DIFFERENTIAL FORMS AND CONNECTIONS

R. W. R. DARLING

This book introduces the tools of modern differential geometry – exterior calculus, manifolds, vector bundles, connections – to advanced undergraduates and beginning graduate students in mathematics, physics, and engineering. It covers both classical surface theory and the modern theory of connections and curvature, and includes a chapter on applications to theoretical physics. The only prerequisites are multivariate calculus and linear algebra; no knowledge of topology is assumed.

The powerful and concise calculus of differential forms is used throughout. Through the use of numerous concrete examples, the author develops computational skills in the familiar Euclidean context before exposing the reader to the more abstract setting of manifolds. There are nearly 200 exercises, making the book ideal for both classroom use and self-study.

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Contents

Preface

Purpose

This book represents an extended version of my lecture notes for a one-semester course on differential geometry, aimed at students without knowledge of topology. Indeed the only prerequisites are a solid grasp of multivariate calculus and of linear algebra. The goal is to train advanced undergraduates and beginning graduate students in exterior calculus (including integration), covariant differentiation (including curvature calculations), and the identification and uses of submanifolds and vector bundles. It is hoped that this will serve both the minority who proceed to study advanced texts in differential geometry, and the majority who specialize in other subjects, including physics and engineering.

Summary of the Contents

Every generation since Newton has seen a richer and deeper presentation of the differential and integral calculus. The nineteenth century gave us vector calculus and tensor analysis, and the twentieth century has produced, among other things, the exterior calculus and the theory of connections on vector bundles. As the title implies, this book is based on the premise that differential forms provide a concise and efficient approach to many constructions in geometry and in calculus on manifolds.

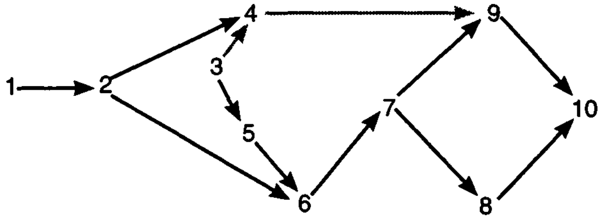
Chapter 1 is algebraic; Chapters 2, 4, 8, and 9 are mostly about differential forms; Chapters 4, 9, and 10 are about connections; and Chapters 3, 5, 6, and 7 are about underlying structures such as manifolds and vector bundles. The reader is not mistaken if he detects a strong influence of Harley Flanders's delightful 1989 text. I would also like to acknowledge that I have made heavy use of ideas from Berger and Gostiaux [1988], and (in Chapters 6 and 9) of my handwritten Warwick University 1981 lecture notes from John Rawnsley, as well as other standard differential geometry texts. Chapter 9 on connections is in the spirit of S. S. Chern [1989], p. ii, who remarks that "the notion of a connection in a vector bundle will soon find its way into a class on advanced calculus, as it is a fundamental notion and its applications are widespread"; these applications include the field theories of physics (see Chapter 10), the study of information loss in parametric statistics, and computer algorithms for recognizing surface deformation. Regrettably the Frobenius Theorem and its applications, and de Rham cohomology, are among many other topics which could not be included; see Flanders [1989] for an excellent treatment of the former, and Berger and Gostiaux [1988] for the latter.

Prerequisites

- **Linear Algebra:** finite-dimensional vector spaces and linear transformations, including the notions of image, kernel, rank, inner product, and determinant.
- **Vector Calculus:** derivative as a linear mapping; grad, div, and curl; line, surface, and volume integrals, including Green's Theorem and Stokes's Theorem; implicit function theorem; and the concept of an open set in Euclidean space.

Advice to the Instructor

In the diagram below, a solid arrow denotes dependency of chapters, and a fuzzy arrow denotes a conceptual relationship. In one semester, an instructor would probably be hard pressed to cover more than six chapters in depth. Chapters 1 and 2 are essential. Some instructors may choose to emphasize the easier and more concrete material in Chapters 3 and 4, which is used in the sequel only as a source of examples, while others may prefer to move rapidly into Chapters 5 and 6 so as to have time for Chapter 8 on integration and/or Chapter 9 on connections. Alternatively one could deemphasize abstract differential manifolds (i.e., skip over Chapter 5), cover only the "local vector bundle" part of Chapter 6, and treat Chapters 7 to 10 in a similarly "local" fashion. As always, students cannot expect to master the material without doing the exercises.



Acknowledgments and Comments

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1 Exterior Algebra

Anyone who has studied linear algebra and vector calculus may have wondered whether the notion of cross product of vectors in 3-dimensional space generalizes to higher dimensions. Exterior algebra, which is a prerequisite for the study of differential forms, shows that the answer is yes. We shall adopt a constructive approach to exterior algebra, following closely the presentation given in Flanders [1989], and we will try to emphasize the connection with the vector algebra notions of cross product and triple product (see Table 1.2 on page 19).

