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Basic Partial Differential Equations

David Bleecker, George Csordas

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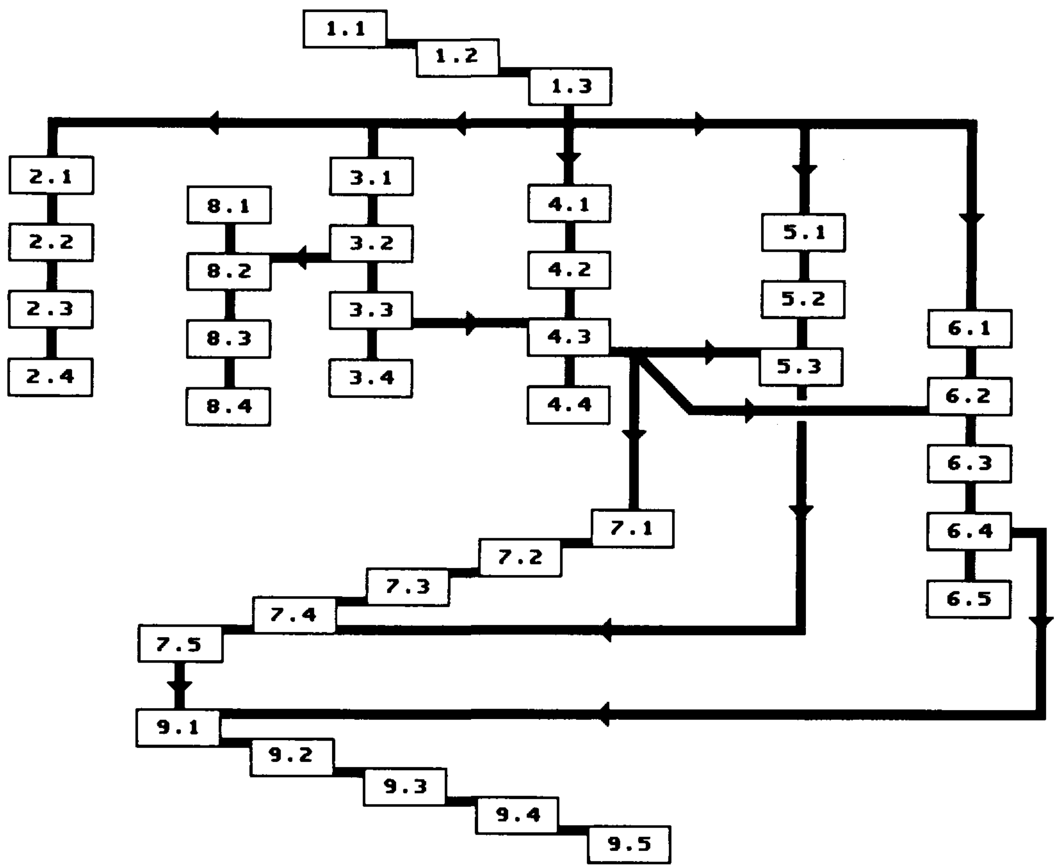


PREFACE

Quantities which depend on space and/or time variables are often governed by differential equations which are based on underlying physical principles. Partial differential equations (PDEs) not only accurately express these principles, but also help to predict the behavior of a system from an initial state of the system and from given external influences. Thus, it is hard to overestimate the relevance of PDEs in all forms of science and engineering, or any endeavor which involves reasonably smooth, predictable changes of measurable quantities.

Having taught from the material in this book for ten years with much feedback from students, we have found that the book serves as a very readable introduction to the subject for undergraduates with a year and a half of calculus, but not necessarily any more. In particular, one need *not* have had a linear algebra course or even a course in ordinary differential equations to understand the material. As the title suggests, we have concentrated only on what we feel are the absolutely essential aspects of the subject, and there are some crucial topics such as systems of PDEs which we only touch on. Yet the book certainly contains more material than can be covered in a single semester, even with an exceptional class. Given the broad relevance of the subject, we suspect that a demand for a second semester surely exists, but has been largely unmet, partly due to the lack of books which take the time and space to be readable by sophomores. A glance at the table of contents or the index reveals some subjects which are regarded as rather advanced (e.g., maximum principles, Fourier transforms, quasi–linear PDEs, spherical harmonics, PDEs on manifolds, complex variable theory, conditions under which Fourier series are uniformly convergent). However, despite general impressions given (perhaps unwittingly) by mathematical gurus, *any* valid mathematical result or concept, regardless of how "advanced" it is, can be broken down into elementary, trivial pieces which are easily understood by all who desire to do so. With regard to the so–called "advanced" topics in this book, we feel that we have accomplished this to a degree which surprised even us. For us it was a constant and worthwhile challenge to make the book completely self–contained for those who have only been through the typical freshman/sophomore calculus sequence, even if they forgot most of it. We have successfully taught students who did not recall how to solve $y'(x) = y(x)$ with $y(0) = 1$ at the beginning of the semester, as was the case with over half of our students according to initial survey tests. However, before the semester's end, these same students could prove and understand the Maximum Principle for the heat equation and could easily deduce the continuous dependence of solutions on initial and boundary data. In essence, "advanced topics" are rarely difficult per se, but they may seem so, if (for the sake of elegance) too little time is spent explaining and motivating them.

We have avoided the temptation to first prove unmotivated results in great generality and then use them to deduce an abundance of particular cases. By and large, we have introduced results and techniques inductively through many solved examples. By the time students have seen enough examples, they can often anticipate, as well as understand, the argument for the general case. In particular, we have found that, in spite of the fact that Sturm–Liouville Theory provides a uniform approach to boundary–value problems, it is not so wise to teach it first to students who are barely familiar with sines and cosines, and then cover the elementary boundary–value problems as special cases. We have proceeded in the opposite manner. After we have handled a variety of simple boundary conditions for the heat equation and treated Fourier series, the student is prepared to study and appreciate Sturm–Liouville Theory as a natural continuation of what has been learned without it. Proceeding from examples to theorems may result in a book which is physically longer, but students learn more rapidly and effectively this way. In short, it is easier to build from the ground up than from the roof down. In the process, we may have sacrificed some degree of elegance, but we have not sacrificed rigor. Nearly every basic result is proved rigorously at some stage, or at least we give a reference (e.g., for the convergence of eigenfunction expansions on manifolds). We certainly do not recommend proving everything in class, since this would severely limit the range of the material covered, but instead the interested student may be directed to the many detailed, thoroughly digestible proofs in the text. On the point of rigor, we mention that many solutions of PDEs are expressible in terms of integrals of Greens functions against boundary and/or initial data. In most PDE texts, such integral formulas are derived (if at all) under the assumption that solutions of the PDEs actually exist. To be honest, one should have the tools to check that the functions defined by such integral formulas actually solve the given problem. This necessarily entails the use of Leibniz's rule for differentiation under an integral, sometimes when the interval of integration is unbounded. One feature of this book, which appears to be absent in other texts, is that there is a complete, elementary proof of Leibniz's rule in the Appendix. To experts, this may be surprising, since many standard proofs entail the use of the Lebesgue Dominated Convergence Theorem. However, in the Appendix, we have proven a suitable version of dominated convergence which avoids the notion of Lebesgue measure and integration. (The idea originated in [Lewin, 1986, 1987].)

Solving problems is the major part of learning any mathematical subject. This book contains many problems which range from the purely routine to those which will challenge even the most brilliant student. Sometimes one finds that although some students can arrive at a solution to a problem through mimicking procedures, they still may not be able to interpret or use the solution or even understand why the expression they have found is actually a solution of the problem. We have tried to counter this tragedy by including many exercises which require the student to think, draw some conclusions, and express themselves, instead of simply implementing purely computational procedures. Since some students will do anything to get the answer in the back of the book, we have been sparing with the answers. However, a solution manual (with complete solutions to all but the most trivial problems) is available to instructors only. We personally worked out each of the problems.

Since the whole book cannot be covered in a single semester, instructors who are limited to a single semester must decide which sections or chapters to cover. Given the demand, instructors might consider the introduction of a second semester of PDEs. Below, we summarize the material covered in the chapters and sections. Following this, some suggestions are given on what sections must, should or could be included in a one–semester or two–quarter course.

Acknowledgements. It is our pleasure to acknowledge the comments and suggestions of our colleagues and students. In particular, we thank Hans Broderson, Karl Heinz Dovermann, Christopher Mawata, Ken Rogers, Mi–Soo Smith, Wayne Smith, David Stegenga, Joel Weiner, George Wilkens, and Les Wilson, who have adopted the notes in their courses. We also acknowledge Paolo Agliano and Paul Kohs who helped us with the typing and the graphics. In addition, a warm *mahalo* is due to the secretarial staff of the Department of Mathematics at the University of Hawaii. A special *mahalo nui loa* is due to Pat Goldstein who cheerfully helped us with much of the clerical work. Last, but not least, we wish to thank our families for their patience and support during the preparation of this work.

Honolulu, 1992 David Bleecker & George Csordas

Chapter–by–chapter synopsis and suggestions for the instructor

Chapter 1 (Review and Introduction): If the students have had a course in ODEs, then Section 1.1 can be skipped, or assigned as reading. Some coverage of Sections 1.2 and 1.3 is necessary for a general overview of PDEs and their applications, and for an introduction to certain topics, such as separation of variables and the superposition principle. These concepts are used often in the sequel.

Chapter 2 (First–Order PDEs): For instructors who regard first–order PDEs as devoid of any real application, we urge them to read the introduction to Chapter 2, before deciding to skip Chapter 2 entirely. Not only are there wide applications to birth and death processes (e.g., the evolution of population densities), continuum mechanics and the development of shocks in traffic flow, but also the student sees how a change of variables can greatly simplify a PDE. Incidentally, we elected not to include examples and drill exercises for putting second–order, linear PDEs (with constant coefficients) into the standard normal forms (e.g, by rotation of axes, etc.), for the simple reason that second–order PDEs which arise in applications are already in a standard form. However, a complete statement of the Classification Theorem is given in Section 1.2, and a complete proof is given in the Appendix A.1. To compensate for lack of practice in change of variables drill for second–order PDEs, there are plenty of change–of–variable problems for first–order PDEs in Chapter 2. First–order PDEs which arise in applications are seldom in the standard form of a parametrized ODE. While Chapters 3–9 do not depend on Chapter 2, instructors should seriously consider doing at least Section 2.1 in which $au_x + bu_y + cu = f(x,y)$ is solved, when a, b, and c are constants. The case of variable coefficients is covered in Section 2.2, and the quasi–linear case is covered in Section 2.3. The fully nonlinear case is covered in the purely optional Section 2.4.

Chapter 3 (The Heat Equation) : Section 3.1 begins with a derivation of the heat equation. The simplest initial/boundary–value problems are solved *without* first introducing Fourier series. Here, we use separation of variables to find product solutions of the heat equation which meet the homogeneous boundary conditions B.C. and then find a linear combination which meets the initial condition. In Chapter 3, initial temperatures are chosen so that they are expressible (via trigonometric identities) as finite linear combinations of sines or cosines of the appropriate form. Students then naturally ask what can be done if this is not the case. In other words, they are naturally motivated for the introduction of Fourier series which is the topic of Chapter 4. In Section 3.2, uniqueness of solutions of various initial/boundary–value problems for the heat equation is proved, by showing that for homogeneous B.C. of the first or second kind, the mean–square of the temperature is non–increasing. The Maximum Principle provides a second approach. We first illustrate the Maximum Principle through a number of examples and we show that it easily leads to continuous (uniform) dependence of solutions on initial/boundary data. The proof of the Maximum Principle is then given at the end of Section 3.2. Section 3.3 deals with the case of various simple B.C. which are time–independent, but possibly inhomogeneous. In Section 3.4, the case of time–dependent B.C. and heat sources are handled by means of Duhamel's principle. Section 3.4 can be skipped or covered later if time permits, and Section 3.3 can be covered quickly and lightly. However, Section 3.1 is certainly part of any first PDE course, and we strongly recommend that Section 3.2 be covered in some detail.

Chapter 4 (Fourier Series and Sturm–Liouville Theory): Students see the need for Fourier series in Chapter 3. In Section 4.1, we introduce the notion of functional orthogonality, and the definition of Fourier series of a function as a formal expression which may or may not converge to the function. Many examples are computed, and the question of convergence is motivated. An estimate for the number of terms needed to uniformly approximate a C^2 function is stated (but the proof is deferred until Section 4.2). We provide a technique for obtaining much sharper estimates by means of integral estimates of the tail of a Fourier series. Section 4.2 contains detailed proofs of the convergence of Fourier series under various assumptions. We gently introduce the difference between pointwise convergence and uniform convergence. Pointwise convergence is proved for piecewise C^1 functions and uniform convergence for *continuous* piecewise C^1 functions. Without the luxury of time, we recommend that the lengthier proofs be skipped or assigned for reading. However, certainly one should get across the general idea that the smoother a function is on a circle, the more rapid is the convergence of its Fourier series. In Section 4.3, we introduce Fourier sine and cosine series which are used to handle (at least formally) the case (left dangling in Chapter 3) that the initial temperature was not a *finite* linear combination of the appropriate form. It is emphasized that infinite sums of C^2 functions need not be C^2, and hence the formal solutions obtained need not be strict solutions. However, by truncating the series at a large enough number of terms one can often meet the I.C. within any positive error, which is all that is needed for applications. The validity of formal solutions under certain assumptions is deferred to Chapter 7. Sturm–Liouville Theory is covered in Section 4.4. At this point the student is ready to savor this subject which extends what is known already to the case of inhomogeneous rods and boundary conditions of the third kind. We provide a convincing sketch of a proof of the infinitude of the eigenvalues for Sturm–Liouville problems, by means of the Sturm Comparison Theorem. Practically none of the rest of the book depends on Section 4.4, except the statement found in Chapter 9 (Section 9.5) that Bessel functions have infinitely many zeros. Thus, in the face of time pressures, it is possible to omit Section 4.4 entirely, although one should at least tell students what it is about. We have found that Section 4.3 can and should be covered rapidly, and that one should stress the statements of the theorems in Section 4.2, but not necessarily the details of the proofs. Section 4.1 should be covered in detail, as it is frequently used later.

Chapter 5 (The Wave Equation): In Section 5.1, the wave equation for a transversely vibrating string is derived from Newton's equation. Some care is taken to explain why the assumption of transverse vibrations actually *implies* a linear wave equation instead of an approximately linear equation. The dubious assumption of "small" vibrations is thus eliminated. The simplest initial/boundary–value problems for a finite string are solved. Uniqueness of solutions of these problems is also proved in Section 5.1, using the energy–integral method. In Section 5.2, we cover D'Alembert's solution of wave problems on the infinite string. Consequences of D'Alembert's solution, such as finite propagation speed are covered, and the method of images for semi–infinite strings is explained. For finite strings, the method of images provides an alternative to the Fourier series approach. The continuous dependence of solutions for the finite string on initial conditions is also an easy consequence of D'Alembert's formula and the method of images. In Section 5.3 a variety of boundary conditions for the string are handled. Also, the inhomogeneous wave equation (i.e., with forcing term) is treated via both Duhamel's principle and the Fourier series approach. Section 5.1 should be covered in some detail, with the complete derivation possibly assigned as reading. Section 5.2 is equally crucial, but if time is running short Section 5.3 can simply be summarized, so that students are aware of what is covered in case they need it.

Chapter 6 (Laplace's Equation): In Section 6.1, Laplace's equation is motivated and it is shown that solutions may be interpreted as steady–state temperature distributions. The Dirichlet and Neumann problems are introduced. Section 6.2 concerns the solution of these problems on a rectangle. Since students are familiar with separation of variables and superposition, this material can be done quickly. Uniqueness and the Maximum Principle are motivated and utilized, but proofs are deferred until Section 6.4. In Section 6.3, we solve Dirichlet and Neumann problems on annuli and disks using polar coordinates. The Mean–Value Theorem and Poisson's Integral Formula are carefully proved, and the regularity of harmonic functions is demonstrated. In

Section 6.4, the Maximum Principle for harmonic functions on bounded domains is proved along with continuous dependence of solutions of the Dirichlet problem on boundary data. The importance of these results has been amply demonstrated to students in the previous sections. Section 6.5 is on the application of complex variable theory to Laplace's equation. We assume *no* knowledge of complex–variables. We do not cover Cauchy's theorem, contour integration, or residue theory, for the simple reason that we do not need it. However, the intimate connection between complex analytic functions and harmonic functions is brought out and exploited. Moreover, the concept and use of conformal mapping to solve problems in steady–state temperatures, fluid flow and electrostatics are handled without any difficulty. All of the material in Chapter 6 is important, and if too much time is spent on material in previous chapters, it may not be possible to cover all of Chapter 6. For a class of mostly engineers, it may be wiser to cover Section 6.5 instead of Section 6.4, if a choice must be made, whereas for mathematics majors the reverse choice is desirable.

Chapter 7 (Fourier Transforms): It will take an exceptional class to reach Chapter 7 in one semester, without skipping all but the most essential material in the previous chapters. However, if students are likely to take a full complex variable course in the future, many concepts in Chapter 6 will be treated in that course. Then, skipping much of Chapter 6 and proceeding with Chapter 7 becomes an attractive possibility. Of course, the possibility of introducing a second semester (or more quarters) of PDEs should be contemplated. The demand is there. In Section 7.1, we introduce complex Fourier series and define the Fourier transform. Many examples are computed. In Section 7.2, we develop the basic properties of Fourier transforms which make them a useful tool for finding solutions of PDEs (i.e., differentiation is carried to a multiplication operator, and multiplication of transforms corresponds to convolution). The idea that the regularity of a function increases the rate of decay of its Fourier transform (and vice versa), is brought out. Although, this is typically regarded as an advanced topic, we treat it in an elementary way, and it is a close relative of the idea (covered in Section 3.2) that the smoothness of a function on a circle increases the rate of decay of its Fourier coefficients. Section 7.3 covers use of the Inversion Theorem, inverse Fourier transforms, and Parseval's equality. The proof of the Inversion Theorem is deferred to a supplement at the end of Chapter 7. In Section 7.4, Fourier transforms are applied to solving PDEs. One may wish to cover Sections 7.1 to 7.3 quickly and concentrate on Section 7.4. Here, we solve the heat problem on the infinite rod, and the Dirichlet problem for the half plane. We felt that it was a good idea to emphasize the fact that Fourier transform methods not only presume that a solution of a problem exists, but also that it has certain decay properties. Thus, integral formulas for solutions obtained in this fashion should be checked independently through a careful application of Leibniz's rule for differentiating under the integral. For a class of mostly engineers, this point can be made, without going through the details of the verification. Although a derivation of D'Alembert's formula for the wave equation is given in Chapter 5, we also show how to get it by Fourier transform techniques and the Dirac delta distribution is discussed. In Section 7.5, heat problems for semi–infinite and finite rods are solved via the method of images. The validity of formal infinite–sum solutions, found in Chapter 4, is now handled with ease. Also, Fourier sine and cosine transformations are introduced and applied.

Chapter 8 (Numerical Solutions of PDEs) : While the solution of PDEs by numerical methods could constitute a whole course, we offer an introduction to the subject in Chapter 8. Our aim is not to present, without proof or motivation, a huge number of algorithms. Instead, we have concentrated on the foundations of the numerical approach, and we work mostly with the familiar heat equation to illustrate the nature and possible pitfalls of the numerical approach. To broaden the horizons, we do provide an optional overview of other numerical methods for other PDEs for the interested reader in Section 8.4. In Section 8.1, the "big O" notation is introduced. There is discussion of Taylor's Theorem which is the basis for the approximation of partial derivatives by finite differences. This allows the approximation of PDE problems by a finite system of equations for the values of the unknown function at grid points. For the heat equation, these systems are easily solved by the explicit method in Section 8.2. Moreover, in the case of the heat equation, the discretization error (i.e., the difference of the numerical solution from the actual solution) is proved to approach zero as the grid point separation goes to zero, at least in the absence of round–off errors. In Section 8.3, we obtain exact solutions for a finite grid by means of the theory of difference equations. We then examine how systematic round–off errors lead to the conclusion that best results are not always obtained by taking the grid size as small as possible. Continuing with the simple case of the heat equation, we obtain theoretical estimates for optimal grid sizes, which are born out to be correct in concrete examples. We believe that it is better to discuss in some depth a number of crucial issues for a single equation, than only briefly comment on a lot of PDEs and techniques. Again, Section 8.4 provides some overview and plenty of references for further study.

