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Paradoxes of the Infinite

Bernard Bolzano

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Paradoxes of the Infinite presents one of the most insightful, yet strangely unacknowledged, mathematical treatises of the 19th century: Dr Bernard Bolzano's *Paradoxien*. This volume contains an adept translation of the work itself by Donald A. Steele S.J., and in addition an historical introduction to the masterpiece, which includes a brief biography as well as an evaluation of Bolzano the mathematician, logician and physicist.

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PARADOXES OF THE INFINITE

by Dr. Bernard Bolzano

TRANSLATED FROM THE GERMAN
OF THE POSTHUMOUS EDITION
BY DR. FR. PŘIHONSKÝ
AND FURNISHED
WITH A HISTORICAL INTRODUCTION
BY DONALD A. STEELE
S.J., D.Ph.

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De Licentia Superiorum Ordinis

Tender-hearted stroke a nettle,
And it stings you for your pains;
Grasp it like a man of mettle,
And it soft as silk remains.

If there be in mathematics a nettle danger out of which has been plucked the flower safety, it is speculation on O and ∞.

Augustus De Morgan, *Transactions of the Cambridge Philosophical Society*, 11 (signed 28 February 1866), 258. The quatrain is taken by De Morgan from the *Verses written on a Window in Scotland*, by Aaron Hill (1685–1750).

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HISTORICAL INTRODUCTION

(1) *means: Footnote* 1 *at end of current division.* (B1) *means: Entry* 1 *in the Bibliography, pages* 176–81. (B1, 1) *means: page* 1 *of entry* 1.

TRANSLATION

Asterisks mark footnotes translated currently from Bolzano.
(1) *means: Translator's note* 1 *at the end of the Translation.*

SELECTED BIBLIOGRAPHY

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SHORT TITLE KEY TO BOLZANO REFERENCES

BD=*Beiträge zu einer begründeteren Darstellung*, Prague, 1810, 152 pp.

BG=*Betrachtungen über einige Gegenstände*, Prague, 1804, 63 pp.

BL=*Binomischer Lehrsatz*, Prague, 1816, 144 pp.

BS=*Begriff des Schönen*, KBGW (5) 3, 1845, 1–92.

DD=*Drei Dimensionen*, Prague, 1843, and KBGW, 15 pp.

DP=*Drei Probleme*, Leipzig, 1817, 80 pp.

EK=*Einteilung der Künste*, KBGW (5) 6, 1851, 133–178.

KBGW=*Abhandlungen der königlich-böhmischen Gesellschaft der Wissenschaften*, Prague.

PU=*Paradoxien des Unendlichen*, Leipzig, 1851, 134 pp.

RB=*Rein analytischer Beweis*, Prague, 1817, and KBGW, 60 pp.

WL=*Wissenschaftslehre*, Sulzbach, 1837, four volumes, 2397 pp.

ZK=*Zusammensetzung der Kräfte*, Prague, 1842, and KBGW, 40 pp.

NOTE ON REFERENCES TO THE 'WISSENSCHAFTSLEHRE'

The Paragraphs are numbered through four volumes as follows: §§1–120 in I, §§121–268 in II, §§269–391 in III, §§392–718 in IV.

For fuller descriptions see the Bibliography on pages 176–9.

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HISTORICAL INTRODUCTION

I. The Historical Context of the *Paradoxien*

BLENDING precision with temperament and a keen scent for fresh sources, Heinrich Scholz is not content in his *Geschichte der Logik* (B43) with the annals of terminology and technique. The first of his three parts is devoted to the changing 'shape' of logic as a whole, to its changing degree of formalism, to its changing frontiers with psychology and epistemology. The historian of mathematics has learnt to do likewise. He has learnt to shift the stress from anecdotes, priorities and descriptions to the emergence, the evolution and the interaction of concepts, problems, methods and theory structures. Two of the tasks to be undertaken in this spirit are, to trace the changing standard of rigour in mathematics and to trace its changing frontiers with physical science.

Hazardous though division into periods is, and doubly hazardous one by centuries, we reach a passable first approximation if we regard the centuries of Eudoxus and Archimedes, of Weierstrass and Hilbert as four centuries of greater rigour, and the century of the Enlightenment and the Encyclopedia as one century of greater laxity, and indeed of a laxity beyond the simple lack of conscientiousness in definition and demonstration. Figures of transition will offer the historian an interest of their own—not only figures like Gauss, the lonely giant, austere and integral, whose Minerva-born precision in real as well as complex function theory only became known long after his death, but also those other figures in whose minds the battle for logical ideals swayed to and fro, with losses as well as gains. Such a figure was De Morgan (1806–1871) in London and such another was Bernard Bolzano (1781–1848) in Prague.

