GLOBAL EDITION

Tenth Edition

PRECALCULUS

GRAPHICAL, NUMERICAL, ALGEBRAIC

Demana • Waits • Foley • Kennedy • Bock

Precalculus

Graphical, Numerical, Algebraic

Tenth Edition
Global Edition

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FOREWORD

We are proud of the fact that earlier editions of *Precalculus: Graphical, Numerical, Algebraic* were among the first to recognize the potential of hand-held graphers for helping students understand function behavior. The power of visualization eventually transformed the teaching and learning of calculus at the college level and in the AP® program, then led to reforms in the high school curriculum articulated in the NCTM *Principles and Standards for School Mathematics* and more recently in the Common Core State Standards. All along the way, this text has kept current with the best practices while continuing to pioneer new ideas in exploration and pedagogy that enhance student learning (for example, the study of function behavior based on the Twelve Basic Functions, an idea that has gained widespread acceptance in the text world).

For those students continuing to a calculus course, this precalculus text concludes with a chapter that prepares students for the two central themes of calculus: instantaneous rate of change and continuous accumulation. This intuitively appealing preview of calculus is both more useful and more reasonable than the traditional, unmotivated foray into the computation of limits, and it is more in keeping with the stated goals and objectives of the AP courses and their emphasis on depth of knowledge.

Recognizing that precalculus is a capstone course for many students, we include *quantitative literacy* topics such as probability, statistics, and the mathematics of finance and integrate the use of data and modeling throughout the text. Our goal is to provide students with the critical-thinking skills and mathematical know-how needed to succeed in college, career, or any endeavor.

Continuing in the spirit of the nine earlier editions, we have integrated graphing technology throughout the course, not as an additional topic but as an essential tool for both mathematical discovery and effective problem solving. Graphing technology enables students to study a full catalog of basic functions at the beginning of the course, thereby giving them insights into function properties that are not seen in many texts until later chapters. By connecting the algebra of functions to the visualization of their graphs, we are even able to introduce students to parametric equations, piecewise-defined functions, limit notation, and an intuitive understanding of continuity as early as Chapter 1. However, the advances in technology and increased familiarity with calculators have blurred some of the distinctions between solving problems and supporting solutions that we had once assumed to be apparent. Therefore, we ask that some exercises be solved without calculators. (See the Technology and Exercises section of the Preface.)

Once students are comfortable with the language of functions, the text guides them through a more traditional exploration of twelve basic functions and their algebraic properties, always reinforcing the connections among their algebraic, graphical, and numerical representations. This text uses a consistent approach to modeling, emphasizing the use of particular types of functions to model behavior in the real world. Modeling is a fundamental aspect of our problem-solving process that is introduced in Section 1.1 and used throughout the text. The text has a wealth of data and range of applications to illustrate how mathematics and statistics connect to every facet of modern life. Each chapter, 1–11, concludes with a modeling project to reinforce and extend students' ability to solve modeling problems.

This text has faithfully incorporated not only the teaching strategies that have made *Calculus: Graphical, Numerical, Algebraic* so popular, but also some of the strategies from the popular Pearson high school algebra series, and thus has produced a seamless pedagogical transition from prealgebra through calculus for students. Although this book can certainly be appreciated on its own merits, teachers who seek coherence and vertical alignment in their mathematics sequence might consider this pedagogical approach to be an additional asset of *Precalculus: Graphical, Numerical, Algebraic*.

This text is written to address current and emerging state curriculum standards. In particular, we embrace NCTM's *Focus in High School Mathematics: Reasoning and Sense Making* and its emphasis on the importance of helping students to make sense of mathematics and to reason using mathematics. The NCTM's *Principles and Standards for School Mathematics* identified five "Process Standards" that should be fundamental in mathematics education. The first of these standards was Problem Solving. Since then, the emphasis on problem solving has continued to grow, to the point that it is now integral to the instructional process in many mathematics classrooms. When the Common Core State Standards for Mathematics detailed eight "Standards for Mathematical Practice" that should be fundamental in mathematics education, again the first of these addressed problem solving. Individual states have also released their own standards over the years, and problem solving is invariably front and center as a fundamental objective. Problem solving, reasoning, sense making, and the related processes and practices of mathematics are central to the approach we use in *Precalculus: Graphical, Numerical, Algebraic*.

We embrace the growing importance and wide applicability of Statistics. Because Statistics is increasingly used in college coursework, the workplace, and everyday life, we include a full chapter on Statistics to help students see that statistical analysis is an investigative process that turns loosely formed ideas into scientific studies. Our five sections on data analysis, probability, and statistical literacy are aligned with the *GAISE* Report published by the American Statistical Association, the College Board's AP® Statistics curriculum, and the Common Core State Standards. Chapter 10 is not intended as a course in statistics but rather as an introduction to set the stage for possible further study.

Dedication

We dedicate this text to the memory of

our eminent colleague, dear friend, and inspirational coauthor

***Bert K. Waits** (1940–2014).*

With his passing, the mathematics community lost a uniquely talented leader.

May he rest in peace, and may the power of visualization*,*

which he passionately promoted, live on!

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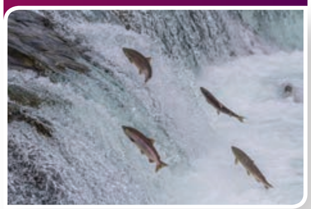

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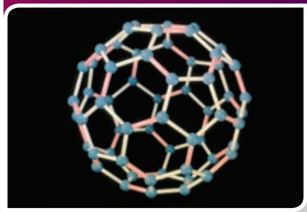

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Dr. Foley coauthored *Precalculus: A Graphing Approach; Precalculus: Functions and Graphs;* and *Advanced Quantitative Reasoning: Mathematics for the World Around Us.*

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Dan Kennedy received his undergraduate degree from the College of the Holy Cross and his master's degree and Ph.D. in mathematics from the University of North Carolina at Chapel Hill. Since 1973 he has taught mathematics at the Baylor School in Chattanooga, Tennessee, where he holds the Cartter Lupton Distinguished Professorship. Dr. Kennedy joined the Advanced Placement® Calculus Test Development Committee in 1986, then in 1990 became the first high school teacher in 35 years to chair that committee. It was during his tenure as chair that the program moved to require graphing calculators and laid the early groundwork for the 1998 reform of the Advanced Placement Calculus curriculum. The author of the 1997 *Teacher's Guide—AP® Calculus*, Dr. Kennedy has conducted more than 50 workshops and institutes for high school calculus teachers. His articles on mathematics teaching have appeared in the *Mathematics Teacher* and the *American Mathematical Monthly*, and he is a frequent speaker on education reform at professional and civic meetings. Dr. Kennedy was named a Tandy Technology Scholar in 1992 and a Presidential Award winner in 1995.

Dr. Kennedy coauthored *Calculus: Graphical, Numerical, Algebraic; Prentice Hall Algebra I; Prentice Hall Geometry;* and *Prentice Hall Algebra 2.*

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Dave Bock holds degrees from the University at Albany (NY) in mathematics (B.A.) and statistics/education (M.S.). Mr. Bock taught mathematics at Ithaca High School for 35 years, including both BC Calculus and AP Statistics. He also taught Statistics at Tompkins-Cortland Community College, Ithaca College, and Cornell University, where he recently served as K–12 Education and Outreach Coordinator and Senior Lecturer for the Mathematics Department. Mr. Bock serves as a Statistics consultant to the College Board, leading numerous workshops and institutes for AP Statistics teachers. He has been a reader for the AP Calculus exam and both a reader and a table leader for the AP Statistics exam. During his career Mr. Bock won numerous teaching awards, including the MAA's Edyth May Sliffe Award for Distinguished High School Mathematics Teaching (twice) and Cornell University's Outstanding Educator Award (three times), and was also a finalist for New York State Teacher of the Year.

Mr. Bock coauthored the AP Statistics textbook *Stats: Modeling the World,* the non-AP text *Stats in Your World,* Barron's *AP Calculus* review book, and Barron's *AP Calculus Flash Cards.*

PREFACE

Our Approach

The Rule of Four—A Balanced Approach

A principal feature of this text is the balance among the algebraic, numerical, graphical, and verbal methods of representing problems: the rule of four. For instance, we obtain solutions algebraically when that is the most appropriate technique to use, and we obtain solutions graphically or numerically when algebra is difficult to use. We urge students to solve problems by one method and then to support or confirm their solutions by using another method. We believe that students must learn the value of each of these methods or representations and must learn to choose the one most appropriate for solving the particular problem under consideration. This approach reinforces the idea that to understand a problem fully, students need to understand it algebraically as well as graphically and numerically.

Problem-Solving Approach

Systematic problem solving is emphasized in the examples throughout the text, using the following variation of Polya's problem-solving process that emphasizes modeling with mathematics:

- *understand* the problem,
- *develop* a mathematical model for the problem,
- *solve* the mathematical model and support or confirm the solution, and
- *interpret* the solution within the problem setting.

Students are encouraged to use this process throughout the text.

Twelve Basic Functions

Twelve basic functions are emphasized throughout the text as a major theme and focus. These functions are

- The Identity Function
- The Squaring Function
- The Cubing Function
- The Reciprocal Function
- The Square Root Function
- The Exponential Function
- The Natural Logarithm Function
- The Sine Function
- The Cosine Function
- The Absolute Value Function
- The Greatest Integer Function
- The Logistic Function

One of the most distinctive features of this text is that it introduces students to the full vocabulary of functions early in the course. Students meet the twelve basic functions graphically in Chapter 1 and are able to compare and contrast them as they learn about concepts like domain, range, symmetry, continuity, end behavior, asymptotes, extrema, and even periodicity—concepts that are difficult to appreciate when the only examples a teacher can refer to are polynomials. With this text, students are able to characterize functions by their behavior within the first month of classes. Once students have a comfortable understanding of functions in general, the rest of the course consists of studying the various types of functions in greater depth, particularly with respect to their algebraic properties and modeling applications.

These functions are used to develop the fundamental analytic skills that are needed in calculus and advanced mathematics courses. A complete gallery of basic functions is included in Appendix C.

Applications, Data, and Modeling

The majority of the applications in the text are based on real data from cited sources, and their presentations are self-contained. As they work through the applications, students are exposed to functions as mechanisms for modeling real-life problems. They learn to analyze and model data, represent data graphically, interpret graphs, and fit curves. Additionally, the tabular representation of data presented in this text highlights the concept that a function is a correspondence between numerical variables. This helps students build the connection between numerical quantities and graphs and recognize the importance of a full graphical, numerical, and algebraic understanding of a problem. For a complete listing of applications, please see the Applications Index on page 947.

Technology and Exercises

The authors of this text have encouraged the use of technology in mathematics education for more than three decades. Our approach to problem solving (pages 92–93) distinguishes between **solving** the problem and **supporting** or **confirming** the solution, and emphasizes how technology figures in each of these processes.

We have come to realize, however, that advances in technology and increased familiarity with calculators have gradually blurred some of the distinctions between solving and supporting that we had once assumed to be apparent. We do not want to retreat in any way from our support of modern technology, but we now provide specific guidance about the intent of the various exercises in our text.

Therefore, as a service to teachers and students alike, exercises in this text that **should be solved without calculators** are identified with gray ovals around the exercise numbers. These usually are exercises that demonstrate how various functions behave algebraically or how algebraic representations reflect graphical behavior and vice versa. Application problems usually have no restrictions, in keeping with our emphasis on **modeling** and on bringing **all representations** to bear when confronting real-world problems.

Incidentally, we continue to encourage the use of calculators to **support** answers graphically or numerically after the problems have been solved with pencil and paper. Any time students can make connections among the graphical, analytical, and numerical representations, they are doing good mathematics.

