The International Library of Philosophy

THE NATURE OF MATHEMATICS

Founded by C. K. Ogden

The International Library of Philosophy

THE NATURE OF MATHEMATICS

Founded by C. K. Ogden

The International Library of Philosophy

PHILOSOPHY OF LOGIC AND MATHEMATICS
In 8 Volumes

I	The Nature of Mathematics	*Black*
II	The Technique of Controversy	*Bogoslovsky*
III	The Limits of Science	*Chwistek*
IV	Foundations of Geometry and Induction	*Nicod*
V	The Foundations of Mathematics	*Braithwaite*
VI	Logical Studies	*von Wright*
VII	A Treatise on Induction and Probability	*von Wright*
VIII	An Examination of Logical Positivism	*Weinberg*

THE NATURE OF MATHEMATICS

A Critical Survey

MAX BLACK

ROUTLEDGE
Routledge
Taylor & Francis Group
LONDON AND NEW YORK

First published in 1933 by
Kegan Paul, Trench, Trubner & Co Ltd

Published 2014 by Routledge
2 Park Square, Milton Park, Abingdon, Oxon OX14 4RN
711 Third Avenue, New York, NY, 10017, USA

Routledge is an imprint of the Taylor & Francis Group, an informa business

British Library Cataloguing in Publication Data
A CIP catalogue record for this book
is available from the British Library

ISBN 13: 978-0-415-22542-7 (hbk)
ISBN 13: 978-0-415-61382-8 (pbk)

CONTENTS

PAGE

PREFACE xiii

INTRODUCTION 1

The task of philosophy, *qua* critic, is to exhibit the structure of the sciences by discriminating between hypotheses and principles, etc.

PRELIMINARY SURVEY OF THREE TYPES OF OPINIONS CONSIDERED 7

LOGISTIC 7

The logistic thesis : pure mathematics is a branch of logic.

FORMALISM 8

The formalist thesis : pure mathematics is the science of the formal structure of symbols.

INTUITIONISM 9

The intuitionist thesis : pure mathematics is founded on a basic intuition of the possibility of constructing an infinite series of numbers.

MUTUAL RELATIONS OF THE THREE SCHOOLS . . 11

SECTION I : LOGISTIC 15

History of Logistic Views of Mathematics . . 15

A notice of the chief logistic writers from Leibniz to Wittgenstein.

The Tasks of a Philosophy of Mathematics . . 19

The finite and infinite problems of a philosophy of mathematics are the investigations of the notions 'integer' and 'continuum' respectively. The subsequent analysis tends to replace these unclear notions by more precise ones with the same formal properties. The plan of such analysis is outlined.

Supplementary Note on Logical Analysis . . 24

A discussion of the general features of all systems of symbols (languages) introduces a definition of logical analysis in terms of the explained notions, multiplicity, significance, and structure. Difficulties arising in the logical analysis of language and here discussed throw light upon analogous difficulties in the analysis of mathematics.

PAGE

The Formal Character of Pure Mathematics . 37

This section describes the ideal arrangement of a branch of pure mathematics as a system of deductions from initial axioms. It is a consequence of the generality of pure mathematics that the subject-matter of such a system is indefinite : the axioms treat of *any* set of objects whose names will fit into the axioms.

The Propositional Calculus 41

The manipulation of propositions, definitions of *implication, equivalence, tautology,* and a typical proof.

The Calculus of Propositional Functions . . . 48

Russell's definition of propositional function described.

Variable and Function in Mathematics . . . 50

Various Usages of Variable Symbols 51

New definitions, often used in the sequel, of the illustrative, formal, determinative, and apparent uses of a variable symbol.

Definitions of Mathematical Functions . . . 53

Two sharply contrasted definitions of mathematical function are current ; in the extensional definition a function is an extended list of pairs of numbers ; the intensional definition is in terms of the relation part and whole between symbols.

Propositional Functions 60

The intensional definition of function adapted to the propositional calculus and contrasted with Russell's definition.

The Calculus of Propositional Functions Resumed . 62

Quantifiers, Truth-Values, etc. 62

Manipulation of the notions *some* and *all* in the calculus of propositional functions. This section and the four short sections which follow deal with the technicalities of manipulating the calculus of propositional calculus.