Chapter 9 (PDEs in Higher Dimensions): In Section 9.1 the fundamental ideas in Chapters 3 though 7 are extended in a straightforward manner to the case of several cartesian spatial coordinates. We solve dynamic heat problems on rectangles and cubes, and consider Laplace's equation on a solid rectangle. Double Fourier transforms and series are easily motivated and introduced. In Section 9.2, it is made clear that the primary objects from which solutions of the heat, wave and potential problems are constructed are the eigenfunctions of the Laplace operator which meet the B.C. . This basic fact is often hidden behind the process of separation of variable and the plethora of special functions which thereby arise in various coordinate systems. A great variety of series expansions for functions all fall into the category of eigenfunction expansions. In Section 9.2, we also prove a uniform convergence result for double Fourier series, and discuss simple properties of double Fourier transforms. In Section 9.3, we begin our study of the standard PDEs in terms of spherical coordinates. The spherical harmonics are defined as eigenfunctions of the Laplace operator on a sphere. They arise as the angular part of eigenfunctions of the Laplace operator on space and can be expressed through associated Legendre functions. We solve a number of heat and wave problems with spherical symmetry. The three–dimensional version of D'Alembert's formula is derived and Huygen's principle is discussed. The determination of all eigenvalues and spherical harmonics, dimensions of eigenspaces, etc. is covered in Section 9.4. There is a complete proof of the uniform convergence of the Laplace series for C^2 functions on a sphere. Moreover, a number of problems with angular dependence (e.g., heat flow in a ball) are solved through the use of spherical harmonics and spherical Bessel functions. In Section 9.5, we consider PDEs in cylindrical coordinate systems and some more PDEs in spherical coordinates, but with nontrivial potentials, such as Schrödinger's equation. The special functions which arise in the process are discussed. While spherical Bessel functions can be expressed in terms of sines and cosines, the cylindrical Bessel functions (of integer order) cannot, which is why we did not handle cylindrical coordinates before spherical ones. We consider a number of applications, ranging from the vibrating circular drum, to the determination of the energy levels and wave functions for the (nonrelativistic) hydrogen atom and the degeneracy of the energy levels which forms the basis for the periodic table. Section 9.6 deals with the standard heat, wave and potential problems on compact submanifolds with boundary in $\mathbb{R}^n$. Laplace operators are defined on these objects in an easily understood way. Although, we do not prove the existence theory for eigenfunctions and eigenvalues in this general setting, some of the more readable references are cited. Admittedly, the eigenfunctions are difficult to concretely compute or approximate, but once the eigenfunctions are given, the solution of the standard heat, wave and potential problems on manifolds proceeds in a way which is quite analogous to the many special cases covered in the rest of the book. This last section essentially unifies and consolidates these special cases into a single framework. Moreover, there is some discussion of Weyl's asymptotic formula for the eigenvalues of the Laplace operator, and the geometric information about the manifold which can be "heard" from the eigenvalues which may be interpreted as frequencies of vibration.

In constructing a one–semester or two–quarter course, we suggest selecting sections from the list below, keeping the indicated priorities in mind. In addition, 1.1 should be covered if your students are weak in ODEs. Sections which are marked with stars can or should be covered in only 2 hours, whereas most instructors will want to spend about 3 hours on the other sections. Leave time for tests and going over some of the homework. Chapters 8 and 9 are probably best left for a second semester or possibly as sources of projects for advanced, gifted and/or highly motivated students. In some schools where students have strong backgrounds or interests in computers one may wish to cover Chapter 8 in lieu of Chapter 7.

crucial sections: 1.2, 1.3*, 3.1, 3.2, 4.1, 4.2, 4.3*, 5.1, 5.2, 6.1*, 6.2*, 6.3

highly desirable sections: 2.1, 3.3*; 5.3*, 6.4, 6.5, 7.1*, 7.2*, 7.3*, 7.4

luxury sections: 2.2, 2.3, 2.4, 3.4, 4.4, 7.5

BASIC PARTIAL DIFFERENTIAL EQUATIONS

CHAPTER 1

REVIEW AND INTRODUCTION

In this chapter, we review those aspects of ordinary differential equations (ODEs) which will be needed in the sequel. We also provide an overview of the applications of partial differential equations (PDEs), and introduce the reader to some elementary techniques, such as separation of variables. The review of ODEs in Section 1.1 is self–contained, since experience dictates that a remedial study of this material is often sorely needed. Even those whose mathematical knowledge of ODEs is sufficient may find the applied examples and problems (dealing with biology, fluid flow, electronics, mechanical vibrations, resonance, etc.) interesting and challenging. Section 1.2 gives the reader a perspective on the uses of PDEs in various scientific applications, such as gravitation, electrostatics, thermodynamics, acoustics, and minimal soap film surfaces. Some of the material (e.g., the use of Green's functions and integral operators), will not be universally appreciated upon a first reading. Indeed, students will find certain aspects of Section 1.2 more illuminating at later stages in their course of study. In Section 1.3, the studies of ODEs and PDEs are contrasted, with regard to the differences in the typical forms for general solutions. We illustrate how side conditions are used to extract particular solutions from general ones. Moreover, the method of separation of variables is also covered in this section.

1.1 A Review of Ordinary Differential Equations

A differential equation is an equation involving an unknown function and its derivatives. If the unknown is a function of more than one variable, then the differential equation is called a **partial differential equation** (henceforth, abbreviated PDE), since the derivatives of the unknown function are partial derivatives. In an ordinary differential equation (ODE), the unknown function depends on a single variable. Before studying PDEs, a review of certain basics of ODEs is desirable, because solutions of PDEs can often be found by solving related ODEs. The following review of first–order ODEs (separable and linear) and homogeneous second–order linear ODEs with constant coefficients will suffice for our purposes.

First–Order ODEs

A first–order ODE is separable, if it can be written in the form

$$f(y)\,\frac{dy}{dx} = g(x)\,, \tag{1}$$

where y is an unknown function of the independent variable x.

One solves such an equation by integrating (if possible) both sides with respect to x. Integrating the left side yields

$$\int f(y)\,\frac{dy}{dx}\,dx = \int f(y)\,dy = F(y) + C_1\,,$$

where F(y) is an antiderivative of f(y) (i.e., $F'(y) = f(y)$) and C_1 is an arbitrary constant. Integrating the right side of (1) also, and letting G(x) denote an antiderivative of g(x), we then obtain

$$F(y) + C_1 = G(x) + C_2 \qquad \text{or} \qquad F(y) = G(x) + C\,, \tag{2}$$

where we have incorporated the arbitrary constants C_1 and C_2 into the single arbitrary constant $C = C_2 - C_1$. In practice, one can obtain (2) by first rewriting (1) in terms of differentials

$$f(y)dy = g(x)dx\,. \tag{3}$$

Then, integrating both sides of (3) yields (2). Note that in (3) the variables x and y are on different sides of the equation, and hence the term "separable equation" is used. If possible, one solves (2) for y in terms of x. However, there may be more than one value (or possibly no value) for y, given x and C. Observe that for a fixed value of C, equation (2) will usually define a curve in the xy–plane, but there is no guarantee that this curve will be the graph of a function of x. Nevertheless, the family of curves obtained by allowing C to vary in (2), is usually considered to adequately represent the set of solutions of (1) or (3).

Example 1. A certain population has $P(t)$ individuals at time t, and its rate of growth is proportional to its size (i.e., $P'(t) = aP(t)$, for some constant $a > 0$). Find $P(t)$ in terms of the initial population $P(0)$ and a.

Solution. The equation $P'(t) = aP(t)$ is separable, since we can write it in the form

$$\frac{dP}{P} = a\,dt.$$

Integrating, we obtain (assuming $P > 0$) $\log(P) = at + C$ or $P(t) = \exp(at + C) = e^C e^{at}$. Since $P(0) = e^C$, the desired solution is

$$P(t) = P(0)e^{at}.$$

Note that the same technique will work in the more general case where $P'(t) = a(t)f(P(t))$ for given functions $a(t)$ and $f(P)$, since this equation is also separable. However, the technique fails for $P'(t) = t + P(t)$ and many other equations which are not separable. □

Example 2. A particle is carried along by a fluid flow in the xy–plane. Suppose that the velocity of the fluid at the arbitrary point (x,y) is $2y\mathbf{i} + 4x\mathbf{j}$ (i.e., the direction and magnitude of the fluid flow varies from point to point). Find the path traced out by the particle, if it is known to pass through the point $(1,3)$.

Solution. The slope of the path of a particle at (x,y) is the ratio $4x/2y$ (assuming that $y \neq 0$) of the components of the fluid velocity vector at (x,y). Assuming that the path is the graph of a function y of x, we then obtain the ODE $y'(x) = 4x/2y$, which is separable $(2y\,dy = 4x\,dx)$. Integrating, we obtain the family of streamlines (cf. Figure 1)

$$y^2 = 2x^2 + C, \tag{4}$$

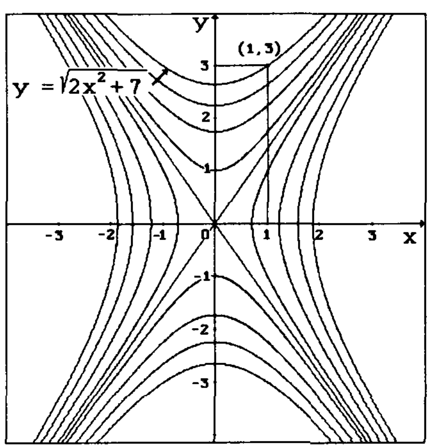


Figure 1

which are hyperbolas. The streamline passing through (1,3) is the upper branch of the hyperbola (4) with $C = 3^2 - 2(1)^2 = 7$, namely $y(x) = (2x^2 + 7)^{\frac{1}{2}}$. □

Another type of first–order ODE which arises in the sequel is the **first–order linear ODE**

$$a(x)y'(x) + b(x)y(x) = c(x) , \tag{5}$$

where $a(x)$, $b(x)$ and $c(x)$ are given continuous functions. Assuming that $a(x) \neq 0$, we may divide (5) by $a(x)$, obtaining an ODE in standard form, in the following sense.

We will say that the first–order linear ODE

$$y'(x) + p(x)y(x) = q(x) \tag{6}$$

is in **standard form.**

If we replace $q(x)$ by 0, the resulting equation

$$y'(x) + p(x)y(x) = 0 \tag{7}$$

is called the **related homogeneous equation** for (6). Unlike (6), (7) is always separable :

$$\frac{dy}{y} = -p(x)\,dx \qquad (y \neq 0) . \tag{8}$$

Thus, by integrating (8), we obtain the following general solution $y_h(x)$ of (7) :

$$y_h(x) = C\cdot\exp[-P(x)] \text{ , where } P(x) \equiv \int p(x)\,dx . \tag{9}$$

The **integrating factor** for equation (6) is defined to be

$$m(x) \equiv \exp[P(x)] = \exp\left[\int p(x)\,dx\right] . \tag{10}$$

Note that by (9) we have $m(x)y_h(x) = C$. Thus,

$$\begin{aligned} 0 = \frac{d}{dx}[m(x)y_h(x)] &= m(x)y_h'(x) + m'(x)y_h(x) \\ &= m(x)y_h'(x) + m(x)p(x)y_h(x) = m(x)[y_h'(x) + p(x)y_h(x)] \end{aligned} \tag{11}$$

where we have used the fact that $m'(x) = \exp[P(x)]P'(x) = m(x)p(x)$. For a solution $y(x)$ of (6) with $q(x) \not\equiv 0$, we do not have $m(x)y(x) = C$, but the computation in (11) yields

$$\frac{d}{dx}[m(x)y(x)] = m(x)[y'(x) + p(x)y(x)] = m(x)q(x) \tag{12}$$

Integrating (12), we obtain

$$m(x)y(x) = \int m(x)q(x)\,dx + C$$

or

$$y(x) = [m(x)]^{-1}\left\{\int m(x)q(x)\,dx + C\right\}. \tag{13}$$

Note that (13) reduces to (9) when $q(x) \equiv 0$. While one may simply use formula (13) to write down the solution of (6), it is preferable to remember the steps of the solution process. (For a summary of these steps see the end of this section.)

Example 3. Sclve $(1+x^2)y' + 2xy = 3x^2$.

Solution. First, put the equation into the standard form (6), namely

$$y' + [2x/(1+x^2)]y = 3x^2/(1+x^2). \tag{14}$$

The integrating factor for equation (14) (cf. (10)) is

$$m(x) = \exp\left[\int \{2x/(1+x^2)\}\,dx\right] = \exp[\log(1+x^2)] = 1+x^2.$$

Equation (12) tells us that if we multiply both sides of (14) by $m(x)$, then we will obtain

$$\frac{d}{dx}[m(x)y(x)] = m(x)q(x) = 3x^2. \tag{15}$$

Integrating both sides of (15), we get $m(x)y(x) = x^3 + C$ or $y(x) = (x^3+C)/(1+x^2)$. □

Example 4. Consider two identical cans, A and B. Assume that syrup will leak out of either can at a rate which is proportional to the volume V of the syrup in the can, say $V'(t) = -kV(t)$, where $k > 0$, due to the leakage. Suppose that the initial volume of syrup in can A is $V_A(0)$, while can B is initially empty. If can A begins leaking into can B at $t = 0$, find the volume $V_B(t)$ of syrup in can B at an arbitrary time $t > 0$.

Solution. The rate of change of $V_B(t)$ is

$$V_B'(t) = -kV_B(t) - V_A'(t). \tag{16}$$

Since $V_A'(t) = -kV_A(t)$, we find, as in Example 1, that $V_A(t) = V_A(0)e^{-kt}$. Thus, by (16)

$$V_B'(t) + kV_B(t) = kV_A(0)e^{-kt} .$$

Multiplying by the integrating factor e^{kt} , we obtain (via (12))

$$\frac{d}{dt}[e^{kt}V_B(t)] = kV_A(0) \quad \text{or} \quad V_B(t) = e^{-kt}[ktV_A(0) + C] .$$

Since $V_B(0) = 0$, we know that $C = 0$. So, $V_B(t) = ktV_A(0)e^{-kt}$. □

Example 5. Suppose that tank A contains salt water with 4 pounds of salt per 100 gallons. Tank B is initially filled with 100 gallons of pure water. Over a period of one hour, the water in tank B is drained at the rate of 3 gallons per minute. The water in tank A flows into tank B at the rate of 5 gallons per minute, as tank B is drained. How many pounds of salt are dissolved in tank B at the end of the hour? Assume that tank B is well–mixed at all times and does not overflow.

Solution. Let S(t) denote the number of pounds of salt in tank B at time t. At time t, tank B loses salt (via draining) at the rate of 3 gallons per minute, times the amount of salt per gallon in tank B, namely, $3S(t)/(100 + (5-3)t)$ lbs./min.. The rate at which tank B gains salt from tank A is 5 times 4/100 lbs./min. . Thus, the net rate of salt increase in tank B is given by $S'(t) = [-3S(t)/(100 + 2t)] + 1/5$. Hence,

$$S'(t) + [3S(t)/(100 + 2t)] = 1/5 . \tag{17}$$

The integrating factor is $m(t) = \exp[\frac{3}{2}\log(100 + 2t)] = (100 + 2t)^{\frac{3}{2}}$. Multiplying (17), on both sides, by the integrating factor, we obtain

$$\frac{d}{dt}[m(t)S(t)] = \frac{1}{5}(100 + 2t)^{\frac{3}{2}} \quad \text{and} \quad S(t) = (100 + 2t)^{-\frac{3}{2}}[\tfrac{1}{25}(100 + 2t)^{\frac{5}{2}} + C] .$$

Since $S(0) = 0$, we have $C = -\frac{1}{25}\cdot 10^5$ and $S(t) = \frac{1}{25}(100 + 2t)^{-\frac{3}{2}}((100 + 2t)^{\frac{5}{2}} - 10^5)$. Finally, $S(60) \approx 7.574$ lbs.. □

Second–Order Linear ODEs with Constant Coefficients

We will need a good understanding of the homogeneous second–order linear equation

$$ay''(x) + by'(x) + cy(x) = 0 , \tag{18}$$

where the coefficients a,b, and c are real constants. If $a = 0$, then (18) is either a linear first–order ODE, or (if also $b = 0$) trivial. Thus, we assume that $a \neq 0$. The usual method of solving (18) is to first assume that a solution is of the form $y(x) = e^{rx}$ for some constant r. Substituting this y(x) into (18), we obtain

$$ae^{rx}r^2 + be^{rx}r + ce^{rx} = e^{rx}(ar^2 + br + c) = 0 .$$

Thus, r must satisfy the quadratic equation $ar^2 + br + c = 0$, known as the **auxiliary equation** for (18). Let $d = b^2 - 4ac$. There are three cases : $d > 0$, $d = 0$ and $d < 0$.

If $d > 0$, then there are two distinct real roots, namely

$$r_1 = \frac{-b + \sqrt{d}}{2a} \quad \text{and} \quad r_2 = \frac{-b - \sqrt{d}}{2a} .$$

In this case, the general solution of (18) is the superposition (or linear combination)

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} , \tag{19}$$

where c_1 and c_2 are arbitrary constants. Recall that the superposition of two solutions of a homogeneous linear equation is also a solution (cf. Problem 7). Moreover, if the equation is second–order and the ratio of two particular solutions is not constant (i.e., they are linearly independent), then any solution is a superposition of these two solutions (cf. Problem 20).

If $d = 0$, then there is only one solution of $ar^2 + br + c = 0$, namely $r = -b/2a$, which is a root of multiplicity 2. However, we recall that, in addition to e^{rx}, there must be another linearly independent solution of (18). By trying a solution of the form $f(x)e^{rx}$, one finds $f''(x) = 0$ (cf. Problem 9). Thus, choosing $f(x) = x$, we obtain another linearly independent solution, xe^{rx}. Hence, when $d = 0$, the general solution of (18) is

$$y(x) = c_1 e^{rx} + c_2 x e^{rx} , \tag{20}$$

where $r = -b/2a$. If $d < 0$, the roots of $ar^2 + br + c = 0$ are complex, namely

$$r_1 = \frac{-b + i\sqrt{|d|}}{2a} \quad \text{and} \quad r_2 = \frac{-b - i\sqrt{|d|}}{2a} , \tag{21}$$

where i has the property that $i^2 = -1$. (Thus, i cannot be a real number.) Now set

$$r_1 = \alpha + i\beta \quad \text{and} \quad r_2 = \alpha - i\beta \tag{22}$$

where α and β are real numbers. Then it can be shown that

$$y(x) = c_3e^{(\alpha+i\beta)x} + c_4e^{(\alpha-i\beta)x}\ , \tag{23}$$

where c_3 and c_4 are arbitrary constants, satisfies (18). In order to construct a more useful form of the solution (23) (for the details see Problem 11) we use **Euler's formula**

$$e^{iy} = \cos(y) + i\,\sin(y)\ . \tag{24}$$

[The Swiss–born mathematician and physicist, Leonhard Euler (1707–1783), made important contributions to many areas of mathematics and celestial mechanics. The number e is named after him.]. Euler's formula can be established by setting $z = iy$ in the power series expansion of the complex exponential e^z :

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots\ . \tag{25}$$

Now using the relation $e^{(\alpha+i\beta)x} = e^{\alpha x}e^{i\beta x}$ and Euler's formula, we can express (23) in the form

$$y(x) = e^{\alpha x}[c_3(\cos(\beta x)+i\sin(\beta x)) + c_4(\cos(\beta x)-i\sin(\beta x))]. \tag{26}$$

Setting $c_1 = c_3 + c_4$ and $c_2 = i(c_3 - c_4)$, (26) becomes

$$y(x) = e^{\alpha x}[c_1\cos(\beta x) + c_2\sin(\beta x)]\ . \tag{27}$$

Finally, with the notation of (22) and (21), we obtain from (27) the following general solution of (18), in the case when $d < 0$:

$$y(x) = e^{-bx/2a}\left[c_1\cos\left[\frac{\sqrt{|d|}\,x}{2a}\right] + c_2\sin\left[\frac{\sqrt{|d|}\,x}{2a}\right]\right]\ . \tag{28}$$

The foregoing results may be summarized as follows :

Consider the ODE

$$ay''(x) + by'(x) + cy(x) = 0 \ , \tag{29}$$

where a,b and c are real constants and $a \neq 0$. Let r_1 and r_2 be the roots of the associated auxiliary equation $ar^2 + br + c = 0$. Let $d = b^2 - 4ac$, and let c_1, c_2 denote arbitrary constants.