1.1 Exterior Powers of a Vector Space

1.1.1 The Second Exterior Power

Let V be an n-dimensional vector space over R. Elements of V will be denoted u, v, w, u_i, v^j, etc., and real numbers will be denoted a, b, c, a_i, b_j, etc. For $p = 0, 1, \ldots, n$, the pth exterior power of V, denoted $\Lambda^p V$, is a real vector space, whose elements are referred to as "***p*-vectors**." For $p = 0, 1$ the definition is straightforward: $\Lambda^0 V = R$, and $\Lambda^1 V = . V$, respectively. $\Lambda^2 V$, consists of formal sums[1]

$$\sum_i a_i (u_i \wedge v_i), \qquad (1.1)$$

where the "wedge product" $u \wedge v$ satisfies the following four rules:

$$(au + w) \wedge v = a(u \wedge v) + w \wedge v; \qquad (1.2)$$

---

[1] A rigorous construction of the second exterior power is given in Section 1.9.

$$u \wedge (bv + w) = b(u \wedge v) + u \wedge w; \tag{1.3}$$

$$u \wedge u = 0; \tag{1.4}$$

$$\{v^1, \ldots, v^n\} \text{ is a basis for } V \Rightarrow \{v^i \wedge v^j : 1 \le i < j \le n\} \text{ is a basis for } \Lambda^2 V. \tag{1.5}$$

Postponing to the end of this chapter the question of whether a vector space with these properties exists, let us note two immediate consequences of (1. 2), (1. 3), and (1. 4). Apply (1. 4) to $(u+v) \wedge (u+v)$, and then express the latter as the sum of four terms using (1. 2) and (1. 3); two of these terms, namely, $u \wedge u$ and $v \wedge v$, are zero, and what remains shows that $u \wedge v + v \wedge u = 0$; hence

$$v \wedge u = -u \wedge v. \tag{1.6}$$

Second (1. 2), (1. 3), and (1. 4) by themselves imply that, for any basis $\{v^1, \ldots, v^n\}$ of V, the set of vectors $\{v^i \wedge v^j : 1 \le i < j \le n\}$ spans $\Lambda^2 V$, because it spans the set of "generators" $\{u \wedge w, u \text{ and } w \in V\}$; to check this, we express u and w in terms of the basis $\{v^1, \ldots, v^n\}$, and apply (1. 2), (1. 3), and (1. 6) to obtain:

$$u \wedge w = \left(\sum a_i v^i\right) \wedge \left(\sum b_j v^j\right) = \sum_{i,j} a_i b_j (v^i \wedge v^j)$$

$$= \sum_{i<j} (a_i b_j - a_j b_i)\,(v^i \wedge v^j).$$

The linear independence of $\{v^i \wedge v^j : 1 \le i < j \le n\}$ cannot, however, be deduced from (1. 2), (1. 3), and (1. 4), and is studied in Section 1.9.

1.1.2 Higher Exterior Powers

The description of $\Lambda^p V$ for any $2 \le p \le n$ follows the same lines; $\Lambda^p V$ is the set of formal sums[2]

$$\sum_{\gamma} a_\gamma (u_{\gamma(1)} \wedge \ldots \wedge u_{\gamma(p)}) \tag{1.7}$$

of "generators" $u_{\gamma(1)} \wedge \ldots \wedge u_{\gamma(p)}$, where each coefficient a_γ is indexed by a multi-index $\gamma = (\gamma(1), \ldots, \gamma(p))$; elements of $\Lambda^p V$ are called "**p-vectors**," and are subject to the rules (1. 8), (1. 9), and (1. 10):

$$(av + w) \wedge u_2 \wedge \ldots \wedge u_p = a(v \wedge u_2 \wedge \ldots \wedge u_p) + w \wedge u_2 \wedge \ldots \wedge u_p, \tag{1.8}$$

---

[2] A rigorous construction is given in Section 1.9.

and similarly if any of the u_i is replaced by such a linear combination;

$$u_i = u_j \text{ for some } i \neq j \Rightarrow u_1 \wedge \ldots \wedge u_p = 0; \tag{1.9}$$

and for any basis $\{v^1, \ldots, v^n\}$ of V, the following set of p-vectors forms a basis for $\Lambda^p V$:

$$\{v^{i(1)} \wedge \ldots \wedge v^{i(p)}, 1 \leq i(1) < \ldots < i(p) \leq n\} \tag{1.10}$$

The expression $u_1 \wedge \ldots \wedge u_{r-1} \wedge (v+w) \wedge u_{r+1} \wedge \ldots \wedge (v+w) \wedge \ldots \wedge u_p$, which is zero by (1. 9), can be expanded using (1. 8) into four terms, two of which are zero; what remains shows that

$$u_1 \wedge \ldots \wedge u_p \text{ changes sign if any two entries are transposed.} \tag{1.11}$$

Also it follows from (1. 8) and (1. 9) that, for any basis $\{v^1, \ldots, v^n\}$ of V, the set of vectors (1. 10) spans $\Lambda^p V$; in order to demonstrate this, we shall need the language of permutations.

1.1.3 Permutations

Let Σ_p denote the set of permutations of the set $\{1, 2, \ldots, p\}$. For example, Σ_3 can be written as $\{e, (1,2), (3,1), (2,3), (1,3,2), (1,2,3)\}$, where $\pi = (3,1)$ means for example that $\pi(1) = 3, \pi(3) = 1$. A **transposition** is an element π of Σ_p that switches i and j for some $i \neq j$, but leaves k fixed for all $k \notin \{i, j\}$; thus in the list for Σ_3 above, the second, third, and fourth elements are transpositions. A result in algebra states that any permutation can be expressed as a composition of transpositions, and that the number m of transpositions is unique modulo 2; we define the **signature** sgn(π) of the permutation π by

$$\operatorname{sgn}(\pi) = (-1)^m. \tag{1.12}$$

It is also true, in the case of the composition $\pi \bullet \pi'$ of two permutations, that $\operatorname{sgn}(\pi \bullet \pi') = \operatorname{sgn}(\pi) \operatorname{sgn}(\pi')$. It follows from (1. 11) that

$$u_{\pi(1)} \wedge \ldots \wedge u_{\pi(p)} = \operatorname{sgn}(\pi) (u_1 \wedge \ldots \wedge u_p). \tag{1.13}$$