The Algebra of Propositional Functions . . . 66

Extensional Propositional Functions 67

Propositional Functions of Functions and the General Functional Calculus 67

PAGE

Extensional Functions of Functions 68

Derivation of Mathematical Functions from Propositional Descriptions 69

In the logistic thesis the problem of adequately symbolizing mathematical expressions reduces in general to the problem of analyzing the interconnections between propositional functions and descriptions. The logistic solution involves difficulties associated with the definition of identity.

Plural Descriptive Phrases 71

Plural descriptive phrases are derived from propositional functions by the technical device of definitions in use involving the use of 'incomplete symbols'. The limitations of this method are noted.

Definitions of Descriptions and Classes 74

Complete and Incomplete Symbols 76

This and the next three sections continue the discussion of incomplete symbols. Russell's definition is stated and rejected in favour of a more precise definition.

Definition of Incomplete Symbol 77

The road from logic to mathematics runs from propositional functions to descriptions to classes to integers; classes have suffered vicissitudes and the change from collections existing in their own right to incomplete symbols manufactured from symbols needs safeguards.

Importance of Incomplete Symbols 78

Ontological Status of Incomplete Symbols . . . 79

Nature of *Principia* Classes 79

Consistency of Definition of Classes as Incomplete Symbols 80

To interpret classes as incomplete symbols is a *tour de force*, needing to be safeguarded against inconsistency by methods here discussed.

The Real Number 85

At a critical point in the logistic system surprising contradictions appear and must be expelled; the problems involved are associated with the mathematical theory of continuity, based upon 'intuitions' whose exact nature is always conveniently vague.

PAGE

Intuitions of Continuity in a Sensory Field . . 87

'Intuitions' of continuity are here analyzed into an apprehension of connectivity and the possibility of indefinitely continued division; neither is directly observed and both must be translated, in any accurate logistic analysis, into statements concerning the multiplicity of symbols denoting portions of the continuum.

Continuity in Geometrical Space 89

The evolution of the notion of continuity from Greek mathematics through Dedekind to *Principia Mathematica*. Uncritical application of the notions of connectivity and infinite divisibility to the space of geometry, conceived as real, provokes paradoxes (Zeno). The connection between continuity and the possibility of measurement leads to the discovery of a mathematical device for comparing incommensurable numbers (Eudoxus) and so eventually to a purely arithmetical conception of the continuum. The Achilles-Tortoise paradox and Eudoxus' construction are discussed.

Dedekind's Definition of Real Number . . 93

Dedekind's analysis of continuity is a natural extension of the mathematical method due to Eudoxus. The Dedekind definition of real number and a few of its consequences are discussed.

The Logistico-Mathematical Paradoxes . . 97

The paradoxes are classified according as they can or cannot be accurately expressed in mathematical symbolism.

Solution of the Paradoxes 100

They are produced by lack of indication of the field of variation of a variable in its determinative usage. Special technical devices (theory of types) are needed to ensure the appropriate restriction of fields of variation.

Note on Types and Orders 101

The method adopted in *Principia Mathematica* for restricting fields of variation involves a hierarchy of types and a subsidiary hierarchy of orders. The latter is untenable and must be abandoned: two methods for dispensing with it are noted.

PAGE

Connection Between the ' Extended Infinite ' and the Paradoxes 104

The use of the ' extended ' infinite is equivalent to a confusion of types. Difficulties inherent in an extended infinity or geometrical continuum are therefore reproduced in the theory of propositional functions and their correction tends to destroy the possibility of adequately symbolizing, by the propositional calculus, of the mathematical theory of functions.

Confusion of Types in the Theorem of the Upper Bound 107

A specific and important example of confusion of types in pure mathematics is here demonstrated.

The Axiom of Reducibility and the Logistic Definition of Real Number 109

Dedekind's definition of real number is based on vague geometrical intuitions and is therefore unsatisfactory ; but the *Principia* substitute needs axioms of infinity and reducibility whose validity is doubtful.