1. If r_1 and r_2 are real and distinct (i.e., $d > 0$), then the general solution of (29) is

$$y(x) = c_1e^{r_1x} + c_2e^{r_2x} \ . \tag{30}$$

2. If $r_1 = r_2 = r$ (i.e. $d = 0$), then the general solution of (29) is

$$y(x) = c_1e^{rx} + c_2xe^{rx} \ . \tag{31}$$

3. If $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$ (i.e., $d < 0$), then the general solution of (29) is

$$y(x) = e^{\alpha x}[c_1\cos(\beta x) + c_2\sin(\beta x)] \ . \tag{32}$$

Example 6. An object of mass m is attached to a spring which lies along the x–axis, as shown in Figure 2 below. With **Hooke's law** in effect, when the object is displaced to the position x, the spring exerts a force $-kx$ (toward the origin, since the constant k is positive) on the object. Let $x(t)$ be the position of the object at time t. The object is also subject to a force, say due to air resistance, which is $-bx'(t)$, for a constant $b > 0$. If the object is released from the position x_0 at time $t = 0$, find the position of the object at any time $t > 0$, using **Newton's second law** of motion $mx''(t) = F(t)$, where $F(t)$ is the total force on the object at time t. [The English scientist Robert Hooke (1635–1703) and mathematician/physicist Isaac Newton (1642–1727) were often at odds, in particular, over the division of credit for the inverse–square law of gravity.]

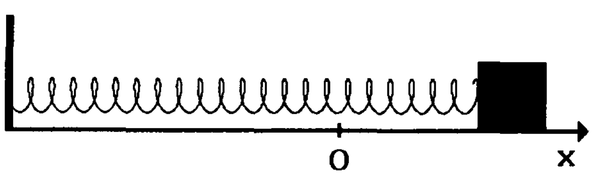


Figure 2

Solution. Since the total force on the object is $F(t) = -kx(t) - bx'(t)$, Newton's second law yields the ODE

$$mx''(t) + bx'(t) + kx(t) = 0 \ . \tag{33}$$

Here $d = b^2 - 4mk$. All three cases $d > 0$, $d = 0$ and $d < 0$ are possible. They are referred to as **over–damped**, **critically–damped** and **under–damped** (or **oscillatory**) respectively. The solutions in these cases are

$$x(t) = e^{-\frac{1}{2}bt/m}[c_1e^{\frac{1}{2}\sqrt{d}\cdot t/m} + c_2e^{-\frac{1}{2}\sqrt{d}\cdot t/m}] \quad (d > 0), \tag{34}$$

$$x(t) = e^{-\frac{1}{2}bt/m}[c_1 + tc_2] \quad (d = 0), \tag{35}$$

$$x(t) = e^{-\frac{1}{2}bt/m}[c_1\cos(\tfrac{1}{2}\sqrt{|d|}\cdot t/m) + c_2\sin(\tfrac{1}{2}\sqrt{|d|}\cdot t/m)] \quad (d < 0). \tag{36}$$

The constants c_1 and c_2 are found from the given initial conditions $x(0) = x_0$ and $x'(0) = 0$. For (34), we have that

$$x(0) = c_1 + c_2 = x_0 \quad \text{and} \quad x'(0) = \frac{1}{2m}[(\sqrt{d} - b)c_1 - (\sqrt{d} + b)c_2] = 0$$

imply $c_1 = \frac{1}{2}[1 + (b/\sqrt{d})]x_0$ and $c_2 = \frac{1}{2}[1 - (b/\sqrt{d})]x_0$.

Hence, (34) becomes

$$x(t) = x_0e^{-\frac{1}{2}bt/m}\left[\tfrac{1}{2}(e^{\frac{1}{2}\sqrt{d}\cdot t/m} + e^{-\frac{1}{2}\sqrt{d}\cdot t/m}) + (b/\sqrt{d})\cdot\tfrac{1}{2}(e^{\frac{1}{2}\sqrt{d}\cdot t/m} - e^{-\frac{1}{2}\sqrt{d}\cdot t/m})\right]$$

$$= x_0e^{-\frac{1}{2}bt/m}[\cosh(\tfrac{1}{2}\sqrt{d}\cdot t/m) + (b/\sqrt{d})\sinh(\tfrac{1}{2}\sqrt{d}\cdot t/m)]. \tag{37}$$

The hyperbolic sine and cosine often occur naturally, when initial conditions are imposed. They are defined by $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$ and $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$. The interested reader who is unfamiliar with these functions and their relation to the usual sine and cosine, should consult Problem 18. The computation of the values for c_1 and c_2 in (35) and (36) is suggested in Problem 12. For certain values of b, m and k, the solutions are graphed in Figure 3 below. □

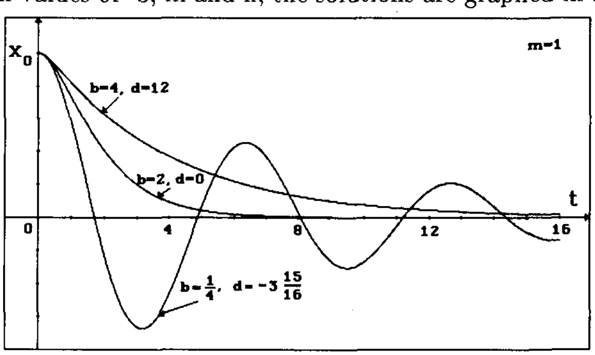


Figure 3

Example 7. Equation (18) also arises in electrical circuit theory. Suppose that a battery of voltage V, a resistor of resistance R, a coil of inductance L and a capacitor of capacitance C are placed in series as shown below in Figure 4. We wish to find the most general expression for the current i(t) in this circuit as a function of time t. [The flow of current in this circuit is governed by the second law of the German physicist Gustav Kirchhoff (1824–1887), who is famous for his contributions in electronics and spectroscopy.]

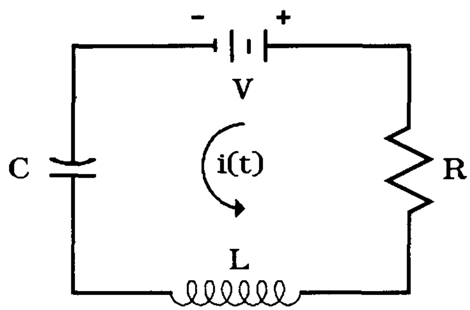


Figure 4

Solution. **Kirchhoff's second law** asserts that the sum of the voltage drops across the elements of any closed loop in a circuit must be zero. At time t, the voltage drop across the resistor is R times the current i(t). The voltage drop across the coil is $L\,i'(t)$. (This drop is due to the fact that an increasing current in a coil creates a changing magnetic field which induces an opposing electric field and induces a voltage drop across the coil.) The voltage drop across a capacitor is $1/C$ times the total amount of charge ($\int i(t)\,dt$) which it has accumulated on one of its plates. By Kirchhoff's second law,

$$R\,i(t) + L\,i'(t) + \frac{1}{C}\int i(t)\,dt - V = 0 \ , \tag{38}$$

and differentiating, we obtain

$$L\,i''(t) + R\,i'(t) + \frac{1}{C}\,i(t) = 0 \ . \tag{39}$$

Hence, the current i(t) behaves just as the displacement of an object attached to a spring as in Example 6, with $m = L$, $b = R$ and $k = 1/C$. In particular, with these new values, formulas (34),(35) and (36) give us the general solutions for i(t) in the three cases. □

We will not cover the general case of the inhomogeneous equation

$$ay''(t) + by'(t) + cy(t) = f(t) \tag{40}$$

which would arise in Example 6 when there is an external driving force f(t), or in Example 7 when the voltage source is variable $[f(t) = V'(t)]$. One could solve (40) by adding a particular solution to the general solution of the related homogeneous equation with f(t) replaced by 0. A particular solution can be obtained by the method of variation of parameters which can be found in most ODE books. However, as an illustration of resonance and the utility of the complex approach, we will find a particular solution of (40) in the important case when f(t) is of the form $A\cos(\omega t)$ or $A\sin(\omega t)$, for a constant amplitude A and angular frequency ω.

Example 8. Find a particular solution of (40) with $abc \neq 0$, in the case when $f(t) = A\cos(\omega t)$ or $f(t) = A\sin(\omega t)$ for a real constant ω, by using the following approach. Determine a complex constant C, such that $y(t) = Ce^{i\omega t}$ solves (40) with $f(t) = Ae^{i\omega t}$. Then show that the real and imaginary parts of $y(t)$ will be the desired particular solutions.

Solution. Substituting the trial solution $y(t) = Ce^{i\omega t}$ into (40) with $f(t) = Ae^{i\omega t}$, we obtain

$$Ce^{i\omega t}[a(i\omega)^2 + bi\omega + c] = Ae^{i\omega t} \quad \text{or} \quad C\cdot[(c - a\omega^2) + ib\omega] = A. \tag{41}$$

Using the identity $(r + is)(r - is) = r^2 + s^2$, we see that

$$[r + is]^{-1} = (r - is)/(r^2 + s^2). \tag{42}$$

Thus, multiplying by $[(c-a\omega^2) + ib\omega]^{-1}$ in (41), we obtain

$$C = A[(c - a\omega^2) - ib\omega]/[(c - a\omega^2)^2 + b^2\omega^2] \tag{43}$$

and

$$\begin{aligned} y(t) &= Ce^{i\omega t} = C[\cos(\omega t) + i\sin(\omega t)] \\ &= A[(c - a\omega^2)\cos(\omega t) + b\omega\sin(\omega t)]/[(c - a\omega^2)^2 + b^2\omega^2] \\ &\quad + iA[(c - a\omega^2)\sin(\omega t) - b\omega\cos(\omega t)]/[(c - a\omega^2)^2 + b^2\omega^2] \\ &= y_R(t) + iy_I(t), \end{aligned}$$

where the last equation defines the real and imaginary parts of the solution $y(t)$ of (40) with $f(t) = Ae^{i\omega t} = A\cos(\omega t) + iA\sin(\omega t)$. Since

$$ay'' + by' + cy = (ay_R'' + by_R' + cy_R) + i(ay_I'' + by_I' + cy_I), \tag{44}$$

we see that $y_R(t)$ solves (40) with $f(t) = A\cos(\omega t)$, while $y_I(t)$ solves (40) with $f(t) = A\sin(\omega t)$. If the frequency ω is allowed to vary, then the amplitude $A[(c - a\omega^2)^2 + b^2\omega^2]^{-\frac{1}{2}}$ (cf. Figure 5 below) of y_R and y_I is largest, when ω is chosen so that $h(\omega) = (c - a\omega^2)^2 + b^2\omega^2$ is minimal.

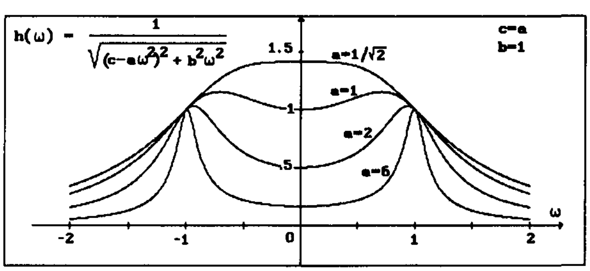


Figure 5

By setting $h'(\omega) = 0$, these resonant frequencies ω_R are found to be

$$\omega_R = \pm \left[\frac{c}{a} - \frac{1}{2}\left[\frac{b}{a}\right]^2 \right]^{\frac{1}{2}}, \tag{45}$$

unless $2ac - b^2 < 0$, in which case $\omega_R = 0$ yields the maximum amplitude. Observe that if $b^2 - 4ac < 0$, then $|\omega_R|$ is less than the natural frequency $\nu \equiv |\sqrt{|d|}\,/2a| = [(c/a) - \frac{1}{4}(b/a)^2]^{\frac{1}{2}}$ which occurs in (32). Note that

$$(\omega_R/\nu)^2 = [1 - \tfrac{1}{2}(b^2/ac)]/[1 - \tfrac{1}{4}(b^2/ac)] ,$$

which shows that $|\omega_R|/\nu \longrightarrow 1$ as $(b^2/ac) \longrightarrow 0$. In the above argument, we have assumed that A does not depend on ω. In applications to electronics, A is usually proportional to ω, in which case (45) does not apply. As an illustration see Problem 13. □

Special Systems of ODEs

Occasionally we will meet a system of linear ODEs of the form

$$x'(t) = ax(t) + by(t) \tag{46a}$$

$$y'(t) = cx(t) + dy(t) , \tag{46b}$$

where a,b,c, and d are given real constants and x(t) and y(t) are unknown functions. We are required to solve the system (46a) (46b) for x(t) and y(t), given the initial values x(0) and y(0). If $b = 0$, we can solve the first order ODE (46a) for x(t). Then we substitute the solution x(t) into (46b), and solve the resulting first–order ODE for y(t). If $b \neq 0$, we differentiate both sides of (46a) and use (46b) as follows :

$$x''(t) = ax'(t) + by'(t) = ax'(t) + b(cx(t) + dy(t))$$

$$= ax'(t) + bcx(t) + d\cdot(x'(t) - ax(t)),$$

where we have used 46(a) for the last equality. Thus, x(t) must satisfy

$$x''(t) - (a + d)x'(t) + (ad - bc)x(t) = 0 . \tag{47}$$

This familiar second–order ODE is solved for x(t), using the initial values x(0) and $x'(0) = ax(0) + by(0)$. There is no need to solve (46b) for y(t), since by (46a)

$$y(t) = (x'(t) - ax(t))/b . \tag{48}$$

The above ideas suffice to solve certain other types of systems of ODEs which arise in the sequel, and there will be no need for differential operator/matrix methods.

Example 9. In Example 2, calculate the position $(x(t),y(t))$ of the particle at any time t, given that $x(0) = 1$ and $y(0) = 3$.

Solution. The velocity vector at time t is $x'(t)\mathbf{i} + y'(t)\mathbf{j}$. Thus, we have the system

$$x'(t) = 2y(t) \tag{49a}$$

$$y'(t) = 4x(t) . \tag{49b}$$

As above, differentiating (49a) and using (49b) we get

$$x''(t) = 2y'(t) = 8x(t) \quad \text{or} \quad x''(t) - 8x(t) = 0 .$$

Since $r^2 - 8 = 0$, we have $r = \pm 2\sqrt{2}$, and the general solution is

$$x(t) = c_1 e^{2\sqrt{2}\cdot t} + c_2 e^{-2\sqrt{2}\cdot t} ,$$

but we need $x(0) = 1$ and $x'(0) = 2y(0) = 6$. These conditions yield

$$\begin{cases} c_1 + c_2 = 1 \\ 2\sqrt{2}\ c_1 - 2\sqrt{2}\ c_2 = 6 \end{cases} \quad \text{or} \quad \begin{cases} c_1 = \frac{1}{2} + \frac{3}{4}\cdot\sqrt{2} \\ c_2 = \frac{1}{2} - \frac{3}{4}\cdot\sqrt{2} \end{cases} .$$

Using (49a),

$$y(t) = \tfrac{1}{2}\, x'(t) = \sqrt{2}\left[c_1 e^{2\sqrt{2}\cdot t} - c_2 e^{-2\sqrt{2}\cdot t}\right] .$$

As t varies, the point $(x(t),y(t))$ traces out a branch of the hyperbola $y^2 - 2x^2 = 7$ (cf. (4) in Example 2 with $C = 7$), because one can verify that $[y(t)]^2 - 2[x(t)]^2 = 7$. The parametric representation $(x(t),y(t))$ for this curve gives us much more information than $y^2 - 2x^2 = 7$ does, since $(x(t),y(t))$ gives us the particle's position at any time t. □

Example 10. The weight $w(t)$ of a certain animal grows at a rate $w'(t) = Cs(t) - K$, where $s(t)$ is the size of the animal's food supply and $K, C > 0$ are constants. We assume that $s(0)$ and $w(0)$ are positive. If $s(t)$ ever becomes 0, then it remains at 0. The animal has starved to death, if $w(t)$ drops to 0. The heavier the animal gets, the more it eats from its food supply which ordinarily would grow at a rate proportional to $s(t)$ in the animal's absence. Thus, while the food supply lasts, $s'(t) = As(t) - Bw(t)$, for constants $A, B > 0$. Show that if $A^2 < 4BC$, then the animal will eventually starve to death (after a number of diet/binge cycles), unless $w(0) = AK/BC$ and $s(0) = K/C$, in which case $w(t)$ and $s(t)$ are constant. (The case where $A^2 > 4BC$ is the subject of Problem 19.)

Solution. We have the system

$$s'(t) = As(t) - Bw(t) \tag{50a}$$

$$w'(t) = Cs(t) - K . \tag{50b}$$

Differentiating the first equation and using the second, we have

$$s''(t) - As'(t) + BCs(t) = BK . \tag{51}$$

The general solution of (51) is the constant particular solution K/C plus the general solution of the related homogeneous equation

$$s''(t) - As'(t) + BCs(t) = 0 . \tag{52}$$

Equation (52) is of the form (18), with $a = 1$, $b = -A$ and $c = BC$. Thus, $d = b^2 - 4ac = A^2 - 4BC$. If $A^2 < 4BC$, then $d < 0$ and the general solution of (51) is

$$s(t) = K/C + e^{\frac{1}{2}At}[c_1\cos(\sqrt{|d|}\cdot t/2) + c_2\sin(\sqrt{|d|}\cdot t/2)] . \tag{53}$$

The function in brackets can be written as $(c_1^2 + c_2^2)^{\frac{1}{2}}\cos[(\sqrt{|d|}\cdot t/2) + \theta]$ for some constant θ (cf. Problem 10). Thus, if c_1 and c_2 are not both zero, the solution will oscillate about K/C with the growing amplitude $(c_1^2 + c_2^2)^{\frac{1}{2}}e^{\frac{1}{2}At}$, as the animal diets and indulges with greater intensity. Eventually this amplitude will be greater than K/C (provided w(t) remains positive), and s(t) must drop to zero at some time, say t_0, during the next cycle. Thus, if the animal is still alive at time t_0, then after t_0, $w'(t) = -K$, and w(t) drops steadily to zero. If c_1 and c_2 are both zero, then $s(t) = K/C$, and (50a) then says that $w(t) = AK/BC$. Figure 6 shows that w(t) might drop to zero while s(t) is still positive. □

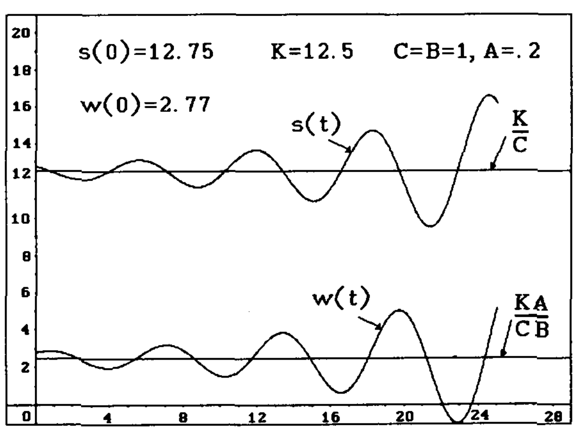


Figure 6

Summary 1.1

1. **First–order separable ODEs** : To solve the first–order separable ODE $f(y)\,\frac{dy}{dx} = g(x)$, write it in the form $f(y)dy = g(x)dx$ and integrate :

$$\int f(y)\,dy = \int g(x)\,dx + C\,.$$

2. **First–order linear ODEs** : The general solution of the first–order linear ODE

$$y'(x) + p(x)y(x) = q(x)\,, \tag{S1}$$

which is in standard form, can be obtained as follows.

(a) Multiply both sides of (S1) by the integrating factor $m(x) = \exp\left\{\int p(x)\,dx\right\}$ and check that

$$\frac{d}{dx}[m(x)y(x)] = m(x)q(x)\,. \tag{S2}$$

(b) Integrate both sides of (S2) to obtain $m(x)y(x) = \int m(x)q(x)\,dx + C$, where C is an arbitrary constant.

(c) The general solution of (S1) is then

$$y(x) = [m(x)]^{-1}\left\{\int m(x)q(x)\,dx + C\right\}.$$

3. **Homogeneous second–order linear ODEs** : To determine the general solution of the homogeneous second–order linear ODE

$$ay''(x) + by'(x) + cy(x) = 0\,, \tag{S3}$$

where a,b,c are real constants and $a \neq 0$, first find the roots r_1 and r_2 of the associated auxiliary equation $ar^2 + br + c = 0$.

Let c_1 and c_2 denote arbitrary constants and let y(x) denote the general solution of (S4).

(i) If r_1 and r_2 are real and distinct, then $y(x) = c_1e^{r_1x} + c_2e^{r_2x}$.