Now we will show how to express an arbitrary generator of $\Lambda^p V$ as a linear combination of the set of vectors (1. 10). We may write

$$\begin{aligned} u_1 \wedge \ldots \wedge u_p &= \Big(\sum_{j(1)} b_{1,j(1)} v^{j(1)}\Big) \wedge \ldots \wedge \Big(\sum_{j(p)} b_{p,j(p)} v^{j(p)}\Big) \\ &= \sum_J c_J (v^{j(1)} \wedge \ldots \wedge v^{j(p)}), \end{aligned}$$

where $J = (j(1), \ldots, j(p))$, and $c_J = b_{1,j(1)} \ldots b_{p,j(p)}$. For any J, there is a unique multi-index $I = (i(1), \ldots, i(p))$ such that $i(1) < \ldots < i(p)$, and a unique $\pi \in \Sigma_p$ such that $J = \pi(I)$, meaning that $(j(1), \ldots, j(p)) = (\pi(i(1)), \ldots, \pi(i(p)))$. Hence by (1. 13), we deduce

$$v^{j(1)} \wedge \ldots \wedge v^{j(p)} = \operatorname{sgn}(\pi)\,(v^{i(1)} \wedge \ldots \wedge v^{i(p)}),$$

and therefore

$$u_1 \wedge \ldots \wedge u_p = \sum_I \Big(\sum_\pi \operatorname{sgn}(\pi)\, c_{\pi(I)}\Big)\,(v^{i(1)} \wedge \ldots \wedge v^{i(p)}), \tag{1. 14}$$

where the first summation is over multi-indices I such that $i(1) < \ldots < i(p)$, and the second summation is over Σ_p. This completes the proof that the vectors (1. 10) span $\Lambda^p V$.

1.1.4 Calculating the Dimension of an Exterior Power

$$\dim(\Lambda^p V) = \frac{n!}{(n-p)!\,p!},\ 0 \le p \le n. \tag{1. 15}$$

Proof: For any basis $\{v^1, \ldots, v^n\}$ of V, the set of p-vectors

$$\{v^{i(1)} \wedge \ldots \wedge v^{i(p)},\ 1 \le i(1) < \ldots < i(p) \le n\}. \tag{1. 16}$$

forms a basis for $\Lambda^p V$, by (1. 10). The number of elements of this set is the number of ways of choosing p objects from n distinct objects, which is the expression shown. ¤

Let us illustrate these ideas by writing down bases for the exterior powers of R^3.

p	Basis for $\Lambda^p V$	Dimension
0	$\{1\}$	1
1	$\{e_1, e_2, e_3\}$	3
2	$\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}$	3
3	$\{e_1 \wedge e_2 \wedge e_3\}$	1

Table 1.1 Exterior powers of Euclidean 3-space

1.2 Multilinear Alternating Maps and Exterior Products

For any set V, the set-theoretic product $V \times \ldots \times V$ (p copies) simply means the set of ordered p-tuples $(u_1, \ldots, u_p)$ where each $u_i \in V$. If V and W are vector spaces, a mapping $h: V \times \ldots \times V \to W$ is called:

- **Multilinear** if $h(au + bu', u_2, \ldots, u_p) = ah(u, u_2, \ldots, u_p) + bh(u', u_2, \ldots, u_p)$, and similarly for the other $(p-1)$ entries of h; h is called **bilinear** if p is 2;
- **Antisymmetric** (or **alternating**) if

$$h(u_{\pi(1)}, \ldots, u_{\pi(p)}) = \operatorname{sgn}(\pi)\, h(u_1, \ldots, u_p), \pi \in \Sigma_p, \tag{1.17}$$

which implies $h(u_1, \ldots, u_p) = 0$ if $u_i = u_j$, some $i \neq j$; for when $u_i = u_j$, some $i \neq j$, transposing the ith and jth entries shows that $h(u_1, \ldots, u_p)$ is the same as its negative.

The student will have encountered the following examples of multilinear alternating maps in linear algebra or vector calculus courses:

$$(u,v) \to u \times v,\ R^3 \times R^3 \to R^3;$$

$$(u,v) \to \left|\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}\right|,\ R^2 \times R^2 \to R;$$

$$(u, v, w) \to u \cdot (v \times w),\ R^3 \times R^3 \times R^3 \to R.$$

The linear maps from V to W will be denoted $L(V \to W)$, and the multilinear alternating maps will be denoted $A_p(V \to W)$. The following property of exterior powers will play a central role in the remainder of this chapter.

1.2.1 Universal Alternating Mapping Property

To every $g \in A_p(V \to W)$, *there corresponds a unique* $\hat{g} \in L(\Lambda^p V \to W)$ *such that*

$$\hat{g}(u_1 \wedge \ldots \wedge u_p) = g(u_1, \ldots, u_p),\ \forall u_1, \ldots, u_p; \tag{1.18}$$

in other words, a unique $\hat{g}$ *such that the following diagram commutes.*[3]

---

[3] A diagram is said to commute if following any sequence of arrows from one set to another yields the same mapping.

$$\begin{tikzcd}
V \times \ldots \times V \arrow[rr, "(u_1, \ldots, u_p) \rightarrow u_1 \wedge \ldots \wedge u_p"] \arrow[dr, "g"'] & & \Lambda^p V \arrow[dl, "\hat{g}"] \\
& W &
\end{tikzcd}$$

Proof: Deferred to Section 1.9.

