin{tikzcd}
\mathcal{D}\mathsf{isk}_{n/M}^B \arrow[r] \arrow[d] & \mathcal{D}\mathsf{isk}_{n/\alpha M}^{B'} \arrow[d] \\
\mathcal{D}\mathsf{isk}_n^B \arrow[r] \arrow[dr, "\alpha A"'] & \mathcal{D}\mathsf{isk}_n^{B'} \arrow[d, "A"] \\
 & \mathcal{V}
\end{tikzcd}.$$

For formal reasons, the top horizontal functor is an equivalence between ∞-categories. It follows that the canonical morphism in $\mathcal{V}$

$$\int_M \alpha A = \mathsf{colim}\Big(\mathcal{D}\mathsf{isk}_{n/M}^B \to \mathcal{D}\mathsf{isk}_n^B \xrightarrow{\alpha A} \mathcal{V}\Big) \longrightarrow \mathsf{colim}\Big(\mathcal{D}\mathsf{isk}_{n/\alpha M}^{B'} \to \mathcal{D}\mathsf{isk}_n^{B'} \xrightarrow{A} \mathcal{V}\Big) = \int_{\alpha M} A$$

is an equivalence, as desired. □

2.3.2 Factorization algebras

We now use factorization homology to construct *factorization algebras* over each B-framed n-manifold, as in the sense used by Costello–Gwilliam (see Definition 2.3.12). Namely, Proposition 2.3.14 shows that $\mathcal{D}\mathsf{isk}_n^B$-algebra A in $\mathcal{V}$ determines a factorization algebra $\mathcal{F}_A$ on each B-framed n-manifold M whose value $\mathcal{F}_A(U)$ on an open subset $U \subset M$

is factorization homology of A,

$$\mathcal{F}_A(U) \simeq \int_U A,$$

over U (with its B-framing inherited from that of M).

For the next definition, for each n-manifold M, we observe the following multi-category structure on the poset $\mathsf{Opens}(M) := \mathsf{Mfld}_{n/M}$ of open subsets in M ordered by inclusion. Namely, an object is an open subset of M, and there is a multi-morphism from $(U_i)_{i\in I}$ to V, which is unique, provided the collection $\{U_i\}_{i\in I}$ is pairwise disjoint and provided $\bigcup_{i\in I} U_i \subset V$. Recall from §2.2.6 the definition of (locally constant) Weiss cosheaves.

Definition 2.3.12. [17] Let $\mathcal{V}$ be a symmetric monoidal ∞-category. Let M be an n-manifold. The ∞-category of ($\mathcal{V}$-valued) *factorization algebras* (on M) is the full ∞-subcategory of algebras in $\mathcal{V}$ over the multi-category $\mathsf{Opens}(M)$ as in the pullback among ∞-categories:

$$\begin{tikzcd}
\mathsf{Alg}_M(\mathcal{V}) \arrow[r] \arrow[d] & \mathsf{Alg}_{\mathsf{Opens}(M)}(\mathcal{V}) \arrow[d] \\
\mathsf{cShv}^{\mathsf{Weiss}}_{\mathcal{V}}(M) \arrow[r] & \mathsf{Fun}\big(\mathsf{Opens}(M),\mathcal{V}\big).
\end{tikzcd}$$

The ∞-category of *locally constant* factorization algebras is the full ∞-subcategory of factorization algebras as in the pullback diagram among ∞-categories:

$$\begin{tikzcd}
\mathsf{Alg}^{\mathsf{l.c.}}_M(\mathcal{V}) \arrow[r] \arrow[d] & \mathsf{Alg}_M(\mathcal{V}) \arrow[d] \\
\mathsf{cShv}^{\mathsf{Weiss,l.c.}}_{\mathcal{V}}(M) \arrow[r] & \mathsf{cShv}^{\mathsf{Weiss}}_{\mathcal{V}}(M).
\end{tikzcd}$$

Remark 2.3.13. In other words, a factorization algebra is a multi-functor $\mathcal{F}\colon \mathsf{Opens}(M) \to \mathcal{V}$ whose restriction to the poset $\mathsf{Opens}(M)$ is a Weiss cosheaf on M. Informally, a factorization algebra is likewise a functor $\mathcal{F}\colon \mathsf{Opens}(M) \to \mathcal{V}$ from the poset of open subsets of M to the underlying ∞-category of $\mathcal{V}$ that satisfies codescent with respect to Weiss covers, together with a system of compatible equivalences in $\mathcal{V}$: for each finite sequence $(U_i)_{i\in I}$ of pairwise disjoint open subsets of M, an equivalence $\mathcal{F}\big(\bigcup_{i\in I} U_i\big) \simeq \bigotimes_{i\in I}\mathcal{F}(U_i)$.

Proposition 2.3.14. *Let $\mathcal{V}$ be a $\otimes$-presentable ∞-category, and let B be a space over* $\mathsf{BO}(n)$*. For each B-framed n-manifold M, factorization homology defines a functor*

$$\mathsf{Alg}_{\mathcal{D}\mathsf{isk}^B_n}(\mathcal{V}) \longrightarrow \mathsf{Alg}^{\mathsf{l.c.}}_M(\mathcal{V}), \qquad A \mapsto \left(U \mapsto \int_U A \right)$$

from $\mathcal{D}\mathsf{isk}^B_n$-algebras to locally constant factorization algebras on M.

Proof. Notice the evident diagram among ∞-operads

$$\mathcal{D}\mathsf{isk}^B_n \xrightarrow{\iota} \mathcal{M}\mathsf{fld}^B_n \longleftarrow \mathsf{Mfld}^B_n \longleftarrow \mathsf{Opens}(M).$$

This diagram determines the diagram among ∞-categories:

$$\begin{tikzcd}
\mathsf{Alg}_{\mathcal{D}\mathsf{isk}_n^B}(\mathcal{V}) \arrow[d, "\text{forget}"'] & \mathsf{Fun}^{\otimes}(\mathcal{M}\mathsf{fld}_n^B, \mathcal{V}) \arrow[l, "\iota^*"'] \arrow[r] \arrow[d, "\text{forget}"'] & \mathsf{Fun}^{\otimes}(\mathsf{Mfld}_n^B, \mathcal{V}) \arrow[r] \arrow[d, "\text{forget}"] & \mathsf{Alg}_{\mathsf{Opens}(M)}(\mathcal{V}) \arrow[d, "\text{forget}"] \\
\mathsf{Fun}(\mathcal{D}\mathsf{isk}_n^B, \mathcal{V}) & \mathsf{Fun}(\mathcal{M}\mathsf{fld}_n^B, \mathcal{V}) \arrow[l, "\iota^*"'] \arrow[r] & \mathsf{Fun}(\mathsf{Mfld}_n^B, \mathcal{V}) \arrow[r] & \mathsf{Fun}\big(\mathsf{Opens}(M), \mathcal{V}\big).
\end{tikzcd}$$

Proposition 2.3.9 gives a commutative diagram involving left adjoints to each instance of ι^*:

$$\begin{tikzcd}
\mathsf{Alg}_{\mathcal{D}\mathsf{isk}_n^B}(\mathcal{V}) \arrow[r, "\iota_!"] \arrow[d, "\text{forget}"'] & \mathsf{Fun}^{\otimes}(\mathcal{M}\mathsf{fld}_n^B, \mathcal{V}) \arrow[r] \arrow[d, "\text{forget}"'] & \mathsf{Fun}^{\otimes}(\mathsf{Mfld}_n^B, \mathcal{V}) \arrow[r] \arrow[d, "\text{forget}"] & \mathsf{Alg}_{\mathsf{Opens}(M)}(\mathcal{V}) \arrow[d, "\text{forget}"] \\
\mathsf{Fun}(\mathcal{D}\mathsf{isk}_n^B, \mathcal{V}) \arrow[r, "\iota_!"] & \mathsf{Fun}(\mathcal{M}\mathsf{fld}_n^B, \mathcal{V}) \arrow[r] & \mathsf{Fun}(\mathsf{Mfld}_n^B, \mathcal{V}) \arrow[r] & \mathsf{Fun}\big(\mathsf{Opens}(M), \mathcal{V}\big).
\end{tikzcd}$$

Corollary 2.2.24 gives the factorization of the bottom horizontal sequence of functors:

$$\begin{tikzcd}
\mathsf{Alg}_{\mathcal{D}\mathsf{isk}_n^B}(\mathcal{V}) \arrow[rrr, "\text{restrict} \circ \iota_!"] \arrow[d, "\text{forget}"'] & & & \mathsf{Alg}_{\mathsf{Opens}(M)}(\mathcal{V}) \arrow[d, "\text{forget}"] \\
\mathsf{Fun}(\mathcal{D}\mathsf{isk}_n^B, \mathcal{V}) \arrow[r, "\iota_!"] & \mathsf{cShv}_{\mathcal{V}}^{\mathsf{Weiss}}(\mathcal{M}\mathsf{fld}_n^B) \arrow[r] & \mathsf{cShv}_{\mathcal{V}}^{\mathsf{Weiss,l.c.}}(\mathsf{Mfld}_n^B) \arrow[r] & \mathsf{cShv}_{\mathcal{V}}^{\mathsf{Weiss,l.c.}}(M).
\end{tikzcd}$$

The result follows. $\square$

Remark 2.3.15. Proposition 2.3.14 grants the commutative diagram among ∞-categories:

$$\begin{tikzcd}
\mathsf{Alg}_{\mathcal{D}\mathsf{isk}_n}(\mathcal{V}) \arrow[rr, "A \mapsto \mathcal{F}_A", "\text{Prop } 2.3.14"'] \arrow[dr, "\int_M"'] & & \mathsf{Alg}_M(\mathcal{V}) \arrow[dl, "\text{global cosections}"] \\
& \mathcal{V} &
\end{tikzcd}$$

In other words, for $\mathcal{F}_A$ the factorization algebra determined by the $\mathcal{D}\mathsf{isk}_n$-algebra A, its global cosections is factorization homology:

$$\int_M A \;\simeq\; \mathcal{F}_A(M).$$

In the case that M is equipped with a framing, A need only be a $\mathcal{E}_n$-algebra (see Remark 2.3.6).

Terminology 2.3.16. For X a stratified space, a factorization algebra $\mathcal{F}$ on X is *constructible* if, for each stratum $X_p \subset X$ of X, the restricted factorization algebra $\mathcal{F}_{|X_p}$ on X_p is locally constant.

Remark 2.3.17. Let X be a stratified space. Consider the ∞-category $\mathcal{B}_X$ of singularity-types in X, and stratified open embeddings among them. Consider the symmetric monoidal ∞-category $\mathcal{D}\mathsf{isk}(\mathcal{B}_X)$ in which an object is a finite-fold disjoint union of objects in $\mathcal{B}_X$, and a morphism between them is a stratified open embedding. Established in [8] is a similar construction to that of Proposition 2.3.14 as it concerns symmetric monoidal functors from $\mathcal{D}\mathsf{isk}(\mathcal{B}_X)$ to constructible factorization algebras on X. See that reference for a thorough development, and §2.8 of this article for a synopsis.

2.3.3 Factorization homology over oriented 1-manifolds with boundary

We show that factorization homology over a closed interval is a two-sided bar construction.

Recall from §2.2.8 the symmetric monoidal ∞-category $\mathcal{M}\mathsf{fld}_1^\partial$ and its symmetric monoidal full ∞-subcategory $\mathcal{D}\mathsf{isk}_1^\partial$ consisting of finite disjoint unions of Euclidean and half-Euclidean spaces. In this subsection we consider, similarly, the symmetric monoidal ∞-category and its symmetric monoidal full ∞-subcategory,

$$\mathcal{D}\mathsf{isk}_1^{\partial,\mathsf{or}} \subset \mathcal{M}\mathsf{fld}_1^{\partial,\mathsf{or}}.$$

An object in the latter is a 1-manifold with boundary equipped with an orientation, while such an object belongs to the smaller if each connected component of its underlying 1-manifold with boundary is diffeomorphic to Euclidean space or half-Euclidean space. The space of morphisms between two such objects therein is the space of smooth open embeddings that preserve orientations, equipped with the compact-open C^∞ topology. The symmetric monoidal structure is disjoint union.

Remark 2.3.18. The 1-manifold with boundary $[-1,1]$, equipped with its standard orientation determined by the non-vanishing vector field ∂_t, is an object in $\mathcal{M}\mathsf{fld}_1^{\partial,\mathsf{or}}$ that does *not* belong to $\mathcal{D}\mathsf{isk}_1^{\partial,\mathsf{or}}$.

We next articulate a sense in which $\mathcal{D}\mathsf{isk}_1^{\partial,\mathsf{or}}$ is entirely combinatorial.

Definition 2.3.19. [8] $\mathsf{Assoc}^{\mathsf{RL}}$ is the ∞-operad corepresenting triples $(A;P,Q)$ consisting of an associative algebra together with a unital right and a unital left module. Specifically, it is a unital multi-category whose space of colors is the three-element set $\{M,R,L\}$, and with spaces of multi-morphisms given as follows. Let $I \xrightarrow{\sigma} \{M,R,L\}$ be a map from a finite set.

- $\mathsf{Assoc}^{\mathsf{RL}}(\sigma,M)$ is the set of linear orders on I for which no element is related to an element of $\sigma^{-1}(\{R,L\})$. In other words, should $\sigma^{-1}(\{R,L\})$ be empty, then there is one multi-morphism from σ to M for each linear order on $\sigma^{-1}(M)$; should $\sigma^{-1}(\{R,L\})$ not be empty, then there are no multi-morphisms from σ to M.

- $\mathsf{Assoc}^{\mathsf{RL}}(\sigma,L)$ is the set of linear orders on I for which each element of $\sigma^{-1}(L)$ is a minimum, and no element is related to an element in $\sigma^{-1}(R)$. In other words, should $\sigma^{-1}(\{R\})$ be empty and $\sigma^{-1}(\{L\})$ have cardinality at most 1, then there is one multi-morphism from σ to M for each linear order on $\sigma^{-1}(M)$; should $\sigma^{-1}(\{R\})$ not be empty or $\sigma^{-1}(\{L\})$ have cardinality greater than 1, then there are no multi-morphisms from σ to M.

- $\mathsf{Assoc}^{\mathsf{RL}}(\sigma, R)$ is the set of linear orders on I for which each element of $\sigma^{-1}(R)$ is a maximum, and no element is related to an element in $\sigma^{-1}(L)$. In other words, should $\sigma^{-1}(\{L\})$ be empty and $\sigma^{-1}(\{R\})$ have cardinality at most 1, then there is one multi-morphism from σ to M for each linear order on $\sigma^{-1}(M)$; should $\sigma^{-1}(\{L\})$ not be empty or $\sigma^{-1}(\{R\})$ have cardinality greater than 1, then there are no multi-morphisms from σ to M.

Composition of multi-morphisms is given by concatenating linearly ordered sets.

The next result references the *symmetric monoidal envelope* of the colored operad $\mathsf{Assoc}^{\mathsf{RL}}$. It is initial among symmetric monodal ∞-categories equipped with an $\mathsf{Assoc}^{\mathsf{RL}}$-algebra. In other words, it is a symmetric monoidal ∞-category $\mathsf{Env}\big(\mathsf{Assoc}^{\mathsf{RL}}\big)$ corepresenting the copresheaf on the ∞-category of symmetric monoidal ∞-categories

$$\mathsf{Map}^{\otimes}\Big(\mathsf{Env}(\mathsf{Assoc}^{\mathsf{RL}}), -\Big)\colon\ \mathsf{Cat}_{\infty}{}^{\otimes} \longrightarrow \mathsf{Spaces}, \qquad \mathcal{V} \mapsto \big(\mathsf{Alg}_{\mathsf{Assoc}^{\mathsf{RL}}}(\mathcal{V})\big)^{\sim},$$

whose value on a symmetric monoidal ∞-category is the moduli space of $\mathsf{Assoc}^{\mathsf{RL}}$-algebras in it.