(ii) If $r_1 = r_2 = r$, then $y(x) = c_1e^{rx} + c_2xe^{rx}$.

(iii) If $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$, then $y(x) = e^{\alpha x}[c_1\cos(\beta x) + c_2\sin(\beta x)]$.

4. Linear systems : To solve the linear system of ODEs

$$x'(t) = ax(t) + by(t) \qquad \text{(A)}$$

$$y'(t) = cx(t) + dy(t) \quad , \qquad \text{(B)}$$

where a,b,c, and d are real constants, with given initial values $x(0)$ and $y(0)$, consider the following cases.

Case 1. If $b = 0$, solve (A) for $x(t)$ and substitute the solution into (B). Then solve the resulting ODE for $y(t)$.

Case 2. If $b \neq 0$, differentiate both sides of (A) with respect to t , and then use (B) to get

$$x'' - (a + d)x' + (ad - bc)x = 0 \ .$$

Using the initial values $x(0)$ and $x'(0) = ax(0) + by(0)$, we first solve the above second–order ODE for $x(t)$, and then set $y(t) = [x'(t) - ax(t)]/b$ by (A).

Exercises 1.1

1. Find the general solutions of the following separable equations :

(a) $\frac{dy}{dx} = xy$ (b) $\frac{dx}{dt} = x(1-x)$ (c) $\frac{dy}{dx} = x^2y^2 + x^2 - y^2 - 1$

(d) $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$ (e) $\frac{dx}{dt} + x^2\sin(t) = 0$ (f) $\frac{dy}{dx} - \frac{x+e^{-x}}{y+e^y} = 0$

(g) $\frac{dx}{dt} = te^{x+t}$ (h) $x\frac{dy}{dx} = 1 + y^2$ (i) $T'(t) + 3T(t) = 0$.

2. A radioactive substance decays at a rate proportional to the amount of the substance present. If 64% of the substance remains after 10 years, what percentage will remain after 15 years ?

3. Torricelli's law states that (under certain ideal circumstances) fluid will leak out of a hole at the base of a container at a rate proportional to the square root of the height of the fluid's surface from the base. Suppose that a cylindrical container is initially filled to a depth of one foot. If it takes one minute for three quarters of the fluid to leak out, how long will it take for all of the fluid to leak out ? [Italian Evangelista Torricelli (1608–1647) succeeded Galileo as professor of mathematics at the Florentine Academy, and following a suggestion of Galileo, invented the mercury barometer.]

4. Solve the following first–order linear equations, subject to the given conditions :

(a) $y'(x) + 2y(x) = e^x$, $y(0) = 1$

(b) $x'(t) - (2/t)x(t) = 1$, $x(1) = 0$

(c) $\sin(x)y'(x) - \cos(x)y(x) = \sin(2x)$, $y(\pi/2) = 0$

(d) $x'(t) + \frac{x(t)}{t} = t^2$, $x(0) = 0$

(e) $3\frac{dy}{dx} + 6xy = 6e^{-x}$, $y(0) = 1$

(f) $\frac{dy}{dx} = 3y + e^{2x}$, $y(0) = 0$

(g) $x'(t) + x(t)\cos(t) = 0$, $x(\pi) = 100$

(h) $\frac{dy}{dx} + (1+2x+3x^2)y = e^{-x-x^2-x^3}$, $y(0) = 3$

(i) $\frac{dx}{dt} + \frac{3x}{2t+100} = 0$, $x(-49.5) = 1$.

5. A population P of bacteria grows at a rate (say $b \cdot P$, $b > 0$) proportional to its size, but it is destroyed at a steadily increasing rate (say $c \cdot t$, $c > 0$) by a spot of mold which starts growing at $t = 0$. Under what circumstances will the mold completely consume the bacteria ?
Hint. Solve $P'(t) = bP(t) - ct$ in terms of b, c, P(0) and t. Under what condition(s) (on P(0), b and c) will P(t) drop to zero for some $t > 0$?

6. Find the general solution, y(x), of the following second–order homogeneous linear ODEs.

(a) $y'' = 0$

(b) $y'' - 3y = 0$

(c) $y'' + 3y = 0$

(d) $y'' + y' = 0$

(e) $y'' - 3y' = 0$

(f) $4y'' + 3y' + 5y = 0$

(g) $2y'' + 5y' + 2y = 0$

(h) $y'' - 6y' + 13y = 0$

(i) $y'' - 4y' + 4y = 0$

(j) $y'' + 10y' + 25y = 0$.

7. Find the particular solutions y(t), meeting the given initial data, of the following second–order homogeneous linear ODEs.

(a) $y'' - 5y' + 6y = 0$; $y(0) = 1$, $y'(0) = 2$

(b) $y'' - 4y' + 4y = 0$; $y(0) = 0$, $y'(0) = 1$

(c) $y'' + y = 0$; $y(0) = a$, $y'(0) = b$

(d) $y'' - y = 0$; $y(0) = a$, $y'(0) = b$

(e) $5y'' + 8y' + 5y = 0$; $y(0) = 0$, $y'(0) = 1$

(f) $5y'' + 8y' + 5y = 0$; $y(0) = 1$, $y'(0) = 0$.

8. (a) Show that if $y_1(x)$ and $y_2(x)$ are solutions of the homogeneous linear ODE $a(x)y'' + b(x)y' + c(x)y = 0$, then the superposition $c_1y_1(x) + c_2y_2(x)$ is also a solution.

(b) If a(x), b(x) and c(x) are continuous with a(x) never zero, then the ODE in part (a) has a unique solution y(x) with given values for $y(x_0)$ and $y'(x_0)$ (cf. [Simmons, Section 57]). Assuming this, show that no solution of this ODE can have a graph which is tangent to the x–axis at some point, unless the solution is identically zero.

9. (a) If $ar^2 + br + c = 0$ has only one root (of multiplicity 2) $r = -b/2a$, show that $f(x)e^{rx}$ is a solution of $ay'' + by' + cy = 0$, if and only if $f''(x) = 0$.

(b) For distinct numbers r_1 and r_2 observe that $\lim_{r_2 \to r_1} \frac{e^{r_2 x} - e^{r_1 x}}{r_2 - r_1} = xe^{r_1 x}$.

How is this observation related to the result in part (a) ?

10. (a) Show that any complex number $z = x + iy$ can be written in "polar form" $re^{i\theta}$, where $r = |z| = [x^2 + y^2]^{\frac{1}{2}}$ and θ are the polar coordinates of the point (x,y) in the Cartesian plane.

(b) For real x, y and ω, note that $x\cos(\omega t) + y\sin(\omega t)$ is the real part of the product $(x + iy)(\cos(\omega t) - i\sin(\omega t)) = re^{i\theta}e^{-i\omega t}$. In view of this show that

$$x\cos(\omega t) + y\sin(\omega t) = r\cos(\omega t - \theta) = r\sin(\omega t - \theta + \pi/2).$$

11. (a) By setting $x = 0$ in the formula $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, $z = x + iy$, and by using the series expansions for $\cos(y)$ and $\sin(y)$, verify that $e^{iy} = \cos(y) + i\sin(y)$.

(b) If $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$, verify that $\frac{1}{2}[e^{r_1 x} + e^{r_2 x}] = e^{\alpha x}\cos(\beta x)$ and $-\frac{1}{2}i[e^{r_1 x} - e^{r_2 x}] = e^{\alpha x}\sin(\beta x)$.

(c) Use the definition $\frac{d}{dx}[f(x) + ig(x)] = f'(x) + ig'(x)$ and the formula

$$e^{(\alpha+i\beta)x} = e^{\alpha x}e^{i\beta x} = e^{\alpha x}[\cos(\beta x) + i\sin(\beta x)]$$

to show that $\frac{d}{dx}e^{rx} = re^{rx}$, where $r = \alpha + i\beta$.

(d) Use part (c) to verify that the function y(x) defined by (23) (see also (21) and (22)) satisfies the differential equation (18).

12. Find the constants c_1 and c_2 in (35) and (36) such that $x(0) = x_0$ and $x'(0) = 0$.

13. Suppose that in Example 7 (with $LRC \neq 0$), the voltage source is alternating, say $V(t) = \sin(\omega t)$. For what value of ω is the amplitude of $i(t)$ the greatest, for large t ?

Hint. In the case of variable $V(t)$, the right side of (39) is replaced by $V'(t)$. Show that any solution of the related homogeneous equation approaches 0 as $t \to \infty$ (such a solution is called **transient**). To find a nontransient particular solution, apply Example 8, with $f(t) = V'(t) = \omega \cdot \cos(\omega t)$, noting that A is ω.

Remark. If $V(t) = V_0e^{i\omega t}$ and $I(t) = I_0e^{i\omega t}$ for complex constants V_0 and I_0, then the complex number I_0/V_0 is called the **admittance** and it is usually denoted by $Y(\omega)$ since it depends on ω, while $Z(\omega) = [Y(\omega)]^{-1}$ is called the **impedance**. The problem is to determine the "low impedance resonance" ω_0 which makes $|Z(\omega)|$ smallest.

14. For the system (46), we showed that $x(t)$ must satisfy $x'' - (a + d)x' + (ad - bc)x = 0$. Show that $y(t)$ must also satisfy $y'' - (a + d)y' + (ad - bc)y = 0$.

15. Solve the following system subject to the given initial data

$$x'(t) = x(t) + y(t) \qquad x(0) = 1$$

$$y'(t) = -x(t) + y(t) \qquad y(0) = 0\,.$$

Draw a rough sketch in the xy–plane of the solution curve $(x(t),y(t))$ as t varies.

16. Consider the system (46). If $(a-d)^2 + 4bc \neq 0$, then show that any complex solution $(x(t),y(t))$ of the system must be of the form

$$x(t) = c_1e^{r_1t} + c_2e^{r_2t} \quad \text{and} \quad y(t) = d_1e^{r_1t} + d_2e^{r_2t}\;,$$

where c_1, c_2, d_1, d_2 are complex constants and r_1 and r_2 are the (possibly nonreal) roots of $r^2 - (a + d)r + (ad - bc) = 0$. What happens if $(a-d)^2 + 4bc = 0$?

Hint. See Problem 14.

17. Solve each of the following systems subject to the given initial data :

(a) $x'(t) = 3x(t) - 4y(t)$, $\quad x(0) = 1$
$y'(t) = x(t) - y(t)$, $\quad y(0) = 1$

(b) $x'(t) = x(t) - 4y(t)$, $\quad x(0) = 1$
$y'(t) = x(t) + y(t)$, $\quad y(0) = 1$

(c) $x'(t) = x(t) + 2y(t)$, $\quad x(0) = 0$
$y'(t) = 3x(t) + 4y(t)$, $\quad y(0) = 1$.

18. For any complex number z, we define the hyperbolic sine and cosine by

$$\sinh(z) = \tfrac{1}{2}(e^z - e^{-z}) \quad \cosh(z) = \tfrac{1}{2}(e^z + e^{-z})\,.$$

(a) Verify that $\cosh^2(z) - \sinh^2(z) = 1$.

(b) For a real variable x, show that $\frac{d}{dx}\sinh(x) = \cosh(x)$ and $\frac{d}{dx}\cosh(x) = \sinh(x)$.

(c) For a real variable y , check that $\sinh(iy) = i\sin(y)$ and $\cosh(iy) = \cos(y)$.

(d) Define sin(z) and cos(z) for any complex number z, by allowing y to be complex in (c). Check that $\sin^2(z) + \cos^2(z) = 1$.

19. In relation to Example 10 , assume that $A^2 > 4BC$ in each of the following parts.

(a) If the animal does not starve to death first, show that its weight eventually grows at an exponential rate, unless the animal maintains the constant weight AK/BC .

(b) Explain intuitively why it is possible to choose positive initial values for w(0) and s(0) such that the animal starves to death.

(c) Give a concrete example to prove your claim in (b).

20. By completing the following steps, show that the general solution of a second–order homogeneous linear equation $ay'' + by' + cy = 0$ [where a,b and c are constants ($a \neq 0$) ; see, however, the final remark after step (f)] is of the form $c_1y_1(x) + c_2y_2(x)$, where $y_1(x)$ and $y_2(x)$ are any two linearly independent solutions (i.e., neither is a constant multiple of the other). We assume that all functions under consideration here have continuous second derivatives everywhere.

(a) Show that two functions f(x) and g(x) (with g(x)/f(x) or f(x)/g(x) differentiable) are linearly dependent on some open interval I, if and only if their Wronskian function W[f,g](x), defined as $f(x)g'(x) - f'(x)g(x)$, is zero for all x in I. [Jozef M. Hoene–Wronski (1778–1853) was a Polish–born, egocentric mathematician and metaphysician. Wronski became later a French citizen. He is best known for the determinants such as $\begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - f'g$, which he used in his "highest law" of mathematics. The term "Wronskian" was coined by Thomas Muir around 1882.]

(b) Show that if y(x) and z(x) are any solutions of $ay'' + by' + cy = 0$, then W[y,z](x) is a solution of $aW'(x) + bW(x) = 0$. Thus $W[y,z](x) = C\exp(-bx/a)$, for some constant C which depends on the choice of solutions y and z. (This is **Abel's formula**.)

(c) Conclude from (b) that if $W[y,z](x) = 0$, for some x, then $W[y,z](x) = 0$ for all x .

(d) In (b) and (c), let $z(x) = d_1y_1(x) + d_2y_2(x)$ for constants d_1 and d_2 (a solution, by Problem 8(a)). Show that $W[y, d_1y_1 + d_2y_2](x) = d_1W[y,y_1](x) + d_2W[y,y_2](x)$. Explain why there must be some constants d_1 and d_2 (not both zero), such that $d_1W[y,y_1](x) + d_2W[y,y_2](x) = 0$ for *some* particular x.

(e) Conclude from (c) and (d) that there are constants d_1 and d_2 (not both zero) such that $W[y, d_1y_1 + d_2y_2](x) = 0$ for *all* x.

(f) Conclude from (a) and (e) that $y(x) = c'[d_1y_1(x) + d_2y_2(x)] = c_1y_1(x) + c_2y_2(x)$ for some constants c', c_1 and c_2 on any interval where $y(x)$ is never 0. (We omit the proof of the fact that if $y(x) = c_1y_1(x) + c_2y_2(x)$ on one interval, then the same is true everywhere.)

Remark. The same proof works in the case where a,b and c are replaced by continuous functions $a(x)$, $b(x)$ and $c(x)$, if one assumes that $a(x)$ is never zero. Then, $W[y,z] = C\exp\left[-\int \frac{b(x)}{a(x)}\,dx\right]$.

1.2 Generalities About PDEs

Let $u = u(x,y,z,...)$ be a function of several unrestricted real variables $x, y, z,...$. (In the remark below, we consider the case where $(x,y,z,...)$ is restricted to some region.) Recall that the partial derivative $\frac{\partial u}{\partial x}$ of u, with respect to the variable x, is just the ordinary derivative of u with respect to x, treating the other variables as constants. We use the following convenient notation

$$u_x = \frac{\partial u}{\partial x}, \quad u_y = \frac{\partial u}{\partial y}, \quad u_z = \frac{\partial u}{\partial z}, \quad ...,$$

$$u_{xx} = \frac{\partial^2 u}{\partial x^2}, \quad u_{yx} = \frac{\partial^2 u}{\partial x \partial y}, \quad u_{xy} = \frac{\partial^2 u}{\partial y \partial x}, \quad ... .$$

The order of a partial derivative is then the same as the number of subscripts. The function u is said to be **continuous at a point** $p = (x_0,y_0,z_0,...)$, if the values of the function can be made arbitrarily close to $u(p)$ by allowing the variables $x, y, z,...$ to vary (simultaneously) only within sufficiently small open intervals about $x_0, y_0, z_0, ...,$ respectively. The function u is **continuous** if it is continuous at all points p.

For a nonnegative integer k, a function u is said to be a $\mathbf{C^k}$ **function**, if every k–th order partial derivative of u exists and is continuous.

A function is a C^0 function, if and only if it is continuous. The notation " $u \in C^k$ " is used to indicate that u is a member of the set of the C^k functions. It is a standard fact that $u \in C^k$ implies $u \in C^{k-1}$ for $k > 0$. For a C^2 function u, recall that $u_{xy} = u_{yx}$. More generally, the order in which one takes k or fewer partial derivatives of a C^k function is immaterial.

Remark. We have assumed above that the function u is defined for all values of the independent variables. The function might only be defined for $(x,y,z,...)$ in a certain region D. The regions which we will encounter are rather simple (e.g., rectangles, strips, discs). If such a region includes some point p of its boundary, then technically the notion of partial derivative of u at p is not defined, unless one wishes to deal with one–sided derivatives. Let us simply say that a function u is C^k on a region D with boundary points, if there is a C^k function v, defined on a larger region without boundary points (i.e., an **open** region) such that $u = v$ at all points of D.

Definition 1. A **partial differential equation (PDE) of order** $k > 0$ is an equation involving an unknown function u, such that k is the greatest of the orders of the partial derivatives of u appearing in the equation.

Definition 2. A **solution of a k–th order PDE**, on a prespecified region D, is a C^k function defined on the region D such that the PDE is satisfied at all interior points of D. If no region is specified in advance, then a solution of a k–th order PDE is a C^k function defined on at least some nonempty open region where it satisfies the PDE.

There are many functions of several variables which arise in practice. At a point (x,y,z) at time t, u(x,y,z,t) might be any one of the following quantities : temperature, electrostatic potential energy, gravitational potential energy, pressure, mass density, energy density, concentration of a certain chemical, etc. . The laws of science are frequently stated in terms of PDEs involving such functions as unknowns. One often has the problem of determining the function u(x,y,z,t) for arbitrary t, for given information about u at time $t = 0$ (i.e., **initial conditions**, abbreviated I.C.). Such problems are referred to as **initial–value problems** . In **steady–state problems**, the function u is independent of t. In this case, one is often interested in solving a PDE for u(x,y,z) in a certain region D in space, where information is given about the behavior of u on the boundary of D (i.e., **boundary conditions**, B.C., are given). Such problems are known as **boundary–value problems**. More generally, one often seeks a solution u(x,y,z,t) of some PDE for points (x,y,z) in a region D at arbitrary time $t > 0$, subject to initial conditions at time $t = 0$, as well as boundary conditions specified at each time $t > 0$. Such a problem is aptly called an **initial/boundary–value problem**. It is important that the initial conditions and boundary conditions be chosen in such a way that the PDE has a unique solution satisfying them. Otherwise, one cannot meet the chief goal of predicting the relevant physical quantity represented by u. Mathematicians tend to be more interested in proving the existence, uniqueness and qualitative behavior of the exact solutions of initial/boundary–value problems, while those who apply the theory are concerned with actually finding functions which satisfy the PDE and initial/boundary conditions, at least within experimental error. In this book, we try to adopt an intermediate stance, believing that each camp can benefit from the considerations of the other. Before continuing our general discussion, we will now present some specific examples. Example 1 is lengthy, but it is well worth understanding.