1.2.2 Exterior Products

There exists a unique bilinear map $(\lambda,\mu) \rightarrow \lambda \wedge \mu$ from $\Lambda^p V \times \Lambda^q V$ to $\Lambda^{p+q} V$, whose effect on generators is that

$$(u_1 \wedge \ldots \wedge u_p) \wedge (w_1 \wedge \ldots \wedge w_q) = u_1 \wedge \ldots \wedge u_p \wedge w_1 \wedge \ldots \wedge w_q. \tag{1.19}$$

To see that this is true, apply 1.2.1 twice: first to the multilinear, alternating map

$$(u_1, \ldots, u_p) \rightarrow u_1 \wedge \ldots \wedge u_p \wedge w_1 \wedge \ldots \wedge w_q,$$

for fixed $w_1 \wedge \ldots \wedge w_q$, so as to obtain a unique $f \in L(\Lambda^p V \rightarrow \Lambda^{p+q} V)$ such that

$$f(u_1 \wedge \ldots \wedge u_p) = u_1 \wedge \ldots \wedge u_p \wedge w_1 \wedge \ldots \wedge w_q,$$

so that we may define

$$\lambda \wedge (w_1 \wedge \ldots \wedge w_q) = f(\lambda);$$

and second to the multilinear, alternating map

$$(w_1, \ldots, w_q) \rightarrow \lambda \wedge (w_1 \wedge \ldots \wedge w_q),$$

for fixed λ, so as to obtain $g_\lambda \in L(\Lambda^q V \rightarrow \Lambda^{p+q} V)$ such that

$$g_\lambda(w_1 \wedge \ldots \wedge w_q) = \lambda \wedge (w_1 \wedge \ldots \wedge w_q). \tag{1.20}$$

Finally the **exterior product** of $\lambda \in \Lambda^p V$ and $\mu \in \Lambda^q V$ is defined by $\lambda \wedge \mu = g_\lambda(\mu)$. The properties of the exterior product, the first two of which are immediate from the preceding construction, are:

- $(\lambda,\mu) \rightarrow \lambda \wedge \mu$ is **distributive** over addition and scalar multiplication;
- **associativity**: $(\lambda \wedge \mu) \wedge \nu = \lambda \wedge (\mu \wedge \nu)$;
- $\mu \wedge \lambda = (-1)^{pq}(\lambda \wedge \mu)$, so two vectors of odd degrees **anticommute**; otherwise the vectors commute.

The last property follows from Exercise 4 below, in the case where λ, μ are generators, and in general from linearity. In order to obtain a practical grasp of exterior products, try Exercises 5 and 6 below.

1.2.3 Example

Suppose V is 4-dimensional with a basis $\{v^1, v^2, v^3, v^4\}$. Then

$$\begin{aligned}(a(v^3 \wedge v^4) + b(v^1 \wedge v^3)) \wedge (c(v^1 \wedge v^2) + d(v^1 \wedge v^4)) &= ac(v^3 \wedge v^4 \wedge v^1 \wedge v^2) \\ &= (-1)^{2(2)} ac(v^1 \wedge v^2 \wedge v^3 \wedge v^4).\end{aligned}$$

1.3 Exercises

1. (a) Repeat Table 1.1 for the case of R^4, using the basis $\{e_1, e_2, e_3, e_4\}$.

   (b) Let $u = ae_1 + ce_3$, $v = be_2 + de_4$; express $u \wedge v$ in terms of your basis of $\Lambda^2 R^4$.

   (c) Let $w = a'e_1 + b'e_2$; express $u \wedge v \wedge w$ in terms of your basis of $\Lambda^3 R^4$.

   (d) Express $u \wedge v \wedge w \wedge e_3$ in terms of your basis of $\Lambda^4 R^4$.

2. Verify that, when $V = R^3$, the cross product $(u,v) \to u \times v$, $R^3 \times R^3 \to R^3$, and the triple product $(u, v, w) \to u \cdot (v \times w)$, $R^3 \times R^3 \times R^3 \to R$, are multilinear, alternating maps.[4]

   Reminder: The cross product of $u = (a_1, a_2, a_3)$ and $v = (b_1, b_2, b_3)$ is

   $$u \times v = \left|\begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix}\right| e_1 - \left|\begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix}\right| e_2 + \left|\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}\right| e_3, \qquad (1.21)$$

   and the triple product satisfies

   $$u \cdot (v \times w) = v \cdot (w \times u) = w \cdot (u \times v) = -(w \cdot (v \times u)). \qquad (1.22)$$

3. Decompose the permutation $(6, 4, 3, 2, 1, 5) \in \Sigma_6$ into a product of transpositions in two different ways, and show that the number of transpositions used is the same modulo 2 in both cases.

4. Prove, by induction or otherwise, that a permutation which sends $(1, 2, \ldots, p+q)$ into $(q+1, \ldots, q+p, 1, 2, \ldots, q)$ has signature

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[4] By the end of this chapter, the reader will realize that, in terms of the star operator discussed in Section 1.7 below, $u \times v = *(u \wedge v)$, and $u \cdot (v \times w) = *(u \wedge v \wedge w)$.

$$\operatorname{sgn}(\pi) = (-1)^{pq}. \tag{1.23}$$

Hint: A possible inductive hypothesis H_k is that whenever $p \geq 1, q \geq 1, p+q \leq k$, then the assertion above holds. To prove H_{k+1} from H_k, start by transposing q and $q+p$, and then rearrange $(q, 1, \ldots, q-1)$ so that H_k can be applied to the first $p+q-1$ entries.