The Axiom of Reducibility 111

The Axiom of Infinity 112

Arguments for the Axiom of Reducibility . . 112

Russell's arguments in favour are criticized but conventions are suggested which may remove some of the objections to the axioms of reducibility and infinity.

Axiom of Reducibility Equivalent to the Assertion of the Existence of *c* Propositional Functions . 115

A mathematical interlude in parenthesis to the main argument.

Other Criticisms of the Axiom of Reducibility . 117

The attempts hitherto made to prove the axiom of reducibility a contingent proposition are fallacious.

F. P. Ramsey 119

This, and the three sections which follow, are a report on various attempts to remedy the worst defects of *Principia Mathematica*. Ramsey's principal contribution was the effort to dispense with an axiom of reducibility by using functions defined in terms of truth values with a minimum of specific reference to symbols.

PAGE

Note on the Thesis of Extensionality 122

A dogma of the Austrian positivists is examined.

H. Weyl 124

In sharp contrast to Ramsey, Weyl attempted to systematize the principles by which symbols, especially propositional functions, are constructed. His method is summarized and its consequences noted.

L. Wittgenstein 129

An (unauthorized) report of some of his views on pure mathematics, which constitute, by implication and explicitly, a thorough repudiation of the logistic thesis.

L. Chwistek 135

Another attempt, which should be compared with the section on Weyl above, to obtain a definition of types of functions in terms of the principles used in their construction.

Conclusions 139

The arguments against the logistic system of *Principia Mathematica* are summarized and a scheme of reconstruction outlined.

SUPPLEMENT A : FORMALISM 147

The Development of Geometry 152

Documentation of the tendency of a science towards increasing abstraction and concern with structure.

The Formalist View of Mathematics 161

Mathematics, if it exhibits structure, does so in complex fashion with the help of ' ideal elements '.

The Formalist Programme in Detail 162

A technical summary, for specialists, of the axioms and symbolic innovations of the formalist school.

Note on Gödel's Theorem 167

A mention of the remarkable theorem which purports to demonstrate the impossibility of proving mathematics to be free from contradictions.

PAGE

SUPPLEMENT B: INTUITIONISM 169

Static and dynamic attitudes to pure mathematics.

The Mathematical Predecessors of the Intuitionists 174

Some account of the opinions of Kronecker and other early intuitionists.

Digression on the Theory of Sets of Points . . 178

The Mathematical Controversy 184

Intuitionism 186

Kant and Brouwer 187

Some striking analogies assist in understanding Brouwer.

The Sociological Basis of Mathematics . . . 191

Brouwer's theory of the evolution of pure mathematics regards the laws of logic as the historical product of man's attempt to organize sets of object finite in number. On examination the same laws are found to apply also, with one exception, to the infinite subject matter of pure mathematics. That exception is the law of the excluded middle.

The Denial of the Law of the Excluded Middle . 195

In 'denying' the law Brouwer is emphatically asserting existence of mathematical entities to be synonymous with the possibility of their construction.

The Intuitionist Continuum 196

'Points' in the continuum are obtained by using free-choice sequences constructed by arbitrary choices of integers at each stage; significant statements concerning such infinite sequences must contain an implicit or explicit indication of the method for testing their truth in a finite number of steps. This is the correct interpretation of the rejection of the law of the excluded middle.

Supplementary Note on the Intuitionist Calculus of Propositions 201

This section and the next are of mainly technical interest; they include the complicated intuitionist definition of sets (classes).

Intuitionist Theory of Cardinal Numbers . . 207

BIBLIOGRAPHY 211

INDEX 215

This page intentionally left blank

PREFACE

THE philosophy of mathematics has suffered from a superfluity of technicalities. This is a pity because it increases the difficulties of acquaintance with that delightful subject. But a more serious consequence is the lack of co-operation and mutual criticism between different groups of experts in this field. In England, for example, the fame of Russell and Whitehead's justly celebrated *Principia Mathematica*, is accompanied by almost complete neglect and ignorance of the equally interesting work of the Formalists and Intuitionists on the Continent. There is much to be said in extenuation for this state of affairs, for the relevant papers are scattered in foreign periodicals, untranslated, often difficult to obtain, and are unintelligible without an extensive acquaintance with the terminology and context of their authors' opinions. To fill this gap in the literature of the nature of mathematics would be a work of many years, and the pages which follow are intended to be no more than an introduction to the whole subject.