Example 1 (Spherically symmetric gravitational potentials). In the Newtonian (pre–Einstein) theory of gravity, at a fixed time, the gravitational acceleration vector field (force per unit mass) is $-\nabla u$, where $\nabla u \equiv u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}$ is the gradient of a function u(x,y,z), called the **gravitational potential**. The function u obeys the second–order PDE

$$u_{xx} + u_{yy} + u_{zz} = 4\pi G\rho \tag{1}$$

where $\rho = \rho(x,y,z)$ is the density (mass per unit volume) of matter at (x,y,z), and G is the gravitational constant, $G \approx 6.668 \times 10^{-11}\ m^3s^{-2}kg^{-1}$. One can also interpret u in other ways, for example, (i) as a steady–state temperature distribution in a solid with internal heat source density proportional to ρ, or (ii) as an electrostatic potential whose negative gradient is the electric field produced by a charge density proportional to ρ. In any case, equation (1) is known as **Poisson's equation.** In the special case when $\rho = 0$, (1) is better known as **Laplace's equation.** Suppose that we seek a solution u(x,y,z) of Laplace's equation which is spherically symmetric in the sense that u(x,y,z) only depends on the distance $r = [x^2 + y^2 + z^2]^{\frac{1}{2}}$ to the origin (0,0,0). In other words, $u(x,y,z) = f(r)$ for some function f of a single variable $r > 0$. Using the chain rule, we have

$$u_x = \frac{df}{dr}\frac{\partial r}{\partial x} = f'(r)\, r_x\,, \quad \text{where } r_x = \frac{\partial}{\partial x}[x^2 + y^2 + z^2]^{\frac{1}{2}} = \tfrac{1}{2}r^{-1}2x = xr^{-1}\,.$$

Then
$$\begin{aligned} u_{xx} &= f''(r)[r_x]^2 + f'(r)\, r_{xx} \\ &= f''(r)(x^2/r^2) + f'(r)[r^{-1} + x(-r^{-2}r_x)] \\ &= f''(r)(x^2/r^2) + f'(r)[r^{-1} - (x^2/r^3)]\,. \end{aligned}$$

We get similar expressions for u_{yy} and u_{zz}. Adding these results, we obtain

$$\begin{aligned} u_{xx} + u_{yy} + u_{zz} &= f''(r)(x^2 + y^2 + z^2)/r^2 + f'(r)[3r^{-1} - (x^2 + y^2 + z^2)/r^3] \\ &= f''(r) + 2r^{-1}f'(r) = 0\,. \end{aligned} \tag{2}$$

Writing $g(r) = f'(r)$, equation (2) becomes the separable (or linear) ODE $g'(r) + 2r^{-1}g(r) = 0$, whose solution is $g(r) = Cr^{-2}$. Thus, $f(r) = -Cr^{-1} + K$, where C and K are arbitrary constants. Hence the general spherically symmetric solution of Laplace's equation is

$$u(x,y,z) = -C[x^2 + y^2 + z^2]^{-\frac{1}{2}} + K = -Cr^{-1} + K\,. \tag{3}$$

If $C \neq 0$, then this solution is not defined at (0,0,0). Thus, the only spherically symmetric solutions that are defined everywhere are the constant solutions $u = K$, which give rise to a zero gravitational (or electric) field ($-\nabla K = \mathbf{0}$). Of course, one does not expect to find any gravity (or static electrical field) when the density ρ is 0 everywhere. When $C \neq 0$, we obtain a solution defined in any region D which excludes (0,0,0). In the gravitational context, take D to be the exterior, $r > r_0$, of some isolated planet. Suppose that the magnitude $|\nabla u| = f'(r) = Cr^{-2}$ of the gravitational acceleration is known to be g at the planet's surface (e.g., for the earth $g \approx 9.8\ \text{m/sec}^2 \approx 32\ \text{ft/sec}^2$). Then we have the boundary condition $Cr_0^{-2} = g$ or $C = gr_0^2$. Thus,

$$u = -gr_0^2r^{-1} + K \quad \text{and} \quad -\nabla u = -g(r_0/r)^2\, \mathbf{e}_r \quad \text{for } r \geq r_0\,, \tag{4}$$

where $\mathbf{e}_r$ is the unit vector field pointing away from (0,0,0). When $r < r_0$, these formulas do not apply, since $\rho > 0$ inside the planet. (In this case we would have to solve Poisson's equation (1) for $r < r_0$). Since $|\nabla u|$ in (4) is proportional to r^{-2}, we have deduced the inverse–square law for gravity from Laplace's equation. □

Remark 1 (Escape velocity). The potential difference $u(\infty) - u(r_0) = gr_0$ is the energy (per unit mass) required to move an object from the planet's surface to arbitrarily far reaches of space.

Thus, ignoring atmospheric resistance, the kinetic energy per unit mass, namely $\frac{1}{2}v^2$ (v = velocity), which is needed for a projectile to completely escape from the planet is gr_0. In other words, the escape velocity is $\sqrt{2gr_0}$. For the earth, this is about 11.2 km/sec ≈ 7 miles/sec . □

Remark 2 (Spherically symmetric solutions in dimension n). In n–dimensional space, the spherically symmetric solutions $u(x_1,\dots,x_n)$ of Laplace's equation $(u_{x_1x_1}+\dots+u_{x_nx_n}=0)$ can be found in the same way as in Example 1 (cf. Problem 6), and are of the form

$$-Cr^{2-n}+K\,, \quad \text{if } n>2 \tag{5a}$$

and

$$C\cdot\log(r)+K\,, \quad \text{if } n=2\,, \tag{5b}$$

where $r=[(x_1)^2+\dots+(x_n)^2]^{\frac{1}{2}}$. Regardless of the dimension, solutions of Laplace's equation are called **harmonic functions.** The formulas (5a) and (5b) show that in dimension n, the magnitude of the force $-\nabla u$ (per unit mass), associated with a spherically symmetric harmonic potential, is proportional to r^{1-n}. In Problem 19, we prove that when $n \geq 4$, a planet subject to such a force cannot have a closed orbit unless the orbit is a perfect circle, a very unstable possibility. (Of course, a wide variety of closed elliptical orbits are possible when $n = 3$). Thus, perhaps it is not so surprising that the space that we live in has no more than 3 dimensions. □

Remark 3 (Other solutions of Laplace's equation). It should be mentioned that there are actually infinitely many independent solutions (not just depending on r) of Laplace's equation in any dimension $n > 1$. For example, consider $u = x,\ y,\ x^2-y^2,\ 2xy,\ x^3-3xy^2,\ e^x\sin(y),\dots$. Chapter 6 is devoted mainly to Laplace's equation in dimension 2 : $u_{xx}+u_{yy}=0$. There, we will consider the boundary–value problem (among others) of determining solutions u of Laplace's equation on a plane region D, given the values of u on the boundary of D. This problem has applications to steady fluid flow, electrostatics and steady–state heat theory in which there is no dependence on the spatial variable z. One reason for the appearance of Laplace's equation in so many contexts is that it is the only homogeneous, linear (cf. Definition 3 below) PDE which involves only partial derivatives of orders strictly between 0 and 4 and retains its form under translations and rotations of coordinates. □

Example 2 (Heat problems). Suppose that $u(x,y,z,t)$ is the temperature at time t at the point (x,y,z) in a homogeneous heat conducting solid without heat sources. Under natural assumptions, one can show that u satisfies the following second–order PDE called the **heat equation**:

$$u_t = k(u_{xx}+u_{yy}+u_{zz})\,, \tag{6}$$

where $k > 0$ is a constant which measures the heat conductivity of the material in the solid. Note that in the case of a steady–state temperature distribution, where u does not depend on t, the left side u_t of (6) vanishes, and thus the steady–state temperature u(x,y,z) satisfies Laplace's equation. In Chapter 3, we will derive and study the heat equation in the simpler one–dimensional setting, where u depends only on x and t . One example of the initial/boundary–value problems which we will consider is

$$\begin{array}{ll} \text{D.E.} & u_t = ku_{xx} \qquad 0 \le x \le 1,\ t \ge 0 \\ \text{B.C.} & u(0,t) = 0 \qquad u(1,t) = 0 \\ \text{I.C.} & u(x,0) = f(x)\ . \end{array} \tag{7}$$

Here u(x,t) is the (uniform) temperature of the cross section at distance x along a solid rod which extends from x = 0 to x = 1. We assume that the rod is covered with heat insulation except at the end cross sections. The B.C. u(0,t) = 0 and u(1,t) = 0 state that the ends of the rod are to be maintained at temperature 0 (e.g., the rod is placed in ice water). The I.C., u(x,0) = f(x), tells us that, at t = 0, the rod has the given temperature distribution f(x). For example, suppose that f(x) = sin(πx). One can easily verify that

$$u(x,t) = e^{-\pi^2 kt}\sin(\pi x) \tag{8}$$

solves the PDE (7) and the initial condition u(x,0) = sin(πx), as well as the boundary conditions. We expect that the rod's temperature will approach the temperature (zero) of its icy environment. Indeed, the factor $\exp[-\pi^2 kt]$ in (8) tells us that, as $t \to \infty$, the temperature of the rod approaches 0, and it does so more rapidly for larger values of the heat conductivity k. More generally, choosing f(x) = sin(nπx) for an arbitrary positive integer n, we get the solution $u(x,t) = \exp[-n^2\pi^2 kt]\sin(n\pi x)$. Note that the rate at which this solution approaches 0 as $t \to \infty$ is faster for larger values of n. Physically, this is so, because the rate of heat transfer from hot to cold regions is greater when these regions are separated by smaller distances, which is the case when n gets larger. Associated with the heat equation, there are many other types of boundary and initial conditions which will be explored and solved in Chapter 3. □

Example 3 (Wave problems). If u(x,y,z,t) is the deviation of air pressure (from its normal value) at (x,y,z) at time t, then (to a good approximation) u satisfies the **wave equation**

$$u_{tt} = a^2(u_{xx} + u_{yy} + u_{zz})\ , \tag{9}$$

where a is the speed of sound. We assume that the elevation is near sea level, so that a can be taken to be the constant 1087 ft/sec . For another interpretation of (9), the scalar potential (as well as the components of the vector potential) of a possibly time–dependent electromagnetic field in vacuum also satisfies the wave equation, where a is the speed of light in a vacuum (≈ 186,000 mi/sec ≈ 2.998×10^8 m/sec). Note that when u is time–independent (e.g., when u is an electrostatic potential), (9) reduces to Laplace's equation, since $u_t = 0$. Returning to the case where u measures air pressure deviations, suppose that we wish to find possible sounds (variations of pressure) inside a closed box. As one nears a wall of the box from inside, one finds that the derivative of the pressure in the direction normal (perpendicular) to the wall approaches 0. This is because wind blows in the direction of the negative pressure gradient, $-\nabla u = u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}$. Since there can be no wind velocity component normal to the wall, the pressure gradient has no normal component. So, for the box $0 \le x \le A$, $0 \le y \le B$, $0 \le z \le C$, we have the following B.C. :

$$\begin{aligned} &u_x(0,y,z) = 0\,, \quad u_y(x,0,z) = 0\,, \quad u_z(x,y,0) = 0 \\ &u_x(A,y,z) = 0\,, \quad u_y(x,B,z) = 0\,, \quad u_z(x,y,C) = 0\,. \end{aligned} \tag{10}$$

There is a large family of solutions of (9) which satisfies the B.C. in (10). For any triplet (m,n,p) of integers, let $\nu(m,n,p) = \frac{1}{2}a[(m/A)^2 + (n/B)^2 + (p/C)^2]^{\frac{1}{2}}$. Then,

$$u(x,y,z,t) = \sin[2\pi\nu(m,n,p)t]\cdot\cos(m\pi x/A)\cdot\cos(n\pi y/B)\cdot\cos(p\pi z/C) \tag{11}$$

satisfies the PDE (9) and the boundary conditions (10) (cf. Problem 9). Moreover, if in (11) we replace the leading factor by $\cos[2\pi\nu(m,n,p)t]$, then we get another solution. Notice that, at points in the box, the pressure (11) oscillates through $\nu(m,n,p)$ cycles per unit time. Hence, $\nu(m,n,p)$ is called the **frequency** of the solution (11). If $A \le B \le C$, the lowest possible nonzero frequency (called the **fundamental pitch**) is $\nu(0,0,1) = \frac{1}{2}a/C$. Taking the box to be an enclosed shower stall with a 7 ft height (and smaller dimension for the base), we have $\frac{1}{2}a/C = 1087/14 \approx 78$ cycles per second (or 78 Hertz), which is the pitch of a rather low voice. □

Remark. In Chapter 5, we concentrate on the one–dimensional wave equation

$$u_{tt} = a^2 u_{xx} \tag{12}$$

for a function $u = u(x,t)$. At a fixed time t, $u(x,t)$ can be interpreted as the transverse displacement at position x of a vibrating string which is stretched along the x–axis when at rest. Here a^2 is T/ρ, where T is the tension at rest and ρ is the mass per unit length of string. In Chapter 5, we provide a derivation of (12) using Newton's second law, and we solve numerous initial/boundary–value problems for the vibrating string. In contrast to the heat equation, in order to determine a unique solution of (12), one needs to know not only the string's initial displacement $u(x,0)$, but also the initial rate of change $u_t(x,0)$. A simple example of an initial/boundary–value problem for the string is

$$\begin{aligned} &\text{D.E.}\quad u_{tt} = a^2 u_{xx} \qquad 0 \le x \le 1\,,\ t \ge 0 \\ &\text{B.C.}\quad u(0,t) = 0\,,\ u(1,t) = 0 \\ &\text{I.C.}\quad u(x,0) = f(x)\,,\ u_t(x,0) = g(x)\,. \end{aligned} \tag{13}$$

The B.C. of (13) state that the ends of the string are held fixed on the x–axis at 0 and 1. Intuitively, the motion of the string is determined only if both the initial transverse displacement f(x) and the initial transverse velocity g(x) are given. For example, if $f(x) = \sin(\pi x)$ and $g(x) = \sin(3\pi x)$, the theory of Chapter 5 yields the solution

$$u(x,t) = \cos(\pi at)\sin(\pi x) + (1/3\pi a)\sin(3\pi at)\sin(3\pi x). \quad \square \tag{14}$$

Linear PDEs, Classification, and the Superposition Principle

All of the PDEs in the above examples are linear. The notion of linearity for PDEs is strictly analogous to linearity for ODEs. Recall that the general n–th order linear ODE is an ODE which is expressible in the form

$$a_n(x)y^{(n)} + \ldots + a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x) , \tag{15}$$

where $a_0(x), a_1(x), \ldots, a_n(x)$ and f(x) are given (possibly constant) functions of the independent variable x. In particular, terms involving y^2 or higher powers of y (or more complicated functions of y or its derivatives), which cannot be eliminated, will make an equation nonlinear. For example, the equations $y' + y^2 = 0$, $(y'')^{-1} - x \cdot \log(y) = x$ and $yy' = 1$ are nonlinear. We say that the left side of (15) is a **linear combination** of $y, y', y'', \ldots$ with coefficients $a_0(x)$, $a_1(x), a_2(x), \ldots$, which are given functions of the independent variable x.

Definition 3. A **linear n–th order PDE** is a PDE which can be put in the following form. The left side of the equation is a linear combination of the unknown function u and its partial derivatives (up to order n) with coefficients which are given functions of the independent variables. The right side must be some given function f of the independent variables. If the function f is identically zero, then the linear PDE is called a **homogeneous** PDE.

Example 4. The general second–order linear PDE for an unknown function $u = u(x,t)$ is

$$q(x,t)u_{xx} + r(x,t)u_{xt} + s(x,t)u_{tt} + a(x,t)u_x + b(x,t)u_t + c(x,t)u = f(x,t) , \tag{16}$$

where q, r, s, a, b, c and f are given functions (possibly constant) of x and t, with q, r, and s not all zero. If $f \equiv 0$, then (16) is the general second–order homogeneous linear PDE. □

Example 5. The one–dimensional heat equation $u_t = ku_{xx}$ can be put in the form (16) with $q = -k$ and $b = 1$, and with zero values for all other coefficients and f. Thus, the heat equation is a homogeneous linear PDE. When heat sources or sinks are present, they can often be represented by the terms cu and h(x,t) in the inhomogeneous heat equation

$$-ku_{xx} + u_t + cu = h(x,t) , \quad k > 0 , \tag{17}$$

which is again a special case of (16). We call (17) the **generalized heat equation.** □

Example 6. As another instance of (16), in the case of a vibrating string with a transverse applied force density proportional to $-cu(x,t) + F(x,t)$, we obtain the one–dimensional inhomogeneous **Klein–Gordon equation**

$$-a^2u_{xx} + u_{tt} + cu = F(x,t) \quad , \quad a > 0 \ . \tag{18}$$

If $F \equiv 0$ and $c = 0$, (18) reduces to the (homogeneous) wave equation $-a^2u_{xx} + u_{tt} = 0$ or (12). We refer to (18) as the **generalized wave equation.** □

Example 7. Usually one does not use t as an independent variable in Poisson's equation (cf. (1)), since t usually connotes time, whereas Poisson's equation is used in a steady–state context. However, using t unconventionally, we obtain Poisson's equation $u_{xx} + u_{tt} = g(x,t)$ in dimension 2. More generally (but still as a special case of (16)), we have the equation

$$a^2u_{xx} + u_{tt} + cu = g(x,t) \quad , \quad a > 0 \ . \tag{19}$$

If t is replaced by y (so that there will be no way of confusing t with time), then equation (19) is known as the **inhomogeneous Helmholtz equation** in dimension 2. Among other things, it is used in the analysis of vibrational modes of the skin of a drum. Roughly speaking, the constant a differs from 1 if the drum has a tension that is higher in one direction than in the other. We refer to (19) as the **generalized Poisson/Laplace equation.** □

It might appear that if we were to concentrate only on the "physical" equations (17),(18) and (19), then we would not make much progress in the study of the more general equation (16). However, in the case where the coefficients in (16) are constants, we have the following result, whose proof is given in Appendix A.1.

The Classification Theorem. Consider the second–order linear PDE

$$aU_{\xi\xi} + bU_{\xi\tau} + cU_{\tau\tau} + dU_{\xi} + eU_{\tau} + kU = F(\xi,\tau)\ , \tag{20}$$

$(a^2+b^2+c^2 \neq 0)$, where the unknown function $U = U(\xi,\tau)$ is C^2 and a, b, c, d, e and k are given real constants and $F(\xi,\tau)$ is a given continuous function. Then there is a change of variables of the form

$$x = \alpha\xi + \beta\tau \quad t = -\beta\xi + \alpha\tau$$
$$u(x,t) = \rho^{-1}\exp(\gamma\xi + \delta\tau)U(\xi,\tau)\ , \tag{21}$$

where α, β, γ, δ and ρ $(\rho \neq 0)$ are real constants with $\alpha^2 + \beta^2 = 1$, such that (20) is transformed into exactly one of the following forms :

1. the form of the generalized wave equation (18), if $b^2 - 4ac > 0$, in which case (20) is called **hyperbolic** ;
2. the form of the generalized Poisson/Laplace equation (19), if $b^2 - 4ac < 0$, in which case (20) is called **elliptic** ;
3. the form of the generalized heat equation (17), if $b^2 - 4ac = 0$, and $2cd \neq be$ or $2ae \neq bd$ in which case (20) is called **parabolic** ;
4. the equation $u_{xx} + cu = g(x,t)$, if $b^2 - 4ac = 0$, $2cd = be$ and $2ae = bd$, in which case (20) is called **degenerate.**

In other words, aside from the degenerate case, equation (20) with constant coefficients is only a disguised version of the generalized wave equation, Poisson/Laplace equation or heat equation, depending on whether (16) is hyperbolic, elliptic or parabolic respectively. While it is good to know the Classification Theorem, it is perhaps not essential to become a virtuoso in performing the required change of variables, because when PDEs are derived from physical considerations in natural coordinates, almost always they are already found to be in a simple standard form. If it is ever needed, the method for the transformation of variables can be gleaned from the proof of the Classification Theorem in the Appendix A.1. Perhaps, the most significant facts to emerge are the following :

(i) Every equation of the form (20) has a physical interpretation, when rewritten in terms of appropriate variables.
(ii) In the general study of (20), there is really no loss of generality in confining one's attention to (17),(18),(19) and the degenerate case which is addressed in Section 1.3 .

The Superposition Principle

A very important fact concerning linear equations is the superposition principle which we will now describe. By definition, a linear PDE can be written in the form $L[u] = f$, where $L[u]$ denotes a linear combination of u and some of its partial derivatives, with coefficients which are given functions of the independent variables. Since $L[u]$ has this form, if we were to replace u by $u_1 + u_2$ the result, namely $L[u_1 + u_2]$, will be the same as $L[u_1] + L[u_2]$. The underlying reason for this is the fact that a partial derivative of the sum of two functions is the sum of their partial derivatives taken separately. More generally, for any constants c_1 and c_2 ,

$$L[c_1u_1 + c_2u_2] = c_1L[u_1] + c_2L[u_2] . \tag{22}$$

As a direct consequence of (22), we have

The Superposition Principle (or **Property**). **Let u_1 be a solution of the linear PDE $L[u] = f_1$ and let u_2 be a solution of the linear PDE $L[u] = f_2$. Then, for any constants c_1 and c_2 , $c_1u_1 + c_2u_2$ is a solution of $L[u] = c_1f_1 + c_2f_2$. In other words,**

$$L[c_1u_1 + c_2u_2] = c_1f_1 + c_2f_2 . \tag{23}$$

In particular, when $f_1 = 0$ and $f_2 = 0$, (23) implies that if u_1 and u_2 are solutions of the homogeneous linear PDE $L[u] = 0$, then $c_1u_1 + c_2u_2$ will also be a solution of $L[u] = 0$.

Proof. By (22) , $L[c_1u_1 + c_2u_2] = c_1L[u_1] + c_2L[u_2] = c_1f_1 + c_2f_2$. □

Example 8. Observe that $u_1(x,y) = x^3$ is a solution of the linear PDE $u_{xx} - u_y = 6x$, and $u_2(x,y) = y^2$ is a solution of $u_{xx} - u_y = -2y$. Find a solution of $u_{xx} - u_y = 18x + 8y$.