5. Let $V = R^3$, with any basis $\{v^1, v^2, v^3\}$; show that

$$(a(v^2 \wedge v^3) + b(v^3 \wedge v^1) + c(v^1 \wedge v^2)) \wedge (\tilde{a}v^1 + \tilde{b}v^2 + \tilde{c}v^3)$$
$$= (a\tilde{a} + b\tilde{b} + c\tilde{c})\, v^1 \wedge v^2 \wedge v^3. \tag{1.24}$$

6. Suppose V is 4-dimensional with a basis $\{v^1, v^2, v^3, v^4\}$. Express the following as multiples of $v^1 \wedge v^2 \wedge v^3 \wedge v^4$:

   (i) $(a(v^1 \wedge v^3) + b(v^2 \wedge v^4)) \wedge (c(v^1 \wedge v^3) + d(v^2 \wedge v^4))$;

   (ii) $(av^1 + bv^4) \wedge (c(v^1 \wedge v^2 \wedge v^3) + d(v^2 \wedge v^3 \wedge v^4))$.

7. The setting is the same as for Exercise 6. Suppose $\mu \in \Lambda^3 V$, $\mu \neq 0$. Characterize the vectors $u \in V$ such that $u \wedge \mu = 0$, and show that the vector space consisting of such u is of dimension 3.

   Hint: Write $u = u_1 v^1 + \ldots + u_4 v^4$, and express μ similarly in terms of the four basis elements of the third exterior power. Obtain a linear relation on the coefficients of u.

8. This is a generalization of Exercise 7. Suppose V is n-dimensional, and μ is an arbitrary nonzero element of $\Lambda^{n-1} V$. Prove that the subspace W^{μ} of elements u of V such that $u \wedge \mu = 0$ is of dimension $n-1$, and deduce from this that there exist vectors $w^1, \ldots, w^{n-1}$ in V such that $\mu = w^1 \wedge \ldots \wedge w^{n-1}$.

   Hint: For the last part, take a basis for W^{μ}, extend it to a basis for V, and express μ in terms of the corresponding basis of $\Lambda^{n-1} V$. **Warning:** This kind of representation does not generally hold for elements of the other exterior powers.

1.4 Exterior Powers of a Linear Transformation

1.4.1 Determinants

Given $A \in L(V \to V)$, define $g_A \colon V^n \to \Lambda^n V \cong R$ by

$$g_A(u_1, \ldots, u_n) = (Au_1) \wedge \ldots \wedge (Au_n). \tag{1.25}$$

It follows immediately from the last equation that g_A is multilinear and antisymmetric, and so, by 1.2.1, there is a unique $f_A \in L(\Lambda^n V \to \Lambda^n V)$ such that

$$f_A(u_1 \wedge \dots \wedge u_n) = (Au_1) \wedge \dots \wedge (Au_n). \tag{1.26}$$

Since $\Lambda^n V$ is one-dimensional and f is linear, it follows that f is simply multiplication by a scalar, which we denote by $|A|$, the **determinant** of A. In other words,

$$|A|\,(u_1 \wedge \dots \wedge u_n) = (Au_1) \wedge \dots \wedge (Au_n). \tag{1.27}$$

It is somewhat surprising to discover that this abstract formulation refers to the same notion of determinant that the student has encountered in matrix algebra:

1.4.2 Formula for the Determinant of a Matrix

Suppose that, in terms of a basis $\{v^1, \dots, v^n\}$ for V, A has the matrix representation $A = (a_{ij})_{1 \le i,j \le n}$ (a_{ij} may also be written $a_{i,j}$). Then taking

$$u_i = \sum_j a_{ij} v^j$$

gives, as in (1. 14),

$$\begin{aligned} u_1 \wedge \dots \wedge u_n &= \Big(\sum_{j(1)} a_{1,j(1)} v^{j(1)}\Big) \wedge \dots \wedge \Big(\sum_{j(n)} a_{n,j(n)} v^{j(n)}\Big), \\ &= \sum_J a_{1,j(1)} \cdots a_{n,j(n)} \,(v^{j(1)} \wedge \dots \wedge v^{j(n)}), \end{aligned}$$

where $J = (j(1), \dots, j(n))$. Any J with two entries the same makes no contribution to the sum, by (1. 9). In all other cases there is a unique $\pi \in \Sigma_n$ such that $(j(1), \dots, j(n)) = (\pi(1), \dots, \pi(n))$. Hence by (1. 13), we deduce

$$v^{j(1)} \wedge \dots \wedge v^{j(n)} = \operatorname{sgn}(\pi)\,(v^1 \wedge \dots \wedge v^n),$$

$$u_1 \wedge \dots \wedge u_n = \left(\sum_{\pi \in \Sigma_n} \operatorname{sgn}(\pi)\, a_{1,\pi(1)} \cdots a_{n,\pi(n)}\right) v^1 \wedge \dots \wedge v^n. \tag{1.28}$$

Thus the formula for the determinant of the matrix is

$$|A| = \sum_{\pi \in \Sigma_n} \operatorname{sgn}(\pi)\, a_{1,\pi(1)} \cdots a_{n,\pi(n)}. \tag{1.29}$$

For example, when $n = 2$,

$$\left|\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right| = a_{11}a_{22} - a_{12}a_{21} = \sum_{\pi \in \Sigma_2} \operatorname{sgn}(\pi)\, a_{1,\pi(1)} a_{2,\pi(2)}.$$

1.4.3 Other Exterior Powers of a Linear Transformation

A generalization of the notion of determinant is that of **exterior powers**[5] **of a linear transformation** $A \in L(V \to W)$. The map $V^p \to \Lambda^p W$ given by

$$(u_1, \ldots, u_p) \to (Au_1) \wedge \ldots \wedge (Au_p)$$

is multilinear and alternating, and so by 1.2.1 it defines an element of $L(\Lambda^p V \to \Lambda^p W)$ denoted $\Lambda^p A$, called the **exterior *p*th power** of A; in other words, $\Lambda^p A$ is specified by its action on generators as follows:

$$\Lambda^p A(u_1 \wedge \ldots \wedge u_p) = (Au_1) \wedge \ldots \wedge (Au_p). \qquad \textbf{(1.30)}$$