As a final note, we will freely admit that different teachers use our text in different ways, and some will probably override our no-calculator recommendations to fit with their pedagogical strategies. In the end, the teachers know what is best for their students, and we are just here to help.

Content Changes to This Edition

Although the table of contents is essentially the same, this edition includes numerous substantial changes. About 5% of the examples have been replaced; another 5% have new data or new contexts. Additionally, 15–20% of the examples have been enhanced or clarified in some way. As for the exercises, again, about 10% have been replaced and another 5% have new data or new contexts. Plus, 5–10% of the exercises have been enhanced or clarified in some way. In particular, to keep the applications of mathematics relevant to our students, we have included the most current data available to us at the time of publication. As an example, look at the Chapter Opener problem on page 166. Not only does this include current data but also an entirely new twist: piecewise modeling.

Several other changes have been made as well. We have updated many of the student and teacher notes. We have updated calculator screenshots to conform to the enhanced capabilities of modern graphing calculators. We have updated and renamed the capstone projects for Chapters 1–11 as Modeling Projects to reflect that they can be used as a bridge to the open-ended modeling recommended in the *GAIMME* report, published in 2016 by the Consortium for Mathematics and Its Applications (COMAP) and the Society for Industrial and Applied Mathematics (SIAM).

Features

Chapter Openers include a general description of an application that can be solved with the concepts learned in the chapter. The application is revisited later in the chapter via a specific problem that is solved.

A **Chapter Overview** begins each chapter to give students a sense of what they are going to learn. This overview provides a roadmap of the chapter and also indicates how the topics in the chapter are connected under one big idea. It is always helpful to remember that mathematics isn't modular, but interconnected, and that the skills and concepts learned throughout the course build on one another to help students understand more complicated processes and relationships. Similarly, the **What you'll learn about . . . and why** feature presents the big ideas in each section and explains their purpose.

Throughout the text, **Vocabulary** is highlighted in yellow for easy reference. Additionally, **Properties**, **Definitions**, and **Theorems** are boxed in purple, and **Procedures** in green, so that they can be easily found. The **Web/Real Data** icon marks the examples and exercises that use data cited from authentic sources.

Each example ends with a suggestion to **Now try** a related exercise. Working the suggested exercise is an easy way for students to check their comprehension of the material while reading each section. Alternatives are provided for these examples in the PowerPoint Slides.

Explorations appear throughout the text and provide students with the perfect opportunity to become active learners and to discover mathematics on their own. This will help hone critical-thinking and problem-solving skills. Some are technology-based; others involve exploring mathematical ideas and connections.

Margin Notes on various topics appear throughout the text. Some of these offer practical advice on using a grapher to obtain the best, most accurate results. Other notes include historical information, give hints about examples, or provide insight to help students avoid common pitfalls and errors.

The **Looking Ahead to Calculus** icon is found throughout the text next to many examples and topics to point out concepts that students will encounter again in calculus. Ideas that foreshadow calculus, such as limits, maximum and minimum, asymptotes, and continuity, are highlighted. Some calculus notation and language are introduced in the early chapters and used throughout the text to establish familiarity.

The review material at the end of each chapter consists of sections dedicated to helping students review the chapter concepts. **Key Ideas** are broken into parts: Properties, Theorems, and Formulas; Procedures; and Gallery of Functions. The **Review Exercises** represent the full range of exercises covered in the chapter and give additional practice with the ideas developed in the chapter. The exercises with red numbers indicate problems that would make up a good chapter test. A **Modeling Project** concludes each chapter and requires students to analyze data. It can be assigned as either individual or group work. Each project expands upon concepts and ideas taught in the chapter and engages students in modeling with mathematics.

Exercise Sets

Each exercise set begins with a **Quick Review** to help students review skills needed in the exercise set and refers them to other sections they can go to for help. Some exercises are designed to be solved *without* a *calculator*; the numbers of these exercises are printed within a gray oval. Students are urged to **support** the answers to these (and all) exercises graphically or numerically, but only after they have solved them with pencil and paper.

There are over 6000 exercises, including 720 Quick Review Exercises. The section exercises have been carefully graded from routine to challenging. The following types of skills are tested in each exercise set:

- Algebraic understanding and procedures
- Applications of mathematics
- Connecting algebra to geometry
- Interpretation of graphs
- Graphical and numerical representations of functions
- Data analysis

The exercise sets include distinctive kinds of thought-provoking exercises:

- **Standardized Test Questions** include two true-false problems with justifications and four multiple-choice questions.
- **Explorations** are opportunities for students to discover mathematics on their own or in groups. These exercises often require the use of critical thinking to explore the ideas involved.
- **Writing to Learn** exercises give students practice at communicating about mathematics and opportunities to demonstrate their understanding of important ideas.
- **Group Activity** exercises ask students to work collaboratively to solve problems while interacting with a few of their classmates.
- **Extending the Ideas** exercises go beyond what is presented in the text. These exercises are challenging extensions of the material in the text.

This variety of exercises provides sufficient flexibility to emphasize the skills and concepts most needed for each student or class.

Technology Resources

The following supplements are available for purchase:

MyLab Math Online Course (optional, for purchase only)—access code required

MyLab Math delivers **proven results** in helping individual students succeed. It provides **engaging experiences** that personalize, stimulate, and measure learning for each student. And it comes from a **trusted partner** with educational expertise and an eye on the future. To learn more about how MyLab Math combines proven learning applications with powerful assessment, visit **https://mlm.pearson.com/global/** or contact your Pearson Sales Representative. In this **MyLab Math** course, you have access to the most cutting-edge, innovative study solutions proven to increase students' success.

Additional Teacher Resources

Most of the teacher supplements and resources available for this text are available electronically for download at the Instructor Resource Center (IRC). Please go to the Pearson Global Editions site, https://media.pearsoncmg.com/intl/ge/abp/resources/

index.html, and select Instructor Resources. Once you register on the resources site, you will be able to access downloadable resources for your textbook.

The following supplements are available to registered adopters:

Online Solutions Manual (Download Only)

Provides complete solutions to all exercises, including Explorations, Quick Reviews, Exercises, Review Exercises, and Modeling Projects.

Online Resource Manual (Download Only)

Provides Major Concepts Review, Group Activity Worksheets, Sample Chapter Tests, Standardized Test Preparation Questions, and Contest Problems.

Online Tests and Quizzes (Download Only)

Provides two parallel tests per chapter, two quizzes for every three to four sections, two parallel midterm tests covering Chapters P–5, and two parallel end-of-year tests covering Chapters 6–11.

TestGen® (Download Only)

TestGen enables teachers to build, edit, print, and administer tests using a computerized bank of questions developed to cover all the objectives of the text. TestGen is algorithmically based, allowing teachers to create multiple but equivalent versions of the same question or test with the click of a button. Teachers can also modify test bank questions or add new questions. Tests can be printed or administered online.

PowerPoint Slides (Download Only)

Features presentations written and designed specifically for this text, including figures, alternative examples, definitions, and key concepts.

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CHAPTER P

Prerequisites

Large distances are measured in *light years*, the distance that light travels in one year. Scientists use the speed of light, which is roughly 299,800 km/sec, to approximate distances within the solar system. For examples, see page 59.

Chapter P Overview

Historically, algebra was used to represent problems with symbols (algebraic models) and solve them by reducing the solution to algebraic manipulation of symbols. This technique is still important today. In addition, graphing calculators are now used to represent problems with graphs (graphical models) and solve them with the numerical and graphical techniques of technology.

We begin with basic properties of real numbers and introduce absolute value, distance formulas, midpoint formulas, and equations of circles. We use the slope of a line to write equations for the line, and we use these equations to solve practical problems. We then explore the basic ideas of complex numbers. We close the chapter by solving equations and inequalities using both algebraic and graphical techniques.

P.1 Real Numbers

What you'll learn about

- Representing Real Numbers
- Order and Interval Notation
- Basic Properties of Algebra
- Integer Exponents
- Scientific Notation

... and why

These topics are fundamental in the study of mathematics and science.

Representing Real Numbers

A **real number** is any number that can be written as a decimal. Real numbers are represented by symbols such as -8, 0, 1.75, $2.333...$, $0.\overline{36}$, $8/5$, $\sqrt{3}$, $\sqrt[3]{16}$, e, and π.

The set of real numbers contains several important subsets:

The **natural (or counting) numbers**:	$\{1, 2, 3, \ldots\}$
The **whole numbers**:	$\{0, 1, 2, 3, \ldots\}$
The **integers**:	$\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

We use braces $\{\ \}$ to enclose the **elements**, or **objects**, of a set. The rational numbers are another important subset of the real numbers. A **rational number** is any number that can be written as a quotient a/b of two integers, where $b \neq 0$. We can use **set-builder notation** to define the rational numbers:

$$\left\{ \frac{a}{b} \,\middle|\, a, b \text{ are integers, and } b \neq 0 \right\}$$

These symbols are read as "the set of all a over b such that a and b are integers, and b is not equal to zero."

The decimal form of a rational number either **terminates** like $7/4 = 1.75$, or is **infinitely repeating** like $4/11 = 0.363636... = 0.\overline{36}$. The bar over the 36 indicates the block of digits that repeats. A real number is **irrational** if it is *not* rational. The decimal form of an irrational number is infinitely nonrepeating. For example,

$$\sqrt{3} = 1.7320508... \text{ and } \pi = 3.14159265....$$

A real number can be approximated by giving a few of its digits. Sometimes we can find the decimal form of rational numbers with calculators, but not very often.

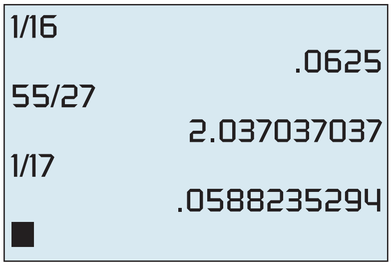


Figure P.1 Calculator decimal representations of $1/16$, $55/27$, and $1/17$ with the calculator set in Floating decimal mode. (Example 1)

EXAMPLE 1 Examining Decimal Forms of Rational Numbers

Determine the decimal form of $1/16$, $55/27$, and $1/17$.

SOLUTION Figure P.1 suggests that the decimal form of $1/16$ terminates and that of $55/27$ repeats in blocks of 037.

$$\frac{1}{16} = 0.0625 \quad \text{and} \quad \frac{55}{27} = 2.\overline{037}$$

We cannot predict the *exact* decimal form of $1/17$ from Figure P.1; however, we can say that $1/17 \approx 0.0588235294$. The symbol $\approx$ is read "*is approximately equal to*." We can use long division (see Exercise 66) to prove that

$$\frac{1}{17} = 0.\overline{0588235294117647}.$$

Now try Exercise 3.

The real numbers and the points of a line can be matched *one-to-one* to form a **real number line**. We start with a horizontal line and match the real number zero with a point O, the **origin**. **Positive numbers** are assigned to the right of the origin, and **negative numbers** to the left, as shown in Figure P.2.

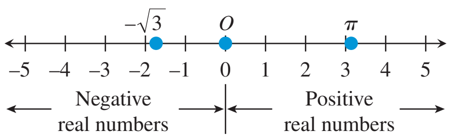


Figure P.2 The real number line.

Every real number corresponds to one and only one point on the real number line, and every point on the real number line corresponds to one and only one real number. Between every pair of real numbers on the number line there are infinitely many more real numbers.

The number associated with a point is **the coordinate of the point**. As long as the context is clear, we will follow the standard convention of using the real number for both the name of the point and its coordinate.

Order and Interval Notation

The set of real numbers is **ordered**. This means that we can use inequalities to compare any two real numbers that are not equal and say that one is "less than" or "greater than" the other.

Order of Real Numbers

Let a and b be any real numbers.