Solution. Here $L[u] = u_{xx} - u_y$, $f_1(x,y) = 6x$ and $f_2(x,y) = -2y$. Note that $18x + 8y = 3f_1(x,y) - 4f_2(x,y)$, and thus $c_1 = 3$ and $c_2 = -4$. The superposition principle then tells us that $3u_1(x,y) - 4u_2(x,y)$ (or $3x^3 - 4y^2$) will be a solution of $u_{xx} - u_y = 18x + 8y$, as can be easily checked directly. □

Example 9. Observe that $u_1(x,t) = \sin(t)\cos(x)$ and $u_2(x,t) = \cos(3t)\sin(3x)$ are solutions of the wave equation $u_{tt} = u_{xx}$. By applying the superposition principle, find infinitely many other solutions, none of which is a constant multiple of any other.

Solution. Note that $u_{tt} = u_{xx}$ can be written in the form of an homogeneous linear PDE $u_{tt} - u_{xx} = 0$. According to the superposition principle, for any constants c_1 and c_2 ,

$$c_1\sin(t)\cos(x) + c_2\cos(3t)\sin(3x)$$

is a solution. For each choice for c_1 and c_2 , we obtain a different solution (cf. Problem 14). By choosing $c_1 = 1$ and letting c_2 vary, we obtain an infinite family of solutions, none of which is a constant multiple of any other. □

A great difficulty in the study of nonlinear equations is the typical failure of the superposition principle for such equations. This failure makes it difficult to form families of new solutions from an original pair of solutions, as the next example illustrates.

Example 10. Consider the nonlinear first–order PDE $u_xu_y - u(u_x + u_y) + u^2 = 0$ or equivalently $(u_x - u)(u_y - u) = 0$. Note that we have two solutions, namely e^x and e^y. However, show that $c_1e^x + c_2e^y$ will not be a solution, unless $c_1 = 0$ or $c_2 = 0$.

Solution. Defining $N[u] = (u_x - u)(u_y - u)$, observe that for any C^1 functions v and w

$$\begin{aligned} N[v + w] &= (v_x + w_x - v - w)(v_y + w_y - v - w) \\ &= N[v] + N[w] + (v_y - v)(w_x - w) + (v_x - v)(w_y - w) . \end{aligned}$$

This computation shows that $N[v + w] \neq N[v] + N[w]$ in general, due to the nonlinearity of the PDE. Taking $v = c_1e^x$ and $w = c_2e^y$ we obtain $N[c_1e^x + c_2e^y] = N[c_1e^x] + N[c_2e^y] + (-c_1e^x)(-c_2e^y) = c_1c_2e^{x+y}$. Thus, $N[c_1e^x + c_2e^y] = 0$ only if $c_1 = 0$ or $c_2 = 0$. □

Although all of the physically relevant PDEs which we have discussed so far are linear, there are many examples of nonlinear PDEs which are of great importance in physics. For example, Einstein's theory of relativity describes the force of gravity in terms of the curvature in the geometry of space–time. The Einstein field equations form a system of nonlinear PDEs. Because of the nonlinearity, solutions of these field equations are difficult to obtain, except in situations where several degrees of symmetry are assumed. Nonlinearity is also found in the PDEs of fluid mechanics, optics and elasticity theory. Nonlinear equations are often approximated by linear equations which hopefully yield solutions that are close to the corresponding solutions of the nonlinear equations. However, many interesting features of the original equations can be lost in the process, and gross errors can arise. In the next example, we illustrate these issues with the nonlinear minimal surface equation whose solutions yield soap film surfaces.

Example 11. Imagine a soap film surface which remains after a (possibly nonplanar) loop of wire is dipped in a soap solution. Due to the surface tension of the film, it will form a surface of least area spanning the loop (i.e. a minimal surface). If the surface is the graph $z = u(x,y)$ of some function u, defined on a bounded region D, then its area is $\iint_D (1 + u_x^2 + u_y^2)^{\frac{1}{2}}\,dxdy$. In 1760, Joseph Louis Lagrange showed that if $u(x,y)$ minimizes this integral among all functions with the same values on the boundary of D, then u must satisfy the (nonlinear) **minimal surface equation**

$$(1 + u_y^2)u_{xx} + (1 + u_x^2)u_{yy} - 2u_xu_yu_{xy} = 0 \ . \tag{24}$$

If one were to assume that the surface $z = u(x,y)$ is nearly level, then u_x and u_y would be small (say compared with 1), and u_x^2 , u_y^2 and u_xu_y would be very small. In this case, it would appear that equation (24) is reasonably approximated by Laplace's equation

$$u_{xx} + u_{yy} = 0 \ . \tag{25}$$

Indeed, if the wire loop is nearly planar, and is held nearly level, then the minimal surface formed will be close (in a sense which is rather difficult to make precise) to the graph of the corresponding solution of Laplace's equation. Troubles arise when the supposition $u_x^2 + u_y^2 << 1$ turns out to be incorrect. As an illustration, we compare the solutions of (24) and (25) in the case where u is assumed to have the form $u = f(r)$, $r = [x^2 + y^2]^{\frac{1}{2}}$. By the computation done in Example 1, it is found that (24) and (25) become

$$rf''(r) + f'(r)(1 + [f'(r)]^2) = 0 \tag{26}$$

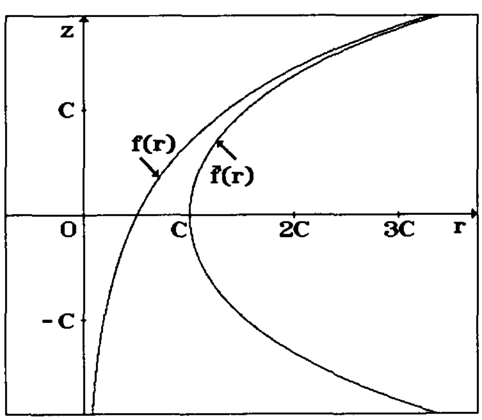
$$rf''(r) + f'(r) = 0 \ , \tag{27}$$

respectively. If we set $g = f'$, then both (26) and (27) are separable first–order ODEs. The corresponding general solutions of (26) and (27) are, respectively,

$$\tilde{f}(r) = C\log(\tfrac{1}{2}[r + (r^2 - C^2)^{\frac{1}{2}}]) + K$$
$$f(r) = C\log(r) + K \ .$$

These solutions agree well for large r, where $\tilde{f}'(r) \approx f'(r) \approx 0$ (i.e., where $u_x^2 + u_y^2 << 1$). However, $\tilde{f}(r)$ and $f(r)$ behave differently as $r \downarrow C$, and $\tilde{f}(r)$ is undefined for $0 \le r \le C$, whereas $f(r)$ is defined for all $r > 0$. In Figure 1 below, we have chosen $C > 0$ and $K = -C\log(C/2)$ so that $\tilde{f}(C) = 0$. The graph of $u(x,y) = \tilde{f}([x^2 + y^2]^{\frac{1}{2}})$ is a minimal surface obtained by revolving the graph of the curve $z = \tilde{f}(r)$ about the z–axis. By joining the curve $z = \tilde{f}(r)$ with $z = -\tilde{f}(r)$, and revolving, we obtain a complete minimal surface running from

$z = -\infty$ to $z = +\infty$. The portion of the surface between the two circles at $z = a$ and $z = b$ is the soap film obtained by dipping those circles (say, wires) in soap solution, provided that $|b - a|$ is not too large (cf. Problem 20). The curve $z = \pm\bar{f}(r)$ is the same as the curve $r = C\cdot\cosh(z/C)$, and the complete minimal surface revolution is called a **catenoid.** The catenoids obtained in this way vary in size but not in shape. The minimum distance of the film to the z–axis is C . If we were to believe the validity of the Laplacian approximation, we might erroneously conclude that the film will continue to approach the z–axis, as $z = f(r)$ does. The reason for the failure of the approximation is that $\bar{f}'(r) = [u_x^2 + u_y^2]^{\frac{1}{2}}$ does not remain small as $r \downarrow C$, but approaches infinity. □



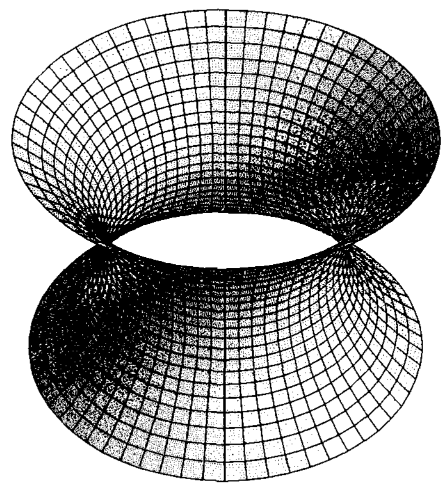


Figure 1

Remark (Black holes). A very similar phenomenon happens when the nonlinear Einstein field equations are used to compute the spherically symmetric geometry for a gravitational field about a ball of mass M and radius r_0. In Example 1, the spherically symmetric Newtonian gravitational potential was found to be $C/r + K$, and it was derived from (the linear) Laplace equation. This formula for the potential is valid as long as $r > r_0$, no matter how small or dense the ball is. However, it is found that Einstein's nonlinear description of the space–time geometry, in terms of the variable r and "time" t , can break down before r reaches r_0. Indeed, if the radius of the ball of mass M is less than the so–called **Schwarzchild radius** $r_M = 2GM/c^2$ (where c is the speed of light and $G = 6.668 \times 10^{-11}\ m^3s^{-2}kg^{-1}$ is the gravitational constant), then the representation of the solution of the field equations becomes undefined as $r \downarrow r_M$. In place of the ball, we then actually have what is known as a **black hole.** As with the soap film, the solution of the Einstein field equations can be mathematically continued, if one changes coordinates from r and t to new variables which can be related to r and t via hyperbolic functions. This continuation of space–time goes inside the throat of the black hole and into "another universe" which is, however, inaccessible by any ordinary means. Indeed, any object that enters the black hole and travels at a speed not greater than c will meet a singular boundary of space–time, never reaching the other universe or returning to our own. These interesting features of Einstein's theory are lost in the linear Newtonian theory which approximates Einstein's theory in less extreme circumstances. □

Operators and Green's Functions

A useful approach to solving linear initial/boundary–value problems for PDEs or ODEs is based on the construction of so–called Green's functions. This concept originated in the memoir, "Essay on the Application of Mathematical Analysis to the Theory of Electricity and Magnetism", published in 1828 by the English mathematician George Green (1793–1841). Green introduced the term "potential" and used what is known now as Green's theorem, to study the properties of electric and magnetic potentials. Here we briefly explain the concepts of linear differential operators, Green's functions and integral operators.

An **operator** is a prescription which assigns to each suitable function some new function. For example, suppose that $L[u] = f$ is a k–th order linear PDE. The operator which assigns, to each C^k function u, the new function $L[u]$, is an example of a differential operator. The concept of such an operator is independent of any particular choice of u, in the same sense that the concept of a certain function, say $\log(x)$, is independent of any particular choice of x. Just as one might prefer to speak of the log function, without any reference to x, it is fashionable to speak of partial differential operators L without any reference to u. For example, the Laplace operator, say in dimension 3, is denoted by

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \ . \tag{28}$$

The operator Δ assigns to each C^2 function u, the new continuous function $\Delta[u]$ or simply Δu, which is $u_{xx} + u_{yy} + u_{zz}$. Thus, Poisson's equation is $\Delta u = f$, where $f = f(x,y,z)$ is a given function. To solve an equation such as $\log(x) = C$ for $x > 0$, recall that we simply operate on both sides by the inverse function exp, obtaining $x = \exp[\log(x)] = \exp(C)$. To solve Poisson's equation, one might attempt to find the inverse operator, say Δ^{-1}, of the Laplace operator. Then we would simply apply Δ^{-1} to both sides of $\Delta u = f$, obtaining the solution $u = \Delta^{-1}[f]$. It turns out that the inverse of the Laplace operator (on a certain class of functions) is the operator which operates on the function f to produce the new function $\Delta^{-1}[f]$ defined by

$$\Delta^{-1}[f](x,y,z) = \iiint G(x,y,z;\bar{x},\bar{y},\bar{z})f(\bar{x},\bar{y},\bar{z})\, d\bar{x}d\bar{y}d\bar{z} \ , \tag{29}$$

where

$$G(x,y,z;\bar{x},\bar{y},\bar{z}) = -\frac{1}{4\pi}[(x-\bar{x})^2 + (y-\bar{y})^2 + (z-\bar{z})^2]^{-\frac{1}{2}}. \tag{30}$$

Observe that when the integration with respect to $\bar{x}$, $\bar{y}$ and $\bar{z}$ is performed, we are left with a function of x, y, and z, which is by definition $\Delta^{-1}[f]$. If the function f is C^1 and is zero outside some ball, then armed with the appropriate tools, one could prove that $\Delta^{-1}[f]$ is a solution of $\Delta[u] = f$. The operator Δ^{-1} is an example of an **integral operator**, i.e. an operator B of the form

$$B[f](p) = \int g(p;q)f(q)\, dq \ . \tag{31}$$

Here p and q range over possibly multidimensional domains. When the solution of an initial/boundary–value problem for a PDE (or ODE) is expressed in the form of an integral operator (e.g., as in (29)) the function $g(p,q)$ is called a **Green's function** for the boundary–value problem. In the case of Poisson's equation (roughly speaking, with the boundary condition that solutions tend to zero at infinity), the Green's function is $G(x,y,z;\bar{x},\bar{y},\bar{z})$ in (30). Once the correct Green's function is found, the problem is reduced to computing the integral (31) for arbitrary p. Such a computation can be quite difficult, and numerical methods might be needed. Of course, when possible, one might prefer an algebraic formula, of the solution of a particular initial/boundary–value problem. For the most part, this is what we strive for in this book. Nevertheless, in general circumstances, Green's functions and their associated integral operators provide a tidy way of presenting solutions which we will exploit occasionally. It should be noted that integral operators (31) are linear, in the sense that $B[c_1f_1 + c_2f_2] = c_1B[f_1] + c_2B[f_2]$, and consequently they can only serve as inverses of *linear* operators. In particular, Green's functions and their integral operators *cannot* be used to express solutions of nonlinear PDEs !

Summary 1.2

1. **C^k functions** : For a nonnegative integer k, a function u is said to be a C^k function, if every k–th order partial derivative of u exists and is continuous.

2. **Linear PDEs** : A linear n–th order PDE is a PDE of the form $L[u] = f$, where $L[u]$ is a linear combination of the unknown function u and its partial derivatives (up to order n), where the coefficients and f are given functions of the independent variables. If $f \equiv 0$, the PDE is called homogeneous.

3. **The Classification Theorem** : The Classification Theorem asserts that every second–order linear PDE (cf. (20)) with constant coefficients, where the unknown function has two independent variables, can be transformed (by a change of variables) into exactly one of the following forms (where $u = u(x,t)$) :

(i) the form of the generalized wave equation

$$-a^2u_{xx} + u_{tt} + cu = F(x,t)\ ,\ a > 0\ ,\ \text{(hyperbolic case)}\ ;$$

(ii) the form of the generalized Poisson/Laplace equation

$$a^2u_{xx} + u_{tt} + cu = g(x,t)\ ,\ a > 0\ ,\ \text{(elliptic case)}\ ;$$

(iii) the form of the generalized heat equation

$$-ku_{xx} + u_t + cu = h(x,t)\ ,\ k > 0\ ,\ \text{(parabolic case)}\ ;$$

(iv) the form

$$u_{xx} + cu = g(x,t) \qquad \text{(degenerate case).}$$

4. **The superposition principle** : The superposition principle (or property) asserts that if u_1 and u_2 are solutions of the linear PDEs $L[u] = f_1$ and $L[u] = f_2$, respectively, then for any constants c_1 and c_2, $c_1u_1 + c_2u_2$ is a solution of $L[u] = c_1f_1 + c_2f_2$. In other words, $L[c_1u_1 + c_2u_2] = c_1L[u_1] + c_2L[u_2]$.

5. **Green's functions** : Green's functions and their associated integral operators are used to represent solutions of initial/boundary–value problems for linear PDEs (or ODEs).

Exercises 1.2

1. Show that the given functions satisfy the accompanying PDE.

(a) $u(x,y) = x + y$; $u_{xx} + u_{yy} = 0$

(b) $u(x,y) = f(x) + g(y)$; $u_{xy} = 0$, where the functions f and g are assumed to be C^2 .

(c) $u(x,y) = f(x+y) + g(x-y)$; $u_{xx} - u_{yy} = 0$.

(d) $u(x,t) = x^2 + 2t$; $u_{xx} = u_t$

(e) $u(x,y) = \sin(x)\cosh(y)$; $u_{xx} + u_{yy} = 0$

(f) $u(x,t) = \sin(x-ct)$; $u_{tt} - c^2u_{xx} = 0$, where c is a real constant.

2. Verify that the following functions are solutions of Laplace's equation $u_{xx} + u_{yy} = 0$.

(a) $u(x,y) = e^y\cos(x)$ (b) $u(x,y) = 3x^2y - y^3$

(c) $u(x,y) = \log(x^2 + y^2)$, $x^2 + y^2 \neq 0$

(d) $u(x,y) = e^y\cos(x) + 3x^2y - y^3 + \log(x^2 + y^2)$, $x^2 + y^2 \neq 0$.

3. Show that the following solve the heat equation $u_t - ku_{xx} = 0$.

(a) $u(x,t) = e^{-kt}\sin(x)$ (b) $u(x,t) = e^{-a^2kt}\cos(ax)$, for any real constant a.

(c) $u(x,t) = e^{kt}\cosh(x)$ (d) $u(x,t) = (1/\sqrt{kt})\exp[-x^2/(4kt)]$.

4. Show that the following are solutions of the wave equation $u_{tt} - c^2u_{xx} = 0$, for some c.

(a) $u(x,t) = x^2 + t^2$ (b) $u(x,t) = \cos(ax)\sin(bt)$, for any real constants a, b.

(c) $u(x,t) = \log(x + t) + (x - t)^2$ (d) $u(x,t) = f(x+2t) + g(x-2t)$, for any C^2 functions f and g.

5. Give the orders of the following PDEs, and classify them as linear or nonlinear. If the PDE is linear, specify whether it is homogeneous or inhomogeneous.

(a) $x^2u_{xxy} + y^2u_{yy} - \log(1 + y^2)u = 0$ (b) $u_x + u^3 = 1$ (c) $u_{xxyy} + e^xu_x = y$

(d) $uu_{xx} + u_{yy} - u = 0$ (e) $u_{xx} + u_t = 3u$.

6. Derive formulas (5a) and (5b) for the most general spherically symmetric solution of Laplace's equation in dimension n .

7. (a) Find a solution of Laplace's equation $u_{xx} + u_{yy} = 0$ of the form $u(x,y) = Ax^2 + Bxy + Cy^2$ $(A^2+B^2+C^2 \neq 0)$ which satisfies the boundary condition $u(\cos(\theta),\sin(\theta)) = \cos(2\theta) + \sin(2\theta)$ for all points $(\cos(\theta),\sin(\theta))$ on the unit circle, $x^2 + y^2 = 1$.

(b) Show that the graph of any solution $u(x,y)$ of Laplace's equation of the form in (a), intersects the xy–plane in a pair of perpendicular lines through (0,0).

8. (a) Show that $u(x,t) = \exp[-n^2\pi^2kt]\sin(n\pi x)$ solves the initial/boundary–value problem given in equations (7) with I.C. $f(x) = \sin(n\pi x)$, if and only if n is an integer.

(b) In how many points does the graph of $\sin(n\pi x)$ intersect the x–axis between 0 and 1 ?

(c) Give a physical reason for why the temperature approaches 0 faster if n is larger.

9. Let $u(x,y,z,t)$ be the solution (11) in Example 3 on wave problems.

(a) Show that $u_{tt} = -[2\pi\nu(m,n,p)]^2u$, $u_{xx} = -(m\pi/A)^2u$, etc. . Use these facts to deduce that $u(x,y,z,t)$ satisfies the wave equation (9).

(b) Verify that $u(x,y,z,t)$ meets the B.C. (10).

(c) The set of points (x,y,z) inside the box, where $u(x,y,z,t)$ is always zero, is the union of a number of intersecting rectangular surfaces which divide the interior of the box into a number of compartments. How many compartments are there ?