Lemma 2.3.20. *Taking connected components defines an equivalence between symmetric monoidal ∞-categories,*

$$[-]\colon\ \mathcal{D}\mathsf{isk}_1^{\partial,\mathsf{or}} \xrightarrow{\ \simeq\ } \mathsf{Env}\big(\mathsf{Assoc}^{\mathsf{RL}}\big) \tag{2.3.1}$$

to the symmetric monoidal envelope, where the values on the symmetric monoidal generators are: $[\mathbb{R}] = M$*,* $[\mathbb{R}_{\geq 0}] = R$*, and* $[\mathbb{R}_{\leq 0}] = L$*.*

Proof. Evidently, this defines a symmetric monoidal functor. We now show that it is an equivalence. Because it is so on symmetric monoidal generators, (2.3.1) is essentially surjective on spaces of objects. It remains to show that (2.3.1) is fully faithful. So let U and V be objects in $\mathcal{D}\mathsf{isk}_1^{\partial,\mathsf{or}}$. We must show that the map between spaces

$$\mathsf{Map}_{\mathcal{D}\mathsf{isk}_1^{\partial,\mathsf{or}}}(U,V) \longrightarrow \mathsf{Map}_{\mathsf{Env}(\mathsf{Assoc}^{\mathsf{RL}})}([U],[V]) \tag{2.3.2}$$

is an equivalence. For $V = \bigsqcup_{\alpha\in[V]} V_\alpha$ the partition as connected components, direct inspection of the definition of these two symmetric monoidal ∞-categories yields an identification of this map is as the $[V]$-indexed product of such maps

$$\begin{aligned}
\mathsf{Map}_{\mathcal{D}\mathsf{isk}_1^{\partial,\mathsf{or}}}(U,V) \quad &\xrightarrow{\simeq} \quad \coprod_{[U]\xrightarrow{f}[V]} \prod_{\alpha\in[V]} \mathsf{Map}_{\mathcal{D}\mathsf{isk}_1^{\partial,\mathsf{or}}}(U_{|f^{-1}\alpha}, V_\alpha) \\
&\longrightarrow \quad \coprod_{[U]\xrightarrow{f}[V]} \prod_{\alpha\in[V]} \mathsf{Map}_{\mathsf{Env}(\mathsf{Assoc}^{\mathsf{RL}})}(f^{-1}\alpha, [V_\alpha]) \\
&\xleftarrow{\simeq} \quad \mathsf{Map}_{\mathsf{Env}(\mathsf{Assoc}^{\mathsf{RL}})}([U],[V]).
\end{aligned}$$

We are therefore reduced to the case that V is non-empty and connected. By direct inspection, the space of morphisms in $\mathcal{D}\mathsf{isk}_1^{\partial,\mathsf{or}}$ from U to V is a 0-type. So we are left to show that the map (2.3.2) is a bijection (on connected components). There are three cases to consider.

- In the case that $V \cong (-1,1)$ is oriented-diffeomorphic to an open interval, this 0-type is empty if U has non-empty boundary, and otherwise it is the set of linear orders on the set $[U]$ of connected components.
- In the case that $V \cong [-1,1)$ is oriented-diffeomorphic to a left-closed/right-open interval, this 0-type is empty if U has non-empty outward-pointing boundary, and otherwise it is the set of linear orders on $[U]$ for which each connected component with inward-pointing boundary is a minimum.
- In the case that $V \cong (-1,1]$ is oriented-diffeomorphic to a left-open/right-closed interval, this 0-type is empty if U has non-empty inward-pointing boundary, and otherwise it is the set of linear orders on $[U]$ for which each connected component with outward-pointing boundary is a maximum.

Inspecting Definition 2.3.19, and the description of the symmetric monoidal functor $[-]$ under examination, reveals that the map (2.3.2) between 0-types is an equivalence, as desired. $\square$

Example 2.3.21. Note the functor $\mathsf{Alg}^{\mathsf{aug}}_{\mathsf{Assoc}}(\mathcal{V}) \to \mathsf{Alg}_{\mathsf{Assoc}^{\mathsf{RL}}}(\mathcal{V})$ from augmented associative algebras, given by $(A \to \mathbb{1}) \mapsto (A; \mathbb{1}, \mathbb{1})$. Concatenating with the equivalence of Lemma 2.3.20 results in a functor $\mathsf{Alg}^{\mathsf{aug}}_{\mathsf{Assoc}}(\mathcal{V}) \to \mathsf{Fun}^{\otimes}\big(\mathcal{D}\mathsf{isk}_1^{\partial,\mathsf{or}}, \mathcal{V}\big)$.

Recall from §2.2.8 the ∞-operad structure on $\mathcal{D}\mathsf{isk}_{n/M}$.

Observation 2.3.22. Consider the ordinary category O^{RL} in which an object is a finite linearly ordered set $(I, \leq)$ together with a pair of disjoint subsets $R \subset I \supset L$ for which each element in R is a minimum and each element in L is a maximum; and in which a morphism $(I, \leq, R, L) \to (I', \leq', R', L')$ is an order preserving map $I \xrightarrow{f} I'$ for which $f(R \sqcup L) \subset R' \sqcup L'$. Concatenating linear orders endows O^{RL} with the structure of a multi-category. Note the evident forgetful morphism between multi-categories $\mathsf{O}^{\mathsf{RL}} \to \mathsf{Env}\big(\mathsf{Assoc}^{\mathsf{RL}}\big)$. By direct inspection, the equivalence (2.3.1) lifts to an equivalence between ∞-operads:

$$\begin{tikzcd}
\mathcal{D}\mathsf{isk}^{\partial,\mathsf{or}}_{1/[-1,1]} \arrow[r, dashed, "\simeq", "{[-]}"'] \arrow[d, "\text{forget}"'] & \mathsf{O}^{\mathsf{RL}} \arrow[d] \\
\mathcal{D}\mathsf{isk}^{\partial,\mathsf{or}}_1 \arrow[r, "\simeq", "{[-]}"'] & \mathsf{Env}(\mathsf{Assoc}^{\mathsf{RL}}).
\end{tikzcd} \tag{2.3.3}$$

Corollary 2.3.23. *Let $(A; P, Q)$ be an $\mathsf{Assoc}^{\mathsf{RL}}$-algebra in $\mathcal{V}$; which is to say an associative algebra A together with a unital left and a unital right A-module. Through Lemma 2.3.20, regard $(A; P, Q)$ as a symmetric monoidal functor $(A; P, Q)\colon \mathcal{D}\mathsf{isk}_1^{\partial,\mathsf{or}} \to \mathcal{V}$. There is a canonical equivalence in $\mathcal{V}$ from the balanced tensor product to factorization homology over the closed interval:*

$$Q \underset{A}{\otimes} P \xrightarrow{\simeq} \int_{[-1,1]} (A; P, Q).$$

Proof. Recognize the opposite of the simplex category as the full subcategory $\Delta^{\mathsf{op}} \subset \mathsf{O}^{\mathsf{RL}}$ consisting of those objects $(I, \leq, R, L)$ for which $R \neq \emptyset \neq L$. Adjoining minima and maxima gives a left adjoint $\mathsf{O}^{\mathsf{RL}} \to \Delta^{\mathsf{op}}$ to the inclusion. Therefore, the inclusion $\Delta^{\mathsf{op}} \to \mathsf{O}^{\mathsf{RL}}$ is final. Concatenating this final functor with the equivalence of Observation 2.3.22 results in a final functor $\mathbf{\Delta}^{\mathsf{op}} \to \mathcal{D}\mathsf{isk}^{\partial,\mathsf{or}}_{1/[-1,1]}$. By inspection, the resulting simplicial object

$$\mathsf{Bar}_\bullet\big(Q,A,P\big)\colon \mathbf{\Delta}^{\mathsf{op}} \to \mathcal{D}\mathsf{isk}^{\partial,\mathsf{or}}_{1/[-1,1]} \to \mathcal{D}\mathsf{isk}^{\partial,\mathsf{or}}_{1} \xrightarrow{[-]} \mathsf{Env}\big(\mathsf{Assoc}^{\mathsf{RL}}\big) \xrightarrow{(A;P,Q)} \mathcal{V}$$

is identified as the two-sided bar construction, as indicated. We conclude the equivalence in $\mathcal{V}$:

$$Q \underset{A}{\otimes} P \;\simeq\; \big|\mathsf{Bar}_\bullet(Q,A,P)\big| \xrightarrow{\simeq} \mathsf{colim}\big(\mathbf{\Delta}^{\mathsf{op}} \xrightarrow{\mathsf{Bar}_\bullet(Q,A,P)} \mathcal{V}\big) \xrightarrow{\simeq} \int_{[-1,1]} (A;P,Q).$$

$\square$

Note that the given multi-functor $\mathsf{O}^{\mathsf{RL}} \to \mathsf{Env}(\mathsf{Assoc}^{\mathsf{RL}})$ witness an equivalence $\mathsf{Env}(\mathsf{O}^{\mathsf{RL}}) \xrightarrow{\simeq} \mathsf{Env}(\mathsf{Assoc}^{\mathsf{RL}})$ between symmetric monoidal ∞-categories. After Observation 2.3.22, this offers the following consequence, which refers to Terminology 2.3.16.

Corollary 2.3.24. *Factorization homology defines an equivalence of ∞-categories,*

$$\int\colon\; \mathsf{Alg}_{\mathsf{Assoc}^{\mathsf{RL}}}(\mathcal{V}) \xrightarrow{\simeq} \mathsf{Alg}_{\mathcal{D}\mathsf{isk}^{\partial,\mathsf{or}}_{1/[-1,1]}}(\mathcal{V}),$$

between associative algebra equipped with a unital left and right module and constructible factorization algebras over the closed interval.

2.3.4 Homology theories: definition

We now define homology theories for B-framed n-manifolds.

Definition 2.3.25 (Collar-gluing)**.** Let M be a B-framed n-manifold. A *collar-gluing* of M is a continuous map

$$f\colon M \to [-1,1]$$

to the closed interval for which the restriction $f_|\colon M_{|(-1,1)} \to (-1,1)$ is a smooth fiber bundle. We will often denote a collar-gluing $M \xrightarrow{f} [-1,1]$ simply as the open cover

$$M_- \bigcup_{M_0\times\mathbb{R}} M_+ \;\cong\; M,$$

where $M_- = f^{-1}[-1,1)$ and $M_+ = f^{-1}(-1,1]$ and $M_0 = f^{-1}\{0\}$.

Remark 2.3.26. We think of a collar-gluing of M as a codimension-1 properly embedded submanifold $M_0 \subset M$ whose complement is partitioned by connected components: $M \smallsetminus M_0 = M_- \sqcup M_+$. Such data is afforded by gluing two manifolds with boundary along a common boundary. The actual data of a collar-gluing specifies that named just above, in addition to a bi-collaring of the common boundary.

Construction 2.2.40 and the results of §2.3.3 give the following.

Corollary 2.3.27. *Let $\mathcal{F}\colon \mathcal{M}\mathsf{fld}_n^B \longrightarrow \mathcal{V}$ be a symmetric monoidal functor. Let M be a B-framed n-manifold. A collar-gluing $M_- \bigcup_{M_0\times\mathbb{R}} M_+ \cong M$ determines an associative algebra $\mathcal{F}(M_0)$ together with a unital left module structure on the object $\mathcal{F}(M_+)$ and a unital right module structure on $\mathcal{F}(M_-)$, as well as a morphism in $\mathcal{V}$:*

$$\mathcal{F}(M_-) \bigotimes_{\mathcal{F}(M_0)} \mathcal{F}(M_+) \longrightarrow \mathcal{F}(M). \tag{2.3.4}$$

Proof. The collar-gluing is a continuous map $M \xrightarrow{f} [-1,1]$. Construction 2.2.40 gives the the first morphism in the composible sequence of morphisms among of ∞-operads:

$$f_*\mathcal{F}\colon\ \mathcal{D}\mathsf{isk}^{\partial,\mathsf{or}}_{1/[-1,1]} \xrightarrow{f^{-1}} \mathcal{M}\mathsf{fld}^B_{n/M} \to \mathcal{M}\mathsf{fld}^B_n \xrightarrow{\mathcal{F}} \mathcal{V}.$$

Through Corollary 2.3.24, this morphism between ∞-operads is equivalent to an algebra in $\mathcal{V}$ over the operad $\mathsf{Assoc}^{\mathsf{RL}}$. Unwinding that equivalence reveals that its underlying associative algebra is $\mathcal{F}(M_0)$, whose underlying object is the value $\mathcal{F}(M_0\times\mathbb{R})$ and whose associative algebra structure is given by oriented embeddings among the $\mathbb{R}$-coordinate, and its underlying unital modules are the values $\mathcal{F}(M_\pm)$. Furthermore, the definition of factorization homology as a colimit supplies the canonical morphism in $\mathcal{V}$:

$$\mathcal{F}(M_-) \bigotimes_{\mathcal{F}(M_0)} \mathcal{F}(M_+) \underset{\text{Cor } 2.3.23}{\simeq} \int_{[-1,1]} f_*\mathcal{F} \longrightarrow \mathcal{F}\big(f^{-1}([-1,1])\big) = \mathcal{F}(M).$$

$\square$

Definition 2.3.28. A symmetric monoidal functor $\mathcal{F} : \mathcal{M}\mathsf{fld}_n^B \to \mathcal{V}$ satisfies $\otimes$-*excision* if, for each collar-gluing $M_- \bigcup_{M_0\times\mathbb{R}} M_+ \cong M$ of B-framed n-manifolds, the canonical morphism in $\mathcal{V}$,

$$\mathcal{F}(M_-) \bigotimes_{\mathcal{F}(M_0\times\mathbb{R})} \mathcal{F}(M_+) \xrightarrow[(2.3.4)]{\simeq} \mathcal{F}(M),$$

is an equivalence. The ∞-category of $\mathcal{V}$-valued *homology theories* for B-framed n-manifolds is the full ∞-subcategory

$$\mathbf{H}(\mathcal{M}\mathsf{fld}_n^B, \mathcal{V}) \ \subset\ \mathsf{Fun}^{\otimes}(\mathcal{M}\mathsf{fld}_n^B, \mathcal{V})$$

consisting of those symmetric monoidal functors that satisfy $\otimes$-excision.

Remark 2.3.29. We emphasize that a $\mathcal{V}$-valued homology theory depends on the symmetric monoidal structure $\otimes$ of $\mathcal{V}$. For instance, let $\Bbbk$ a field and consider the ∞-category $\mathsf{Mod}_\Bbbk$ of $\Bbbk$-modules. There are two natural symmetric monoidal structures on $\mathsf{Mod}_\Bbbk$: direct sum $\oplus$; tensor product $\otimes_\Bbbk$. As established in §2.5.1, a homology theory valued in $(\mathsf{Mod}_\Bbbk, \oplus)$ evaluates on an n-manifold M as the $\Bbbk$-chains, $\mathsf{C}_*(M;\Bbbk)$. On the other hand, in §2.5.2, specifically Remark 2.5.17, shows that a homology theory valued in $(\mathsf{Mod}_\Bbbk, \otimes_\Bbbk)$ typically does *not* factor through the forgetful functor $\mathcal{M}\mathsf{fld}_n \to \mathsf{Spaces}$ to the underlying homotopy type of manifolds.

2.3.5 Pushforward

We prove that factorization homology satisfies $\otimes$-excision in the sense of Definition 2.3.28. We realize this as an instance of a general construction of a pushforward.