Symbol	Definition	Read
$a > b$	$a - b$ is positive	a is greater than b
$a < b$	$a - b$ is negative	a is less than b
$a \geq b$	$a - b$ is positive or zero	a is greater than or equal to b
$a \leq b$	$a - b$ is negative or zero	a is less than or equal to b

The symbols $>$, $<$, $\geq$, and $\leq$ are **inequality symbols**.

Unordered Systems

Not all number systems are ordered. For example, the complex number system, introduced in Section P.6, has no natural ordering.

Opposites and Number Line

$a < 0 \Leftrightarrow -a > 0$

If $a < 0$, then a is to the left of 0 on the real number line, and its opposite, $-a$, is to the right of 0. Thus, $-a > 0$. This logic can be reversed: If $-a > 0$, then $a < 0$.

Geometrically, $a > b$ means that a is to the right of b (equivalently, b is to the left of a) on the real number line. For example, $6 > 3$ implies that 6 is to the right of 3 on the real number line. Note also that $a > 0$ means that $a - 0$, or simply a, is positive, and $a < 0$ means that a is negative.

We are able to compare any two real numbers because of the following important property of the real numbers.

Trichotomy Property

Let a and b be any real numbers. Exactly one of the following is true:

$$a < b, \quad a = b, \quad \text{or} \quad a > b$$

Inequalities can be used to specify **intervals** of real numbers, as illustrated in Example 2.

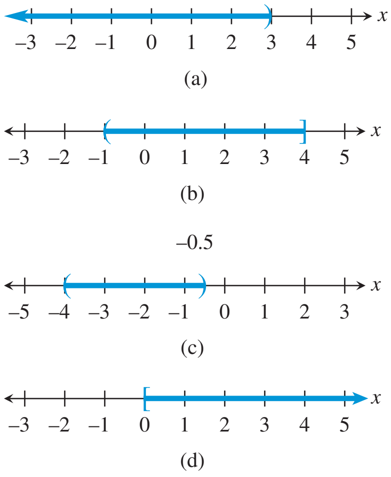


Figure P.3 In graphs of inequalities, parentheses correspond to $<$ and $>$, and brackets correspond to $\le$ and $\ge$. (Examples 2 and 3)

EXAMPLE 2 Interpreting Inequalities

Interpret the meaning of, and graph, the interval of real numbers for the inequality.

(a) $x < 3$ **(b)** $-1 < x \le 4$

SOLUTION

(a) The inequality $x < 3$ describes all real numbers less than 3 (Figure P.3a).

(b) The *double inequality* $-1 < x \le 4$ represents all real numbers between -1 and 4, excluding -1 and including 4 (Figure P.3b). **Now try Exercise 5.**

EXAMPLE 3 Writing Inequalities

Write an inequality based on the description, and draw its graph.

(a) The real numbers between -4 and -0.5

(b) The real numbers greater than or equal to zero

SOLUTION

(a) $-4 < x < -0.5$ (Figure P.3c)

(b) $x \ge 0$ (Figure P.3d) **Now try Exercise 15.**

As shown in Example 2, inequalities define *intervals* on the real number line. We often use $[2, 5]$ to describe the *bounded interval* determined by $2 \le x \le 5$. This interval is **closed** because it contains its *endpoints* 2 and 5. There are four types of **bounded intervals**.

Bounded Intervals of Real Numbers

Let a and b be real numbers with $a < b$.

Interval Notation	Interval Type	Inequality Notation	Graph
$[a, b]$	Closed	$a \le x \le b$	
(a, b)	Open	$a < x < b$	
$[a, b)$	Half-open	$a \le x < b$	
$(a, b]$	Half-open	$a < x \le b$	

The numbers a and b are the **endpoints** of each interval.

We use the interval notation $(-\infty, \infty)$ to represent the entire set of real numbers. The symbols $-\infty$ (*negative infinity*) and ∞ (*positive infinity*) allow us to use interval notation for unbounded intervals and are *not* real numbers.

Interval Notation Using $\pm\infty$

Because $-\infty$ is *not* a real number, we use $(-\infty, 2)$ instead of $[-\infty, 2)$ to describe $x < 2$. Similarly, we use $[-1, \infty)$ instead of $[-1, \infty]$ to describe $x \geq -1$.

The interval of real numbers determined by the inequality $x < 2$ can be described by the *unbounded interval* $(-\infty, 2)$. This interval is **open** because it does *not* contain its endpoint 2. In addition to $(-\infty, \infty)$, there are four types of **unbounded intervals**.

Unbounded Intervals of Real Numbers

Let a and b be real numbers.

Interval Notation	Interval Type	Inequality Notation	Graph
$[a, \infty)$	Closed	$x \geq a$	
(a, ∞)	Open	$x > a$	
$(-\infty, b]$	Closed	$x \leq b$	
$(-\infty, b)$	Open	$x < b$	

Each of these intervals has exactly one endpoint, namely a or b.

EXAMPLE 4 Converting Between Intervals and Inequalities

Convert interval notation to inequality notation, or vice versa. State whether the interval is bounded or unbounded, and open or closed. Graph the interval and identify its endpoints.

(a) $[-6, 3)$ **(b)** $(-\infty, -1)$ **(c)** $-2 \leq x \leq 3$

SOLUTION

(a) The interval $[-6, 3)$ corresponds to $-6 \leq x < 3$ and is bounded and half-open (Figure P.4a). The endpoints are -6 and 3.

(b) The interval $(-\infty, -1)$ corresponds to $x < -1$ and is unbounded and open (Figure P.4b). The only endpoint is -1.

(c) The inequality $-2 \leq x \leq 3$ corresponds to the closed, bounded interval $[-2, 3]$ (Figure P.4c). The endpoints are -2 and 3. **Now try Exercise 29.**

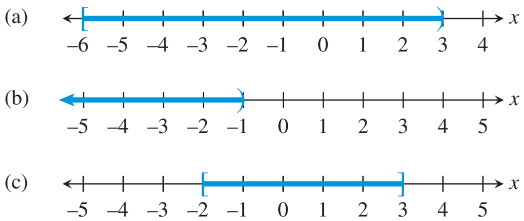


Figure P.4 Graphs of the intervals of real numbers in Example 4.

Basic Properties of Algebra

Algebra involves the use of letters and other symbols to represent real numbers. A **variable** is a letter or symbol (for example, x, y, t, θ) that represents an unspecified real number. A **constant** is a letter or symbol (for example, $-2, 0, \sqrt{3}, \pi$) that represents a specific real number. An **algebraic expression** is a combination of variables and constants involving addition, subtraction, multiplication, division, powers, and roots.

We state some of the properties of the arithmetic operations of addition, subtraction, multiplication, and division, represented by the symbols $+$, $-$, $\times$ (or $\cdot$) and $\div$ (or $/$), respectively. Addition and multiplication are the primary operations. Subtraction and division are defined in terms of addition and multiplication.

Subtraction: $a - b = a + (-b)$

Division: $\dfrac{a}{b} = a\left(\dfrac{1}{b}\right),\ b \neq 0$

Subtraction vs. Negative Numbers

On many calculators, there are two "–" keys, one for subtraction and one for negative numbers or opposites. Be sure you know how to use both keys correctly. Misuse can lead to incorrect results.

In the above definitions, $-b$ is the **additive inverse** or **opposite** of b, and $1/b$ is the **multiplicative inverse** or **reciprocal** of b. Perhaps surprisingly, additive inverses are not always negative numbers. The additive inverse of 5 is the negative number -5. However, the additive inverse of -3 is the positive number 3.

The following properties hold for real numbers, variables, and algebraic expressions.

Properties of Algebra

Let u, v, and w be real numbers, variables, or algebraic expressions.

1. **Commutative properties**
   Addition: $u + v = v + u$
   Multiplication: $uv = vu$
2. **Associative properties**
   Addition:
   $(u + v) + w = u + (v + w)$
   Multiplication: $(uv)w = u(vw)$
3. **Identity properties**
   Addition: $u + 0 = u$
   Multiplication: $u \cdot 1 = u$
4. **Inverse properties**
   Addition: $u + (-u) = 0$
   Multiplication: $u \cdot \dfrac{1}{u} = 1,\ u \neq 0$
5. **Distributive properties**
   Multiplication over addition:
   $u(v + w) = uv + uw$
   $(u + v)w = uw + vw$
   Multiplication over subtraction:
   $u(v - w) = uv - uw$
   $(u - v)w = uw - vw$

For each distributive property, the left-hand side of the equation shows the **factored form** of the algebraic expression, and the right-hand side shows the **expanded form**.

EXAMPLE 5 Using the Distributive Property

(a) Write the expanded form of $(a + 2)x$.

(b) Write the factored form of $3y - by$.

SOLUTION

(a) $(a + 2)x = ax + 2x$

(b) $3y - by = (3 - b)y$

Now try Exercise 37.

Here are some properties of the additive inverse, together with examples that help illustrate their meanings.

Properties of the Additive Inverse

Let u and v be real numbers, variables, or algebraic expressions.

Property	Example
1. $-(-u) = u$	$-(-3) = 3$
2. $(-u)v = u(-v) = -(uv)$	$(-4)3 = 4(-3) = -(4 \cdot 3) = -12$
3. $(-u)(-v) = uv$	$(-6)(-7) = 6 \cdot 7 = 42$
4. $(-1)u = -u$	$(-1)5 = -5$
5. $-(u + v) = (-u) + (-v)$	$-(7 + 9) = (-7) + (-9) = -16$

Integer Exponents

Exponential notation is used to shorten products of factors that repeat. For example,

$$(-3)(-3)(-3)(-3) = (-3)^4 \quad \text{and} \quad (2x + 1)(2x + 1) = (2x + 1)^2.$$

Exponential Notation

Let a be a real number, variable, or algebraic expression and n be a positive integer. Then

$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}},$$

where n is the **exponent**, a is the **base**, and a^n is the ***n*th power of *a***, read as "a to the nth power."

The two exponential expressions in Example 6 have the same value but have different bases. Be sure you understand the distinction.

Understanding Notation

$(-3)^2 = 9$

$-3^2 = -9$

Be careful!

EXAMPLE 6 Identifying the Base

(a) In $(-3)^5$, the base is -3.

(b) In -3^5, the base is 3.

Now try Exercise 43.

Here are the basic properties of exponents, together with examples to illustrate their meanings.

Properties of Exponents

Let u and v be real numbers, variables, or algebraic expressions and m and n be integers. All bases are assumed to be nonzero.

Property	Example
1. $u^m u^n = u^{m+n}$	$5^3 \cdot 5^4 = 5^{3+4} = 5^7$
2. $\dfrac{u^m}{u^n} = u^{m-n}$	$\dfrac{x^9}{x^4} = x^{9-4} = x^5$
3. $u^0 = 1$	$8^0 = 1$
4. $u^{-n} = \dfrac{1}{u^n}$	$y^{-3} = \dfrac{1}{y^3}$
5. $(uv)^m = u^m v^m$	$(2z)^5 = 2^5 z^5 = 32z^5$
6. $(u^m)^n = u^{mn}$	$(x^2)^3 = x^{2 \cdot 3} = x^6$
7. $\left(\dfrac{u}{v}\right)^m = \dfrac{u^m}{v^m}$	$\left(\dfrac{a}{b}\right)^7 = \dfrac{a^7}{b^7}$

To simplify an expression involving powers means to rewrite it so that each factor appears only once, all exponents are positive, and exponents and constants are combined as much as possible.