I have had two aims in mind : to present a considered critical exposition of *Principia Mathematica* and to give supplementary accounts of the formalist and intuitionist doctrines in sufficient detail to lighten the paths of all who may be provoked to read the original papers. Various innovations have been introduced and, though I have not avoided technicalities where they were necessary, all technical terms and symbols have been as far as possible defined. So I hope this book may be of use not only to specialists in mathematical logic but to philosophers and others who begin to read it with less knowledge of the complexities of symbolism. In order to assist readers who may wish to omit sections chiefly concerned with technicalities or familiar definitions, I have adopted the device of adding to many sections a summary or comment, printed in small type immediately after the corresponding subheadings; and I would encourage readers new to the subject to read the introduction and these scattered comments before reading the remainder of the text.

I wish to express my thanks to Professor Bernays for much helpful information concerning the formalists, to Dr. Chwistek for copies of his papers, to the Aristotelian Society for permission to incorporate part of a paper read in 1933, to Professor L. S. Stebbing, Dr. J. H. Woodger, and Miss M. MacDonald for reading the following pages in proof and for much encouragement, and to S. Black, J. M. Burnett, and L. E. R. Mowat for assistance with the transcription of manuscript.

M. B.

May, 1933.

THE NATURE OF MATHEMATICS

A CRITICAL SURVEY

INTRODUCTION

The task of philosophy, *qua* critic, is to exhibit the structure of the sciences by discriminating between hypotheses and principles, etc.

THE successes of the scientific method have led philosophers to dream of a scientific philosophy which, by borrowing the technique of the established sciences, might hope to reach something of their certainty and cumulative success. Philosophy, however, in its function of critic—and it is with that aspect of philosophy that we shall be here concerned—cannot desire to compete with the sciences. The discovery of empirical generalization is the work of the experimental sciences, the formulation of self-evident laws belongs to mathematics, and both are outside the scope of critical philosophy. Its object is to clarify by criticizing knowledge already organized into systems; and of these it prefers the older, more developed, sciences, which combine extreme complexity of theory with consistency in practice. For these qualities are associated with a high degree of utility in practical applications and induce in the creators and admirers of the science a state of self-consciousness inviting the apologetic services of philosophy. In each of these respects the science of mathematics is a most admirable field for the exercise of applied philosophy.

The implied assertion that the established sciences are highly consistent needs to be qualified by the explicit recognition that no science which is still in the process of developing is more than partially self-consistent. For scientific research is characterized by the choice between mutually inconsistent theories, lack of relevant data leading to the postulation of provisional hypotheses which subsequently require to be limited in their application or even totally abandoned.

Postulates need to be distinguished as hypotheses and principles ; for, of those postulates which are not ultimately rejected but are incorporated into the main body of the science as knowledge accumulates, some become theorems or laws while others, through their success in stimulating fruitful research, gradually acquire the character of general principles, which embody concepts fundamental to the science. Hypotheses, that is postulates which may become laws, can be disproved casually enough, but principles, since they control the manner in which problems are formulated or difficulties resolved, are formally not susceptible to disproof, and their rejection requires a violent revolution in the methods of the science.

Vagueness of the concepts which occur in the normative principles makes their exact formulation an ideal which is approached by gradual approximation ; clear understanding of the concepts used occurs late in the history of a science.

Postulates and concepts are created not by the common agreement of scientists but by scattered individuals or small groups. At the moment of conception concepts are formless, implications of theories are only partially understood ; later, theories produced by specialists in one department of the science are found to conflict with the postulates of other departments, in themselves equally plausible or as firmly established. The necessity of resolving such discords reacts upon the concepts of the science, leads to more exact formulation of the postulates and clearer understanding of the concepts involved. Even at moments of apparently extreme stability, the equilibrium of scientific opinion is the immobility of a body under the action of mutually opposing forces.