Through Leibniz and Newton and other substantial contributors, and not least through the resuscitated influence of Eudoxus and Archimedes, powerful new methods brought mathematical analysis into being: yet these methods could not be pursued at once to their logical foundations or to their full generality. They were applied instead to particular problems inspired by the growth of physical science. Intoxicating success in this field not only tended to relax logic, but also to mask problems of existence and uniqueness. It was hard at that time, taking a simple example, to see in velocity not a physical entity reached by our senses, but an indirect and sophisticated construction, legitimate and significant to be sure, but only at the price of the same disciplined reflection which fashioned at last, with diamond-cutter's patience, the four Dini derivates of a function. It was harder to understand, in consequence, that 'velocities' ingenuously accepted as existent and bilateral are no secure basis for a 'differential calculus' (1).

Such seeming generality as lay in the use of infinite series tempted even Newton himself to speak of 'aliarum omnium quadratura' (2), and was a stronger temptation still to Euler, whose adventurous computations, checked by transient scruples, produced numerous results that happened to be true and very valuable and eventually demonstrable, together with other results patently absurd. At length, the genius who inverted elliptic integrals also saw that the bare correction of fallacies was necessary but not sufficient. One '*Fandenskab*,' one impish puzzle, remained to stimulate Abel: why had the invalid methods not more often led to false results (3)? Precisely the same question crossed the mind of Bolzano (4) and led him to do his own work on some of the fundamental problems of mathematics.

Nor was laxity in ratiocination the only evil inviting reform at the hands of Bolzano and Abel, of Gauss more privately and deeply, of Cauchy and Weierstrass more publicly and efficaciously. The mere hardihoods of Euler (5) are a commonplace of mathematical history; but he went farther. After defining a function for integer arguments, he tried to 'deduce' its value for fractional ones by a quasi-magical interpolation (6). He openly advises the reader to trust his calculations rather than his judgement (7) and makes in the *Mechanica* a suggestion best regarded as a joke at his own expense: on arriving at a point where mathematical difficulties occur, a particle is blandly said (8) to suffer annihilation! Mathematics might, in short, be a world of natural phenomena parallel to that of physics, with entities awaiting discovery as America awaited Leif Ericsson or Columbus. The identification of $\log(-1)$ was hotly disputed before ever the function $\log(x)$ had been defined over $x=-1$ at all, and this sterile controversy (9) was only laid to rest by Gauss (10), though then very thoroughly. Even the sense of aptness in notation must have been dull when D'Alembert denoted (11) the first three derivates of $f(x)$ by $\Delta(x)$, $\Gamma(x)$ and $\Psi(x)$. The genial poet-mathematician or mathematician-poet Abraham Gotthelf Kaestner wrote more truly than he realised when, with a touch of Hogarth and a sly quotation from Ovid, he compared the analysis of his day (12) to a beautiful maiden clad in slatternly garments; and that nursing-mother of subtleties in algebra and analysis, and of ideas so beautiful and fertile and ordinative as those of group and ring and ideal, the theory of numbers, suffered the gibe of Voltaire (13) harmlessly but not unrepresentatively.

Nevertheless, history is not discontinuous enough for the eighteenth century to lack misgivings. Within three years Benjamin Robins (14) castigated the vagaries quoted from Euler's *Mechanica* and a third vagary which our Bolzano was also to reject (PU, §70), if for reasons of his own. The terms 'convergent' and 'divergent' had been introduced by James Gregory (15), one definition had been interestingly refined by Robert Simson (16), Varignon had sobered Leibniz' quasi-magical summation of the Grandi series (17) and Leibniz himself, questioning Newton's right to use series so freely, openly desiderated criteria of 'advergence' (18), now called 'convergence.' Euler himself addresses reproaches to those who neglect this or that precaution (19). Laplace begins (20) to protest at the summation of the Grandi series by 'probability considerations,' Clairaut begins to rate faults of procedure as more serious than faults of calculation and Lagrange begins (21) to demur at uncontrolled approximations and the failure to estimate remainders. For one clouded moment in which D'Alembert says 'allez en avant, la foi vous viendra' (22), we have two lucid ones in which the same D'Alembert says that simplicity and rigour are not mutually exclusive but mutually helpful, and that results obtained from divergent series remain suspect even when they harmonise with facts otherwise known (23).