The following result is the technical crux of the later results of this article; through this result one can access the values of factorization homology. We state the the result now, and prove it at the end of this section.

Lemma 2.3.30. ([21], [3]) *Let M be a B-framed n-manifold, and let A be a $\mathcal{D}\mathsf{isk}_n^B$-algebra in $\mathcal{V}$, for $\mathcal{V}$ a $\otimes$-presentable ∞-category. For each collar-gluing $M_- \bigcup_{M_0\times\mathbb{R}} M_+ \cong M$, the canonical morphism in $\mathcal{V}$,*

$$\int_{M_-} A \bigotimes_{\int_{M_0\times\mathbb{R}} A} \int_{M_+} A \xrightarrow[(2.3.4)]{\simeq} \int_M A$$

is an equivalence.

Lemma 2.3.30 specializes to to the following special case of $n=1$, $B\simeq *$, and $M=S^1$, which demonstrates the utility of $\otimes$-excision.

Corollary 2.3.31. ([3], [43]) *Let A be an associative algebra in a $\otimes$-presentable ∞-category $\mathcal{V}$. There is a canonical equivalence between objects in $\mathcal{V}$:*

$$\mathsf{HH}_\bullet(A) \ \simeq\ \int_{S^1} A$$

between the Hochschild homology of A and factorization homology of A over the circle.

Proof. Consider the collar-gluing $\mathbb{R}^{\mathsf{op}} \bigcup_{\mathbb{R}^{\mathsf{op}}\sqcup\mathbb{R}} \mathbb{R} \cong S^1$, where the 'op' superscripts denote the opposite orientation from the standard orientation on Euclidean space, which is the map $\mathsf{prom}\colon S^1 \to [-1,1]$ which projects unit vectors in $\mathbb{R}^2$ to the first coordinate. Lemma 2.3.30, applied to this collar-gluing, states the last of the equivalences in the sequence of equivalences in $\mathcal{V}$:

$$\mathsf{HH}_\bullet(A) \ \simeq\ A \bigotimes_{A\otimes A^{\mathsf{op}}} A \ \simeq\ \int_{\mathbb{R}} A \bigotimes_{\int_{S^0\times\mathbb{R}} A} \int_{\mathbb{R}} A \xrightarrow{\simeq} \int_{S^1} A.$$

The middle equivalence is by inspecting values, while the first is definitional. $\square$

The next definition makes use of the multi-functor $f^{-1}\colon \mathcal{D}\mathsf{isk}^{\partial}_{k/N} \to \mathcal{M}\mathsf{fld}^B_{n/M}$ of Construction 2.2.40 associated to each continuous map $M\to N$ for which each of the restrictions, $M_{|N\smallsetminus\partial N} \to N\smallsetminus\partial N$ and $M_{|\partial N}\to\partial N$, are smooth manifold bundles.

Definition 2.3.32. [3] Let M be an B-framed n-manifold, and let N be an oriented k-manifold, possibly with boundary. For $f: M\to N$ a map such that the restrictions of f over both the interior of N and the boundary of N are fiber bundles, then the ∞-category

$\mathcal{D}\mathsf{isk}_f$ is the limit of the diagram among ∞-categories

$$\begin{tikzcd}
\mathcal{D}\mathsf{isk}^B_{n/M} \arrow[dr] & & \mathsf{Ar}(\mathcal{M}\mathsf{fld}^B_{n/M}) \arrow[dl, "\mathsf{ev}_0"'] \arrow[dr, "\mathsf{ev}_1"] & & \mathcal{D}\mathsf{isk}^\partial_{k/N} \arrow[dl, "f^{-1}"'] \\
& \mathcal{M}\mathsf{fld}^B_{n/M} & & \mathcal{M}\mathsf{fld}^B_{n/M} &
\end{tikzcd},$$

where $\mathsf{Ar}(\mathcal{M}\mathsf{fld}^B_{n/M})$ is the ∞-category of functors $[1] \to \mathcal{M}\mathsf{fld}^B_{n/M}$.

Remark 2.3.33. Informally, an object in the ∞-category $\mathcal{D}\mathsf{isk}_f$ is a triple (V, U, β) for which: U is an open subset of N that is diffeomorphic to a disjoint union of Euclidean spaces; V is an open submanifold of M that is diffeomorphic to a disjoint union of Euclidean spaces; and β is an isotopy among embeddings of V into M from the given inclusion to one that factors through $f^{-1}U \hookrightarrow M$.

The following result is stated and proved in §3 of [3]. Its proof amounts to a partition-of-unity argument (specifically Theorem 1.1 [19] or Theorem A.3.1 [43]), applied to unordered configuration spaces.

Lemma 2.3.34. [3] *In the situation of Definition 2.3.32, the functor*

$$\mathsf{ev}_0 : \mathcal{D}\mathsf{isk}_f \to \mathcal{D}\mathsf{isk}^B_{n/M}, \qquad (U, V, \beta) \mapsto V,$$

is final.

Lemma 2.3.34 applied to the fold map $M \sqcup M \to M$ gives the following important technical property to the ∞-category of disks in M.

Corollary 2.3.35 (Corollary 3.22 in [3], Proposition 5.5.2.16 in [43])**.** *For M a B-framed n-manifold, the ∞-overcategory $\mathcal{D}\mathsf{isk}^B_{n/M}$ is sifted.*

We now apply Lemma 3.32 to factorization homology in the next result. In the statement of this result we use the notation

$$f_*A : \ \mathcal{D}\mathsf{isk}^\partial_{k/N} \xrightarrow{f^{-1}} \mathcal{M}\mathsf{fld}^B_{n/M} \xrightarrow{\int A} \mathcal{V}$$

for the composite functor, where f^{-1} is as in Construction 2.2.40.

Proposition 2.3.36 (Proposition 3.23 in [3])**.** *Fix a map of spaces $B \to \mathsf{BO}(n)$. Let M be a B-framed manifold, and let A be a $\mathcal{D}\mathsf{isk}^B_n$-algebra in a $\otimes$-presentable ∞-category $\mathcal{V}$. Let $f : M \to N$ be a continuous map to a smooth k-manifold with boundary whose restriction over the boundary and interior of N is a smooth fiber bundle. There is a canonical map in $\mathcal{V}$,*

$$\int_N f_*A := \mathsf{colim}\Big(\mathcal{D}\mathsf{isk}^\partial_{k/N} \xrightarrow{f_*A} \mathcal{V}\Big) \xrightarrow{\simeq} \int_M A,$$

which is an equivalence.

Proof of Lemma 2.3.30. Let $f : M \to [-1,1]$ be a collar-gluing. There are canonical morphisms in $\mathcal{V}$

$$\int_{M_-} A \bigotimes_{\int_{M_0\times\mathbb{R}} A} \int_{M_+} A \xrightarrow[\text{Cor } 2.3.23]{\simeq} \int_{[-1,1]} f_* A \xrightarrow[\text{Prop } 2.3.36]{\simeq} \int_M A.$$

Corollary 2.3.23 states that the first arrow is an equivalence. Proposition 2.3.36 states that the second morphism is an equivalence. □

2.3.6 Homology theories: characterization

We characterize factorization homology, analogously to Eilenberg–Steenrod's characterization of homology theories for spaces.

Theorem 2.3.37. [3] *Let $\mathcal{V}$ be $\otimes$-presentable ∞-category. Let $B \to \mathsf{BO}(n)$ be a map from a space. There is a canonical equivalence between ∞-categories,*

$$\int : \mathsf{Alg}_{\mathcal{D}\mathsf{isk}^B_n}(\mathcal{V}) \overset{\simeq}{\rightleftarrows} \mathbf{H}(\mathcal{M}\mathsf{fld}^B_n, \mathcal{V}) : \mathsf{ev}_{\mathbb{R}^n},$$

between $\mathcal{D}\mathsf{isk}^B_n$-algebras in $\mathcal{V}$ and homology theories for B-framed n-manifolds with coefficients in $\mathcal{V}$. This equivalence is implemented by the factorization homology functor $\int$ and the functor of evaluation on $\mathbb{R}^n$.

Proof. Proposition 2.3.9 identifies factorization homology as the values of symmetric monoidal left Kan extension, thereby implementing the left adjoint in an adjunction

$$i_!\colon \mathsf{Alg}_{\mathcal{D}\mathsf{isk}^B_n}(\mathcal{V}) \rightleftarrows \mathsf{Fun}^{\otimes}\big(\mathcal{M}\mathsf{fld}^B_n, \mathcal{V}\big)\colon i^*.$$

The unit of this adjunction is an equivalence because $\mathcal{D}\mathsf{isk}^B_n \to \mathcal{M}\mathsf{fld}^B_n$ is fully faithful, and Kan extension along a fully faithful functor restricts as the original functor.

Now let $\mathcal{F}\colon \mathcal{M}\mathsf{fld}^B_n \to \mathcal{V}$ be a symmetric monoidal functor. The counit of this adjunction evaluates on $\mathcal{F}$ as a morphism $\int A \to \mathcal{F}$, where $A = \mathcal{F}_{|\mathbb{R}^n}$ is the $\mathcal{D}\mathsf{isk}^B_n$-algebra defined by the values of $\mathcal{F}$ on disjoint unions of B-framed Euclidean n-spaces. It remains to verify that this counit is an equivalence. To that end, consider the full ∞-subcategory $\mathcal{M} \subset \mathcal{M}\mathsf{fld}^B_n$ consisting of those objects M for which this counit $\int_M A \xrightarrow{\simeq} \mathcal{F}(M)$ is an equivalence. We wish to show the inclusion $\mathcal{M} \hookrightarrow \mathcal{M}\mathsf{fld}^B_n$ is an equivalence.

Both $\int A$ and $\mathcal{F}$ are symmetric monoidal functors. So the full ∞-subcategory $\mathcal{M} \subset \mathcal{M}\mathsf{fld}^B_n$ is symmetric monoidal. By definition, $\mathcal{M}$ contains B-framed Euclidean spaces. We conclude the containment $\mathcal{D}\mathsf{isk}^B_n \subset \mathcal{M}$.

Now, for each $0 \leq k \leq n$, consider the base change B_k of $B \to \mathsf{BO}(n)$ along the map $\mathsf{BO}(k) \times \mathsf{BO}(n-k) \xrightarrow{\oplus} \mathsf{BO}(n)$ then projected $B_k \to \mathsf{BO}(k) \times \mathsf{BO}(n-k) \xrightarrow{\mathsf{proj}} \mathsf{BO}(k)$. We will show, by induction on k, that $\mathcal{M}$ contains the image of the evident functor $\mathcal{M}\mathsf{fld}^{B_k}_k \to \mathcal{M}\mathsf{fld}^B_n$. The previous paragraph gives that this is the case for $k = 0$.

By assumption, $\mathcal{F}$ satisfies $\otimes$-excision, while Lemma 2.3.30 states that $\int A$ satisfies $\otimes$-excision as well. We conclude that $\mathcal{M}$ is closed under collar-gluings. Therefore $\mathcal{M}$

contains the image of any B_k-framed k-manifold that can be witnessed by a finite iteration of collar-gluings from disjoint unions of B_{k-1}-framed $(k-1)$-manifolds. By induction on k, we are therefore reduced to showing each B_k-framed k-manifold can be witnessed as a finite iteration of collar-gluings involving B_k-framed k-manifolds in the image of $\mathcal{M}\mathsf{fld}_{k-1}^{B_{k-1}} \to \mathcal{M}\mathsf{fld}_k^{B_k} \to \mathcal{M}\mathsf{fld}_n^{B}$.

Well, the reigning assumption that manifolds be finitary guarantees, for each B_k-framed k-manifold M, the existence of a compactification $\overline{M}$ of the underlying manifold of M as a smooth k-manifold with boundary. Choose a proper Morse function $f\colon \overline{M} \to [-R, R]$ for which $f^{-1}(\{\pm R\}) = \partial\overline{M}$. Such a Morse function witnesses M as a finite collar-gluing involving B_k-framed k-manifolds of whose underlying manifold is diffeomorphic to $M_0 \times \mathbb{R}$ for some $(k-1)$-manifold M_0. Such a $(k-1)$-manifold is therefore isomorphic to one that is equipped with a B_{k-1}-framing. □

Remark 2.3.38. In [8] a version of Theorem 2.3.37 is established for structured singular manifolds equipped with a stratified tangential structure – this will be surveyed in §2.8. The result, as it appears there, is slightly more general than Theorem 2.3.37 above: the assumption that $\mathcal{V}$ be $\otimes$-presentable can be weakened to the assumption that the underlying ∞-category of $\mathcal{V}$ is presentable yet the symmetric monoidal structure distributes only over geometric realizations and filtered colimits (i.e., over sifted colimits).

Remark 2.3.39. In [3] a version of Theorem 2.3.37 is established for topological manifolds in place of smooth manifolds. In essence, the only feature of the smooth category used in the proof of Theorem 2.3.37 was the existence of open handlebody decompositions, granted by Morse functions. But such open handlebody decompositions are guaranteed in the topological setting, through the results of [48] for $n = 3$, of [50] as it concerns the complement of a point for $n = 4$ together with smoothing theory [32], of [50] for $n = 5$, and of [32] for $n > 5$.

For much of this article, we will primarily concern ourselves with homology theories in the sense of Definition 2.3.28. We point out, however, that there are very interesting functors symmetric monoidal functors $\mathcal{M}\mathsf{fld}_n \to \mathcal{V}$ that are not $\otimes$-excisive. For instance, as detailed in [12], for X a derived (commutative) stack over a fixed field $\Bbbk$, cotensoring to X followed by taking ring of functions defines a symmetric monoidal functor:

$$\mathcal{M}\mathsf{fld}_n \xrightarrow{M \mapsto X^M} \mathsf{Stacks}(\Bbbk) \xrightarrow{\mathcal{O}} \mathsf{Mod}_{\Bbbk}\,.$$

As the case $n = 1$, the value of this functor on the circle is the Hochschild homology of X: $\mathcal{O}(X^{S^1}) \simeq \mathsf{HH}_\bullet(X)$ (compare with Corollary 2.3.31). If X is not affine, the above functor will generically fail to be $\otimes$-excisive. Now, the cotensor X^M only depends on the underlying homotopy type of M, as we shall see in Proposition 2.5.7. This is a feature of X being a derived *commutative* stack. Should X be a derived stack over $\mathcal{D}\mathsf{isk}_n$-algebras as in [20], there is a refined version of the cotensor X^M that is sensitive to the manifold structure of M. Derived stacks over $\mathcal{D}\mathsf{isk}_n$-algebras arise as the outcome of deformation quantization of a shifted symplectic derived (commutative) stacks, such as in Rozansky–Witten theory.

2.4 Nonabelian Poincaré duality

Applying Theorem 2.3.37, we offer a slightly different perspective, and proof, of the nonabelian Poincaré duality of Salvatore [51], Segal [56], and Lurie [43], which calculates factorization homology of iterated loop spaces as compactly-supported mapping spaces.

Definition 2.4.1. For a space B, $\mathsf{Spaces}_B := (\mathsf{Spaces}_{/B})^{B/}$ is the ∞-category of *retractive* spaces over B, i.e., spaces over B equipped with a section. The ∞-category $\mathsf{Spaces}_B^{\geq n}$ is the full ∞-subcategory of Spaces_B consisting of those $X \rightleftarrows B$ for which the retraction is n-connective, that is, the homomorphism $\pi_q X \to \pi_q B$ is an isomorphism for $q < n$ with any choice of base-point of B.

Remark 2.4.2. Equivalently, an object $X \in \mathsf{Spaces}_B^{\geq n}$ may be thought of as a fibration $X \to B$ equipped with a section for which, for each $b \in B$, the fiber X_b is n-connective.