Moving Factors

Be sure you understand how exponent property 4 permits us to move factors from the numerator to the denominator, and vice versa:

$$\frac{v^{-m}}{u^{-n}} = \frac{u^n}{v^m}$$

EXAMPLE 7 Simplifying Expressions Involving Powers

(a) $(2ab^3)(5a^2b^5) = 10(aa^2)(b^3b^5) = 10a^3b^8$

(b) $\dfrac{u^2v^{-2}}{u^{-1}v^3} = \dfrac{u^2u^1}{v^2v^3} = \dfrac{u^3}{v^5}$

(c) $\left(\dfrac{x^2}{2}\right)^{-3} = \left(\dfrac{2}{x^2}\right)^3 = \dfrac{2^3}{(x^2)^3} = \dfrac{8}{x^6}$

Now try Exercise 47.

Scientific Notation

Any positive number can be written in **scientific notation**,

$$c \times 10^m, \text{ where } 1 \le c < 10 \text{ and } m \text{ is an integer.}$$

This notation provides a way to work with very large and very small numbers. For example, the distance between Earth and the Sun is about 93,000,000 miles. In scientific notation,

$$93{,}000{,}000 \text{ mi} = 9.3 \times 10^7 \text{ mi.}$$

The *positive exponent* 7 indicates that moving the decimal point in 9.3 to the right 7 places produces the decimal form of the number.

The mass of an oxygen molecule is about

$$0.000\ 000\ 000\ 000\ 000\ 000\ 000\ 054 \text{ g.}$$

In scientific notation,

$$0.000\ 000\ 000\ 000\ 000\ 000\ 000\ 054 \text{ g} = 5.4 \times 10^{-23} \text{ g.}$$

The *negative exponent* -23 indicates that moving the decimal point in 5.4 to the left 23 places produces the decimal form of the number.

EXAMPLE 8 Converting to and from Scientific Notation

(a) $2.375 \times 10^8 = 237{,}500{,}000$

(b) $0.000000349 = 3.49 \times 10^{-7}$

Now try Exercises 57 and 59.

EXAMPLE 9 Using Scientific Notation

Simplify $\dfrac{(360{,}000)(4{,}500{,}000{,}000)}{18{,}000}$.

SOLUTION

$$\begin{aligned}\frac{(360{,}000)(4{,}500{,}000{,}000)}{18{,}000} &= \frac{(3.6 \times 10^5)(4.5 \times 10^9)}{1.8 \times 10^4} \\ &= \frac{(3.6)(4.5)}{1.8} \times 10^{5+9-4} \\ &= 9 \times 10^{10} \\ &= 90{,}000{,}000{,}000\end{aligned}$$

Now try Exercise 63.

Using a Calculator Figure P.5 shows two ways to perform the computation. In the first, the numbers are entered in decimal form. In the second, the numbers are entered in scientific notation. The calculator uses “9E10” to stand for 9×10^{10}.

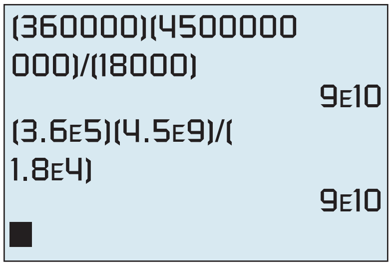


Figure P.5 Be sure you understand how your calculator displays scientific notation. (Example 9)

QUICK REVIEW P.1

1. List the positive integers between -3 and 7.
2. List the integers between -3 and 7.
3. List all negative integers greater than -4.
4. List all positive integers less than 5.

In Exercises 5 and 6, use a calculator to evaluate the expression. Round the value to two decimal places.

5. (a) $4(-3.1)^3 - (-4.2)^5$ (b) $\dfrac{2(-5.5) - 6}{7.4 - 3.8}$
6. (a) $5[3(-1.1)^2 - 4(-0.5)^3]$ (b) $5^{-2} + 2^{-4}$

In Exercises 7 and 8, evaluate the algebraic expression for the given values of the variables.

7. $x^3 - 2x + 1, x = -2, 1.5$
8. $a^2 + ab + b^2, a = -3, b = 2$

In Exercises 9 and 10, list the possible remainders.

9. When the positive integer n is divided by 7
10. When the positive integer n is divided by 13

SECTION P.1 Exercises

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator*.

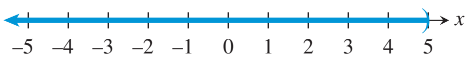
In Exercises 1–4, find the decimal form for the rational number. State whether it repeats or terminates.

1. $-37/8$
2. $15/99$
3. $-13/6$
4. $5/37$

In Exercises 5–10, interpret the meaning of, and graph, the interval of real numbers.

5. $x \le 2$
6. $-2 \le x < 5$
7. $(-\infty, 7)$
8. $[-3, 3]$
9. x is negative.
10. x is greater than or equal to 2 and less than or equal to 6.

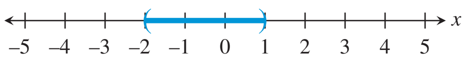
In Exercises 11–16, write an inequality for the interval of real numbers.

11. $[-1, 1)$
12. $(-\infty, 4]$
13. 

14. 

15. x is between -1 and 2.
16. x is greater than or equal to 5.

In Exercises 17–22, use interval notation to describe the interval of real numbers.

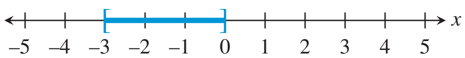
17. $x > -3$
18. $-7 < x < -2$
19. 

20. 

21. x is greater than -3 and less than or equal to 4.
22. x is positive.

In Exercises 23–28, use words to describe the interval of real numbers.

23. $4 < x \le 9$
24. $x \ge -1$
25. $[-3, \infty)$
26. $(-5, 7)$
27. 

28. 

In Exercises 29–32, convert to inequality notation. State whether the interval is bounded or unbounded and whether it is open or closed. Identify the endpoints.

29. $(-3, 4]$ **30.** $(-3, -1)$

31. $(-\infty, 5)$ **32.** $[-6, \infty)$

In Exercises 33–36, use both inequality and interval notation to describe the set of numbers. State the meaning of any variables you use.

33. Writing to Learn Bill is at least 29 years old.

34. Writing to Learn No item at Sarah's Variety Store costs more than \$2.00.

35. Writing to Learn The price of a gallon of gasoline varies from \$3.099 to \$4.399.

36. Writing to Learn Salary raises at California State University at Chico will be between 2% and 6.5% this year.

In Exercises 37–40, use the distributive property to write the factored form or the expanded form of the given expression.

37. $a(x^2 + b)$ **38.** $(y - z^3)c$

39. $ax^2 + dx^2$ **40.** $a^3z + a^3w$

In Exercises 41 and 42, find the additive inverse of the number.

41. $6 - \pi$ **42.** -7

In Exercises 43 and 44, identify the base of the exponential expression.

43. -5^2 **44.** $(-2)^7$

45. Group Activity Discuss which algebraic property or properties are illustrated by the equation. Try to reach a consensus.

(a) $(3x)y = 3(xy)$ (b) $a^2b = ba^2$

(c) $a^2b + (-a^2b) = 0$ (d) $(x + 3)^2 + 0 = (x + 3)^2$

(e) $a(x + y) = ax + ay$

46. Group Activity Discuss which algebraic property or properties are illustrated by the equation. Try to reach a consensus.

(a) $(x + 2)\dfrac{1}{x + 2} = 1$ (b) $1 \cdot (x + y) = x + y$

(c) $2(x - y) = 2x - 2y$

(d) $2x + (y - z) = 2x + (y + (-z))$
$= (2x + y) + (-z) =$
$(2x + y) - z$

(e) $\dfrac{1}{a}(ab) = \left(\dfrac{1}{a}a\right)b = 1 \cdot b = b$

In Exercises 47–52, simplify the expression. Assume that the variables in the denominators are nonzero.

47. $\dfrac{x^4y^3}{x^2y^5}$ **48.** $\dfrac{(3x^2)^2y^4}{3y^2}$

49. $\left(\dfrac{4}{x^2}\right)^2$ **50.** $\left(\dfrac{2}{xy}\right)^{-3}$

51. $\dfrac{(x^{-3}y^2)^{-4}}{(y^6x^{-4})^{-2}}$ **52.** $\left(\dfrac{4a^3b}{a^2b^3}\right)\left(\dfrac{3b^2}{2a^2b^4}\right)$

The data in Table P.1 give the expenditures in millions of dollars for U.S. public schools for the 2013–2014 school year.

Table P.1 U.S. Public School Expenditures

Category	Amount (millions of \$)
Current expenditures	535,665
Capital outlay	45,474
Interest on school debt	17,247
Total	606,490

Source: National Center for Education Statistics, U.S. Department of Education, as reported in The World Almanac and Book of Facts 2017.

In Exercises 53–56, write the amount of expenditures in dollars obtained from the category in scientific notation.

53. Current expenditures

54. Capital outlay

55. Interest on school debt

56. Total

In Exercises 57 and 58, write the number in scientific notation.

57. The mean distance from Jupiter to the Sun is about 483,900,000 miles.

58. The electric charge, in coulombs, of an electron is about $-0.000\,000\,000\,000\,000\,000\,16$.

In Exercises 59–62, write the number in decimal form.

59. 3.33×10^{-8}

60. 6.73×10^{11}

61. The distance that light travels in 1 year (*one light year*) is about 5.87×10^{12} mi.

62. The mass of a neutron is about 1.6747×10^{-24} g.

In Exercises 63 and 64, use scientific notation to simplify.

63. $\dfrac{(1.3 \times 10^{-7})(2.4 \times 10^8)}{1.3 \times 10^9}$ without using a calculator

64. $\dfrac{(3.7 \times 10^{-7})(4.3 \times 10^6)}{2.5 \times 10^7}$

Explorations

65. Investigating Exponents For positive integers m and n, we can use the definition to show that $a^m a^n = a^{m+n}$.

(a) Examine the equation $a^m a^n = a^{m+n}$ for $n = 0$ and explain why it is reasonable to define $a^0 = 1$ for $a \neq 0$.

(b) Examine the equation $a^m a^n = a^{m+n}$ for $n = -m$ and explain why it is reasonable to define $a^{-m} = 1/a^m$ for $a \neq 0$.

66. Decimal Forms of Rational Numbers Here is the third step when we divide 1 by 17. (The first two steps are not shown because the quotient is 0 in each case.)

$$\begin{array}{r} 0.05 \\ 17\overline{)1.00} \\ \underline{85} \\ 15 \end{array}$$

By convention we say that 1 is the first remainder in the long division process, 10 is the second, and 15 is the third remainder.

(a) Continue this long division process until a remainder is repeated, and complete the following table:

Step	Quotient	Remainder
1	0	1
2	0	10
3	5	15
⋮	⋮	⋮

(b) Explain why the digits that occur in the quotient between the pair of repeating remainders determine the infinitely repeating portion of the decimal representation. In this case

$$\frac{1}{17} = 0.\overline{0588235294117647}.$$

(c) Explain why this procedure will always determine the infinitely repeating portion of a rational number whose decimal representation does not terminate.

Standardized Test Questions

67. True or False The additive inverse of a real number must be negative. Justify your answer.

68. True or False The reciprocal of a positive real number must be less than 1. Justify your answer.

In Exercises 69–72, solve these problems without using a calculator.

69. Multiple Choice Which of the following inequalities corresponds to the interval $[-2, 1)$?

(A) $x \leq -2$ (B) $-2 \leq x \leq 1$

(C) $-2 < x < 1$ (D) $-2 < x \leq 1$

(E) $-2 \leq x < 1$

70. Multiple Choice What is the value of $(-2)^4$?

(A) 16 (B) 8

(C) 6 (D) -8

(E) -16

71. Multiple Choice What is the base of the exponential expression -7^2?

(A) -7 (B) 7

(C) -2 (D) 2

(E) 1

72. Multiple Choice Which of the following is the simplified form of $\frac{x^6}{x^2}$, $x \neq 0$?

(A) x^{-4} (B) x^2

(C) x^3 (D) x^4

(E) x^8

Extending the Ideas

The **magnitude** of a real number is its distance from the origin.