The matrix representation of $\Lambda^p A$ may be obtained as follows. If $\{v^1, \ldots, v^n\}$ is a basis for V, and $\{w^1, \ldots, w^m\}$ for W, then $\{\sigma^I\}$ and $\{\tau^K\}$ are bases for $\Lambda^p V$ and $\Lambda^p W$, respectively, where

$$\sigma^I = v^{i(1)} \wedge \ldots \wedge v^{i(p)},\ 1 \le i(1) < \ldots < i(p) \le n; \qquad \textbf{(1.31)}$$

$$\tau^K = w^{k(1)} \wedge \ldots \wedge w^{k(p)},\ 1 \le k(1) < \ldots < k(p) \le m. \qquad \textbf{(1.32)}$$

If $Av^i = \sum_k a^i_k w^k$, then

$$\begin{aligned}(\Lambda^p A)\sigma^I &= (Av^{i(1)}) \wedge \ldots \wedge (Av^{i(p)}) \\ &= \sum_J a^{i(1)}_{j(1)} \cdots a^{i(p)}_{j(p)} (w^{j(1)} \wedge \ldots \wedge w^{j(p)}),\end{aligned}$$

where J runs through the set of all multi-indices. As usual, summands where $j(r) = j(s)$ for some $r \ne s$ are zero, and we express the other summands as in the steps preceding (1. 14): there is a unique $K = (k(1), \ldots, k(p))$ such that $k(1) < \ldots < k(p)$, and a unique $\pi \in \Sigma_p$ such that $J = \pi(K)$, meaning that $(j(1), \ldots, j(p)) = (\pi(k(1)), \ldots, \pi(k(p)))$. Since

$$w^{j(1)} \wedge \ldots \wedge w^{j(p)} = \operatorname{sgn}(\pi)\,(w^{k(1)} \wedge \ldots \wedge w^{k(p)}),$$

we obtain:

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[5] This idea is needed in calculations related to the pullback of differential forms in Chapter 2, and is also relevant to Stokes's Theorem in Chapter 8.

$$(\Lambda^p A)\,\sigma^I = \sum_K \left(\sum_{\pi \in \Sigma_p} \operatorname{sgn}(\pi)\, a^{i(1)}_{\pi(k(1))} \cdots a^{i(p)}_{\pi(k(p))}\right)(w^{k(1)} \wedge \ldots \wedge w^{k(p)}) \tag{1.33}$$

$$= \sum_K a^I_K \tau^K,$$

and so $\Lambda^p A$ is represented by the matrix (a^I_K) of all the $p \times p$ minors of A, where

$$a^I_K = \sum_{\pi \in \Sigma_p} \operatorname{sgn}(\pi)\, a^{i(1)}_{\pi(k(1))} \cdots a^{i(p)}_{\pi(k(p))}. \tag{1.34}$$

An opportunity to evaluate this matrix when $m = n = 3, p = 2$, is provided in Exercise 10 below. This construction generalizes the notion of determinant because, when $V = W$ and $p = n$, then $\Lambda^n A$ has the effect of multiplication by $|A|$.

1.4.4 The Isomorphism $\Lambda^p(V^*) \cong (\Lambda^p V)^*$

Recall that the **dual space** $V^* = L(V \to R)$ of the n-dimensional vector space V is another n-dimensional vector space, consisting of the linear mappings from V to R, which are called **linear forms**. It is often helpful, though not necessary, to conceptualize elements of V as n-dimensional column vectors, and elements of V^* as n-dimensional row vectors which act on the column vectors by usual matrix multiplication.

Given linear forms $\psi_1, \ldots, \psi_p \in V^*$, where $p \le n$, the isomorphism (constructed below) will show that $\psi_1 \wedge \ldots \wedge \psi_p \in \Lambda^p(V^*)$ acts linearly on $\Lambda^p V$ as follows:

$$(\psi_1 \wedge \ldots \wedge \psi_p) \cdot (u_1 \wedge \ldots \wedge u_p) = \sum_{\pi \in \Sigma_p} \operatorname{sgn}(\pi)\, \psi_1(u_{\pi(1)}) \cdots \psi_p(u_{\pi(p)}). \tag{1.35}$$

1.4.4.1 Examples

When $p = 2$ and when $p = 3$, respectively,

$$(\varphi \wedge \psi) \cdot (u \wedge v) = \varphi(u)\psi(v) - \varphi(v)\psi(u); \tag{1.36}$$

$$(\psi_1 \wedge \psi_2 \wedge \psi_3) \cdot (u_1, u_2, u_3) = |(\psi_i(u_j))|. \tag{1.37}$$

1.4.4.2 Constructing the Isomorphism

Given linear forms $\psi_1, \ldots, \psi_p \in V^*$, where $p \le n$, consider the mapping $A \in L(V \to R^p)$ given by

$$Au = \psi_1(u)\,e_1 + \ldots + \psi_p(u)\,e_p, \tag{1.38}$$

where $\{e_1, \ldots, e_p\}$ is the standard basis for R^p. Referring to (1. 30), we see that the range of $\Lambda^p A$ is the one-dimensional space $\Lambda^p R^p$ spanned by $e_1 \wedge \ldots \wedge e_p$; therefore there exists a unique linear form, temporarily denoted

$$\psi_1 \Diamond \ldots \Diamond \psi_p \in (\Lambda^p V)^*, \tag{1.39}$$

given by the equation

$$\Lambda^p A(\lambda) = (\psi_1 \Diamond \ldots \Diamond \psi_p)(\lambda)(e_1 \wedge \ldots \wedge e_p), \lambda \in \Lambda^p V. \tag{1.40}$$