Definition 2.4.3. Fix $X \in \mathsf{Spaces}_B$. Let M be a manifold, equipped with a map $M \to B$. The space of *compactly supported sections* of X over M is the colimit in the ∞-category Spaces:

$$\Gamma_{\mathsf{c}}(M, X) \ := \underset{\underset{\text{compact}}{K \subset M}}{\mathsf{colim}} \mathsf{Map}_{/B}\big((M \smallsetminus K \to M), (B \to X)\big),$$

where the colimit is indexed by the filtered poset of compact codimension-0 submanifold with boundary in M, and where the space indexed by a compact subset $K \subset M$ is the pullback:

$$\begin{tikzcd}
\mathsf{Map}_{/B}\big((M \smallsetminus K \to M), (B \to X)\big) \arrow[r] \arrow[d] & \mathsf{Map}_{/B}(M, X) \arrow[d] \\
\ast \simeq \mathsf{Map}_{/B}(M \smallsetminus K, B) \arrow[r] & \mathsf{Map}_{/B}(M \smallsetminus K, X).
\end{tikzcd}$$

For $B \to \mathsf{BO}(n)$ a map between spaces, note that $\Gamma_{\mathsf{c}}(-, X)$ defines a covariant functor $\mathcal{M}\mathsf{fld}_n^B \to \mathsf{Spaces}$. By inspection, this functor carries finite disjoint unions to finite products of spaces, which is to say that $\Gamma_{\mathsf{c}}(-, X)$ is symmetric monoidal with respect to the Cartesian monoidal structure on the ∞-category Spaces of spaces. In particular, the restriction

$$\Omega^n_B X \colon \mathcal{D}\mathsf{isk}_n^B \hookrightarrow \mathcal{M}\mathsf{fld}_n^B \xrightarrow{\Gamma_{\mathsf{c}}(-, X)} \mathsf{Spaces} \tag{2.4.1}$$

is a symmetric monoidal functor, i.e., a $\mathcal{D}\mathsf{isk}_n^B$-algebra in Spaces. Each point $\ast \xrightarrow{\{b\}} B$ determines a B-framed n-dimensional vector space (V, b), which can be regarded as an object $(V, b) \in \mathcal{D}\mathsf{isk}_n^B$. The value of the above functor $\Omega^n_B X$ on this object is equivalent to the n-fold loop space $\Omega^n X_b$ of the fiber X_b of the map $X \to B$ over $b \in B$.

We introduce the following terminology in order to state Theorem 2.4.5. Consider the base point $\ast \xrightarrow{\{\mathbb{R}^n\}} \mathsf{BO}(n)$. Each lift of this point $\ast \xrightarrow{\{b\}} B$, which is a point in the fiber $\mathsf{fiber}\big(B \to \mathsf{BO}(n)\big)$, canonically determines a symmetric monoidal functor $\mathcal{D}\mathsf{isk}_n^{\mathsf{fr}} \to \mathcal{D}\mathsf{isk}_n^B$. Thereafter, each symmetric monoidal functor $A \colon \mathcal{D}\mathsf{isk}_n^B \to \mathsf{Spaces}$ restricts as an associative monoid

$$\pi_0(A,b)\colon \mathcal{D}\mathsf{isk}_1^{\mathsf{fr}} \xrightarrow{\mathbb{R}^{n-1}\times -} \mathcal{D}\mathsf{isk}_n^{\mathsf{fr}} \longrightarrow \mathcal{D}\mathsf{isk}_n^B \xrightarrow{A} \mathsf{Spaces} \xrightarrow{\pi_0} \mathsf{Set}.$$

We say A is *group-like* if, for each point $b \in \mathsf{fiber}\big(B \to \mathsf{BO}(n)\big)$, this monoid $\pi_0(A,b)$ is a group.

Example 2.4.4. In the case $B \simeq *$, a B-framing is a framing in the standard sense and $\mathcal{D}\mathsf{isk}_n^B = \mathcal{D}\mathsf{isk}_n^{\mathsf{fr}}$. A symmetric monoidal functor $A\colon \mathcal{D}\mathsf{isk}_n^{\mathsf{fr}} \to \mathsf{Spaces}$ is then the data of an $\mathcal{E}_n$-algebra, and this $\mathcal{E}_n$-algebra is group-like in the sense of [45] if and only if A is group-like in the the above sense.

Theorem 2.4.5. [3] *Restricting the functor* $\Gamma_{\mathsf{c}} : \mathsf{Spaces}_B \to \mathsf{Fun}^{\otimes}(\mathcal{M}\mathsf{fld}_n^B, \mathsf{Spaces})$ *defines a fully-faithful inclusion*

$$\mathsf{Spaces}_B^{\geq n} \hookrightarrow \mathbf{H}(\mathcal{M}\mathsf{fld}_n^B, \mathsf{Spaces})$$

of n-connective retractive spaces over B into homology theories. The essential image consists of those $\mathcal{F}$ for which the restriction $\mathcal{F}_{|\mathcal{D}\mathsf{isk}_n^B}$ is group-like.

Remark 2.4.6. The fully faithfulness of the above functor is given by Theorem 5.1.3.6 of [43]. This is a parametrized form of May's theorem from [45], identifying n-connective spaces as $\mathcal{D}\mathsf{isk}_n^{\mathsf{fr}}$-algebras.

The following result is the technical crux in the proof of Theorem 2.4.5; it asserts that, for each n-connective retractive space $X \rightleftarrows B$, the assignment of compactly supported sections $\Gamma_{\mathsf{c}}(-,X)$ is $\otimes$-excisive.

Lemma 2.4.7. [3] *Let $X \in \mathsf{Spaces}_B^{\geq n}$ be an n-connective retractive space over B. Let M be B-framed n-manifold. For $M' \bigcup_{M_0\times\mathbb{R}} M'' \cong M$ a collar-gluing, the canonical map between spaces,*

$$\Gamma_{\mathsf{c}}(M',X) \times_{\Gamma_{\mathsf{c}}(M_0\times\mathbb{R},X)} \Gamma_{\mathsf{c}}(M'',X) \xrightarrow{\simeq} \Gamma_{\mathsf{c}}(M,X),$$

from the quotient of the product $\Gamma_{\mathsf{c}}(M',X)\times\Gamma_{\mathsf{c}}(M'',X)$ by the diagonal action of $\Gamma_{\mathsf{c}}(M_0\times\mathbb{R},X)$, is an equivalence.

Proof given in [3]. Since $M_0 \hookrightarrow M$ is a proper embedding, a compactly supported section over M can be restricted to obtain a compactly-supported section over M_0, as well as over $M \smallsetminus M'$ and over $M \smallsetminus M''$. Namely, there is a diagram among spaces of compactly-supported sections

$$\begin{tikzcd}
\Gamma_{\mathsf{c}}(M',X)\times\Gamma_{\mathsf{c}}(M'',X) \arrow[rr] \arrow[dr] \arrow[dd] & & \Gamma_{\mathsf{c}}(M'',X) \arrow[d] \\
& \Gamma_{\mathsf{c}}(M,X) \arrow[r] \arrow[d] \arrow[dr] & \Gamma_{\mathsf{c}}(M\smallsetminus M',X) \arrow[d] \\
\Gamma_{\mathsf{c}}(M',X) \arrow[r] & \Gamma_{\mathsf{c}}(M\smallsetminus M'',X) \arrow[r] & \Gamma_{\mathsf{c}}(M_0,X).
\end{tikzcd}$$

By inspection, the bottom horizontal sequence is a fiber sequence, as is the right vertical sequence, as is the diagonal sequence. Also, the inner square is pullback because

$M' \bigcup_{M_0 \times \mathbb{R}} M'' \cong M$ is a pushout. Because $M_0 \subset M$ is equipped with a regular neighborhood, these fiber sequences are in fact Serre fibration sequences, and so the inner square is a weak homotopy pullback square. In particular, there is a right homotopy coherent action of $\Omega\,\Gamma_{\mathsf{c}}(M_0, X)$ on $\Gamma_{\mathsf{c}}(M', X)$, a left homotopy coherent action of $\Omega\,\Gamma_{\mathsf{c}}(M_0, X)$ on $\Gamma_{\mathsf{c}}(M'', X)$, and a continuous map of topological spaces

$$\Gamma_{\mathsf{c}}(M', X) \underset{\Omega\,\Gamma_{\mathsf{c}}(M_0, X)}{\times} \Gamma_{\mathsf{c}}(M'', X) \longrightarrow \Gamma_{\mathsf{c}}(M, X) \tag{2.4.2}$$

from the balanced homotopy coinvariants. Because $X \to B$ is n-connective and M_0 is $(n-1)$-dimensional, the base $\Gamma_{\mathsf{c}}(M_0, X)$ is connected. It follows that the map (2.4.2) is in fact a weak homotopy equivalence. The assertion follows after the canonical identification $\Omega\,\Gamma_{\mathsf{c}}(M_0, X) \cong \Gamma_{\mathsf{c}}(M_0 \times \mathbb{R}, X)$ as group-like $\mathcal{E}_1$-spaces. □

Theorem 2.3.37 implies the nonabelian Poincaré duality theorem of Salvatore [51], Segal [56], and Lurie [43]. First, recall from (2.4.1) the $\mathcal{D}\mathsf{isk}_n^B$-algebra $\Omega^n_B X$ determined by a retractive space $X \rightleftarrows B$.

Corollary 2.4.8 (Nonabelian Poincaré duality [3])**.** *Let $B \to \mathsf{BO}(n)$ be a map from a space. Let $X \rightleftarrows B$ be an n-connective retractive space over B. For each B-framed n-manifold M, there is a canonical equivalence*

$$\int_M \Omega^n_B X \xrightarrow{\simeq} \Gamma_{\mathsf{c}}(M, X)$$

from the factorization homology over M of $\Omega^n_B X$ to the space of compactly-supported sections of X over M.

We now explain how this result specializes to familiar Poincaré duality between homology and compactly-supported cohomology. Let A be an abelian group. Consider the product space $X \simeq \mathsf{BO}(n) \times K(A, n)$. Observe that the values of the n-disk algebra

$$\Omega^n_{\mathsf{BO}(n)} X : \mathcal{D}\mathsf{isk}_n \longrightarrow \mathsf{Spaces}, \quad \bigsqcup_{0 \le i \le k} V_i \mapsto \prod_{1 \le i \le k} \mathsf{Map}_*\big(V_i^+, K(A, n)\big) \underset{\text{noncanonical}}{\simeq} A^{\times k},$$

are non-canonically identified as products of A. Furthermore, the restriction of this n-disk algebra along the standard symmetric monoidal functor $\mathcal{D}\mathsf{isk}_n^{\mathsf{fr}} \to \mathcal{D}\mathsf{isk}_n$, which selects Euclidean spaces, is the $\mathcal{E}_n$-algebra forgotten from the commutative algebra structure on A. The $\mathsf{O}(n)$-action on A is that inherited from the infinite-loop structure of the Eilenberg–Mac Lane spectrum $\mathsf{H}A$ through the J-homomorphism. Let M be an n-manifold. An A-orientation of M defines a natural transformation making the diagram of ∞-categories commute:

$$\begin{tikzcd}
\mathcal{D}\mathsf{isk}_{n/M} \arrow[r] \arrow[d, "\pi_0"'] & \mathcal{D}\mathsf{isk}_n \arrow[d, "\Omega^n_{\mathsf{BO}(n)} X"] \\
\mathsf{Fin} \arrow[r, "I \mapsto A^I"] & \mathsf{Spaces};
\end{tikzcd} \tag{2.4.3}$$

here, the bottom horizontal functor is the symmetric monoidal functor from finite sets with disjoint union, which is the free symmetric monoidal ∞-category generated by a point, determined by the commutative algebra A in the Cartesian symmetric monoidal ∞-category Spaces. So choose an A-orientation of M (supposing one exists). Corollary 2.4.8 then yields an equivalence between spaces

$$\int_M A \underset{(2.4.3)}{\simeq} \int_M \Omega^n_{\mathsf{BO}(n)} X \underset{\text{Cor } 2.4.8}{\simeq} \mathsf{Map_c}\big(M, K(A,r)\big). \tag{2.4.4}$$

Note that each space in this display is equipped with a base point, and the maps in this display are canonically based maps. As the $\mathcal{E}_n$-algebra A is forgotten from a commutative algebra, we recognize $\int_M A$ as the free A-module in the ∞-category Spaces generated by the underlying space of M. Therefore, applying homotopy groups to (2.4.4) gives an isomorphism between graded abelian groups,

$$\overline{\mathsf{H}}_*(M;A) \;\cong\; \pi_* \int_M A \;\cong\; \pi_* \mathsf{Map_c}\big(M, K(A,n)\big) \;\cong\; \overline{\mathsf{H}}^{n-*}_{\mathsf{c}}(M;A),$$

from reduced homology to reduced cohomology.

Remark 2.4.9. The factorization homology $\int_M \Omega^n_B X$ is built from configuration spaces of disks in M with labels defined by $X \to B$, and the preceding result thereby has roots in the configuration space models of mapping spaces dating to the work of Segal, May, McDuff and others in the 1970s; see [53], [45], [46], and [14]. However, the classical configuration-space-with-labels, as described in [14], models a mapping space with target the n-fold suspension of X, rather than into X itself. Factorization homology thus more closely generalizes the configuration spaces with *summable* or *amalgamated* labels of Salvatore [51] and Segal [56].

Proof of Theorem 2.4.5. Corollary 2.4.8 identifies the functor

$$\Gamma_{\mathsf{c}}\colon\ \mathsf{Spaces}_B^{\geq n} \to \mathbf{H}(\mathcal{M}\mathsf{fld}_n^B, \mathsf{Spaces})$$

as the composition $\Gamma_{\mathsf{c}}\colon\ \mathsf{Spaces}_B^{\geq n} \xrightarrow{\Omega^n_B} \mathcal{D}\mathsf{isk}_n^B \xrightarrow{\int} \mathbf{H}(\mathcal{M}\mathsf{fld}_n^B, \mathsf{Spaces})$. Theorem 2.3.37 gives that $\int$ is fully faithful, so it remains to argue that Ω^n_B is fully faithful with essential image the group-like $\mathcal{D}\mathsf{isk}_n^B$-algebras in spaces. This is immediate because, for instance, $\mathsf{Spaces}_{/B}$ is an ∞-topos (Theorem 5.1.3.6 of [43]). $\square$

2.5 Calculations

In §2.4, we identified values of factorization homology of $\mathcal{D}\mathsf{isk}_n$-algebras in spaces, with its Cartesian monoidal structure, as twisted compactly supported mapping spaces. Here we identify more values of factorization homology of $\mathcal{D}\mathsf{isk}_n$-algebras in chain complexes, and spectra, notably with monoidal structure given by tensor product – these cases are closest to the physical motivation given in the introduction.

In this section, we fix a map $B \to \mathsf{BO}(n)$ from a space. Two cases of notable interest are $* \xrightarrow{\{\mathbb{R}^n\}} \mathsf{BO}(n)$ and $\mathsf{BO}(n) \xrightarrow{\mathsf{id}} \mathsf{BO}(n)$.

2.5.1 Factorization homology for direct sum

We now show that in the simple case of $\mathcal{D}\mathsf{isk}_n$-algebras in chain complexes with direct sum, factorization homology recovers ordinary homology.