This state of affairs is a commonplace in the experience of any scientific researcher, yet it is more than that private conflict of ideas in the inventor's mind which is part of the process of invention. The contradictions inhere in the very principles of the science, produced by the inevitable vagueness of the concepts it employs. However much reflection and experiment by the inventors of theories may mitigate the opposition of mutually contradictory opinions by modification and elimination of obscurity, contradictions remain even in scientific theories which find widespread acceptance. In the theories of all branches of science where progress is still being made, in biology, physics, chemistry, mathematics, there are striking paradoxes and contradictions to be found, and those sciences alone are completely consistent which, like anatomy, have degenerated into catalogues. It is important to recognize and distinguish contradictions produced by imprecise formulation of concepts; they are often a sign of vitality and indicate that the scientist's capacity for recognizing relevance and unity in a confusing multiplicity of heterogeneous phenomena is ahead of the careful expression of its discoveries.

Nowhere have such contradictions been more frequent than in mathematics, nor has progress in any science been more steady. Gauss and Fermat, among scores of other famous names, are sufficient illustrations of famous mathematicians who were able to obtain, by apparently fallacious reasoning, valid results of the highest importance in subsequent mathematical researches.

The title of "The Foundations of Mathematics" which the philosophical analysis of mathematics has often received is therefore a misleading one if, taken in conjunction with these contradictions, it suggests that the traditional certainty of mathematics is in question. It is a fallacy to which the philosopher is particularly liable to imagine that the mathematical edifice is in danger through weak foundations, or that philosophy must be invited like a newer Atlas to carry the burden of the disaster on its shoulders.

The progressive elimination of contradictions in mathematics is the work of mathematical insight, a continuous process which can be clearly traced in successive mathematical researches. Philosophical analysis has the equally valuable aim of exhibiting the structure of mathematics: first, the internal structure, by showing the interdependence of theorems, axioms, and definitions, distinguishing between hypotheses and principles, etc.; secondly, the external structure, the relation of mathematical knowledge to non-mathematical.

Exhibition of internal structure has technical importance for mathematics by leading to the rejection of unnecessary postulates and again to the recognition of unexpected analogies between the anatomies of different mathematical disciplines. Such morphological investigations require mathematical technique, and particularly the extensive use of symbols. For mathematics is the study of all structures whose form can be expressed in symbols, it is the grammar of all symbolic systems and, as such, its methods are peculiarly appropriate to the investigation of its own internal structure. But the structure of mathematics, though implicit in its theorems, is not clearly shown and tends to be confused even by those who are most familiar with it. It is the philosopher's task to exhibit the inherent structure and to invent a suitable symbolism for its expression. Elimination of unnecessary postulates and the explicit exhibition of the structure of mathematics prevents confusion of purpose within the science and adds to the æsthetic satisfaction of contemplating it.

The technique required for this type of analysis does not, in the present writer's opinion, require the acceptance of any metaphysical dogmas ; in its systematic aspect it can be correctly regarded as a branch of applied mathematics if that science is not restricted to physical applications but is allowed to include any subject-matter amenable to mathematical investigations ; in its philosophic aspect it is a branch of applied logic.[1] The details of such a technique must, however, be reserved for future exposition. The purpose of this essay is only to report and criticize attempts that have already been made to analyze mathematics.

Philosophical analysis must take into account lack of structure for, in so far as a science contains inconsistencies, it cannot be considered as a system, it is to that extent in process of acquiring a form and not in possession of one. Philosophers, however, under scholastic influences, have too often overlooked this fact and have been suspected in consequence by the practising scientist. For, when faced with the difficulty of clarifying existing knowledge, the temptation is great to find compensation in admiring the complex structure which represents partial success and to supplement it by unwarranted extrapolation. In the case of one's own philosophic system familiarity or the inertia of habitual thought processes inspires exaggerated respect and tempts the philosopher to bring the technique of theology to the help of the analytic method. God arrives to solve the difficulties of Berkeleian idealism or Earl Russell in less ambitious times invokes the Axiom of Reducibility.

In no branch of critical philosophy is this danger greater than in the analysis of mathematics, a discipline which acquires from its subject-matter a dangerous facility in the manufacture of vast systems of symbols whose architectonic

[1] For definition of the distinction between the philosophic and systematic aspects of any study cf. *infra*, p. 141.

complexity is occasionally of the same order as the labour required for their intelligent manipulation.