73. List the whole numbers whose magnitudes are less than 7.

74. List the natural numbers whose magnitudes are less than 7.

75. List the integers whose magnitudes are less than 7.

76. Writing to Learn Combining Rational and Irrational Numbers In each case, write an explanation to justify your answer.

(a) When two rational numbers are added, is the sum a rational number?

(b) When two rational numbers are multiplied, is the product a rational number?

(c) When a rational number and an irrational number are added, is the sum a rational number?

(d) When a *nonzero* rational number and an irrational number are multiplied, is the product a rational number?

P.2 Cartesian Coordinate System

What you'll learn about

- Cartesian Plane
- Absolute Value of a Real Number
- Distance Formulas
- Midpoint Formulas
- Equations of Circles
- Applications

... and why

These topics provide the foundation for the material that will be addressed in this text.

Cartesian Plane

The points in a plane correspond to ordered pairs of real numbers, just as the points on a line are associated with individual real numbers. This correspondence creates the **Cartesian plane**, or the **rectangular coordinate system** in the plane.

To construct a rectangular coordinate system (Cartesian plane), draw a pair of perpendicular real number lines, one horizontal and the other vertical, with the lines intersecting at their respective origins (Figure P.6). Their point of intersection, O, is the **origin** of the Cartesian plane. The horizontal line is usually the **x-axis**, and the vertical line is usually the **y-axis**. The positive direction on the x-axis is to the right, and the positive direction on the y-axis is up. The two axes divide the Cartesian plane into four **quadrants**, as shown in Figure P.7.

Each point P of the plane is associated with an **ordered pair (x, y)** of real numbers, the **(Cartesian) coordinates of the point**. The **x-coordinate** is the coordinate of the point on the x-axis that intersects with the vertical line from P. The **y-coordinate** is the coordinate of the point on the y-axis that intersects with the horizontal line from P (Figure P.7). Figure P.6 shows the points P and Q with coordinates $(4, 2)$ and $(-6, -4)$, respectively. As long as the context is clear, we use ordered pairs of real numbers to name points, not just their coordinates. For example, we can use $(-6, -4)$ to name point Q.

Not always x and y

In applications, the horizontal axis often represents time, typically denoted by the variable t. The vertical axis can represent any attribute of interest. For example, if the vertical axis represents force, we may use F as the variable.

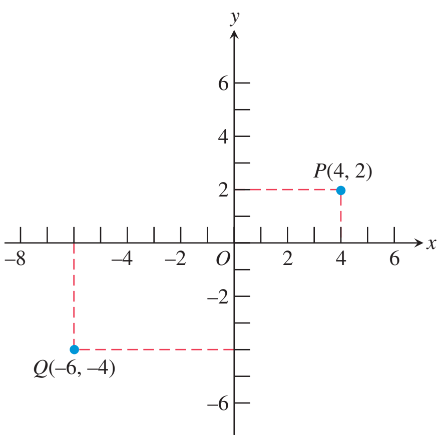


Figure P.6 The Cartesian coordinate plane.

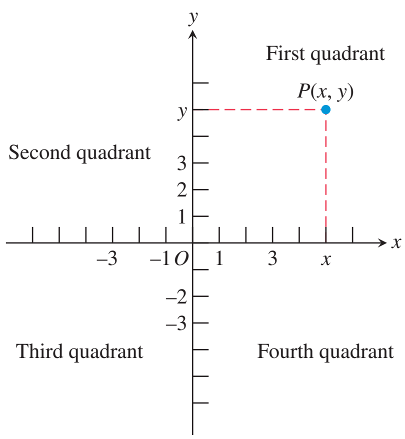


Figure P.7 The four quadrants. Points on the x- or y-axis are not in any quadrant.

Table P.2 U.S. Exports to Mexico

Time (years)	U.S. Exports (billions of $)
2005	120.2
2010	163.7
2012	215.9
2013	226.0
2014	240.3
2015	235.7

Source: U.S. Census Bureau, The World Almanac and Book of Facts 2017.

EXAMPLE 1 Plotting Data on U.S. Exports to Mexico

The values in billions of dollars of U.S. exports to Mexico for selected years from 2005 through 2015 are given in Table P.2. Plot the (time, export value) ordered pairs on a rectangular coordinate system.

SOLUTION The points are plotted in Figure P.8 on page 37. **Now try Exercise 31.**

A **scatter plot** is a graph of (x, y) data pairs on a Cartesian plane. Figure P.8 is a scatter plot of the data from Table P.2.

Absolute Value of a Real Number

The *absolute value of a real number* is its **magnitude** (size). For example, the absolute value of 3 is 3, and the absolute value of -5 is 5.

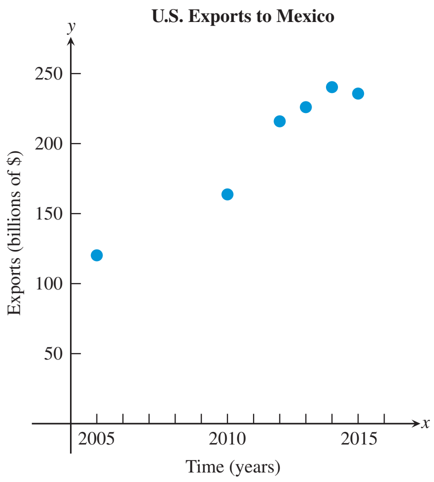


Figure P.8 The graph for Example 1.

DEFINITION Absolute Value of a Real Number

The **absolute value of a real number a** is

$$|a| = \begin{cases} a, \text{ if } a > 0 \\ 0, \text{ if } a = 0 \\ -a, \text{ if } a < 0. \end{cases}$$

EXAMPLE 2 Using the Definition of Absolute Value

Evaluate:

(a) $|-4|$

(b) $|\pi - 6|$

SOLUTION

(a) Because $-4 < 0$, $|-4| = -(-4) = 4$.

(b) Because $\pi \approx 3.142$, $\pi - 6$ is negative, so $\pi - 6 < 0$. Thus, $|\pi - 6| = -(\pi - 6) = 6 - \pi \approx 2.858$.

Now try Exercise 9.

Here is a summary of some important properties of absolute value.

Properties of Absolute Value

Let a and b be real numbers.

1. $|a| \geq 0$

2. $|-a| = |a|$

3. $|ab| = |a||b|$

4. $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}, b \neq 0$

Distance Formulas

The *distance* between -1 and 4 on the number line is 5 (Figure P.9). This distance may be found by subtracting the smaller number from the larger: $4 - (-1) = 5$.
If we use absolute value, the order of subtraction does not matter:

$$|4 - (-1)| = |-1 - 4| = 5$$

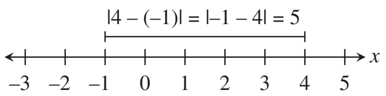


Figure P.9 Finding the distance between -1 and 4.

Distance Formula (Number Line)

Let a and b be real numbers. The **distance between a and b** is

$$|a - b|.$$

Note that $|a - b| = |b - a|$.

Absolute Value and Distance

If we let $b = 0$ in the distance formula, we see that the distance between a and 0 is $|a|$. Thus, the absolute value of a number is its distance from zero.

To find the *distance* between two points that lie on the same horizontal or vertical line in the Cartesian plane, we use the distance formula for points on a number line. For example, the distance between points x_1 and x_2 on the x-axis is $|x_1 - x_2| = |x_2 - x_1|$ and the distance between points y_1 and y_2 on the y-axis is $|y_1 - y_2| = |y_2 - y_1|$.

To find the distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ that do not lie on the same horizontal or vertical line, we form the right triangle determined by P, Q, and $R(x_2, y_1)$ (Figure P.10).

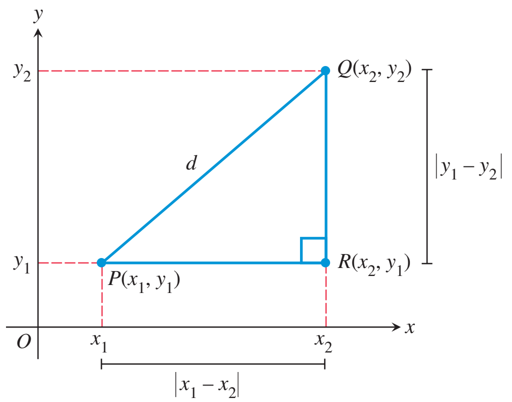


Figure P.10 Forming a right triangle with hypotenuse $\overline{PQ}$.

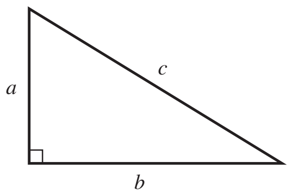


Figure P.11 The Pythagorean Theorem: In a right triangle, $c^2 = a^2 + b^2$.

The distance from P to R is $|x_1 - x_2|$, and the distance from R to Q is $|y_1 - y_2|$. By the **Pythagorean Theorem** (Figure P.11), the distance d between P and Q is

$$d = \sqrt{|x_1 - x_2|^2 + |y_1 - y_2|^2}.$$

Because $|x_1 - x_2|^2 = (x_1 - x_2)^2$ and $|y_1 - y_2|^2 = (y_1 - y_2)^2$, we obtain the following formula.

Distance Formula (Cartesian Plane)

The **distance d between points $P(x_1, y_1)$ and $Q(x_2, y_2)$** in a Cartesian plane is

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

EXAMPLE 3 Finding the Distance Between Two Points

Find the distance d between the points $(1, 5)$ and $(6, 2)$.

SOLUTION

$$\begin{aligned} d &= \sqrt{(1-6)^2 + (5-2)^2} && \text{The distance formula} \\ &= \sqrt{(-5)^2 + 3^2} \\ &= \sqrt{25 + 9} \\ &= \sqrt{34} \approx 5.831 && \text{Using a calculator} \end{aligned}$$

Now try Exercise 13.

Midpoint Formulas

When the endpoints of a segment on a number line are known, we take the average of their coordinates to find the midpoint of the segment.

Midpoint Formula (Number Line)

The **midpoint of the line segment with endpoints a and b** is

$$\frac{a + b}{2}.$$

EXAMPLE 4 Finding the Midpoint of a Line Segment

The midpoint of the line segment with endpoints -9 and 3 on a number line is

$$\frac{(-9) + 3}{2} = \frac{-6}{2} = -3.$$

See Figure P.12. **Now try Exercise 23.**

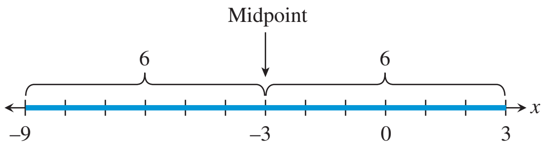


Figure P.12 Notice that the distance from the midpoint, -3, to 3 or to -9 is 6. (Example 4)

Just as with number lines, the midpoint of a line segment in the Cartesian plane involves averaging. Each coordinate of the midpoint is the average of the corresponding coordinates of its endpoints.

Midpoint Formula (Cartesian Plane)

The **midpoint of the line segment with endpoints (a, b) and (c, d)** is

$$\left(\frac{a + c}{2}, \frac{b + d}{2}\right).$$

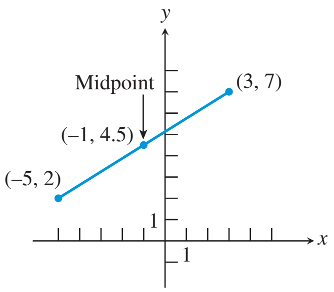


Figure P.13 Midpoint of a line segment. (Example 5)

EXAMPLE 5 Finding the Midpoint of a Line Segment

The midpoint of the line segment with endpoints $(-5, 2)$ and $(3, 7)$ is

$$(x, y) = \left(\frac{-5 + 3}{2}, \frac{2 + 7}{2}\right) = (-1, 4.5).$$

See Figure P.13. **Now try Exercise 25.**

Equations of Circles

A **circle** is the set of points in a plane at a fixed distance (**radius**) from a fixed point (**center**) in the plane. Figure P.14 shows the circle with center (h, k) and radius r. If (x, y) is any point on the circle, the distance formula gives

$$\sqrt{(x - h)^2 + (y - k)^2} = r.$$

Squaring both sides, we obtain the following equation for a circle.