Fix a presentable ∞-category $\mathcal{V}$, such as the ∞-category $\mathsf{Spectra}$ of spectra, or the ∞-category $\mathsf{Mod}_{\Bbbk}$ of chain complexes over a fixed commutative ring $\Bbbk$. Coproduct, which in the case that $\mathcal{V}$ is stable is direct sum, endows $\mathcal{V}$ with the coCartesian symmetric monoidal structure: $(\mathcal{V}, \amalg)$. Consider the commutative diagram among ∞-categories, in which each functor is implemented by the evident restriction:

$$\begin{tikzcd}
\mathsf{Alg}_{\mathcal{D}\mathsf{isk}_n^B}(\mathcal{V},\amalg) \arrow[d, "\simeq"'] & \mathsf{Fun}^{\otimes}\big(\mathcal{M}\mathsf{fld}_n^B, (\mathcal{V},\amalg)\big) \arrow[l] \arrow[dl] \\
\mathsf{Fun}(B,\mathcal{V}) &
\end{tikzcd}.$$

In this case of a coCartesian symmetric monoidal structure, the downward functor is an equivalence, as indicated. In other words, a $\mathcal{D}\mathsf{isk}_n^B$-algebra in $(\mathcal{V}, \amalg)$ is precisely a functor $B \to \mathcal{V}$. [9]

Example 2.5.1. Let G be a topological group equipped with a continuous representation $G \xrightarrow{\rho} \mathsf{GL}(n) \simeq \mathsf{O}(n)$. Then a $\mathcal{D}\mathsf{isk}_n^{\mathsf{B}G}$-algebra in $(\mathcal{V}, \amalg)$ is a G-module $\mathsf{B}G \to \mathcal{V}$. In the case that $G = e$ is the trivial group, such a G-module is simply an object in $\mathcal{V}$.

Through Proposition 2.3.9, which identifies factorization homology as a left adjoint to the top horizontal functor in the preceding diagram, we conclude that factorization is identified as a left adjoint to the downleftward functor, which is restriction along the fully faithful inclusion $B \to \mathcal{M}\mathsf{fld}_n^B$ of B-framed Euclidean spaces. Left adjoints to restriction functors evaluate as left Kan extensions. The standard formula for left Kan extensions therefore identifies, for each B-framed n-manifold $M = (M, \varphi)$ and each $\mathcal{D}\mathsf{isk}_n^B$-algebra $B \xrightarrow{V} \mathcal{V}$, the value of factorization homology as the colimit in $\mathcal{V}$:

$$\int_M V \;\simeq\; \mathsf{colim}\big(B_{/M} \to B \xrightarrow{V} \mathcal{V}\big).$$

Corollary 2.2.31 identifies the functor $B_{/M} \to B$ appearing in this expression as the given functor $M \xrightarrow{\varphi} B$ from the underlying ∞-groupoid, or homotopy type, of M:

$$\int_M V \;\simeq\; \mathsf{colim}\big(M \xrightarrow{\varphi} B \xrightarrow{V} \mathcal{V}\big).$$

Remark 2.5.2. Note that the value of factorization homology in $(\mathcal{V}, \amalg)$ over a B-framed manifold $M \xrightarrow{\varphi} B$ only depends on the underlying space of M equipped with its given map to B.

Example 2.5.3. Let G be a topological group equipped with a continuous representation $G \xrightarrow{\rho} \mathsf{GL}(n) \simeq \mathsf{O}(n)$. Let V be a G-module in $\mathcal{V}$. Let M be an n-manifold. A $\mathsf{B}G$-framing

[9] Here and always, the space B is taken as an ∞-groupoid and thereafter as an ∞-category. In this way, we make sense of a functor $B \to \mathcal{V}$, which is simply a B-indexed local system of objects in $\mathcal{V}$.

on M is a principal G-bundle $P \to M$ equipped with a ρ-equivariant map $P \to \mathsf{Fr}(M)$ over M to the frame-bundle of M. Given such a $\mathsf{B}G$-framing on M, we identify the factorization homology

$$\int_M V \ \simeq\ P \underset{G}{\otimes} V$$

as the diagonal G-quotient of the tensor of $V \in \mathcal{V}$ with the space P. In particular, in the case that $G \xrightarrow{=} \mathsf{O}(n)$, the factorization homology is the diagonal $\mathsf{O}(n)$-quotient with the frame-bundle of M tensored with the $\mathsf{O}(n)$-module V in $\mathcal{V}$:

$$\int_M V \ \simeq\ \mathsf{Fr}(M) \underset{\mathsf{O}(n)}{\otimes} V.$$

In the case that $G = e$ is a trivial group, then $V \in \mathcal{V}$ is simply an object and the principal bundle $P \xrightarrow{=} M$ is identical to M, so that factorization homology

$$\int_M V \ \simeq\ M \otimes V$$

is simply the tensor of the object $V \in \mathcal{V}$ with the underlying space of M.

Example 2.5.4. Suppose $B = *$, so that a B-framing on an n-manifold is exactly a framing. Suppose $\mathcal{V} = \mathsf{Mod}_{\Bbbk}$ is the ∞-category of chain complexes over a fixed commutative ring $\Bbbk$. As discussed above, a $\mathcal{E}_n$-algebra in $(\mathsf{Mod}_{\Bbbk}, \oplus)$ is simply a chain complex, $V \in \mathsf{Mod}_{\Bbbk}$. Furthermore, the factorization homology

$$\int_M V \ \simeq\ M \otimes V \ \simeq\ \mathsf{C}_*(M; V) \ \in \mathsf{Mod}_{\Bbbk}$$

is the singular chain complex of M with coefficients in V.

Example 2.5.5. Again suppose $B = *$, so that a B-framing on an n-manifold is exactly a framing. Suppose $\mathcal{V} = \mathsf{Spectra}$ is the ∞-category of spectra. As discussed above, an $\mathcal{E}_n$-algebra in $(\mathsf{Spectra}, \oplus)$ is simply a spectrum $V \in \mathsf{Spectra}$. Furthermore, the factorization homology

$$\int_M V \ \simeq\ M \otimes V \ \simeq\ \Sigma^\infty_+ M \wedge V \ \in \mathsf{Spectra}$$

is the smash product of V with the suspension spectrum of M.

Remark 2.5.6. Taking products with Euclidean spaces defines a sequence of symmetric monoidal ∞-categories:

$$\mathcal{M}\mathsf{fld}_0^{\mathsf{fr}} \longrightarrow \mathcal{M}\mathsf{fld}_1^{\mathsf{fr}} \longrightarrow \cdots \longrightarrow \mathcal{M}\mathsf{fld}_n^{\mathsf{fr}} \longrightarrow \cdots.$$

Forgetting to underlying spaces defines, for each k, a symmetric monoidal functor $\mathcal{M}\mathsf{fld}_k^{\mathsf{fr}} \to \mathsf{Spaces}^{\mathsf{fin}}$ to the ∞-category of finite spaces (i.e., finite CW complexes) and coproduct among them. Via the canonical equivalence between underlying spaces $M \simeq M \times \mathbb{R}$ for each framed manifold M, these forgetful functors assemble as a symmetric monoidal functor from the colimit

$$\underset{k \geq 0}{\mathsf{colim}}\, \mathcal{M}\mathsf{fld}_k^{\mathsf{fr}} \longrightarrow \mathsf{Spaces}^{\mathsf{fin}}\,. \tag{2.5.1}$$

We argue that this symmetric monoidal functor is an equivalence.

- **Essentially surjective.** Each finite space is equivalent to the underlying space of the geometric realization of a finite simplicial complex. Such a geometric realization can be piecewise-linearly embedded into Euclidean space. A regular open neighborhood of such an embedding is a finitary smooth manifold that is homotopy equivalent to this geometric realization. In this way we conclude that the functor (2.5.1) is essentially surjective.

- **Fully faithful.** Let M and N be n-manifolds. It follows from Whitney's embedding theorem that the map between spaces

$$\operatorname{colim}_{k\geq 0} \mathsf{Emb}^{\mathsf{fr}}(M \times \mathbb{R}^{k-n}, N \times \mathbb{R}^{k-n}) \longrightarrow \mathsf{Map}_{\mathsf{Spaces}}(M, N)$$

  is a weak homotopy equivalence.

Through the equivalence (2.5.1), Theorem 2.3.37 applied to $\mathcal{V} = (\mathsf{Mod}_{\Bbbk}, \oplus)$ recovers to the formulation of the Eilenberg-Steenrod axioms given in the introduction. If one sets $\mathcal{V}$ to be the opposite $(\mathsf{Mod}_{\Bbbk}^{\mathsf{op}}, \oplus)$, then one likewise recovers the Eilenberg–Steenrod axioms for cohomology.

2.5.2 Factorization homology with coefficients in commutative algebras

Here, we examine factorization homology of commutative algebras, otherwise known as $\mathcal{E}_\infty$-algebras.

Fix a symmetric monoidal ∞-category $\mathcal{V}$ that is $\otimes$-presentable.

There is a canonical identification of the symmetric monoidal envelope of the commutative ∞-operad

$$\mathsf{Env}(\mathsf{Comm}) \ \simeq \ \mathsf{Fin} = (\mathsf{Fin}, \amalg)$$

with the coCartesian symmetric monoidal category of finite sets. [10] This is to say that restriction along the commutative algebra in Fin whose underlying object is $* \in \mathsf{Fin}$ defines an equivalence between ∞-categories:

$$\mathsf{Fun}^{\otimes}(\mathsf{Fin}, \mathcal{V}) \xrightarrow{\simeq} \mathsf{Alg}_{\mathsf{Com}}(\mathcal{V}).$$

So restriction along the symmetric monoidal functor given by taking connected components

$$[-] : \mathcal{D}\mathsf{isk}_n^B \to \mathcal{D}\mathsf{isk}_n \xrightarrow{[-]} \mathsf{Fin}$$

defines a forgetful functor

$$\mathsf{fgt}\colon\ \mathsf{Alg}_{\mathsf{Com}}(\mathcal{V}) \longrightarrow \mathsf{Alg}_{\mathcal{D}\mathsf{isk}_n^B}(\mathcal{V}).$$

[10] See the discussion just above Lemma 2.3.20 for a quick description of a *symmetric monoidal envelope*.

Via Corollary 3.2.3.3 of [43], the assumed $\otimes$-presentability of $\mathcal{V}$ implies the ∞-category $\mathsf{Alg}_{\mathsf{Com}}(\mathcal{V})$ is presentable. In particular, it is tensored over the ∞-category of spaces:

$$\mathsf{Spaces} \times \mathsf{Alg}_{\mathsf{Com}}(\mathcal{V}) \xrightarrow{\otimes} \mathsf{Alg}_{\mathsf{Com}}(\mathcal{V}), \qquad (X, A) \mapsto \mathsf{colim}\big(X \xrightarrow{!} * \xrightarrow{\{A\}} \mathsf{Alg}_{\mathsf{Com}}(\mathcal{V})\big).$$

Proposition 2.5.7. *The following diagram among ∞-categories canonically commutes:*

$$\begin{tikzcd}
\mathcal{M}\mathsf{fld}_n^B \times \mathsf{Alg}_{\mathsf{Com}}(\mathcal{V}) \arrow[r, "\mathrm{forget}\times\mathsf{id}"] \arrow[d, "\mathsf{id}\times\mathrm{fgt}"'] & \mathsf{Spaces} \times \mathsf{Alg}_{\mathsf{Com}}(\mathcal{V}) \arrow[r, "\otimes"] & \mathsf{Alg}_{\mathsf{Com}}(\mathcal{V}) \arrow[d, "\mathrm{forget}"] \\
\mathcal{M}\mathsf{fld}_n^B \times \mathsf{Alg}_{\mathcal{D}\mathsf{isk}_n^B}(\mathcal{V}) \arrow[rr, "\int"] & & \mathcal{V}.
\end{tikzcd}$$

In particular, for each B-framed n-manifold M, and each commutative algebra A, there is a canonical equivalence

$$\int_M A \;\simeq\; M \otimes A$$

between the factorization homology of A over M and the tensor of the commutative algebra A with the underlying space of M.

Proof. Observe the commutative diagram among symmetric monoidal ∞-categories:

$$\begin{tikzcd}
\mathcal{D}\mathsf{isk}_n^B \arrow[r] \arrow[d, "{[-]}"'] & \mathcal{M}\mathsf{fld}_n^B \arrow[d, "\mathrm{forget}"] \\
\mathsf{Fin} \arrow[r] & \mathsf{Spaces}
\end{tikzcd}$$

in which the bottom two are endowed with their coCartesian symmetric monoidal structures. Restriction defines the commutative diagram among ∞-categories of symmetric monoidal functors

$$\begin{tikzcd}
\mathsf{Alg}_{\mathcal{D}\mathsf{isk}_n^B}(\mathcal{V}) & \mathsf{Fun}^{\otimes}(\mathcal{M}\mathsf{fld}_n^B, \mathcal{V}) \arrow[l] \\
\mathsf{Alg}_{\mathsf{Com}}(\mathcal{V}) \arrow[u, "\mathsf{fgt}"] & \mathsf{Fun}^{\otimes}(\mathsf{Spaces}, \mathcal{V}). \arrow[l] \arrow[u]
\end{tikzcd} \tag{2.5.2}$$

It follows from Proposition 3.2.4.7 of [43] that the forgetful functor $\mathsf{Alg}_{\mathsf{Com}}(\mathcal{V}) \to \mathcal{V}$ canonically lifts as a symmetric monoidal functor $\big(\mathsf{Alg}_{\mathsf{Com}}(\mathcal{V}), \amalg\big) \to \big(\mathcal{V}, \otimes\big)$ from the coCartesian symmetric monoidal structure on $\mathsf{Alg}_{\mathsf{Com}}(\mathcal{V})$. Postcomposing by this symmetric monoidal functor determines the commutative diagram among ∞-categories:

$$\begin{tikzcd}
\mathsf{Alg}_{\mathsf{Com}}\big(\mathsf{Alg}_{\mathsf{Com}}(\mathcal{V}), \amalg\big) \arrow[d] & \mathsf{Fun}^{\otimes}\big(\mathsf{Spaces}, (\mathsf{Alg}_{\mathsf{Com}}(\mathcal{V}), \amalg)\big) \arrow[l] \arrow[d] \\
\mathsf{Alg}_{\mathsf{Com}}(\mathcal{V}) & \mathsf{Fun}^{\otimes}(\mathsf{Spaces}, \mathcal{V}). \arrow[l]
\end{tikzcd} \tag{2.5.3}$$

It follows from §2.5.1 that the right downward functor in this diagram is an equivalence. From the same proposition cited in [43] above, the right downward functor preserves colimits.