Quietly, in the last year of that century, three presage-laden sheets were being written, but were to remain unprinted until 1912. Starting with a bounded sequence (a_n) the writer arrived by clearcut steps and one then pardonable gap at a number L_0 such that each higher number was exceeded by no term, and each lower number by at least one term. Successive termwise decapitations then gave monotone sequences of 'fines superiores' (L_n) and corresponding 'inferiores' (M_n), whereupon the former had in turn a 'finis inferior' L and the latter a 'superior' M. When L and M coincided, the original sequence was convergent and had their common value for its hitherto cruder 'limit.' Notwithstanding the gap and the absence of the modern Latin names, we here stand closer to the 'limites superior et inferior' than we do in the *Cours d'Analyse* of Cauchy (24). These three sheets (the turn of the tide back to logic and so back from manipulation to mathematics) were the work of Gauss (25) at the age of twenty-four; but to notice the selfsame gap, and to attempt its bridging, was the work of Bolzano.

When Gauss in 1812 deliberately caters for lovers of rigour and appeals with justice to Greek ideals, initiates deeper convergence testing, warns against the snags of divergence and plans a dissertation expressly on that theme (26); when Abel in 1826 corrects Cauchy's belief that a sequence of continuous functions must needs have, if any, then a continuous limit function, dispenses with absolute convergence at a boundary and uses uniform convergence even if without isolating it as a concept (27); when Dirichlet warns Steiner that sundry 'identifications' of maxima repose on existence hypotheses (28), then the great watershed is fairly behind us. And more recent, more competent examinations of Bolzano's mathematical remains reveal that he, like Gauss, was eremitically and uninfluentially yet distinctly ahead of his contemporaries in surprisingly many of these delicate points.

The battle for logical conscientiousness (not as barren pedantry, but for rigour in mathematics as the shield of its autonomy, the spring of its fruitfulness, the condition of its beauty) was now joined. Weierstrass and Cantor, Kronecker and Frege, Brouwer and Weyl could carry it on each in his own time and fashion. Whitehead could look back (29) on the easier satisfaction of our mathematical forefathers, Russell could dismiss infinitesimals (30) with four scarifying adjectives and Hardy's *Pure Mathematics* could preach the epsilontic gospel to Littlewood's 'cannibals.' But victories came slowly at first. The logically and hellenistically preciser research on Greek mathematics published during the short career of *Quellen und Studien* (31) will best prepare the reader to savour the exquisite shallowness of Lacroix's dictum (32) that 'we have no further need of the subtleties with which the Greeks plagued themselves'! Subtleties which touch again and again on problems but recently rediscovered! After learning from Bolzano in 1817, save for a real number preconstruction, that $|a_m - a_n| < \varepsilon$ for all $m > n > \nu(\varepsilon)$ is sufficient as well as necessary for the convergence of (a_n), we are somewhat disconcerted to find Laurent in 1862 bereft of counsel (33) at this point: not missing what we miss, but muttering that we must bypass an 'obstacle which it would be too difficult to overturn.' Again, when Bertrand's *Traité* of 1864 simply takes the convergence of a bounded monotone sequence for granted (34), our minds travel involuntarily forty-seven years back to the treatise in which a proof was attempted with insight and courage, and achieved less incompletely than was to be expected: namely, to the *Rein analytischer Beweis* of Bolzano. Two years afterwards, in 1866, the same climate reigns in Duhamel as in Bertrand, and a work professing by title to discuss the 'methods of the sciences of ratiocination' proves the Bolzano-Weierstrass Theorem for large intervals by the simple device of assuming it for smaller ones (35). Even De Morgan, amid many sallies towards rigour, is once so exuberant in language as to say that he could believe anything of a function which became infinite (36): presumably less a counsel of despair than a humorous warning to expect the unexpected.

In brief, it would be a simplification unjust in one direction and unmerited in the other if we qualified the eighteenth century, for all its fecundity in verificanda, as one of unrelieved laxity in mathematical reasoning, and the nineteenth as one of unsullied rigour. Nevertheless, the transition was markedly in that sense. Euler and D'Alembert, Boscovich and Hindenburg, Kaestner and Laplace belong to the one epoch, and Legendre with Lagrange prepare the next: and all these mathematicians died during Bolzano's lifetime. Cauchy and Jacobi, Dirichlet and Weierstrass, Hamilton and De Morgan, Grassmann and Boole, Riemann, Kronecker and Dedekind, Dini and Cantor belong to another and very different epoch: and all these mathematicians were born during Bolzano's lifetime. When we reflect too that Daniel Bernoulli's life just overlapped into Bolzano's, that the life of Gauss embraced it, that Bolzano's life embraced those of Abel and Galois and just overlapped into that of Frege, we see that even in the bare chronological sense Bolzano marks a distinct point of historical inflection. That he does so with his actual work in mathematics as well must be shown after we have given a short sketch of his life.

NOTES TO I: THE HISTORICAL CONTEXT OF THE *PARADOXIEN*

(1) An opposite and no less characteristic view underlies the anecdote of the student reproved for defining a derivate as a limit and not as a velocity. See Alexander Russell: *Lord Kelvin*. London and Glasgow 1938, pp. 30–31.