(d) At what points in the box does the pressure experience the greatest changes ?

10. Refer to Example 3, and assume that the box is cubical with $A = B = C = 1$ and $a = 2$.

(a) By giving an example, show that it is possible for two independent solutions of the form of $u(x,y,z,t)$ in (11) to have the same frequency.

(b) List the ten lowest positive *distinct* frequencies for the box.

11. (a) Show that if $f(x) = \sin(\pi x)$ and $g(x) = \sin(3\pi x)$, then $u(x,t)$ in (14) solves the initial/boundary–value problem (13).

(b) Find two solutions $u(x,t)$ of the D.E. and B.C. in (13) such that these two solutions have the same initial profile $u(x,0)$, but have different initial velocity distribution $u_t(x,0)$.

12. For what values of the positive constants m and n will the second–order PDE $u_{xx} + u_{yy} + mu_{xy} + u_x + nu_y = 0$ be (a) hyperbolic, (b) elliptic, (c) parabolic or (d) degenerate ?

13. Observe that $u_1(x,y) = x^3$ solves $u_{xx} + u_{yy} = 2$ and $u_2(x,y) = cx^3 + dy^3$ solves $u_{xx} + u_{yy} = 6cx + 6dy$ for real constants c and d.

(a) Find a solution of $u_{xx} + u_{yy} = Ax + By + C$ for given real constants A, B and C.

(b) How can many more solutions of the problem in (a) be produced ?

14. In relation to Example 9, show that if $c_1\sin(t)\cos(x) + c_2\cos(3t)\sin(3x) = d_1\sin(t)\cos(x) + d_2\cos(3t)\sin(3x)$ for all (x,t) , then $c_1 = d_1$ and $c_2 = d_2$.

15. By direct computation, verify that by revolving the curve $y = \cosh(x)$ about the x–axis, we obtain a solution $u(x,y) = [\cosh^2(x) - y^2]^{\frac{1}{2}}$ of the minimal surface equation (24) on the domain $|y| < \cosh(x)$. In view of the solution $\Upsilon(r)$ found in Example 11, give a purely geometrical reason for why $u(x,y)$ must be a solution.

16. Suppose that $u(x,y)$ is any solution of the minimal surface equation (24), for (x,y) in some open region D in the plane.

(a) Show that it is not always true that $cu(x,y)$ will be a solution for all real c.

(b) Show that if $c \neq 0$, then $cu(x/c,y/c)$ will be a solution on the new region consisting of all (x,y) with (x/c,y/c) in D .

(c) Explain the results of (a) and (b) geometrically in terms of similarity between the shapes of the surfaces.

17. Let u(x) be an arbitrary C^1 function defined for $x \geq 0$, such that $u(0) = 0$. Consider the ordinary differential operator d/dx which assigns to each such function u the new continuous function $u'(x)$. Show that the inverse operator, say B, assigns to each continuous function f(x), defined for $x \geq 0$, the function

$$B[f](x) \equiv \int_0^\infty g(x,z)f(z)\,dz \ , \ \text{where } g(x,z) = \begin{cases} 1 & 0 \leq z \leq x \\ 0 & z > x \end{cases} .$$

Consequently, the solution of the problem $u'(x) = f(x)$ $(x \geq 0)$ with boundary condition $u(0) = 0$, is given in terms of the integral operator B with Green's function g(x,z).

18. Let p(x) and q(x) be given continuous functions. Show that the solution of the linear ODE $u'(x) + p(x)u(x) = q(x)$ $(x \geq 0)$, with B.C. $u(0) = 0$, is given by

$$A[q](x) = \int_0^\infty G(x,z)q(z)\,dz \quad , (x \geq 0),$$

where the Green's function is $G(x,z) = \exp[P(z) - P(x)]g(x,z)$, and $P(x)$ is an antiderivative of the function $p(x)$ and $g(x,z)$ is defined in Problem 17. What is the inverse operator of A ? **Hint.** Use Leibniz's rule in Appendix A.3.

19. Here we demonstrate the instability of planetary orbits in dimension greater than 3, assuming a spherically symmetric harmonic potential. In Example 1 (or Problem 6), we showed that such a potential in dimension $n > 2$ is of the form $-Cr^{2-n}$ (where $C > 0$, as we assume that the force is attractive). The angular momentum for the path $(r(t),\theta(t))$ in polar coordinates of a planet of mass m is $mr^2\dot{\theta}$ (where $\dot{\theta} \equiv \frac{d\theta}{dt}$) which is some constant, say A, for a central force. Thus, $\dot{\theta} = A/(mr^2)$. The kinetic energy of the planet is then $\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) = \frac{1}{2}m\dot{r}^2 + A^2/(2mr^2)$, where $\dot{r} \equiv \frac{dr}{dt}$. The total energy (kinetic + potential) is a constant

$$E = \tfrac{1}{2}m\dot{r}^2 + [A^2/(2mr^2) - Cmr^{2-n}] \ . \qquad (*)$$

Let $f(r)$ be the function in brackets in (*). Assume that the planet's orbit has at least two consecutive local extrema for r, say r_1 and r_2 (with $r_1 < r_2$). Of course, this assumption is possible when $n = 3$, since then there are elliptical orbits. For $n > 3$, we now show that this assumption leads to a contradiction. At such extreme points on the orbit, we have $\dot{r} = 0$, and thus $f(r_1) = f(r_2) = E$ by (*). Since $\frac{1}{2}m\dot{r}^2 > 0$ while the planet moves between the two consecutive extrema, we must have $f(r) < E$ for $r_1 < r < r_2$ by (*). Hence f(r) must have a local minimum which is strictly less than E at some point r_0 between r_1 and r_2 .

(a) When $n = 4$, show that there is no r_0 such that $f'(r_0) = 0$, unless $f(r) \equiv E = 0$, but then $f(r_0)$ is not strictly less than E .

(b) When $n > 4$, show that there is only one positive value r_0 where $f'(r_0) = 0$, and this value is a local maximum instead of a local minimum, as can be deduced from the fact that $\lim_{r\to 0} f(r) = -\infty$ and $\lim_{r\to\infty} f(r) = 0$.

(c) Show that there is no such contradiction when $n = 3$, since $\lim_{r\to 0} f(r) = +\infty$ and $\lim_{r\to\infty} f(r) = 0$, when $n = 3$.

(d) A circular orbit is possible for $n \geq 4$, but such an orbit is unstable, since the slightest nudge will throw the planet out of a circular orbit. Assume that the orbit is not a perfect circle.
(i) If $n = 4$, show that either $r(t) \to \infty$ as $t \to \infty$, or $r(t) \to 0$ as t approaches some *finite* value.
(ii) If $n \geq 5$, show that in addition to the two possibilities in (i), it can also rarely happen that the orbit will spiral toward a circular orbit. Show that this can only occur if the maximum value of f(r) is E. Why is this a rare occurance ?

20. In the the following, we deduce that a minimal soap film cannot be formed between two coaxial rings of radius R which are separated by a distance of more than $1.3255 \cdot R$.

(a) *For* $C > 0$, *consider the curve* $r = C \cdot \cosh(z/C)$ *in the zr–plane.* Show that the tangent line through the point (z_0, r_0) on this curve passes through the origin only when $\cosh(z_0/C)/\sinh(z_0/C) = z_0/C$ (i.e., $\coth(z_0/C) = z_0/C$).

(b) Show that there is a unique positive solution of $\coth(x) = x$, say $\alpha \approx 1.200$.
Hint. Let $g(x) = \coth(x) - x$. For small $x > 0$, show that $g(x) > 0$, while $g(x) < 0$ for large $x > 0$. Show that $g(x)$ is strictly decreasing for $x > 0$, by computing $g'(x)$.

(c) Show that the tangent lines in part (a) must be of the form $r = \pm\sinh(\alpha) \cdot z$, where α is defined in (b). Hence, regardless of the value of C, these lines are tangent to each of the curves $r = C \cdot \cosh(z/C)$.

(d) From Part (c) and the convexity of the curves $r = C \cdot \cosh(z/C)$ $(C > 0)$, deduce that all of these curves are contained in the wedge $r \geq \sinh(\alpha) \cdot |z|$.

(e) Conclude that there is no minimal surface joining two coaxial rings of radius R, if the rings are separated by a distance of more than $2R/\sinh(\alpha) < 1.3255 \cdot R$.

Remark. If the separation distance is less than $2R/\sinh(\alpha)$, then there are actually two surfaces of the form $r = C \cdot \cosh(z/C)$ that join the rings. The surface with the larger value of C is the one which actually has the minimum area (i.e., the one which arises physically).

1.3 General Solutions and Elementary Techniques

Ideally, one would like to have a general technique that could be used to find all of the solutions of an arbitrary PDE, or at least a relevant solution that satisfies certain initial/boundary conditions. Such a general technique does not exist even for the class of first–order ODEs. Recall that for such equations, there is a variety of techniques which work when the first order ODE is of a particular form (e.g., separable, homogeneous, exact, linear, etc.). Moreover, it is easy to find first–order ODEs that do not have any of these forms. The situation for PDEs is similar. It is easy to find PDEs for which there is no known method which will yield a single solution. Fortunately, the PDEs which arise in practice are not completely arbitrary. Indeed, there are few different kinds of PDEs, or systems of PDEs, which regularly appear in applications. Although there are some procedures that apply to more than one relevant equation, it is better not to develop excessively such procedures apart from the specific PDEs to which they will be applied. Instead, we prefer to handle each relevant equation separately. When a pattern of techniques emerges, we will note it and appreciate it, but we see no advantage in trying to force the solution process into a preconceived mold which could be motivated only with a great deal of hindsight. Also, unlike the theory of ODEs, the methods for solving PDEs often depend more on the form of the imposed boundary conditions than on the PDE itself. This makes it even more difficult to develop a unified theoretical edifice, if that were our goal. Nevertheless, in this section, we discuss some elementary techniques. One technique, known as "separation of variables", is a preliminary step used in solving a wide variety of PDE problems. However, first we shall illustrate some of the differences between PDEs and ODEs. We also explore some of the difficulties in determining the form of the general solution of a PDE, and in finding particular solutions which meet given side conditions.

General Solutions and Particular Solutions of PDEs

Recall that the general solution of an n–th order linear ODE involves n arbitrary constants. These constants are determined when the solution is required to satisfy n initial conditions. For example, the general solution, y(t), of the second–order ODE

$$y'' + y = 0 \tag{1}$$

is

$$y(t) = c_1\cos(t) + c_2\sin(t)\ , \tag{2}$$

where c_1 and c_2 are arbitrary constants. If we also specify the initial conditions $y(0) = 0$ and $y'(0) = 1$, then the only solution of (1) which meets these conditions is $y(t) = \sin(t)$. Recall (cf. Definition 2 of Section 1.2) that a solution of PDE of order n is required to be a C^n function on the open set (possibly prespecified) where it is defined.

> The **general solution** of a PDE is the collection of all solutions of the PDE.

As with ODEs, it is usually not possible to list all of the solutions, but rather one specifies the form of the general solution as in (2). However, the form of the general solution of an n–th order PDE typically involves n arbitrary functions, rather than arbitrary constants. The following Example illustrates this.

Example 1. Find the general solution of the first–order linear PDE for $u = u(x,y)$

$$u_x(x,y) = 2xy\ , \quad \text{for all } (x,y). \tag{3}$$

Solution. If we hold y fixed and integrate with respect to x, we obtain

$$u(x,y) = x^2y + f(y)\ . \tag{4}$$

Note that the constant of integration may depend on y, and indeed any function of the form (4) satisfies (3). As a technical point, recall that in Definition 2 of Section 1.2, we require that the function f(y) in (4) be C^1 (i.e., f has a continuous first derivative). If in place of (3) we had the PDE $u_x(x,y,z) = 2xy$, then the form of the general solution would be $u(x,y,z) = x^2y + g(y,z)$ for an arbitrary C^1 function $g(y,z)$. □

> Whenever integrating with respect to one variable, remember to add an arbitrary function of the other variables.

Example 2. Find the general solution of the third–order PDE

$$u_{xyy} = 2\sin(x)\ , \quad u = u(x,y,z)\ , \quad \text{for all } (x,y,z). \tag{5}$$

Solution. Integrating (5) once with respect to y, we get $u_{xy}(x,y,z) = 2y\sin(x) + f(x,z)$. Integrating again with respect to y, we obtain $u_x(x,y,z,) = y^2\sin(x) + yf(x,z) + g(x,z)$. Finally, integrating with respect to x, we obtain the general solution

$$u(x,y,z) = -y^2\cos(x) + yF(x,z) + G(x,z) + h(y,z)\ , \tag{6}$$

where $F(x,z)$ and $G(x,z)$ are antiderivatives (with respect to x) of $f(x,z)$ and $g(x,z)$, respectively. Since we want the solution to be C^3, we require that F, G and h be C^3 functions, and except for this requirement, these functions are arbitrary. □

Of course, as with the ODEs, it is not always possible to find the general solution of a PDE simply by integrating a few times. Nevertheless, the above examples suggest that typically the general solution of an n–th order PDE, for an unknown function u of m independent variables, involves n arbitrary functions of m–1 variables. However, it is easy to find examples which violate this rule. For instance, consider the following example.

Example 3. Find the general solution of

$$(u_{xx})^2 + (u_{yy})^2 = 0\ , \quad u = u(x,y)\ , \quad \text{for all } (x,y). \tag{7}$$

Solution. A C^2 function $u(x,y)$ solves this equation, if and only if $u_{xx} = 0$ and $u_{yy} = 0$. Since $u_{xx} = 0$, u must have the form $u(x,y) = f(y)x + g(y)$. However, since $u_{yy} = 0$, u must also have the form $u(x,y) = h(x)y + k(x)$. The only functions which have both of these forms are of the form

$$u(x,y) = axy + bx + cy + d\ , \tag{8}$$

where a, b, c and d are arbitrary constants. Thus, the general solution of (7) involves four arbitrary constants instead of two arbitrary functions of a single variable. Note also that the superposition of two solutions of the form (8) is also a solution. Hence, (7) also provides us with an example of a nonlinear PDE whose solutions obey a superposition principle. □

In Example 3 of Section 2.2, we show that the homogeneous first–order linear equation $xu_x - yu_y + yu = 0$ has a general solution which depends on *two* arbitrary functions, instead of one. Thus, there are really no precise rules concerning the form of the general solution of (even) linear PDEs. However, it will be convenient to introduce the following notion of a "generic" solution of a PDE. Such a solution has the *expected* form of a general solution, a form which might not be realized for certain PDEs as we have just seen.

Definition 1. A **generic solution** of an n–th order PDE for an unknown function of m independent variables is a solution which involves n arbitrary C^n functions of m–1 variables. Moreover, we require that none of these arbitrary functions can be eliminated or combined without losing solutions in the process.

Remark. The last requirement ensures that one cannot simply increase the number of arbitrary functions by replacing some arbitrary function by a sum of two new arbitrary functions, or by some similar artifice. For instance, the solution (6) is generic, but $-y^2\cos(x) + y[k(x,z) - j(x,z)] + g(x,z)$ is not generic (even though there are three arbitrary functions), since $k(x,z) - j(x,z)$ can be replaced by $f(x,z)$. □

While the general solutions (4) and (6) for the PDEs in Examples 1 and 2 above are generic, according to Definition 1, the general solution (8) of the PDE (7) is not generic. It is also possible to have a generic solution which is not a general solution as the following example shows.

Example 4. Find a generic solution of the nonlinear first–order PDE

$$u_x(x,y) = [u(x,y)]^2\ . \tag{9}$$

Solution. By fixing y, we may regard (9) as a first–order separable ODE, namely $u^{-2}du = dx$, assuming that $u \neq 0$. Integrating, we get the solution $-u^{-1} = x + g(y)$, or

$$u(x,y) = -[x + g(y)]^{-1}\ , \tag{10}$$

where g is an arbitrary C^1 function and $u(x,y)$ is defined everywhere except for points (x,y) on the curve $x = -g(y)$. The solution (10) is generic. However, (10) is not the most general solution, because there are solutions of (9) which are not of the form (10). Indeed, $u(x,y) \equiv 0$ is such a solution. One can produce other solutions, by "pasting" two solutions together (see Problem 11). Now suppose that an open region D is given beforehand, and suppose that only solutions which are defined throughout D are allowed, then the function $g(y)$ must satisfy the requirement that the curve $x = -g(y)$ does not intersect D. Since no such region was specified here, we regard all functions of the form (10) as solutions. If we had required that the solution be defined everywhere, then the only solution would be $u(x,y) \equiv 0$. □

Example 5. Consider the first–order linear PDE

$$xu_x - 2xu_y = u \ . \tag{11}$$

Show that

$$u(x,y) = xf(2x + y) \tag{12}$$

is a generic solution of (11), where f is an arbitrary C^1 function.

Solution. First note that despite the involvement of both x and y in $f(2x + y)$, the function f is still really a function of one variable, since f has only one "slot", unlike say $g(x,y)$. Thus, by definition, $u(x,y)$ in (12) will define a generic solution if it satisfies (11). The product and chain rules yield $u_x = f(2x + y) + xf'(2x + y)\cdot 2$ and $u_y = xf'(2x + y)$. Hence, $xu_x - 2xu_y = xf(2x + y) + 2x^2f'(2x + y) - 2x^2f'(2x + y) = xf(2x + y) = u$. Thus, (12) defines a generic solution. Using the theory of Chapter 2, one can prove that the general solution of (11) has the form (12). □

Usually one wants to find a particular solution of a PDE which meets a side condition. The next two examples show how such a solution may be extracted from a generic solution.

Example 6. Find a solution of (11) which satisfies the condition $u(1,y) = y^2$ for all y.

Solution. The condition $u(1,y) = y^2$ specifies the values of the solution $u(x,y)$ for points (x,y) on the line $x = 1$, parallel to the y–axis. Since (12) is a generic solution, it suffices to find a function f such that $u(1,y) = 1\cdot f(2 + y) = y^2$ or $f(2 + y) = y^2$. To find such a function, let $r = 2 + y$. Since $y = r - 2$, we have $f(r) = (r - 2)^2$. Thus, f is the function which takes a number, subtracts 2, and squares the result. In particular, $f(2x + y) = (2x + y - 2)^2$. Hence, $u(x,y) = x(2x + y - 2)^2$. One should check directly that this u satisfies the PDE (11) and $u(1,y) = y^2$. □

Example 7. Show that the wave equation $u_{tt} = c^2u_{xx}$ has a generic solution of the form

$$u(x,t) = f(x + ct) + g(x - ct) , \tag{13}$$

where f and g are arbitrary C^2 functions. Find a particular solution meeting the initial conditions

$$\text{I.C.} \quad u(x,0) = h(x) \quad \text{and} \quad u_t(x,0) = 0 , \tag{14}$$

where $h(x)$ is a given C^2 function.

Solution. One can directly verify that (13) is a solution of $u_{tt} = c^2u_{xx}$, as in Problem 4(d) of Section 1.2 . Since the wave equation is second–order and there are two arbitrary functions in (13), neither of which can be eliminated without losing solutions (cf. Problem 12), we conclude that (13) is a generic solution. By setting $t = 0$ in (13) and using $u(x,0) = h(x)$, we get that $f(x) + g(x) = h(x)$. By differentiating (13) with respect to t, we obtain $u_t(x,t) = f'(x + ct)c + f'(x - ct)(-c)$, whence $u_t(x,0) = 0$ yields $f'(x) - g'(x) = 0$. Thus, (14) gives us two conditions on the two unknown functions f and g, namely

$$f(x) + g(x) = h(x) \quad \text{and} \quad f(x) - g(x) = K. \tag{15}$$

Adding corresponding sides of the equations (15), we obtain $f(x) = \frac{1}{2}[h(x) + K]$. Similarly $g(x) = \frac{1}{2}[h(x) - K]$. These identities determine the functions f and g in terms of the given function h. Thus, we obtain the following solution of $u_{tt} = c^2u_{xx}$, which meets the initial conditions (14) :

$$u(x,t) = \tfrac{1}{2}[h(x + ct) + K + h(x - ct) - K] = \tfrac{1}{2}[h(x + ct) + h(x - ct)] . \tag{16}$$

In Problem 12, the reader is asked to check directly the validity of (16) . □

Elementary Techniques

We have already seen in Example 1 and 2 that PDEs, which simply set a partial derivative of the unknown function equal to a given function, can be solved by direct integration. The PDE $u_x = u^2$ in Example 4 cannot be solved by integrating both sides with respect to x, because the right side involves the unknown function u(x,y). However, we were able to solve this equation by ODE techniques.