The reader may verify using (1. 33) (see Exercise 11 below) that

$$(\psi_1 \Diamond \ldots \Diamond \psi_p)(u_1 \wedge \ldots \wedge u_p) = \sum_{\pi \in \Sigma_p} \operatorname{sgn}(\pi) \psi_1(u_{\pi(1)}) \ldots \psi_p(u_{\pi(p)}), \tag{1.41}$$

and also that the map $\psi_1 \wedge \ldots \wedge \psi_p \to \psi_1 \Diamond \ldots \Diamond \psi_p$ is linear and one-to-one (see Exercise 12). Since the dimension of $\Lambda^p(V^*)$ is the same as that of $(\Lambda^p V)^*$, this establishes an isomorphism from $\Lambda^p(V^*)$ to $L(\Lambda^p V \to R)$. ¤

In subsequent chapters, we shall drop the $\Diamond$ notation, and identify $\psi_1 \Diamond \ldots \Diamond \psi_p$ with $\psi_1 \wedge \ldots \wedge \psi_p$. Thus equation (1. 35) replaces (1. 41).

1.5 Exercises

9. (a) Show using (1. 27) that if $A, B \in L(V \to V)$, then $|AB| = |A||B|$.

(b) Show using (1. 30) that if $B \in L(V \to W)$ and $A \in L(W \to Y)$, then

$$\Lambda^p(AB) = \Lambda^p(A)\Lambda^p(B).$$

10. Suppose $V = R^3$, and $A \in L(R^3 \to R^3)$ is expressible in terms of the usual basis $\{e_1, e_2, e_3\}$ as the matrix

$$\begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

for some real number φ. Express $\Lambda^2 A$ as a 3×3 matrix with respect to the basis $\{e_2 \wedge e_3, e_3 \wedge e_1, e_1 \wedge e_2\}$.

11. Verify the formula (1. 41), using (1. 40).

12. Show that the map $\psi_1 \wedge \ldots \wedge \psi_p \to \psi_1 \Diamond \ldots \Diamond \psi_p$, appearing in (1. 40), is linear and one-to-one.

Hint: To show the map is one-to-one, note that by (1. 41), $\psi_1 \Diamond \ldots \Diamond \psi_p$ is zero if and only if $\{\psi_1, \ldots, \psi_p\}$ is linearly independent; now appeal to (1. 10).

13. Show that, for general finite-dimensional vector spaces V and W, the spaces $\Lambda^p L(V \to W)$ and $L(\Lambda^p V \to W)$ are not necessarily isomorphic.

14. Show that the exterior powers of a linear transformation $A \in L(V \to W)$ satisfy

$$(\Lambda^{p+q}A)(\lambda \wedge \mu) = \Lambda^p A(\lambda) \wedge \Lambda^q A(\mu) \tag{1.42}$$

for any $p + q \le n$, $\lambda \in \Lambda^p V$, $\mu \in \Lambda^q V$, by applying (1. 30) to generators, and using the associativity of the exterior product.

15. For $p, q \ge 1$, let $\Sigma_{p,q}$ denote the set of permutations π of $(1, 2, \ldots, p+q)$ such that $\pi(1) < \ldots < \pi(p)$, $\pi(p+1) < \ldots < \pi(p+q)$ (think of splitting the top p cards from a deck of $p+q$ cards, and shuffling them in the usual way into the bottom q cards — there are $(p+q)!/(p!q!)$ such permutations). Notice that associativity of the exterior product implies that the image of the exterior product of

$$(\varphi_1 \wedge \ldots \wedge \varphi_p) \in \Lambda^p(V^*) \text{ and } (\psi_1 \wedge \ldots \wedge \psi_q) \in \Lambda^q(V^*)$$

under the isomorphism 1.4.4 must satisfy

$$\begin{aligned}(\varphi_1 \wedge \ldots \wedge \varphi_p) \wedge (\psi_1 \wedge \ldots \wedge \psi_q) &\to (\varphi_1 \lozenge \ldots \lozenge \varphi_p) \lozenge (\psi_1 \lozenge \ldots \lozenge \psi_q) \\ &= \varphi_1 \lozenge \ldots \lozenge \varphi_p \lozenge \psi_1 \lozenge \ldots \lozenge \psi_q\end{aligned}$$

(see (1. 39) for the notation). Prove that $\lozenge$ extends to a map

$$\lozenge : L(\Lambda^p V \to R) \times L(\Lambda^q V \to R) \to L(\Lambda^{p+q} V \to R);$$

$$(h \lozenge l)(u_1 \wedge \ldots \wedge u_{p+q}) \tag{1.43}$$

$$= \sum_{\pi \in \Sigma_{p,q}} \operatorname{sgn}(\pi) h(u_{\pi(1)} \wedge \ldots \wedge u_{\pi(p)}) l(u_{\pi(p+1)} \wedge \ldots \wedge u_{\pi(p+q)}).$$

Hint: Use Exercise 14.

1.6 Inner Products

1.6.1 Definition of an Inner Product

An **inner product** on a vector space V is a map $V \times V \to R$, denoted $\langle \cdot | \cdot \rangle$, with:

- **Bilinearity:** $u \to \langle u|v\rangle$ is linear for every v, and $v \to \langle u|v\rangle$ is linear for every u;
- **Symmetry:** $\langle u|v\rangle = \langle v|u\rangle$;
- **Nondegeneracy:** If z satisfies $\langle z|u\rangle = 0, \forall u$, then $z = 0$.