From §2.5.1, we conclude that the left adjoint to the bottom horizontal functor in (2.5.2) is the composite functor

$$\mathsf{Alg}_{\mathsf{Com}}(\mathcal{V}) \xrightarrow{A\mapsto(X\mapsto X\otimes A)} \mathsf{Fun}^{\otimes}\big(\mathsf{Spaces}, \mathsf{Alg}_{\mathsf{Com}}(\mathcal{V})\big) \xrightarrow{\text{forget}} \mathsf{Fun}^{\otimes}(\mathsf{Spaces}, \mathcal{V})$$

of first tensoring followed by the forgetful functor. Now, taking left adjoints of the horizontal functors results in a lax-commutative diagram among ∞-categories:

$$\begin{tikzcd}
\mathsf{Alg}_{\mathcal{D}\mathsf{isk}_n^B}(\mathcal{V}) \arrow[rr, bend left, "\int"] & \Downarrow & \mathsf{Fun}^{\otimes}(\mathcal{M}\mathsf{fld}_n^B, \mathcal{V}) \\
\mathsf{Alg}_{\mathsf{Com}}(\mathcal{V}) \arrow[u, "\mathsf{fgt}"] \arrow[r, "\otimes"] & \mathsf{Fun}^{\otimes}\big(\mathsf{Spaces}, (\mathsf{Alg}_{\mathsf{Com}}(\mathcal{V}), \amalg)\big) \arrow[r, "\text{forget}"] & \mathsf{Fun}^{\otimes}(\mathsf{Spaces}, \mathcal{V}). \arrow[u]
\end{tikzcd}$$

The proof is complete once we show the 2-cell in this diagram is invertible. This 2-cell is invertible if and only if, for each commutative algebra $A \in \mathsf{Alg}_{\mathsf{Com}}(\mathcal{V})$ and each B-framed n-manifold M, the resulting morphism in $\mathcal{V}$,

$$\int_M A \longrightarrow M \otimes A, \tag{2.5.4}$$

is an equivalence. (Here, and in what follows, we are not giving notation to the functor fgt.) Because the underlying space of each B-framed Euclidean n-dimensional space is contractible, this morphism (2.5.4) is an equivalence in such cases. By construction, the tensoring functor $\mathsf{Spaces} \xrightarrow{X\mapsto X\otimes A} \mathsf{Alg}_{\mathsf{Com}}(\mathcal{V})$ preserves colimits. Because collar-gluings among B-framed n-manifolds forget as pushouts among their underlying spaces, it follows that the functor $\mathcal{M}\mathsf{fld}_n^B \xrightarrow{M\mapsto M\otimes A} \mathsf{Alg}_{\mathsf{Com}}(\mathcal{V}) \xrightarrow{\text{forget}} \mathcal{V}$ is $\otimes$-excisive, and in particular it is symmetric monoidal. It follows from Theorem 2.3.37 that the morphism (2.5.4) is an equivalence, as desired. □

Proposition 2.5.7, together with the universal property of tensoring over spaces, yields the following.

Corollary 2.5.8. *Let $A \in \mathsf{Alg}_{\mathsf{Com}}(\mathcal{V})$ be a commutative algebra in a symmetric monoidal ∞-category that is $\otimes$-presentable. Let M be a B-framed n-manifold. There is a canonical commutative algebra structure on the factorization homology $\int_M A$. Furthermore, as a commutative algebra, it corepresents the copresheaf*

$$\mathsf{Map}_{\mathsf{Com}}\Big(\int_M A, -\Big)\colon\ \mathsf{Alg}_{\mathsf{Com}}(\mathcal{V}) \longrightarrow \mathsf{Spaces}, \qquad C \mapsto \mathsf{Map}_{\mathsf{Com}}(A, C)^M,$$

whose value on a commutative algebra is the the cotensor, or mapping space, of M to the space of morphisms from A to C.

Proposition 2.5.7 and Corollary 2.5.8, together with the fact that left adjoints compose, yields the following.

Corollary 2.5.9. *In the situation of Corollary 2.5.8, for $V \in \mathcal{V}$ an object with $\mathsf{Sym}(V) \in \mathsf{Alg}_{\mathsf{Com}}(\mathcal{V})$ the free commutative algebra on V, there is a canonical equivalence between commutative algebras in $\mathcal{V}$:*

$$\int_M \mathsf{Sym}(V) \;\simeq\; \mathsf{Sym}(M \otimes V).$$

Example 2.5.10. In the case that $\mathcal{V} = (\mathsf{Mod}_{\Bbbk}, \otimes_{\Bbbk})$, Corollary 2.5.9 specializes as an equivalence

$$\int_M \mathsf{Sym}(V) \;\simeq\; \mathsf{Sym}\big(\mathsf{C}_*(M;V)\big),$$

the graded-commutative algebra on the chain complex of singular chains on the underlying space of M with coefficients in V.

The next result, proved as Proposition 5.3 in [3], further identifies factorization homology of commutative algebras, in a suitably infinitesimal case: the cohomology of suitably nilpotent spaces. Its proof amounts to using Proposition 2.5.7, then arguing via an Eilenberg–Moore spectral sequence.

Proposition 2.5.11. [3] *Let M be an n-manifold. Let X be a nilpotent space whose first n homotopy groups are finite, $|\pi_i X| < \infty$ for $i \leq n$, and let $\Bbbk$ be a commutative ring. There is a natural equivalence of $\Bbbk$-modules*

$$\int_M \mathsf{C}^*(X;\Bbbk) \;\simeq\; \mathsf{C}^*(X^M;\Bbbk)$$

between the factorization homology of M with coefficient in the $\Bbbk$-cohomology of X and the $\Bbbk$-cohomology of the space of maps to X from the underlying space of M.

Remark 2.5.12. In the work [25], the authors lay out a similar approach, by way of Hochschild homology-type invariants, to the study of the cohomology of mapping spaces.

2.5.3 Factorization homology from Lie algebras

We now identify factorization homology of $\mathcal{D}\mathsf{isk}_n$-algebras coming from Lie algebras. The results that follow are closely analogous to those above concerning the factorization homology of $\mathcal{D}\mathsf{isk}_n$-algebras coming from based topological spaces. For simplicity, we assume our Lie algebras are defined over a fixed field $\Bbbk$ of characteristic zero.

As we proceed, we use that the ∞-category $\mathsf{Alg}_{\mathsf{Lie}}(\mathsf{Mod}_{\Bbbk})$ of Lie algebras over $\Bbbk$ are cotensored over spaces:

$$\mathsf{Spaces}^{\mathsf{op}} \times \mathsf{Alg}_{\mathsf{Lie}}(\mathsf{Mod}_{\Bbbk}) \xrightarrow{(X,\mathfrak{g}) \mapsto \mathfrak{g}^X} \mathsf{Alg}_{\mathsf{Lie}}(\mathsf{Mod}_{\Bbbk}),\;\; (X,\mathfrak{g}) \mapsto \lim\Big(X \xrightarrow{!} * \xrightarrow{\{\mathfrak{g}\}} \mathsf{Alg}_{\mathsf{Lie}}(\mathsf{Mod}_{\Bbbk})\Big).$$

This limit indeed exists because, for instance, the ∞-category $\mathsf{Alg}_{\mathsf{Lie}}(\mathsf{Mod}_{\Bbbk})$ admits products, cofiltered limits, and totalizations. More explicitly, $\mathfrak{g}^X \simeq \mathsf{C}^*(X;\mathfrak{g}) = \mathsf{Hom}_{\Bbbk}\big(\mathsf{C}_*(X;\Bbbk),\mathfrak{g}\big)$, with a canonically inherited Lie algebra structure. For M an n-manifold, we define the kernel Lie algebra,

$$\begin{tikzcd} \mathsf{Map}_{\mathsf{c}}(M,\mathfrak{g}) \arrow[r] \arrow[d] & \mathfrak{g}^{M^+} \arrow[d, "\mathsf{ev}_+"] \\ 0 \arrow[r] & \mathfrak{g}, \end{tikzcd}$$

where M^+ is the 1-point compactification of M. More explicitly, $\mathsf{Map}_{\mathsf{c}}(M,\mathfrak{g}) \simeq \mathsf{C}^*_{\mathsf{c}}(M;\mathfrak{g})$, the compactly-supported cochains on M with coefficients in $\mathfrak{g}$. In the case that $M = \mathbb{R}^n$, we notate $\Omega^n\mathfrak{g} := \mathsf{Map}_{\mathsf{c}}(\mathbb{R}^n,\mathfrak{g})$, and observe that this Lie algebra assembles as a $\mathcal{D}\mathsf{isk}_n$-algebra,

$$\Omega^n\mathfrak{g}\colon\ \mathcal{D}\mathsf{isk}_n \longrightarrow \big(\mathsf{Alg}_{\mathsf{Lie}}(\mathsf{Mod}_{\Bbbk}),\times\big),$$

to the Cartesian symmetric monoidal structure on Lie algebras. Postcomposing this symmetric monoidal functor by that of Lie algebra chains, which indeed carries finite products of Lie algebras to finite tensor products (over $\Bbbk$) of $\Bbbk$-modules, results in a composite symmetric monoidal functor

$$\mathsf{C}^{\mathsf{Lie}}_*\big(\Omega^n\mathfrak{g}\big)\colon\ \mathcal{D}\mathsf{isk}_n \xrightarrow{\Omega^n\mathfrak{g}} \big(\mathsf{Alg}_{\mathsf{Lie}}(\mathsf{Mod}_{\Bbbk}),\times\big) \xrightarrow{\mathsf{C}^{\mathsf{Lie}}_*} \big(\mathsf{Mod}_{\Bbbk},\underset{\Bbbk}{\otimes}\big).$$

The next identification is also discussed in [27] and [17], and appears in the present form in [3]. Its proof amounts to applying Lie algebra chains, which is a geometric realization-preserving functor, to the Poincaré/Koszul duality equivalence $\int_M \Omega^n\mathfrak{g} \simeq \mathsf{Map}_{\mathsf{c}}(M,\mathfrak{g})$ of [4].

Proposition 2.5.13. *Let M be an n-manifold. Let $\mathfrak{g}$ be a Lie algebra over $\Bbbk$. There is a canonical identification between $\Bbbk$-modules,*

$$\int_M \mathsf{C}^{\mathsf{Lie}}_*\big(\Omega^n\mathfrak{g}\big) \ \simeq\ \mathsf{C}^{\mathsf{Lie}}_*\big(\mathsf{Map}_{\mathsf{c}}(M,\mathfrak{g})\big).$$

Remark 2.5.14. Let $\mathfrak{g}$ be a Lie algebra over $\Bbbk$. The $\mathcal{D}\mathsf{isk}_n$-algebra $\mathsf{C}^{\mathsf{Lie}}_*\big(\Omega^n\mathfrak{g}\big)$ appearing above has an interesting interpretation, established by Knudsen [35]. There is a forgetful functor from $\mathcal{E}_n$-algebras to Lie algebras,

$$\mathsf{Alg}_{\mathsf{Lie}}(\mathsf{Mod}_{\Bbbk}) \longleftarrow \mathsf{Alg}_{\mathcal{E}_n}(\mathsf{Mod}_{\Bbbk})\ :\mathsf{fgt}.$$

For $n = 1$, there is the standard forgetful functor; for $n > 1$, Kontsevich's formality result (see [37] for a sketched outline, and [39] for a more detailed account) identifies the operad $\mathsf{C}_*(\mathcal{E}_n;\mathbb{R})$ in $\mathsf{Mod}_{\Bbbk}$ is equivalent to the Poisson operad P_n, to which the Lie operad in $\mathsf{Mod}_{\mathbb{R}}$ maps.[11] (See [16] for an account at the level of homology.) The adjoint functor theorem (Corollary 5.5.2.9 of [42]) applies to this forgetful functor, thereby offering a left adjoint:

$$\mathsf{U}_n\colon\ \mathsf{Alg}_{\mathsf{Lie}}(\mathsf{Mod}_{\Bbbk}) \ \rightleftarrows\ \mathsf{Alg}_{\mathcal{E}_n}(\mathsf{Mod}_{\Bbbk})\ :\mathsf{fgt}.$$

In the case $n = 1$, this left adjoint U_1 agrees with the familiar universal enveloping algebra functor. In general, there is an identification between $\mathcal{E}_n$-algebras,

$$\mathsf{U}_n\mathfrak{g}\ \simeq\ \mathsf{C}^{\mathsf{Lie}}_*\big(\Omega^n\mathfrak{g}\big)$$

[11] The operad P_n in $\mathsf{Mod}_{\Bbbk}$ represents: a $\mathsf{Mod}_{\Bbbk}$-module with a commutative and associative multiplication $\cdot$; an $(n-1)$-shifted binary operation $[-,-]$ that satisfies the Jacobi identity; the structure of this Lie bracket $[-,-]$ being a derivation over $\cdot$ in each variable.

proved by Knudsen [35]. Through this identification, Proposition 2.5.13 can be reformulated as an equivalence of chain complexes over $\Bbbk$,

$$\int_M \mathsf{U}_n\mathfrak{g} \;\simeq\; \mathsf{C}_*^{\mathsf{Lie}}\big(\mathsf{Map_c}(M,\mathfrak{g})\big),$$

in the case that M is equipped with a framing.

2.5.4 Factorization homology of free $\mathcal{D}\mathsf{isk}_n^B$-algebras

We now identify the factorization homology of free $\mathcal{D}\mathsf{isk}_n^B$-algebras. The results here are extracted from §5.2 of [3], §4.3 of [8], and §2.4 of [5].

For this section, we fix a symmetric monoidal ∞-category $\mathcal{V}$ that is $\otimes$-presentable. Also, for simplicity restrict our attention to the case the space B is connected, and equipped with a base point. This condition, and structure, on B identify the tangential structure

$$(B \to \mathsf{BO}(n)) \;=\; (\mathsf{B}G \xrightarrow{\mathsf{B}\rho} \mathsf{BO}(n))$$

as the classifying space of a morphism $\rho\colon G \to \mathsf{O}(n)$ between group-objects in Spaces.

For each B-framed n-manifold M, and for each finite cardinality $i \geq 0$, consider the configuration space

$$\mathsf{Conf}_i(M) \;:=\; \Big\{\{1,\dots,i\} \xrightarrow{c} M \mid c \text{ is injective}\Big\} \;\subset\; M^{\times i},$$

which is an open subspace of the i-fold product of M. As an open subspace, it inherits the structure of a smooth ni-manifold. The $\mathsf{B}G$-framing on M then determines a $\mathsf{B}(\Sigma_i \wr G)$-framing on $\mathsf{Conf}_i(M)$:

$$\begin{tikzcd} & \mathsf{B}(\Sigma_i \wr G) \arrow[d] \\ \mathsf{Conf}_i(M) \arrow[ur] \arrow[r, "\tau_{\mathsf{Conf}_i(M)}"] & \mathsf{BO}(ni). \end{tikzcd}$$

This lift of the tangent classifier of $\mathsf{Conf}_i(M)$ selects a principal $\Sigma_i \wr G$-bundle

$$\mathsf{Conf}_i^G(M) \longrightarrow \mathsf{Conf}_i(M),$$

which we refer to as a *G-framed* configuration space. In the extremal case of $G \xrightarrow{=} \mathsf{O}(n)$, then we have $\mathsf{Conf}_i^G(M) = \mathsf{Conf}_i(M)$. In the extremal case that $G = e$ is the trivial group then $\mathsf{Conf}_i^G(M) = \mathsf{Conf}_i^{\mathsf{fr}}(M)$ is the *framed* configuration space, a point in which is an injection $c\colon \{1,\dots,i\} \hookrightarrow M$ together with a choice of the basis for each tangent space $T_{c(j)}M$.

The *free $\mathcal{D}\mathsf{isk}_n^B$-algebra* functor is the left adjoint

$$\mathsf{Free}_n^B\colon\ \mathsf{Mod}_G(\mathcal{V}) \;:=\; \mathsf{Fun}(B,\mathcal{V}) \;\rightleftarrows\; \mathsf{Alg}_{\mathcal{D}\mathsf{isk}_n^B}(\mathcal{V})$$

to the restriction along the fully faithful inclusion $B \to \mathcal{D}\mathsf{isk}_n^B$ of the B-framed Euclidean spaces.

The next result identifies the factorization homology of free $\mathcal{D}\mathsf{isk}_n^{\mathsf{B}G}$-algebras in terms of G-framed configuration spaces.

Proposition 2.5.15. *Let $\rho\colon G \to \mathsf{O}(n)$ be a morphism between group-objects in* Spaces. *Let M be a* $\mathsf{B}G$*-framed n-manifold. Let $V \in \mathsf{Mod}_G(\mathcal{V})$ be a G-module in $\mathcal{V}$. There is a canonical equivalence in $\mathcal{V}$,*

$$\int_M \mathsf{Free}_n^{\mathsf{B}G}(V) \;\simeq\; \coprod_{i\geq 0} \mathsf{Conf}_i^G(M) \underset{\Sigma_i \wr G}{\otimes} V^{\otimes i}$$

between the factorization homology of the free $\mathcal{D}\mathsf{isk}_n^{\mathsf{B}G}$-algebra and the coproduct of G-framed configuration spaces of M labeled by V.