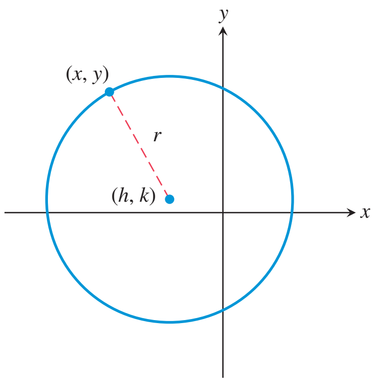


Figure P.14 The circle with center (h, k) and radius r.

DEFINITION Standard Form Equation of a Circle

The **standard form equation of a circle** with center (h, k) and radius r is

$$(x - h)^2 + (y - k)^2 = r^2.$$

EXAMPLE 6 Finding Standard Form Equations of Circles

Find the standard form equation of the circle.

(a) Center $(-4, 1)$, radius 8 **(b)** Center $(0, 0)$, radius 5

SOLUTION

(a) $(x - h)^2 + (y - k)^2 = r^2$ Standard form equation

$(x - (-4))^2 + (y - 1)^2 = 8^2$ Substitute $h = -4$, $k = 1$, $r = 8$.

$(x + 4)^2 + (y - 1)^2 = 64$

(b) $(x - h)^2 + (y - k)^2 = r^2$ Standard form equation

$(x - 0)^2 + (y - 0)^2 = 5^2$ Substitute $h = 0$, $k = 0$, $r = 5$.

$x^2 + y^2 = 25$

Now try Exercise 41.

Applications

EXAMPLE 7 Using an Inequality to Express Distance

We can state that "the distance between x and -3 is less than 9" using the inequality

$$|x - (-3)| < 9 \quad \text{or} \quad |x + 3| < 9.$$

Now try Exercise 51.

The converse of the Pythagorean Theorem is true. That is, if the sum of squares of the lengths of the two sides of a triangle equals the square of the length of the third side, then the triangle is a right triangle.

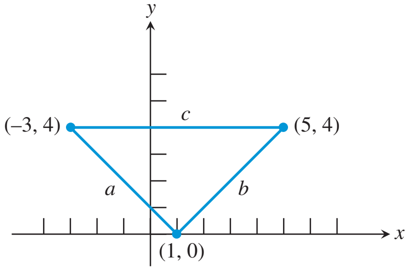


Figure P.15 The triangle in Example 8.

EXAMPLE 8 Verifying Right Triangles

Use the converse of the Pythagorean Theorem and the distance formula to prove that the points $(-3, 4)$, $(1, 0)$, and $(5, 4)$ determine a right triangle.

SOLUTION The three points are plotted in Figure P.15. We need to show that the lengths of the sides of the triangle satisfy the Pythagorean relationship $a^2 + b^2 = c^2$. Applying the distance formula, we find that

$$a = \sqrt{(-3 - 1)^2 + (4 - 0)^2} = \sqrt{32}$$
$$b = \sqrt{(1 - 5)^2 + (0 - 4)^2} = \sqrt{32}$$
$$c = \sqrt{(-3 - 5)^2 + (4 - 4)^2} = \sqrt{64}$$

The triangle is a right triangle because

$$a^2 + b^2 = \left(\sqrt{32}\right)^2 + \left(\sqrt{32}\right)^2 = 32 + 32 = 64 = c^2.$$

Now try Exercise 39.

Properties of geometric figures can sometimes be confirmed using analytic methods such as the midpoint formulas.

EXAMPLE 9 Using the Midpoint Formula

It is a fact from geometry that the diagonals of a parallelogram bisect each other. Prove this with a midpoint formula.

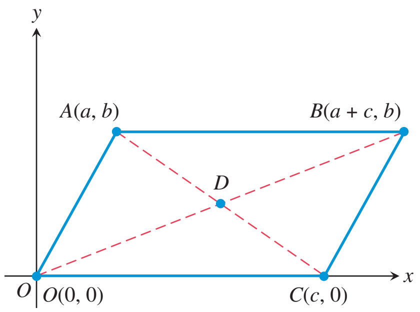


Figure P.16 The coordinates of B must be $(a + c, b)$ in order for CB to be parallel to OA. (Example 9)

SOLUTION We can position a parallelogram in the rectangular coordinate plane as shown in Figure P.16. Applying the midpoint formula for the Cartesian plane to segments OB and AC, we find that

$$\text{midpoint of segment } OB = \left(\frac{0 + a + c}{2}, \frac{0 + b}{2}\right) = \left(\frac{a + c}{2}, \frac{b}{2}\right)$$

$$\text{midpoint of segment } AC = \left(\frac{a + c}{2}, \frac{b + 0}{2}\right) = \left(\frac{a + c}{2}, \frac{b}{2}\right)$$

The midpoints of segments OA and AC are the same, so the diagonals of the parallelogram $OABC$ meet at their midpoints and thus bisect each other.

Now try Exercise 37.

QUICK REVIEW P.2

In Exercises 1 and 2, plot the two numbers on a number line. Then find the distance between them.

1. $\sqrt{7}, \sqrt{2}$ **2.** $-\frac{5}{3}, -\frac{9}{5}$

In Exercises 3 and 4, plot the real numbers on a number line.

3. $-3, 4, 2.5, 0, -1.5$ **4.** $-\frac{5}{2}, -\frac{1}{2}, \frac{2}{3}, 0, -1$

In Exercises 5 and 6, plot the points.

5. $A(3, 5), B(-2, 4), C(3, 0), D(0, -3)$

6. $A(-3, -5), B(2, -4), C(0, 5), D(-4, 0)$

In Exercises 7–10, use a calculator to evaluate the expression. Round your answer to two decimal places.

7. $\frac{-17 + 28}{2}$ **8.** $\sqrt{13^2 + 17^2}$

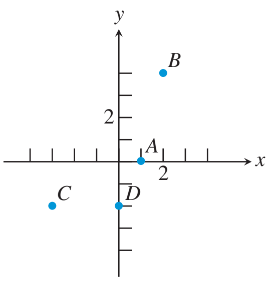
9. $\sqrt{6^2 + 8^2}$ **10.** $\sqrt{(17 - 3)^2 + (-4 - 8)^2}$

SECTION P.2 Exercises

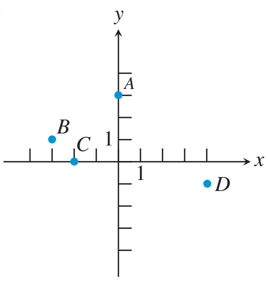
Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator*.

In Exercises 1 and 2, estimate the coordinates of the points.

1.



2.



In Exercises 3 and 4, name the quadrants containing the points.

3. (a) $(2, 4)$ (b) $(0, 3)$ (c) $(-2, 3)$ (d) $(-1, -4)$

4. (a) $\left(\frac{1}{2}, \frac{3}{2}\right)$ (b) $(-2, 0)$ (c) $(-1, -2)$ (d) $\left(-\frac{3}{2}, -\frac{7}{3}\right)$

In Exercises 5–8, evaluate the expression.

5. $3 + |-3|$ **6.** $2 - |-2|$

7. $|(-2)3|$ **8.** $\frac{-2}{|-2|}$

In Exercises 9 and 10, rewrite the expression without using absolute value symbols.

9. $|\pi - 4|$ **10.** $|\sqrt{5} - 5/2|$

In Exercises 11–18, find the distance between the points.

11. $-9.3, 10.6$ **12.** $-5, -17$

13. $(-4, -3), (1, 1)$ **14.** $(-3, -1), (5, -1)$

15. $(0, 0), (3, 4)$ **16.** $(-1, 2), (2, -3)$

17. $(-2, 0), (5, 0)$ **18.** $(0, -8), (0, -1)$

In Exercises 19–22, find the perimeter and area of the figure determined by the points.

19. $(-5, 3), (0, -1), (4, 4)$

20. $(-2, -2), (-2, 2), (2, 2), (2, -2)$

21. $(-3, -1), (-1, 3), (7, 3), (5, -1)$

22. $(-2, 1), (-2, 6), (4, 6), (4, 1)$

In Exercises 23–28, find the midpoint of the line segment with the given endpoints.

23. $-9.3, 10.6$

24. $-5, -17$

25. $(-1, 3), (5, 9)$

26. $(3, \sqrt{2}), (6, 2)$

27. $(-7/3, 3/4), (5/3, -9/4)$

28. $(5, -2), (-1, -4)$

In Exercises 29–34, draw a scatter plot of the data given in the table.

29. U.S. Motor Vehicle Production The total number of motor vehicles in millions (y) produced by the United States each year from 2009 through 2015 is given in the table. *(Source: Automotive News Data Center and R. L. Polk Marketing Systems as reported in The World Almanac and Book of Facts 2017.)*

x	2009	2010	2011	2012	2013	2014	2015
y	5.59	7.63	8.46	10.14	11.07	11.66	12.10

30. World Motor Vehicle Production The total number of motor vehicles in millions (y) produced in the world each year from 2009 through 2015 is given in the table. *(Source: Automotive News Data Center and R. L. Polk Marketing Systems as reported in The World Almanac and Book of Facts 2017.)*

x	2009	2010	2011	2012	2013	2014	2015
y	59.1	73.3	76.0	81.1	87.5	89.8	90.8

31. U.S. Imports from Mexico The total in billions of dollars of U.S. imports from Mexico for selected years is given in Table P.3.

Table P.3 U.S. Imports from Mexico

Time (years)	U.S. Imports (billions of $)
2005	170.1
2010	230.0
2012	277.6
2013	280.6
2014	295.7
2015	296.4

Source: U.S. Census Bureau, The World Almanac and Book of Facts 2017.

32. U.S. Agricultural Exports The total in billions of dollars of U.S. agricultural exports for selected years is given in Table P.4.

Table P.4 U.S. Agricultural Exports

Time (years)	U.S. Exports (billions of $)
2005	62.5
2010	108.5
2012	135.9
2013	141.1
2014	152.3
2015	139.7

Source: U.S. Department of Agriculture, The World Almanac and Book of Facts 2017.

33. U.S. Exports to China The total in billions of dollars of U.S. exports to China for selected years is given in Table P.5.

Table P.5 U.S. Exports to China

Time (years)	U.S. Exports (billions of $)
2005	41.2
2010	91.9
2012	110.5
2013	121.7
2014	123.6
2015	116.1

Source: U.S. Department of Agriculture, The World Almanac and Book of Facts 2017.

34. U.S. Exports to Canada The total in billions of dollars of U.S. exports to Canada for selected years is given in Table P.6.

Table P.6 U.S. Exports to Canada

Time (years)	U.S. Exports (billions of $)
2005	211.9
2010	249.3
2012	292.7
2013	300.8
2014	312.8
2015	280.6

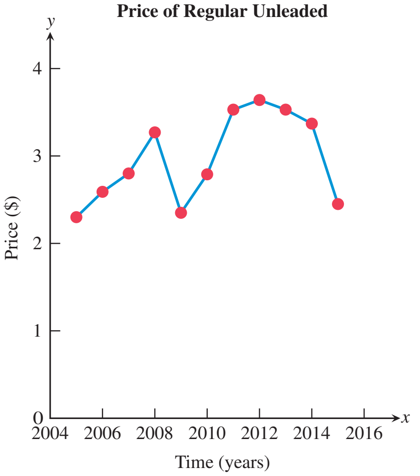
Source: U.S. Census Bureau, The World Almanac and Book of Facts 2017.