Note that this definition is a little more general than the one often given in linear algebra courses, since it is **not** assumed that $\langle u|u\rangle \geq 0$.

1.6.1.1 Characterization of Nondegeneracy

If $\{v^1, \ldots, v^n\}$ *is any basis for V, the nondegeneracy condition is equivalent to:*

$$\left|\begin{bmatrix} \langle v^1|v^1\rangle & \ldots & \langle v^1|v^n\rangle \\ \ldots & \ldots & \ldots \\ \langle v^n|v^1\rangle & \ldots & \langle v^n|v^n\rangle \end{bmatrix}\right| \neq 0. \tag{1.44}$$

Proof: To check that this condition is sufficient, take any z which satisfies $\langle z|u\rangle = 0, \forall u$. Let us expand z in terms of the basis as $z = a_1 v^1 + \ldots + a_n v^n$. Taking inner products with each v^i in turn gives the system of linear equations:

$$\sum_j a_j \langle v^j|v^i\rangle = 0, i = 1, \ldots, n. \tag{1.45}$$

Condition (1. 44) implies that the matrix $(\langle v^j|v^i\rangle)$ is invertible, and hence the only solution to (1. 45) is for all the a_j to be zero, showing that $z = 0$. Proof of the converse is left as an exercise. ¤

1.6.2 Examples

- The **dot product** in R^n.

$$\langle (a_1, \ldots, a_n) | (b_1, \ldots, b_n) \rangle = (a_1, \ldots, a_n) \cdot (b_1, \ldots, b_n) = a_1 b_1 + \ldots + a_n b_n.$$

- The **Lorentz inner product** in R^4: if c denotes the speed of light,

$$\langle (a_1, \ldots, a_4) | (b_1, \ldots, b_4) \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3 - c^2 a_4 b_4. \tag{1.46}$$

1.6.3 Orthonormal Bases and Their Signatures

It follows from the axioms that every inner product space contains an element v such that

$$\langle v|v\rangle = \pm 1; \tag{1.47}$$

for if $\langle z|z\rangle = 0$ for all $z \in V$, then

$$2\langle u|w\rangle = \langle u + w|u + w\rangle - \langle u|u\rangle - \langle w|w\rangle = 0$$

for every u and w, which contradicts nondegeneracy; so take some z with $a = \langle z|z\rangle \neq 0$, and let $v = |a|^{-1/2} z$. A basis $\{v^1, \ldots, v^n\}$ for V is called an **orthonormal basis** if

$$\langle v^i|v^j\rangle = 0, i \neq j \; ; \langle v^i|v^i\rangle = \pm 1, i = 1, \ldots, n. \tag{1. 48}$$

An induction argument, suggested in Exercise 23 below, shows that every inner product space has an orthonormal basis. Moreover if there are r plus signs and $s = n - r$ minus signs in (1. 48), $t = r - s$ is called the **signature** of the inner product space; this does not depend on the choice of orthonormal basis (see Exercise 24).

A useful property of inner product spaces is the following.

1.6.4 Linear Forms on an Inner Product Space

Every $f \in L(V \to R)$ *is of the form* $f(\cdot) = \langle \cdot|u\rangle$ *for some* $u \in V$.

Proof: Take $u = f(v^1)v^1 + \ldots + f(v^n)v^n$, using the orthonormal basis in (1. 48); then for any $w = a_1 v^1 + \ldots + a_n v^n$,

$$f(w) = \sum_j a_j f(v^j) = \sum_j (a_j)\langle v^j|u\rangle = \langle w|u\rangle. \qquad ¤$$

1.6.5 Inner Products on Exterior Powers

Suppose V has an inner product $\langle\cdot|\cdot\rangle$. Then there exists a bilinear mapping $\langle\cdot|\cdot\rangle_p$ from $\Lambda^p V \times \Lambda^p V$ to R, characterized by the formula

$$\langle u_1 \wedge \ldots \wedge u_p | v_1 \wedge \ldots \wedge v_p\rangle_p = \left\| \begin{bmatrix} \langle u_1|v_1\rangle & \ldots & \langle u_1|v_p\rangle \\ \ldots & \ldots & \ldots \\ \langle u_p|v_1\rangle & \ldots & \langle u_p|v_p\rangle \end{bmatrix} \right\| . \tag{1. 49}$$

To see that this is so, note that the determinant on the right is multilinear and alternating in $(u_1, \ldots, u_p)$ and in $(v_1, \ldots, v_p)$, respectively, and use 1.2.1 twice as in the construction of the exterior product in Section 1.2.2. Clearly $\langle\cdot|\cdot\rangle_p$ is symmetric, because transposing the matrix in (1. 49) does not change the value of its determinant.

1.6.5.1 An Orthonormal Basis for an Exterior Power

$\langle\cdot|\cdot\rangle_p$ *is an inner product on* $\Lambda^p V$. *If* $\{v^1, \ldots, v^n\}$ *is an orthonormal basis for V, I is an ascending multi-index (i.e.,* $1 \le i(1) < \ldots < i(p) \le n$*), and*

$$\sigma^I = v^{i(1)} \wedge \ldots \wedge v^{i(p)}, \tag{1. 50}$$

then $\{\sigma^I\}$*, as I ranges over ascending multi-indices, is an orthonormal basis of* $\Lambda^p V$.

Proof: To show that $\langle\cdot|\cdot\rangle_p$ is an inner product on $\Lambda^p V$, it only remains to show that it is nondegenerate. We know from (1. 10) that the $\{\sigma^I\}$ form a basis for $\Lambda^p V$, where now $\{v^1, \ldots, v^n\}$ is an orthonormal basis for V. Nondegeneracy follows from (1. 44) once we