The proof of Proposition 2.5.15, which appears in the above-mentioned citations, amalgamates the following key observations. First, the result is true for M a Euclidean space. Second, a hypercover-type argument (as in Theorem A.3.1 of [43]), applied to V-labeled configuration spaces, gives that the right-hand term satisfies $\otimes$-excision.

Using that left adjoints compose, Proposition 2.5.15 applied to a free G-module on an object $W \in \mathcal{V}$ gives the following result.

Corollary 2.5.16. *In the case that $V \simeq G \otimes W$ is the free G-module on an object $W \in \mathcal{V}$, there is a canonical equivalence in $\mathcal{V}$:*

$$\int_M \mathsf{Free}_n^{\mathsf{B}G}(G \otimes W) \;\simeq\; \coprod_{i\geq 0} \mathsf{Conf}_i(M) \underset{\Sigma_i}{\otimes} W^{\otimes i}.$$

Remark 2.5.17 (On homotopy invariance)**.** The work of Longoni–Salvatore [41] shows that configuration spaces of a manifold are sensitive to simple-homotopy type of the manifold. Based on Proposition 2.5.15, this has the following consequence, that factorization homology is not a homotopy invariant of manifolds, even in the weakest sense. The strongest form of homotopy invariance for the functor $\int A$, for a fixed $\mathcal{D}\mathsf{isk}_n$-algebra A, would be the structure of a factorization

$$\begin{tikzcd}
\mathcal{M}\mathsf{fld}_n \arrow[r, "\int A"] \arrow[d, "\text{forget}"'] & \mathcal{V} \\
\mathsf{Spaces} \arrow[ur, dashed] &
\end{tikzcd}$$

of $\int A$ through the forgetful functor $\mathcal{M}\mathsf{fld}_n \to \mathsf{Spaces}$. This structure of homotopy invariance is equivalent to the $\mathcal{D}\mathsf{isk}_n$-algebra A being commutative. Likewise, a strong form of proper-homotopy invariance would be a factorization

$$\begin{tikzcd}
\mathcal{M}\mathsf{fld}_n \arrow[r, "\int A"] \arrow[d, "(-)^+"'] & \mathcal{V} \\
\mathsf{Spaces}_*^{\mathsf{op}} \arrow[ur, dashed] &
\end{tikzcd}$$

where $(-)^+\colon \mathcal{M}\mathsf{fld}_n \to \mathsf{Spaces}_*^{\mathsf{op}}$ is 1-point compactification: sending a manifold M to the 1-point compactification M^+, and an open embedding $M \hookrightarrow N$ to the collapse map $N^+ \to M^+$. The existence of this second factorization is roughly equivalent to A being the n-disk enveloping algebra of a Lie algebra.

2.6 Filtrations

In this section, we establish two filtrations of factorization homology: in the manifold variable, one can filter by bounding the cardinality of the number of embedded disks; in the algebra variable, one can filter by bounding the number of multiplications allowed by elements of the algebra. Filtrations of the first kind generalize the Goodwillie–Weiss embedding calculus [59]; filtrations of the second kind are an instance of the Goodwillie functor calculus [24]. These filtrations are exchanged under Koszul duality, as is described in the following section. For simplicity, we have written these sections for the case of framed n-manifolds—see [5] and [4] for the case of unoriented n-manifolds, and the case of B-framed n-manifolds is analogous.

For the rest of this section, we fix a symmetric monoidal ∞-category $\mathcal{V}$ which is stable and $\otimes$-presentable.

2.6.1 Cardinality filtrations

Definition 2.6.1. The ∞-category $\mathcal{M}\mathsf{fld}_n^{\mathsf{fr},\mathsf{surj}}$ is the ∞-subcategory of $\mathcal{M}\mathsf{fld}_n^{\mathsf{fr}}$ whose objects are nonempty framed n-manifolds and whose morphisms consist of those embeddings which induce surjections on the set of connected components. $\mathcal{M}\mathsf{fld}_n^{\mathsf{fr},\leq k} \subset \mathcal{M}\mathsf{fld}_n^{\mathsf{fr},\mathsf{surj}}$ is the full ∞-subcategory whose objects have at most k components. The ∞-categories $\mathcal{D}\mathsf{isk}_n^{\mathsf{fr},\mathsf{surj}}$ and $\mathcal{D}\mathsf{isk}_n^{\mathsf{fr},\leq k}$ are the further ∞-subcategories of $\mathcal{M}\mathsf{fld}_n^{\mathsf{fr},\mathsf{surj}}$ whose objects are nonempty finite disjoint unions of Euclidean spaces.

That is, an embedding $f\colon M \hookrightarrow N$ belongs to $\mathcal{M}\mathsf{fld}_n^{\mathsf{fr},\mathsf{surj}}$ if and only if $\pi_0 f\colon \pi_0 M \to \pi_0 N$ is surjective.

Definition 2.6.2. For a functor $A : \mathcal{D}\mathsf{isk}_n^{\mathsf{fr}} \to \mathcal{V}$, the kth truncation of factorization homology is the colimit of the composite functor

$$\tau_{\leq k} \int_M A := \mathsf{colim}\Big(\mathcal{D}\mathsf{isk}_{n/M}^{\mathsf{fr},\leq k} \xrightarrow{A} \mathcal{V}\Big).$$

The difference between the subsequent truncations has a concrete description in terms of configuration spaces:

Lemma 2.6.3. [5] *Let A be an $\mathcal{E}_n$-algebra in $\mathcal{V}$. For each closed framed n-manifold M, there is a canonical cofiber sequence in $\mathcal{V}$:*

$$\tau_{\leq k-1} \int_M A \longrightarrow \tau_{\leq k} \int_M A \longrightarrow \mathsf{Conf}_k(M)^+ \bigotimes_{\Sigma_k} A^{\otimes k}.$$

Here, $\mathsf{Conf}_k(M)^+ \in \mathsf{Spaces}_*$ is the 1-point compactification of $\mathsf{Conf}_k(M)$, regarded as a based Σ_k-space; the cofiber is the reduced tensor with this based Σ_k-space.

Example 2.6.4. Consider the case that $\mathcal{V} = \mathsf{Mod}_{\Bbbk}$ and $n = 1$, so that $\int_{S^1} A \simeq \mathsf{HC}_*(A)$ is the Hochschild chain complex of the associative algebra A. After Lemma 2.6.3, Poincaré duality applied to the the framed k-manifold $\mathsf{Conf}_k(S^1)$ with coefficients in the local system $A^{\otimes k}$, identifies, for $k>0$, the cofibers of the cardinality filtration as

$$\mathsf{Conf}_k(S^1)^+ \bigotimes_{\Sigma_k} A^{\otimes k} \;\simeq\; \mathsf{cInd}_{\mathsf{C}_k}^{\mathbb{T}}(A[1]^{\otimes k}).$$

Here, $A[1]$ is chain complex over $\Bbbk$ which is the suspension of A; C_k is the cyclic group of order k, regarded as the closed subgroup of kth roots of unity in the Lie group $\mathbb{T}$ of unit complex numbers; $\mathsf{cInd}_{\mathsf{C}_k}^{\mathbb{T}}$ is the coinduction from the standard action of C_k on the k-fold tensor product $A[1]^{\otimes k}$.

The next series of results lead to the proof of Proposition 2.6.9, that factorization homology can be computed as a sequential colimit of its truncated values. To do this, we will use a comparison of our disk categories with the Ran space $\mathsf{Ran}(M)$. This is the space of finite subsets of M, topologized so that points can collide.

Definition 2.6.5. $\mathsf{Fin}^{\mathsf{surj}}$ is the category of finite nonempty sets and surjections among them. The Ran space of a topological space M is the topological space

$$\mathsf{Ran}(M) := \mathsf{colim}\Big(\mathsf{Fin}^{\mathsf{surj},\mathsf{op}} \xrightarrow{M} \mathsf{Top}\Big)$$

which is the colimit of the diagram sending a set J to the space M^J, and a surjection $J \to I$ to the diagonal map $M^I \hookrightarrow M^J$. For finite subset $S \subset M$, the topological space

$$\mathsf{Ran}(M)_S \subset \mathsf{Ran}(M)$$

is the subspace consisting of those finite subsets of M which contain S. The cardinality stratification of $\mathsf{Ran}(M)_S \to \mathbb{N}$ sends a finite subset T to its cardinality $|T|$.

We will make use of the exit-path ∞-category of a stratified space: see [43] or [7].

Lemma 2.6.6. [5] *There is contravariant equivalence*

$$\mathcal{D}\mathsf{isk}_{n/M}^{\mathsf{fr},\mathsf{surj}} \simeq \mathsf{Exit}\big(\mathsf{Ran}(M)\big)^{\mathsf{op}}$$

with the opposite of the exit-path ∞-category of the Ran space of M. More generally, fix an embedding $S \subset \coprod_S \mathbb{R}^n \hookrightarrow M$ for $S \subset M$. Then there is an equivalence

$$\mathcal{D}\mathsf{isk}_{n/M}^{\mathsf{fr},\mathsf{surj}} \underset{\mathcal{D}\mathsf{isk}_{n/M}^{\mathsf{fr}}}{\times} \mathcal{D}\mathsf{isk}_{n/M}^{\mathsf{fr},\coprod_S \mathbb{R}^n/^{\mathsf{inj}}} \simeq \mathsf{Exit}\big(\mathsf{Ran}(M)_S\big)^{\mathsf{op}}$$

where $\mathcal{D}\mathsf{isk}_{n/M}^{\mathsf{fr},\coprod_S \mathbb{R}^n/^{\mathsf{inj}}}$ is the ∞-subcategory of $(\mathcal{D}\mathsf{isk}_{n/M})^{\coprod_S \mathbb{R}^n/}$ whose objects are those $\coprod_S \mathbb{R}^n \hookrightarrow U \hookrightarrow M$ such that $S \to \pi_0 U$ is injective.

Lemma 2.6.7 (Lemma 2.3.2 [5])**.** *The functor*

$$\mathcal{D}\mathsf{isk}_{n/M}^{\mathsf{fr},\mathsf{surj}} \longrightarrow \mathcal{D}\mathsf{isk}_{n/M}^{\mathsf{fr}}$$

is final.

Proof. By Quillen's Theorem A, it suffices to show that for each $g : \coprod_k \mathbb{R}^n \hookrightarrow M$, an object of $\mathcal{D}\mathsf{isk}_{n/M}$, the ∞-undercategory

$$\mathcal{D}\mathsf{isk}^{\mathsf{fr,surj}}_{n/M} \underset{\mathcal{D}\mathsf{isk}^{\mathsf{fr}}_{n/M}}{\times} \left(\mathcal{D}\mathsf{isk}^{\mathsf{fr}}_{n/M}\right)^{\coprod_k \mathbb{R}^n/}$$

has a contractible classifying space. There exists an adjunction

$$\mathcal{D}\mathsf{isk}^{\mathsf{fr,surj}}_{n/M} \underset{\mathcal{D}\mathsf{isk}^{\mathsf{fr}}_{n/M}}{\times} \mathcal{D}\mathsf{isk}^{\mathsf{fr},\coprod_S \mathbb{R}^n/^{\mathsf{inj}}}_{n/M} \leftrightarrows \mathcal{D}\mathsf{isk}^{\mathsf{fr,surj}}_{n/M} \underset{\mathcal{D}\mathsf{isk}^{\mathsf{fr}}_{n/M}}{\times} \left(\mathcal{D}\mathsf{isk}^{\mathsf{fr}}_{n/M}\right)^{\coprod_k \mathbb{R}^n/}$$

hence an equivalence between their classifying spaces. By Lemma 2.6.7, we obtain

$$\mathsf{B}\Big(\mathcal{D}\mathsf{isk}^{\mathsf{fr,surj}}_{n/M} \underset{\mathcal{D}\mathsf{isk}^{\mathsf{fr}}_{n/M}}{\times} \mathcal{D}\mathsf{isk}^{\mathsf{fr},\coprod_S \mathbb{R}^n/^{\mathsf{inj}}}_{n/M}\Big) \simeq \mathsf{B}\,\mathsf{Exit}\big(\mathsf{Ran}(M)_S\big) \simeq \mathsf{Ran}(M)_S.$$

The result now follows from the contractibility of this form of the Ran space—to see this, we use a now standard argument from [9]: the Ran space carries a natural H-space structure, given by taking unions of subsets, for which the composition of the diagonal and the H-space multiplication is the identity. Consequently, its homotopy groups must be zero. $\square$

Remark 2.6.8. Combining the equivalence

$$\mathcal{D}\mathsf{isk}^{\mathsf{fr,surj}}_{n/M} \simeq \mathsf{Exit}\big(\mathsf{Ran}(M)\big)^{\mathsf{op}}$$

together with the finality of Lemma 2.6.7, we obtain that factorization homology (or any colimit indexed by $\mathcal{D}\mathsf{isk}^{\mathsf{fr}}_{n/M}$) is equivalent to the global sections of an associated cosheaf on the Ran space of M.

Together with Lemma 2.6.3, the following is the main conclusion of this section, describing a complete filtration of factorization homology with layers given by twisted homologies of configuration spaces.

Proposition 2.6.9. *The natural map from the sequential colimit of truncations*

$$\varinjlim_k \tau_{\leq k} \int_M A \longrightarrow \int_M A$$

is an equivalence.

Proof. The finality of Lemma 2.6.7 implies the equivalence

$$\int_M A := \mathsf{colim}\Big(\mathcal{D}\mathsf{isk}^{\mathsf{fr}}_{n/M} \xrightarrow{\int A} \mathcal{V}\Big) \simeq \mathsf{colim}\Big(\mathcal{D}\mathsf{isk}^{\mathsf{fr,surj}}_{n/M} \xrightarrow{A} \mathcal{V}\Big).$$

There is then a sequence of equivalences

$$\begin{aligned}\mathsf{colim}\Big(\mathcal{D}\mathsf{isk}^{\mathsf{fr,surj}}_{n/M} \xrightarrow{A} \mathcal{V}\Big) &\simeq \mathsf{colim}\Big(\Big(\varinjlim_k \mathcal{D}\mathsf{isk}^{\mathsf{fr},\leq k}_{n/M}\Big) \xrightarrow{A} \mathcal{V}\Big) \simeq \varinjlim_k \mathsf{colim}\Big(\mathcal{D}\mathsf{isk}^{\mathsf{fr},\leq k}_{n/M} \xrightarrow{A} \mathcal{V}\Big)\\ &= \varinjlim_k \tau_{\leq k} \textstyle\int_M A\end{aligned}$$

from which the result follows. $\square$

A dual result holds for factorization cohomology with coefficients in an $\mathcal{E}_n$-coalgebra.

Definition 2.6.10. For C an $\mathcal{E}_n$-coalgebra in $\mathcal{V}$ (i.e., an $\mathcal{E}_n$-algebra in $\mathcal{V}^{\mathsf{op}}$), the factorization cohomology of M with coefficients in C is the limit

$$\int^M C := \lim\Big(\mathcal{D}\mathsf{isk}^{\mathsf{op}}_{n/M} \xrightarrow{C} \mathcal{V}\Big).$$

Proposition 2.6.11. *For M a closed framed n-manifold and C an $\mathcal{E}_n$-coalgebra in $\mathcal{V}$, there exists a cofiltration of $\int^M C$,*

$$\int^M C \longrightarrow \tau^{\leq \bullet} \int^M C$$

whose associated graded is

$$\coprod_k \Big(\big(C^{\otimes k}\big)^{\mathsf{Conf}_k(M)^+} \Big)^{\Sigma_k}.$$

2.6.2 Goodwillie filtrations

Fixing a manifold M, one can apply the Goodwillie functor calculus, as developed in [24], [38], and [43], to the functor

$$\int_M : \mathsf{Alg}^{\mathsf{aug}}_{\mathcal{E}_n}(\mathcal{V}) \longrightarrow \mathcal{V}.$$

The output is a cofiltration of factorization homology whose kth term is the polynomial approximation $P_k \int_M$.