35. Reading from Graphs Using the graph below, estimate the price of gasoline (in dollars) for

(a) 2006 (b) 2010 (c) 2014

36. Percent Increase Using the graph below, estimate the percent increase (or decrease) in the price of gasoline from

(a) 2005 to 2010 (b) 2010 to 2015



Source: U.S. Census Bureau, The World Almanac and Book of Facts 2017.

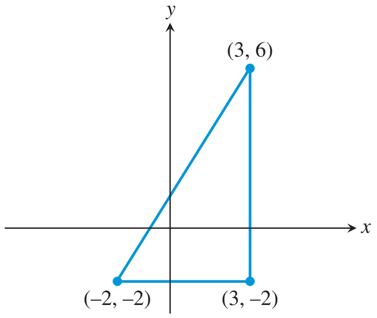
37. Prove that the figure determined by the points is an isosceles triangle: $(1, 3)$, $(4, 7)$, $(8, 4)$

38. **Group Activity** Prove that the diagonals of the figure determined by the points bisect each other.

(a) Square $(-7, -1)$, $(-2, 4)$, $(3, -1)$, $(-2, -6)$

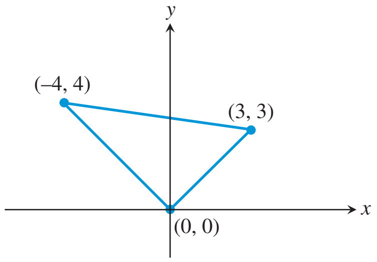
(b) Parallelogram $(-2, -3)$, $(0, 1)$, $(6, 7)$, $(4, 3)$

39. (a) Find the lengths of the sides of the triangle in the figure.



(b) **Writing to Learn** Prove that the triangle is a right triangle.

40. (a) Find the lengths of the sides of the triangle in the figure.



(b) **Writing to Learn** Prove that the triangle is a right triangle.

In Exercises 41–44, find the standard form equation for the circle.

41. Center $(1, 2)$, radius 5

42. Center $(-3, 2)$, radius 1

43. Center $(-1, -4)$, radius 3

44. Center $(0, 0)$, radius $\sqrt{3}$

In Exercises 45–48, find the center and radius of the circle.

45. $(x - 3)^2 + (y - 1)^2 = 36$

46. $(x + 4)^2 + (y - 2)^2 = 121$

47. $x^2 + y^2 = 5$

48. $(x - 2)^2 + (y + 6)^2 = 25$

In Exercises 49–52, write the statement using absolute value notation.

49. The distance between x and 4 is 3.

50. The distance between y and -2 is greater than or equal to 4.

51. The distance between x and c is less than d units.

52. y is more than d units from c.

53. Let $(4, 4)$ be the midpoint of the line segment determined by the points $(1, 2)$ and (a, b). Determine a and b.

54. **Writing to Learn** **Isosceles but Not Equilateral** Prove that the triangle determined by the points $(3, 0)$, $(-1, 2)$, and $(5, 4)$ is isosceles but not equilateral.

55. **Writing to Learn** **Equidistant Point** Prove that the midpoint of the hypotenuse of the right triangle with vertices $(0, 0)$, $(5, 0)$, and $(0, 7)$ is equidistant from the three vertices.

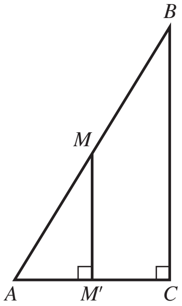
56. **Writing to Learn** Describe the set of real numbers that satisfy $|x - 2| < 3$.

57. **Writing to Learn** Describe the set of real numbers that satisfy $|x + 3| \geq 5$.

Standardized Test Questions

58. **True or False** If a is a real number, then $|a| \geq 0$. Justify your answer.

59. **True or False** Let ΔABC and $\Delta AMM'$ be right triangles as shown in the figure. If M is the midpoint of segment AB, then M' is the midpoint of segment AC. Justify your answer.



In Exercises 60–63, solve these problems without using a calculator.

60. **Multiple Choice** Which of the following is equal to $|1 - \sqrt{3}|$?

(A) $1 - \sqrt{3}$ (B) $\sqrt{3} - 1$

(C) $(1 - \sqrt{3})^2$ (D) $\sqrt{2}$

(E) $\sqrt{1/3}$

61. **Multiple Choice** Which of the following is the midpoint of the line segment with endpoints -3 and 2?

(A) $5/2$ (B) 1

(C) $-1/2$ (D) -1

(E) $-5/2$

62. **Multiple Choice** Which of the following is the center of the circle $(x - 3)^2 + (y + 4)^2 = 2$?

(A) $(3, -4)$ (B) $(-3, 4)$

(C) $(4, -3)$ (D) $(-4, 3)$

(E) $(3/2, -2)$

63. **Multiple Choice** Which of the following points is in the third quadrant?

(A) $(0, -3)$ (B) $(-1, 0)$

(C) $(2, -1)$ (D) $(-1, 2)$

(E) $(-2, -3)$

Explorations

64. Dividing a Line Segment into Thirds

(a) Find the coordinates of the points one-third and two-thirds of the way from $a = 2$ to $b = 8$ on a number line.

(b) Repeat (a) for $a = -3$ and $b = 7$.

(c) Find the coordinates of the points one-third and two-thirds of the way from a to b on a number line.

(d) Find the coordinates of the points one-third and two-thirds of the way from the point $(1, 2)$ to the point $(7, 11)$ in the Cartesian plane.

(e) Find the coordinates of the points one-third and two-thirds of the way from the point (a, b) to the point (c, d) in the Cartesian plane.

Extending the Ideas

65. Writing to Learn Equidistant Point from Vertices of a Right Triangle Prove that the midpoint of the hypotenuse of any right triangle is equidistant from the three vertices.

66. Comparing Areas Consider the four points $A(0, 0)$, $B(0, a)$, $C(a, a)$, and $D(a, 0)$. Let P be the midpoint of the line segment CD and Q the point one-fourth of the way from A to D on segment AD.

(a) Find the area of triangle BPQ.

(b) Compare the area of triangle BPQ with the area of square $ABCD$.

In Exercises 67–69, let $P(a, b)$ be a point in the first quadrant.

67. Find the coordinates of the point Q in the fourth quadrant so that the x-axis is the perpendicular bisector of PQ.

68. Find the coordinates of the point Q in the second quadrant so that the y-axis is the perpendicular bisector of PQ.

69. Find the coordinates of the point Q in the third quadrant so that the origin is the midpoint of the segment PQ.

70. Writing to Learn Prove that the distance formula for the number line is a special case of the distance formula for the Cartesian plane.

P.3 Linear Equations and Inequalities

What you'll learn about

- Equations
- Solving Equations
- Linear Equations in One Variable
- Linear Inequalities in One Variable

... and why

These topics provide the foundation for algebraic techniques needed throughout this text.

Equations

An **equation** is a statement of equality between two expressions. Here are some properties of equality that we use to solve equations algebraically.

Properties of Equality

Let u, v, w, and z be real numbers, variables, or algebraic expressions.

1. **Reflexive** $u = u$
2. **Symmetric** If $u = v$, then $v = u$.
3. **Transitive** If $u = v$ and $v = w$, then $u = w$.
4. **Additive** If $u = v$ and $w = z$, then $u + w = v + z$.
5. **Multiplicative** If $u = v$ and $w = z$, then $uw = vz$.

Solving Equations

A **solution of an equation in *x*** is a value of x for which the equation is true. To **solve an equation in *x*** means to find all values of x for which the equation is true, that is, to find all solutions of the equation.

EXAMPLE 1 Confirming a Solution

Prove that $x = -2$ is a solution of the equation $x^3 - x + 6 = 0$.

SOLUTION Let $x = -2$. Then

$$\begin{aligned} x^3 - x + 6 &= (-2)^3 - (-2) + 6 \\ &= -8 + 2 + 6 \\ &= 0. \end{aligned}$$

Thus, by the transitive property of equality, -2 is a value of x for which the equation $x^3 - x + 6 = 0$ is true. Hence, $x = -2$ is a solution of the equation $x^3 - x + 6 = 0$.

Now try Exercise 1.

Linear Equations in One Variable

The most basic equation in algebra is a *linear equation*.

DEFINITION Linear Equation in *x*

A **linear equation in *x*** is one that can be written in the form

$$ax + b = 0,$$

where a and b are real numbers and $a \neq 0$.

The equation $2z - 4 = 0$ is linear in the variable z. Because of the exponent 2, the equation $3u^2 - 12 = 0$ is *not* linear in the variable u. A linear equation in one variable has exactly one solution. We solve such an equation by transforming it into an *equivalent equation* whose solution is obvious. Two or more equations are **equivalent** if they have the same solutions. For example, the equations $2z - 4 = 0$, $2z = 4$, and $z = 2$ are all equivalent equations.

Operations for Equivalent Equations

An equivalent equation is obtained if one or more of the following operations are performed.

Operation	**Given Equation**	**Equivalent Equation**
1. Combine like terms, reduce fractions, and remove grouping symbols.	$2x + x = \frac{3}{9}$	$3x = \frac{1}{3}$
2. Perform the same operation on both sides.		
(a) Add (-3).	$x + 3 = 7$	$x = 4$
(b) Subtract $(2x)$.	$5x = 2x + 4$	$3x = 4$
(c) Multiply by a nonzero constant $(1/3)$.	$3x = 12$	$x = 4$
(d) Divide by a nonzero constant (3).	$3x = 12$	$x = 4$

The next two examples illustrate how to use equivalent equations to solve linear equations.

EXAMPLE 2 Solving a Linear Equation

Solve $2(2x - 3) + 3(x + 1) = 5x + 2$. Support the result with a calculator.

SOLUTION

$$\begin{aligned} 2(2x - 3) + 3(x + 1) &= 5x + 2 & \\ 4x - 6 + 3x + 3 &= 5x + 2 & \text{Distributive properties} \\ 7x - 3 &= 5x + 2 & \text{Combine like terms.} \\ 2x &= 5 & \text{Add 3, and subtract } 5x. \\ x &= 2.5 & \text{Divide by 2.} \end{aligned}$$

To support our algebraic work we can evaluate the expressions in the original equation for $x = 2.5$. Figure P.17 shows that each side of the original equation is equal to 14.5 if $x = 2.5$. **Now try Exercise 23.**

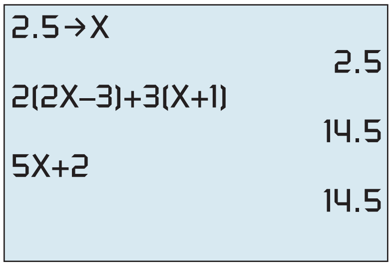


Figure P.17 The top line stores the number 2.5 into the variable x. (Example 2)

If an equation involves fractions, find the least common denominator (LCD) of the fractions and multiply both sides by the LCD. This is sometimes referred to as *clearing the equation of fractions*. Example 3 illustrates this method.

Integers and Fractions

Notice in Example 3 that $2 = \frac{2}{1}$.