(2) *Analysis per aequationes numero terminorum infinitas*. Extant 1669. published 1704. See heading of Chapter III.

(3) Letter to Holmboe, 16 January 1826.

(4) *Paradoxien des Unendlichen*, §37, p. 65.

(5) In the *Institutiones Calculi Differentialis* (1775), 500, the summatory formula for a series differs according as n is odd or even, but despite the reference to different n, successive couples are added, n apparently eliminates itself and the average is declared to be the sum to infinity. Another drastic example occurs in §19 of *Considerationes de seriebus quibusdam*, Werke (1) 14 (1925), 427.

(6) *Sur le temps de la chute d'un corps attiré vers un centre de forces en raison réciproque des distances*. Histoire de l'Académie de Berlin (1760), 250–260. See p. 256.

(7) *Mechanica I*, 108, Werke (2) 1 (1912), 88.

(8) *Mechanica I*, 315, Werke (2) 1 (1912), 258–259.

(9) The literature of this notorious controversy on the logarithms of negative and complex numbers fills an entire page in F. W. A. Murhard's *Bibliotheca Mathematica*, 2 (1798), 286; and a particular account of it was composed by B. F. Thibaut, Göttingen, 1797.

(10) Letter to Bessel, 18 December 1811, Auwers, p. 157. Gauss defines the logarithmic function as a complex integral and explains its multiple value by differing paths of integration.

(11) *Recherches sur différents points importants du systême du monde.* Paris 1 (1754), 50.

(12) *Anfangsgründe der Analysis des Unendlichen.* Göttingen, 1799. See the peroration to the preface, signed March 1761.

(13) *Dictionnaire philosophique*, article 'Géométrie,' at end.

(14) *Remarks on Mr. Euler's Treatise of Motion.* London 1739, pp. 1–29 on Euler, pp. 30–112 on others named in title.

(15) *Vera circuli et hyperbolae quadratura*, Padua 1668.

(16) *De limitibus quantitatum et rationum fragmentum.* Opera quaedam relicta, Glasgow 1776. See pp. 3–4.

(17) Letter to Leibniz, 19 November 1712. Gerhardt, *Leibnizens mathematische Schriften*, 4 (1859), 188.

(18) Letter to Johann Bernoulli, 25 October 1713. Gerhardt, *Leibnizens mathematische Schriften*, 3 (1885), 922–923.

(19) Opera posthuma, edited by F. H. and N. Fuss. Petrograd 1 (1862), 301.

(20) *Théorie des probabilités.* Paris 1820, p. cix.

(21) *Théorie des fonctions.* Paris 1796, §53.

(22) J. Bertrand: *D'Alembert.* Paris 1889, p. 56.

(23) *Opuscles mathématiques.* Paris 5 (1768), 183. The notion that Cesàro and Hölder and similar means be methods of capturing an elusive but pre-existent limit is a relic of eighteenth-century misplaced concreteness.

(24) Paris 1821, Chapter VI, pp. 114–152. Oeuvres (2) 3 (1897). The entry of 'la plus grande des limites' is, in comparison with Gauss, rather surreptitious.

(25) L. Schlesinger: *Ueber Gauss' Arbeiten zur Funktionentheorie.* Materialen für eine wissenschaftliche Biographie von Gauss, Heft III. Berlin, 1912. Full text on pp. 136–140.

(26) *Disquisitiones generales circa seriem hypergeometricam.* Göttinger Kommentarien 1812, Werke 2 (1876), 125–162. See in particular pp. 139 and 152.

(27) In his memoir on the binomial series, Crelle 1 (1826), 311–339. See pp. 316 and 337.

(28) E. Lampe: *Zur Biographie von Jakob Steiner.* Bibliotheca Mathematica, (3) 1 (1899), 129–141. See p. 134.

(29) *Introduction to Mathematics*. London 1919, pp. 155–157.

(30) 'Irrelevant, unnecessary, erroneous and self-contradictory': *Principles of Mathematics*. Cambridge 1903 (reprinted 1938), p. 345.

(31) *Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik*. Berlin (Springer) 1–4 (1929–1938). Contains some twenty-five historiographically rigorous studies in Greek mathematics by O. Becker, J. Klein, O. Neugebauer, J. Stenzel, O. Toeplitz and others.

(32) *Traité de calcul différentiel*. Paris 1 (1810), 11.

(33) *Théorie des séries*. Paris 1862, p. 6.

(34) *Traité de calcul différentiel*. Paris 1864, §229.

(35) J. M. C. Duhamel: *Des méthodes dans les sciences du raisonnement*. Paris 2 (1866), 413.

(36) *Transactions of the Cambridge Philosophical Society*, 11 (signed 28 February 1866), 242.