> If a PDE involves only partial derivatives with respect to one of the independent variables, then such an equation may be regarded as an ODE for an unknown function of a single variable, where the other variables are held fixed. In the solution, the arbitrary constants are replaced by arbitrary functions of these remaining variables.

By way of illustration, we solve here the homogeneous version of the degenerate equation $u_{xx} + cu = g(x,t)$ which arose in the Classification Theorem of Section 1.2 .

Example 8. Find the general solution of the PDE

$$u_{xx} + cu = 0 , \quad u = u(x,t) \tag{17}$$

in the three cases $c > 0$, $c = 0$ and $c < 0$.

Solution. For fixed t , (17) is a second–order linear ODE with constant coefficients (discussed in Section 1.1) for u, regarded as a function of x. If $c > 0$, then for each fixed t, the solution is of the form $c_1\sin(\sqrt{c}\cdot x) + c_2\cos(\sqrt{c}\cdot x)$. However, as t varies, the choices for c_1 and c_2 may change (i.e., they may be functions of t). Consequently, the general solution of (17) is

$$u(x,t) = f_1(t)\sin(\sqrt{c}\cdot x) + f_2(t)\cos(\sqrt{c}\cdot x) ,$$

where f_1 and f_2 are arbitrary C^2 functions. The general solution in the cases $c = 0$ and $c < 0$ are, respectively,

$$u(x,t) = f_1(t)x + f_2(t) \quad \text{and} \quad u(x,t) = f_1(t)e^{\sqrt{|c|}\cdot x} + f_2(t)e^{-\sqrt{|c|}\cdot x} \quad \square.$$

Example 9. Find the general solution $u = u(x,y)$ of

$$u_{yy} + u_y = x . \tag{18}$$

Solution. By fixing x, we can regard (18) as a linear, inhomogeneous, second–order ODE with y as the independent variable. A particular solution is $u(x,y) = xy$. The auxiliary equation for the related homogeneous equation is $r^2 + r = 0$, which has roots 0 and −1. Remembering that the arbitrary constants may depend on x, we add the general solution of the homogeneous equation to the particular solution, and thus obtain the following general solution of (18) :

$$u(x,y) = xy + f(x) + g(x)e^{-y} , \tag{19}$$

where $f(x)$ and $g(x)$ are arbitrary C^2 functions. One can also solve (18) by first integrating with respect to y, obtaining the first–order linear ODE (where x fixed)

$$u_y + u = xy + h(x) . \tag{20}$$

We multiply each side of (20) by the integrating factor e^y, obtaining

$$\frac{\partial}{\partial y}[e^y u] = xye^y + e^y h(x) \quad \text{or} \quad e^y u = x(ye^y - e^y) + e^y h(x) + k(x) .$$

Thus, another form of the general solution of (18), is given by

$$u(x,y) = xy - x + h(x) + k(x)e^{-y}, \quad h, k \in C^2. \tag{21}$$

Note that (19) and (21) appear to be different, but they are actually equivalent. Indeed, adding the function $-x$ to the arbitrary function $h(x)$ simply gives us another arbitrary function which may be identified with $f(x)$ in (19). Often, different methods yield general solutions which appear to be different, but which are actually equivalent in the sense that they generate the same family of solutions as the arbitrary functions vary. □

Separation of Variables

> The method of **separation of variables** is used to find those solutions (if any) of a PDE which are products of functions, each of which depends on just one of the independent variables. Such solutions are called **product solutions.**

The following examples illustrate the method of separation of variables.

Example 10. Using separation of variables, find the product solutions of the heat equation with temperature–dependent sink, namely

$$u_t - u_{xx} = -u, \quad u = u(x,t) \tag{22}$$

Solution. Substituting a product solution of the form $u(x,t) = f(x)g(t)$ into (22), we get

$$f(x)g'(t) - f''(x)g(t) = -f(x)g(t). \tag{23}$$

Then we separate the variables, so that functions in the variable x only appear on one side, and functions in the variable t only appear on the other side. If this is possible, it can usually be accomplished by first dividing by $f(x)g(t)$ and then rearranging :

$$[g'(t)/g(t)] + 1 = f''(x)/f(x). \tag{24}$$

The only way that a function of x can equal a function of t is for both functions to be the same constant, say λ. Thus, (24) splits into two ODEs, namely

$$[g'(t)/g(t)] + 1 = \lambda \quad \text{or} \quad g'(t) + (1-\lambda)g(t) = 0 \tag{25}$$

and

$$f''(x)/f(x) = \lambda \quad \text{or} \quad f''(x) - \lambda f(x) = 0 \tag{26}$$

The general solution of (25) is $g(t) = C\exp[(\lambda - 1)t]$. The form of the general solution of (26) depends on whether $\lambda > 0$, $\lambda < 0$ or $\lambda = 0$. If $\lambda < 0$, then $f(x) = c_1\sin(\sqrt{|\lambda|}\cdot x) + c_2\cos(\sqrt{|\lambda|}\cdot x)$, and in this case the product solution $f(x)g(t)$ is

$$u(x,t) = [c_1\sin(\sqrt{|\lambda|}\cdot x) + c_2\cos(\sqrt{|\lambda|}\cdot x)]\exp[(\lambda - 1)t]. \tag{27}$$

Note that the arbitrary constant C in $g(t)$ has been absorbed into c_1 and c_2, without loss of generality. (For the cases where $\lambda > 0$ and $\lambda = 0$, see Problem 5). □

Remark 1. Note that (27) is not a generic solution, since there are no arbitrary functions involved. Thus, solutions obtained by separation of variables are usually far from being general solutions. However, if the PDE is linear and homogeneous, then the linear combinations of product solutions (for various values of λ) will also be solutions according to the superposition principle in Section 1.2 . Often, solutions obtained in this way are sufficiently general for applications, as will be seen repeatedly in Chapter 3 onwards. □

Remark 2. A seasoned separatist, say Dr. XX, will realize in advance that undesirable square roots and absolute value signs will appear in the solution of (27). To avoid this, Dr. XX (by second nature) will write the negative separation constant λ in the form $-\lambda^2$ for some $\lambda > 0$. Then Dr. XX arrives not only at the prettier solution

$$u(x,t) = [c_1\sin(\lambda x) + c_2\cos(\lambda x)]\exp[-(\lambda^2 + 1)t]$$

which is equivalent to (27), but also dazzles fledgling students with her brilliance. We hope that this remark will spare the reader such bewilderment. □

In the case of more than two independent variables, separation of variables involves a number of stages, as we illustrate next.

Example 11. Find some nontrivial product solutions of the following wave equation for the amplitude $u(x,y,t)$ of a transversely vibrating membrane at (x,y) at time t

$$u_{tt} = u_{xx} + u_{yy} . \tag{28}$$

Solution. Let $u(x,y,t)$ be of the form $X(x)Y(y)T(t)$ for functions X, Y and T. This notation for the function is helpful in keeping track of the variables which correspond to the functions. Substituting u into (28), we get $XYT'' = X''YT + XY''T$. Separating t from x and y, we get

$$T''/T = X''/X + Y''/Y .$$

A function of t can only equal a function of x and y when these functions are constant. Thus,

$$T''/T = \lambda \quad \text{or} \quad T'' - \lambda T = 0 \tag{29}$$

and

$$X''/X + Y''/Y = \lambda \quad \text{or} \quad X''/X = \lambda - Y''/Y . \tag{30}$$

Both sides of the last equation in (30) must be a constant, say μ (Why ?). Thus we obtain

$$T'' - \lambda T = 0 \ , \ X'' - \mu X = 0 \ , \ Y'' + (\mu - \lambda)Y = 0 . \tag{31}$$

There are a number of possibilities, depending on the signs of λ, μ and $\mu - \lambda$. Since our aim is not to produce every conceivable product solution, we will make some choices that will produce a popular family of solutions. For constants a, b and c, let $\lambda = -a^2 - b^2$, $\mu = -a^2$, $\mu - \lambda = b^2$. Then selecting some particular solutions of (31), we obtain a nontrivial family of product solutions

$$\cos([a^2 + b^2]^{\frac{1}{2}}t)\sin(ax)\sin(by)\ . \tag{32}$$

Of course, in (32) one can replace the cosine by a sine and any of the sines by cosines; there are eight possibilities. By forming a linear combination of the eight possibilities, we obtain an even larger family of solutions, by the superposition principle. One can even replace all of the sines and cosines by hyperbolic sines and cosines, say in (32), and still get a valid family of product solutions. Indeed, such families would result from setting $\lambda = a^2 + b^2$, $\mu = a^2$ and $\mu - \lambda = -b^2$ (see Problem 18 of Section 1.1). □

Summary 1.3

1. **General solutions** : The general solution of a PDE is the collection of all solutions of the PDE.

2. **Generic solutions** : A generic solution of an n–th order PDE for an unknown function of m independent variables is a solution which involves n arbitrary C^n functions of m–1 variables. Examples 3 and 4 show that a general solution need not be generic, and a generic solution need not be general.

3. **ODE technique** : If a PDE involves only partial derivatives with respect to *one* of the independent variables, then such an equation may be regarded as an ODE for an unknown function of a single variable, where the other variables are held fixed. In the solution, the arbitrary constants are replaced by arbitrary functions of these remaining variables.

4. **Separation of variables** : The method of separation of variables is used to find those solutions u(x,y) (if any) of the form f(x)g(y). Such solutions are called product solutions. Upon substituting the form of the product solution into the PDE, one tries to get expressions involving x on one side of the equation and those involving y on the other (i.e., one tries to separate variables). If this is possible, then both sides can be set equal to a constant, and one obtains an ODE for f(x) and an ODE for g(y). For unknown functions of three or more variables, several stages of the separation process are carried out. Solutions obtained in this way are usually far from being general solutions of the PDE.

Exercises 1.3

1. Find the general solution of each of the following PDEs by means of direct integration.

(a) $u_x = 3x^2 + y^2$, $u = u(x,y)$ (b) $u_{xy} = x^2y$, $u = u(x,y)$
(c) $u_{xyz} = 0$, $u = u(x,y,z)$ (d) $u_{xtt} = \exp[2x + 3t]$, $u = u(x,t)$.

2. Find general solutions of the following PDEs for $u = u(x,y)$ by using ODE techniques.
(a) $u_x - 2u = 0$ (b) $yu_y + u = x$
(c) $u_x + 2xu = 4xy$ (d) $yu_{xy} + 2u_x = x$ (**Hint.** First integrate with respect to x.)
(e) $u_{yy} - x^2u = 0$.

3. For the PDEs (a) through (e) of Problem 2, find a particular solution satisfying the following respective side conditions.

(a) $u(0,y) = y$ (b) $u(x,1) = \sin(x)$

(c) $u(x,x) = 0$ (i.e., $u = 0$ on the line $y = x$) (d) $u(x,1) = 0$ and $u(0,y) = 0$

(e) $u(x,0) = 1$ and $u_y(x,0) = 0$.

4. Find a nontrivial family of solutions of the following PDEs by the method of separation of variables. You need not find the most general solution obtainable in this way.

(a) $u_t = 2u_{xx}$, $u = u(x,t)$ (b) $u_x = 4u_y$, $u = u(x,y)$

(c) $u_{tt} = 16u_{xx}$, $u = u(x,t)$ (d) $u_t = u_{xx} + u_{yy}$, $u = u(x,y,t)$

(e) $u_{xx} + u_{yy} + u_{zz} = 0$, $u = u(x,y,z)$.

5. Find the product solutions of the PDE in Example 10, in the cases where the separation constant (i.e., λ) is positive or zero. When the separation constant is positive, find an equivalent product solution (as in Remark 2) which does not involve square roots.

6. In Section 1.2 we have used trial solutions of the form e^{rx} to find particular solutions of certain ODEs. The higher dimensional analogue of this substitution (as, for example, $u(x,y) = \exp(rx + sy)$, where r and s are constants) is called the **exponential substitution**. Use the exponential substitution to find a nontrivial family of solutions of each of the following PDEs.

(a) $2u_x + 3u_y - 2u = 0$, $u = u(x,y)$ (b) $4u_{xx} - 4u_{xy} + u_{yy} = 0$, $u = u(x,y)$

(c) $u_{xyz} - u = 0$, $u = u(x,y,z)$ (d) $u_{xx} + u_{yy} = 14\exp(2x + y)$, $u = u(x,y)$

(e) $u_{xx} + u_{yy} = u$, $u = u(x,y)$.

7. Consider the problem $u_{xx} + u_{xy} + u_{yy} = 0$, $u = u(x,y)$, and attempt to use the method of separation of variables to arrive at $f''(x)g(y) + f'(x)g'(y) + f(x)g''(y) = 0$.

(a) If $f(x)g(y) \neq 0$, verify that $-f''(x)/f(x) = [g'(y)/g(y)][f'(x)/f(x)] + g''(y)/g(y)$.

(b) Deduce from (a) that if $f'(x)/f(x)$ is not constant, then $g'(y)/g(y)$ is a constant, say s.

(c) Deduce from (b) that $g(y) = ce^{sy}$, and $g''(y)/g(y) = s^2$.

(d) Show that $f''(x) + sf'(x) + s^2f(x) = 0$. Solving this ODE for f(x), obtain the solution

$$u(x,y) = [c_1\cos(\tfrac{1}{2}\sqrt{3}\cdot sx) + c_2\sin(\tfrac{1}{2}\sqrt{3}\cdot sx)]\exp[s(y - \tfrac{1}{2}x)] .$$

8. For each of the following PDEs, find some constants a and b (not both zero), such that $u(x,y) = f(ax + by)$ is a generic solution, where f is an arbitrary C^1 function.

(a) $u_x + 2u_y = 0$ (b) $5u_x + 6u_y = 0$ (c) $cu_x + du_y = 0$, for any constants c and d.

9. Use the technique of Problem 8 to solve the following PDEs, subject to the given side conditions. Explain why one cannot obtain the solutions by using separation of variables.

(a) $u_x + 2u_y = 0$, $u(x,0) = x$ (b) $u_x + 3u_y = 0$, $u(x,2x + 1) = x^2$

(c) $3u_x - 4u_y = 0$, $u(x,x) = x^2 - x$ (d) $u_x + 2u_y = 2x + 4y$, $u(0,y) = y^2 + 1$.

Hint. For (d), first find a particular solution $u_p(x,y)$ of the form $ax^2 + by^2$.

10. For given real constants A, B and C, consider the second-order PDE $Au_{xx} + Bu_{xy} + Cu_{yy} = 0$. Show that if $B^2 - 4AC > 0$ (i.e., the PDE is hyperbolic), then this PDE has a generic solution of the form $u(x,y) = f(ax + by) + g(cx + dy)$, where a, b, c and d are real constants, and where f and g are C^2 functions.

Hint. Assume $u(x,y) = h(rx + sy)$, obtain $Ar^2 + Brs + Cs^2 = 0$, fix r and solve for s.

11. In relation to Example 4, where the PDE $u_x = u^2$ was considered, define

$$u(x,y) = \begin{cases} -[x + g(y)]^{-1} & \text{for } y > 0 \text{ , } x \neq -g(y) \\ 0 & \text{for } y \leq 0 \end{cases}$$

(a) Show that if $g(y) = y^{-2}$, then $u(x,y)$, $u_x(x,y)$ and $u_y(x,y)$ are continuous at points of the x–axis. Deduce that u is C^1 (and a solution of $u_x = u^2$), except at points on the curve $x = -y^{-2}$, $y > 0$.

(b) Let $g(y) = y^{-1}$. Show that $u_y(x,y)$ is not continuous at points on the x–axis, because in this case $u_y(x,y)$ jumps as y passes through 0. Why does this imply that $u(x,y)$ is not a solution of the PDE $u_x = u^2$ in the region consisting of the whole plane except for points on the curve $x = -y^{-1}$, $y > 0$?

12. The following parts concern the solution $u(x,t) = f(x + ct) + g(x - ct)$ of the wave equation $u_{tt} = c^2u_{xx}$, where f and g are C^2 functions.

(a) Let $u(x,t) = f(x + ct)$. Suppose that for each fixed time t we graph u as a function of x. Show that as t advances, the graph moves to the left with velocity c. What about $u(x,t) = g(x - ct)$?

(b) Show that if $f(x + ct) = g(x - ct)$ for all x and t, then f and g must be constant.

(c) Deduce from (b) that neither $f(x + ct)$ nor $g(x - ct)$ can be eliminated from the solution $u(x,t) = f(x + ct) + g(x - ct)$ without losing solutions in the process.

(d) Check directly that (16) solves the PDE $u_{tt} = c^2 u_{xx}$ with the I.C. given by (14).

CHAPTER 2

FIRST-ORDER PDEs

In most PDE textbooks, first–order PDEs usually receive only a brief treatment. One reason for this is that the PDEs which have the most obvious applications are the standard second–order PDEs, namely the heat, wave and Laplace equations. Moreover, the theory of first–order PDEs locally reduces to the study of systems of first–order ODEs, which is presumably a subject of another course. Here we will find that first–order PDEs have a variety of applications. Also, there are certain global topological considerations which arise in the study of first–order PDEs which make the theory more than just a study of systems of ODEs.

In Section 2.1, we solve first–order, linear PDEs with constant coefficients by introducing a linear change of variables, which converts the PDE into a family of ODEs depending on a parameter. We apply this theory to population and inventory analysis. In Section 2.2, we handle the case of first–order, linear PDEs with nonconstant coefficients. This is done by making a nonlinear change of variables, so that when all but one of the new variables is held fixed, one obtains a characteristic curve along which the PDE becomes an ODE in the remaining new variable. By piecing together the solutions of the ODEs on these curves, we indicate how certain global considerations may arise. Applications to gas flow and differential geometry are provided. In Section 2.3, we show how this method of characteristics extends to first–order linear PDEs in three dimensions, which we use to solve related first–order quasi–linear PDEs in two dimensions. Among many possible applications, we show how quasi–linear PDEs arise in the study of traffic flow and nonlinear continuum mechanics, particularly with regard to the phenomenon of shock waves. In the optional Section 2.4, the more involved theory of arbitrary nonlinear first–order PDEs is introduced, and there is an application to the study of the motion of wave fronts in an inhomogeneous medium with a variable wave propagation speed. Moreover, in this application, we see the wave/particle duality in the Hamilton–Jacobi theory which foreshadows the analogous duality which lies at the foundations of quantum mechanics.

2.1 First–Order Linear PDEs (Constant Coefficients)

Perhaps the simplest nontrivial type of PDE is the first–order linear PDE

$$au_x + bu_y + cu = f(x,y) , \quad u = u(x,y) , \ a^2 + b^2 > 0 , \tag{1}$$

where a, b and c are given constants and f(x,y) is a given continuous function. Our first main goal will be to find the general solution of (1). In the easy case, when b = 0, (1) is

$$au_x(x,y) + cu(x,y) = f(x,y) , \tag{2}$$

which (for each fixed y) is a first–order, linear ODE for u(x,y) regarded as a function of x. Following the procedure in the Summary of Section 1.1, we can solve (2), by first dividing by a (a ≠ 0) and multiplying by the integrating factor $e^{cx/a}$. Thus,

$$e^{cx/a}\frac{\partial u}{\partial x}(x,y) + e^{cx/a}\frac{c}{a}u(x,y) = \frac{1}{a}f(x,y)e^{cx/a}$$

or

$$\frac{\partial}{\partial x}\Big[e^{cx/a}u(x,y)\Big] = \frac{1}{a}f(x,y)e^{cx/a} .$$

Integrating both sides with respect to x and multiplying by $e^{-cx/a}$, we obtain the general solution of (2), namely

$$u(x,y) = e^{-cx/a}\Big[\frac{1}{a}\int f(x,y)e^{cx/a}\,dx + C(y)\Big] , \tag{3}$$

where C(y) is an arbitrary C^1 function of y. The success of this method depends heavily on the fact that u_y is not present in (2). This is what enabled us to treat (2) as an ODE.

To handle the more general case when b ≠ 0 , we begin with the observation that $au_x + bu_y$ is the dot product of the vector **ai** + b**j** with the gradient $\nabla u = u_x\mathbf{i} + u_y\mathbf{j}$, and hence $au_x + bu_y$ is essentially the derivative of u in the direction of the vector a**i** + b**j** . If we introduce a new coordinate system for the xy–plane, so that one of the new axes is pointing in the direction a**i** + b**j** , then $au_x + bu_y$ will be proportional to the partial derivative of u with respect to the new variable labeling that axis, and we will have reduced (1) to the form of (2) in terms of new coordinates. To find an appropriate change of variables, first note that the family of