Recent research in the philosophy of mathematics has shown that each of the three principal theories of the nature of mathematics which are discussed in this book contains serious imperfections, some of which may be attributed to the causes indicated above. With this warning to the reader we may conclude these generalities and proceed to a preliminary summary of the three main types of theories which are to be the objects of our investigation.

Preliminary Survey of Three Types of Opinions Considered

Before commencing a detailed account a short description of the general features of the three main schools of mathematical philosophy with which we shall be concerned and their relations to one another may facilitate the orientation of the reader who is unfamiliar with the subject.

The three schools of thought chosen on account of their importance and influence are usually distinguished as Logistic, Formalistic, and Intuitionist, their best known living exponents being Earl Russell, Professor Hilbert, and Professor Brouwer respectively. Their doctrines differ as much in methods of approaching problems as in their conclusions.

Logistic

The logistic thesis : pure mathematics is a branch of logic.

The programme of the logistic school has been expressed by Russell as follows : " Pure Mathematics is the class of all propositions of the form 'p implies q' where p and q are propositions containing one or more variables, the same in the two propositions, and neither p nor q contains any constants except logical constants. And logical constants are all notions definable in terms of the following : implication, the relation of a term to a class of which it is a member, the notion of *such that,* the notion of relation, and such further notions as may be involved in the general notion of propositions of the above form. In addition to these mathematics *uses* a notion which is not a constituent of the propositions which it considers, namely the notion of truth" (*Principles of Mathematics,* p. 3). In other words, the propositions of mathematics are propositions of logic, they state relations between propositions whose content has been abstracted to leave only their form, shown by the logical constants *and*, *or*, etc.

On this view, all mathematical concepts such as *number*, *differential coefficient*, etc., must be capable of definition in terms of logical concepts, pure mathematics becomes a branch of logic and the distinction between the two subjects is merely one of practical convenience. Much of Russell's work, like that of his collaborator, Professor Whitehead, and his great predecessors, Frege and Peano, was devoted to performing the reduction of mathematical concepts to logical concepts. The culminating achievement of this school is Russell and Whitehead's *Principia Mathematica*, a massive work of bewildering complexity but great logical beauty, which purports to be a detailed reduction of the whole of pure mathematics to logic.

Formalism

The formalist thesis : pure mathematics is the science of the formal structure of symbols.

The formalists, on the other hand, deny that mathematical concepts can be reduced to logical concepts and assert that the many difficulties of logic which beset the path of the logistic philosophies have nothing to do with mathematics. They see in mathematics the science of the structure of objects. Numbers are the simplest structural properties of objects and are themselves objects with new properties. The mathematician can study the properties of objects only by making a system of signs which stand for them and by recognizing and allowing for the irrelevant features of the signs he uses. But provided he has an adequate system of signs he need no longer worry about their meaning since he can see in the signs themselves those structural properties which interest him. Hence the formalists emphasize the importance of the formal characteristics of the mathematician's sign-language, those which are independent of the meaning he may want to attach to them. This is not to say that mathematics is a meaningless game as the formalists have often been accused of asserting; they say that mathematics is concerned with the structural properties of symbols (and hence of all objects) independent of their meaning. This view has proved very fruitful in geometry and its success in that field is largely responsible for its widespread popularity. The formalists naturally lay a greater value upon a consistent symbolism than the logisticians; the contradictions in pure mathematics can be removed, they say, only by the provision of a symbolism which has been demonstrated to be foolproof. The demonstration itself cannot be carried through by the use of symbols independently of their meaning, for these symbols in turn would have to be legitimized and so *ad infinitum*; but they demand a demonstration using no process of thought essentially more complicated than that by which we see that two things and two things together make four. Most of the recent work of the formalists has been directed towards an elementary proof of the validity of mathematics from this angle. So far their success has been only partial, and there are grave doubts whether their programme can be consistently carried through.

Intuitionism

The intuitionist thesis: pure mathematics is founded on a basic intuition of the possibility of constructing an infinite series of numbers.

The formalists lay the emphasis on symbolism, the intuitionists on thought. For the latter the body of