Proposition 2.6.12. [5] *Let A an augmented $\mathcal{E}_n$-algebra in $\mathcal{V}$ with cotangent space LA. For each closed framed n-manifold M, the following is a fiber sequence*

$$\begin{tikzcd}
\mathsf{Conf}_k(M) \underset{\Sigma_k}{\otimes} LA^{\otimes k} \arrow[r] \arrow[d] & P_k \int_M A \arrow[d] \\
\mathbb{1} \arrow[r] & P_{k-1} \int_M A
\end{tikzcd}$$

where P_k is Goodwillie's kth polynomial approximation [24] applied to the functor $\int_M$, and LA is the cotangent space of A at the augmentation (see [20]). In particular, the Goodwillie derivative $\partial_k \int_M$ is canonically equivalent with $\mathsf{Conf}_k(M) \otimes \mathbb{1}$, the tensor of the space $\mathsf{Conf}_k(M)$ with the unit object in the symmetric monoidal ∞-category $\mathcal{V}$.

2.7 Poincaré/Koszul duality

Koszul duality, in its most basic form, interchanges two forms of algebra, determined by the form of algebraic structure that the cotangent space at an augmentation naturally inherits. There are two classical forms of Koszul duality: commutative algebra is Koszul dual to Lie algebra; associative algebra is dual to associative algebra. In particular, associative algebra is Koszul self-dual. This second assertion generalizes to the situation of $\mathcal{E}_n$-algebra, where the Koszul self-duality dates to Getzler–Jones [23]. This Koszul duality for $\mathcal{E}_n$-algebra can be expressed in terms of factorization homology for manifolds with boundary. See [44] for another treatment of Koszul self-duality of $\mathcal{E}_n$-algebra, in terms of twisted arrow categories.

Theorem 2.7.1. ([5], [4]) *Let $\mathcal{V}$ be a stable $\otimes$-presentable ∞-category. There is a functor*

$$\int_{\mathbb{D}^n/\partial\mathbb{D}^n} : \mathsf{Alg}^{\mathsf{aug}}_{\mathcal{E}_n}(\mathcal{V}) \longrightarrow \mathsf{cAlg}^{\mathsf{aug}}_{\mathcal{E}_n}(\mathcal{V})$$

sending an augmented $\mathcal{E}_n$-algebra A to an $\mathcal{E}_n$-coalgebra structure on

$$\int_{\mathbb{D}^n/\partial\mathbb{D}^n} A \simeq A \bigotimes_{\int_{S^{n-1}\times\mathbb{R}} A} \mathbb{1}\,.$$

There is a natural transformation, the Poincaré/Koszul duality map, in $\mathsf{Fun}(\mathbb{N}^{\triangleright},\mathcal{V})$:

$$\begin{tikzcd} P_\bullet \int_M A \arrow[r] \arrow[d] & \tau^{\leq\bullet}\int^M\Big(\int_{\mathbb{D}^n/\partial\mathbb{D}^n} A\Big) \arrow[d] \\ \int_M A \arrow[r] & \int^M\Big(\int_{\mathbb{D}^n/\partial\mathbb{D}^n} A\Big). \end{tikzcd}$$

This map is an equivalence when restricted to $\mathsf{Fun}(\mathbb{N},\mathcal{V})$.

Remark 2.7.2. The coalgebra structure of $\int_{\mathbb{D}^n/\partial\mathbb{D}^n} A$ is given in [4] by showing that factorization homology with coefficients in an augmented $\mathcal{E}_n$-algebra is functorial with respect to the fold map

$$\mathbb{D}^n/\partial\mathbb{D}^n \longrightarrow \mathbb{D}^n/\partial\mathbb{D}^n \vee \mathbb{D}^n/\partial\mathbb{D}^n.$$

More generally, the theory of zero-pointed manifolds of [4] exhibits an additional functoriality for factorization homology of augmented algebras, of extension-by-zero maps.

In particular, the Goodwillie completion $\varprojlim P_\bullet \int_M A$ is equivalent to factorization cohomology with the coefficients in the coalgebra which is Koszul-dual to A. The natural map to the completion

$$\int_M A \longrightarrow \varprojlim P_\bullet \int_M A$$

need not be an equivalence, however.[12] A generalization of factorization homology, taking coefficients in formal moduli problems, gives a way to correct the failure of Poincaré/Koszul duality to be an equivalence in general.

[12]One should only expect this map to be an equivalence if $\mathcal{V}$ carries a t-structure, suitably compatible with the monoidal structure, for which the augmentation ideal of A is either connected or n-coconnected.

Definition 2.7.3. Let $\mathcal{V}$ be a $\otimes$-presentable ∞-category. Let $B \to \mathsf{BO}(n)$ be a map from a space. Let $X\colon \mathsf{Alg}_{\mathcal{D}\mathsf{isk}_n^B}(\mathcal{V}) \to \mathsf{Spaces}$ be a functor. Let M be a B-framed n-manifold M. The factorization homology of M with coefficients in X is the object in $\mathcal{V}$

$$\int_M X \ := \lim_{A\in \mathsf{Aff}^{\mathsf{op}}_{/X}} \int_M A,$$

where $\mathsf{Aff} := \mathsf{Alg}_{\mathcal{D}\mathsf{isk}_n^B}(\mathcal{V})^{\mathsf{op}} \subset \mathsf{Fun}\big(\mathsf{Alg}_{\mathcal{D}\mathsf{isk}_n^B}(\mathcal{V}), \mathsf{Spaces}\big)$ is the image of the Yoneda embedding.

Remark 2.7.4. Intuitively, the object $\int_M X$ is $\Gamma(X, \int_M \mathcal{O})$, the global sections of the presheaf on X obtained by applying factorization homology of M to the structure sheaf of X. We interpret this form of factorization homology as the observables in a topological quantum field theory: a sigma-model with algebraic target X.

There is a modification of the above definition for a *formal* moduli problem, i.e., a space-valued functor on an ∞-category of Artin $\mathcal{E}_n$-algebras.

Definition 2.7.5 ($\mathsf{Artin}_{\mathcal{E}_n}$)**.** The full ∞-subcategory $\mathsf{Triv}_{\mathcal{E}_n} \subset \mathsf{Alg}^{\mathsf{aug},\geq 0}_{\mathcal{E}_n}(\mathsf{Mod}_{\Bbbk})$ is the essential image of connective perfect $\Bbbk$-modules $\mathsf{Perf}^{\geq 0}_{\Bbbk}$ under the functor that assigns a complex V the associated square-zero extension $\Bbbk \oplus V$. The ∞-category $\mathsf{Artin}_{\mathcal{E}_n}$ of local Artin $\mathcal{E}_n$-algebras is the smallest full ∞-subcategory of $\mathsf{Alg}^{\mathsf{aug},\geq 0}_{\mathcal{E}_n}(\mathsf{Mod}_{\Bbbk})$ that contains $\mathsf{Triv}_{\mathcal{E}_n}$ and is closed under small extensions. That is:

If B is in $\mathsf{Artin}_{\mathcal{E}_n}$, $\Bbbk \oplus V$ is in $\mathsf{Triv}_{\mathcal{E}_n}$, and the following diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ \Bbbk \arrow[r] & \Bbbk \oplus V \end{tikzcd}$$

forms a pullback square in $\mathsf{Alg}^{\mathsf{aug},\geq 0}_{\mathcal{E}_n}(\mathsf{Mod}_{\Bbbk})$, then A is in $\mathsf{Artin}_{\mathcal{E}_n}$.

Definition 2.7.6. Fix a field $\Bbbk$, and let $X\colon \mathsf{Artin}_{\mathcal{E}_n} \to \mathsf{Spaces}$ be a functor of local Artin $\mathcal{E}_n$-algebras over $\Bbbk$. The factorization homology of X over M is the $\Bbbk$-module

$$\int_M X \ := \lim_{R\in \mathsf{Artin}^{\mathsf{op}}_{\mathcal{E}_n/X}} \int_M R.$$

Such formal moduli problems X are defined by the generalization of the Maurer–Cartan functor from classical Lie theory. In the following definition, we use the Koszul duality functor $\mathbb{D}^n$ given as the composite of linear duality with factorization homology of $\mathbb{D}^n/\partial\mathbb{D}^n$.

$$\mathbb{D}^n : \mathsf{Alg}^{\mathsf{aug}}_{\mathcal{E}_n}(\mathsf{Mod}_{\Bbbk}) \longrightarrow \mathsf{cAlg}^{\mathsf{aug}}_{\mathcal{E}_n}(\mathsf{Mod}_{\Bbbk}) \longrightarrow \mathsf{Alg}^{\mathsf{aug}}_{\mathcal{E}_n}(\mathsf{Mod}_{\Bbbk})^{\mathsf{op}} .$$

That is, the Koszul-dual $\mathcal{E}_n$-algebra is

$$\mathbb{D}^n A = \Big(\int_{\mathbb{D}^n/\partial\mathbb{D}^n} A\Big)^{\vee}.$$

Definition 2.7.7. For an Artin $\mathcal{E}_n$-algebra R and an augmented $\mathcal{E}_n$-algebra A over a field $\Bbbk$, the Maurer–Cartan space

$$\mathsf{MC}_A(R) := \mathsf{Alg}^{\mathsf{aug}}_{\mathcal{E}_n}(\mathbb{D}^n R, A)$$

is the space of maps from the Koszul dual of R to A. The Maurer–Cartan functor $\mathsf{MC}\colon \mathsf{Alg}^{\mathsf{aug}}_{\mathcal{E}_n} \to \mathsf{Fun}(\mathsf{Artin}_{\mathcal{E}_n}, \mathsf{Spaces})$ is the adjoint of the pairing

$$\mathsf{Artin}_{\mathcal{E}_n} \times \mathsf{Alg}^{\mathsf{aug}}_{\mathcal{E}_n}(\mathsf{Mod}_\Bbbk) \xrightarrow{\mathbb{D}^n \times \mathsf{id}} \mathsf{Alg}^{\mathsf{aug}}_{\mathcal{E}_n}(\mathsf{Mod}_\Bbbk)^{\mathsf{op}} \times \mathsf{Alg}^{\mathsf{aug}}_{\mathcal{E}_n}(\mathsf{Mod}_\Bbbk) \xrightarrow{\mathsf{Map}(-,-)} \mathsf{Spaces}\,.$$

MC sends A to the functor $\mathsf{MC}_A : \mathsf{Artin}_{\mathcal{E}_n} \to \mathsf{Spaces}$.

The following is the framed version of the main theorem of [5].

Theorem 2.7.8 (Poincaré/Koszul duality [5])**.** *Let M be a closed framed n-manifold; let A be an augmented $\mathcal{E}_n$-algebra over a field $\Bbbk$ with MC_A the associated formal moduli functor of $\mathcal{E}_n$-algebras. There is a natural equivalence*

$$\Big(\int_M A\Big)^\vee \simeq \int_M \mathsf{MC}_A$$

between the $\Bbbk$-linear dual of the factorization homology of M with coefficients in A and the factorization homology of M with coefficients in the moduli functor MC_A.

Remark 2.7.9. One can regard the failure of the duality map in Theorem 2.7.1 to be an equivalence to be related to the non-affineness of the formal moduli problem MC_A. Were MC_A to be affine, it would be equivalent to the formal spectrum of the Koszul dual, $\mathsf{Spf}(\mathbb{D}^n A)$, in which the case the duality map of Theorem 2.7.1 would be an equivalence. In terms of topological quantum field theory, we regard MC_A as determining a sigma-model whose target $X = \mathsf{MC}_A$ is formal, but not necessarily affine. Theorem 2.7.8 then describes the observables $\int_M X$ in this sigma-model in terms of the factorization homology with coefficients in a shift of the cotangent space of the point $\mathsf{Spec}(\Bbbk) \to X$.

2.8 Factorization homology for singular manifolds

We briefly outline an extension of factorization homology to singular manifolds, such as manifolds equipped with *defects*. The notion of singular manifolds, and structured versions thereof, is developed in [7]. The development of factorization homology of such can be found in [8].

2.8.1 Singular manifolds

Heuristically, a singular manifold is a stratified space in which each stratum is endowed with the structure of a smooth manifold, and in which each stratum has a canonically associated link which is itself a singular manifold. Informally, the ∞-category $\mathcal{S}\mathsf{nglr}$ of singular manifolds and open (stratified) embeddings among such is minimal with respect to the following features.

1. The empty set $\emptyset$ is a singular manifold.

2. For L a compact singular manifold, the open cone

$$\mathsf{C}(L) \ := \ * \coprod_{L\times\{0\}} L \times [0,1)$$

is a singular manifold, with the cone-point $*$ a stratum.

3. For X and Y singular manifolds, their product $X \times Y$ is a singular manifold.

4. For $U \subset X$ an open subset of a singular manifold, then U is canonically a singular manifold.

5. For $\mathcal{U}$ an open cover by singular manifolds of a paracompact Hausdorff space X, then X is a singular manifold.

6. For L a compact singular manifold, the continuous homomorphism

$$\mathsf{Aut}(L) \xrightarrow{\simeq} \mathsf{Aut}\big(\mathsf{C}(L)\big), \qquad f \mapsto \mathsf{C}(f),$$

is equipped with a deformation retraction whose retraction $D_*\colon \mathsf{Aut}\big(\mathsf{C}(L)\big) \to \mathsf{Aut}(L)$ is a continuous homomorphism.

A rigorous definition of $\mathcal{S}\mathsf{nglr}$ is given in §1 of [7]. That definition is distractingly inductive, and so we elect to not spell it out here.

Terminology 2.8.1. In this article, by "singular manifold" we will mean "finitary singular manifold with boundary," in the sense of [7].

Remark 2.8.2. The first two axioms give that $*$ is a singular manifold, and thereafter that $\mathsf{C}(*) = [0,1)$ is a singular manifold with strata $\{0\}$ and $(0,1)$. The third and fifth axiom gives that $(0,1)^{\times n} \cong \mathbb{R}^n$ is a singular manifold. The fifth axiom thereafter gives that each manifold is a singular manifold. The second and third axiom then give that $\mathbb{R}^k \times \mathsf{C}(L)$ is a singular manifold for each compact manifold L. In general, an arbitrary singular manifold is a paracompact Hausdorff topological space that admits a basis for its topology consisting of open embeddings from singular manifolds of the form $\mathbb{R}^k \times \mathsf{C}(L)$, where k is an integer and L is a compact singular manifold.

A *basic* is a singular manifold of the form $\mathbb{R}^k \times \mathsf{C}(L)$ for k an integer and L a compact singular manifold.

Remark 2.8.3. We follow up on the previous remark. Namely, the definition of a singular manifold detailed in [7] is in analogy with that of a smooth manifold: it is a paracompact Hausdorff topological space equipped with a maximal atlas by *basics*.

Remark 2.8.4. The sixth axiom above reflects the regularity of singular manifolds. The retraction map can be interpreted as taking the 'derivative at the cone-point'. A consequence of this structure is that one can 'resolve singularities' of a singular manifold. Specifically, for X a singular manifold, and for $X_d \subset X$ a stratum that is closed as a subspace, there