EXAMPLE 3 Solving a Linear Equation Involving Fractions

Solve

$$\frac{5y - 2}{8} = 2 + \frac{y}{4}.$$

SOLUTION The denominators are 8, 1, and 4. The LCD of the fractions is 8. (See Appendix A.3 if necessary.)

$$\begin{aligned}
\frac{5y - 2}{8} &= 2 + \frac{y}{4} && \\
8\left(\frac{5y - 2}{8}\right) &= 8\left(2 + \frac{y}{4}\right) && \text{Multiply by the LCD 8.} \\
8 \cdot \frac{5y - 2}{8} &= 8 \cdot 2 + 8 \cdot \frac{y}{4} && \text{Distributive property} \\
5y - 2 &= 16 + 2y && \text{Simplify.} \\
5y &= 18 + 2y && \text{Add 2.} \\
3y &= 18 && \text{Subtract } 2y. \\
y &= 6 && \text{Divide by 3.}
\end{aligned}$$

We leave it to you to check the solution using either paper and pencil or a calculator.

Now try Exercise 25.

Linear Inequalities in One Variable

We used inequalities to describe order on the number line in Section P.1. For example, if x is to the left of 2 on the number line, or if x is any real number less than 2, we write $x < 2$. The most basic inequality in algebra is a *linear inequality*.

DEFINITION Linear Inequality in *x*

A **linear inequality in *x*** is one that can be written in the form

$$ax + b < 0, \quad ax + b \le 0, \quad ax + b > 0, \quad \text{or} \quad ax + b \ge 0,$$

where a and b are real numbers and $a \neq 0$.

To **solve an inequality in *x*** means to find all values of x for which the inequality is true. A **solution of an inequality in *x*** is a value of x for which the inequality is true. The set of all solutions of an inequality is the **solution set** of the inequality. We **solve an inequality** by finding its solution set. Here is a list of properties we use to solve inequalities.

Direction of an Inequality

Multiplying (or dividing) an inequality by a positive number preserves the direction of the inequality. Multiplying (or dividing) an inequality by a negative number reverses the direction.

Properties of Inequalities

Let u, v, w, and z be real numbers, variables, or algebraic expressions, and c a real number.

1. **Transitive** If $u < v$ and $v < w$, then $u < w$.
2. **Additive** If $u < v$, then $u + w < v + w$.
   If $u < v$ and $w < z$, then $u + w < v + z$.
3. **Multiplicative** If $u < v$ and $c > 0$, then $uc < vc$.
   If $u < v$ and $c < 0$, then $uc > vc$.

There are similar properties for $\le$, $>$, and $\ge$.

The set of solutions of a linear inequality in one variable is an interval of real numbers. Just as with linear equations, we solve a linear inequality by transforming it into an *equivalent inequality* whose solutions are obvious. Two or more inequalities are **equivalent** if they have the same solution set. The properties of inequalities listed on the previous page describe operations that transform an inequality into an equivalent one.

EXAMPLE 4 Solving a Linear Inequality

Solve $3(x - 1) + 2 \leq 5x + 6$.

SOLUTION

$$\begin{aligned}
3(x - 1) + 2 &\leq 5x + 6 \\
3x - 3 + 2 &\leq 5x + 6 && \text{Distributive property} \\
3x - 1 &\leq 5x + 6 && \text{Combine like terms.} \\
3x &\leq 5x + 7 && \text{Add 1.} \\
-2x &\leq 7 && \text{Subtract } 5x. \\
\left(-\frac{1}{2}\right) \cdot -2x &\geq \left(-\frac{1}{2}\right) \cdot 7 && \text{Multiply by } -1/2. \text{ (The inequality reverses.)} \\
x &\geq -3.5
\end{aligned}$$

The solution set of the inequality is the set of all real numbers greater than or equal to -3.5. In interval notation, the solution set is $[-3.5, \infty)$.

Now try Exercise 41.

Because the solution set of a linear inequality is an interval of real numbers, we can display the solution set with a number line graph as illustrated in Example 5.

EXAMPLE 5 Solving a Linear Inequality Involving Fractions

Solve the inequality, and graph its solution set.

$$\frac{x}{3} + \frac{1}{2} > \frac{x}{4} + \frac{1}{3}$$

SOLUTION The LCD of the fractions is 12.

$$\begin{aligned}
\frac{x}{3} + \frac{1}{2} &> \frac{x}{4} + \frac{1}{3} \\
12 \cdot \left(\frac{x}{3} + \frac{1}{2}\right) &> 12 \cdot \left(\frac{x}{4} + \frac{1}{3}\right) && \text{Multiply by the LCD 12.} \\
4x + 6 &> 3x + 4 && \text{Simplify.} \\
x + 6 &> 4 && \text{Subtract } 3x. \\
x &> -2 && \text{Subtract 6.}
\end{aligned}$$

The solution set is the interval $(-2, \infty)$. Its graph is shown in Figure P.18.

Now try Exercise 37.

Figure P.18 The graph of the solution set of the inequality in Example 5.

Sometimes two inequalities are combined in a *double inequality,* which is solved by isolating x as the middle expression. Example 6 illustrates this.

EXAMPLE 6 Solving a Double Inequality

Solve the inequality, and graph its solution set.

$$-3 < \frac{2x+5}{3} \leq 5$$

SOLUTION

$$-3 < \frac{2x+5}{3} \leq 5$$

$$-9 < 2x + 5 \leq 15 \quad \text{Multiply by 3.}$$

$$-14 < 2x \leq 10 \quad \text{Subtract 5.}$$

$$-7 < x \leq 5 \quad \text{Divide by 2.}$$

The solution set is the set of all real numbers greater than -7 and less than or equal to 5. In interval notation, the solution set is $(-7, 5]$. Its graph is shown in Figure P.19.

Now try Exercise 47.

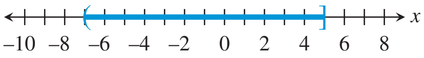


Figure P.19 The graph of the solution set of the double inequality in Example 6.

QUICK REVIEW P.3

In Exercises 1 and 2, simplify the expression by combining like terms.

1. $2x + 5x + 7 + y - 3x + 4y + 2$

2. $4 + 2x - 3z + 5y - x + 2y - z - 2$

In Exercises 3 and 4, use the distributive property to expand the products. Simplify the resulting expression by combining like terms.

3. $3(2x - y) + 4(y - x) + x + y$

4. $5(2x + y - 1) + 4(y - 3x + 2) + 1$

In Exercises 5–10, use the LCD to combine the fractions. Simplify the resulting fraction.

5. $\frac{2}{y} + \frac{3}{y}$ **6.** $\frac{1}{y-1} + \frac{3}{y-2}$

7. $2 + \frac{1}{x}$ **8.** $\frac{1}{x} + \frac{1}{y} - x$

9. $\frac{x+4}{2} + \frac{3x-1}{5}$ **10.** $\frac{x}{3} + \frac{x}{4}$

SECTION P.3 Exercises

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator*.

In Exercises 1–4, which values of x are solutions of the equation?

1. $2x^2 + 5x = 3$

(a) $x = -3$ (b) $x = -\frac{1}{2}$ (c) $x = \frac{1}{2}$

2. $\frac{x}{2} + \frac{1}{6} = \frac{x}{3}$

(a) $x = -1$ (b) $x = 0$ (c) $x = 1$

3. $\sqrt{1 - x^2} + 2 = 3$

(a) $x = -2$ (b) $x = 0$ (c) $x = 2$

4. $(x - 2)^{1/3} = 2$

(a) $x = -6$ (b) $x = 8$ (c) $x = 10$

In Exercises 5–10, determine whether the equation is linear in x.

5. $5 - 3x = 0$ **6.** $5 = 10/2$

7. $x + 3 = x - 5$ **8.** $x - 3 = x^2$

9. $2\sqrt{x} + 5 = 10$ **10.** $x + \frac{1}{x} = 1$

In Exercises 11–24, solve the equation without using a calculator.

11. $3x = 24$ **12.** $4x = -16$

13. $3t - 4 = 8$ **14.** $2t - 9 = 3$

15. $2x - 3 = 4x - 5$ **16.** $4 - 2x = 3x - 6$

17. $4 - 3y = 2(y + 4)$ **18.** $4(y - 2) = 5y$

19. $\frac{1}{2}x = \frac{7}{8}$ **20.** $\frac{2}{3}x = \frac{4}{5}$

21. $\frac{1}{2}x + \frac{1}{3} = 1$

22. $\frac{1}{3}x + \frac{1}{4} = 1$

23. $2(3 - 4z) - 5(2z + 3) = z - 17$

24. $3(5z - 3) - 4(2z + 1) = 5z - 2$

In Exercises 25–28, solve the equation. Support your answer with a calculator.

25. $\frac{2x - 3}{4} + 5 = 3x$

26. $2x - 4 = \frac{4x - 5}{3}$

27. $\frac{t + 5}{8} - \frac{t - 2}{2} = \frac{1}{3}$

28. $\frac{t - 1}{3} + \frac{t + 5}{4} = \frac{1}{2}$

29. Writing to Learn Write a statement about a solution of an equation suggested by the computations in the figure.

(a)
```
-2→X
                  -2
2X²+X-6
                   0
```

(b)
```
3/2→X
                 1.5
2X²+X-6
                   0
```

30. Writing to Learn Write a statement about a solution of an equation suggested by the computations in the figure.

(a)
```
2→X
                   2
7X+5
                  19
4X-7
                   1
```

(b)
```
-4→X
                  -4
7X+5
                 -23
4X-7
                 -23
```

In Exercises 31–34, which values of x are solutions of the inequality?

31. $2x - 3 < 7$

(a) $x = 0$ (b) $x = 5$ (c) $x = 6$

32. $3x - 4 \geq 5$

(a) $x = 0$ (b) $x = 3$ (c) $x = 4$

33. $-1 < 4x - 1 \leq 11$

(a) $x = 0$ (b) $x = 2$ (c) $x = 3$

34. $-3 \leq 1 - 2x \leq 3$

(a) $x = -1$ (b) $x = 0$ (c) $x = 2$

In Exercises 35–42, solve the inequality, and draw a number line graph of the solution set.

35. $x - 4 < 2$

36. $x + 3 > 5$

37. $2x - 1 \leq 4x + 3$

38. $3x - 1 \geq 6x + 8$

39. $2 \leq x + 6 < 9$

40. $-1 \leq 3x - 2 < 7$

41. $2(5 - 3x) + 3(2x - 1) \leq 2x + 1$

42. $4(1 - x) + 5(1 + x) > 3x - 1$

In Exercises 43–54, solve the inequality.

43. $\frac{5x + 7}{4} \leq -3$

44. $\frac{3x - 2}{5} > -1$

45. $4 \geq \frac{2y - 5}{3} \geq -2$

46. $1 > \frac{3y - 1}{4} > -1$

47. $0 \leq 2z + 5 < 8$

48. $-6 < 5t - 1 < 0$

49. $\frac{x - 5}{4} + \frac{3 - 2x}{3} < -2$

50. $\frac{3 - x}{2} + \frac{5x - 2}{3} < -1$

51. $\frac{2y - 3}{2} + \frac{3y - 1}{5} < y - 1$

52. $\frac{3 - 4y}{6} - \frac{2y - 3}{8} \geq 2 - y$

53. $\frac{1}{2}(x - 4) - 2x \leq 5(3 - x)$

54. $\frac{1}{2}(x + 3) + 2(x - 4) < \frac{1}{3}(x - 3)$

In Exercises 55–58, find the solutions of the equation or inequality that are displayed in Figure P.20.

55. $x^2 - 2x < 0$

56. $x^2 - 2x = 0$

57. $x^2 - 2x > 0$

58. $x^2 - 2x \leq 0$

X	Y1	
0	0	
1	-1	
2	0	
3	3	
4	8	
5	15	
6	24	
Y1 = X²-2X		

Figure P.20 The second column gives values of $y_1 = x^2 - 2x$ for $x = 0, 1, 2, 3, 4, 5,$ and 6.

59. Writing to Learn Explain how the second equation was obtained from the first.

$$x - 3 = 2x + 3, \quad 2x - 6 = 4x + 6$$

60. Writing to Learn Explain how the second equation was obtained from the first.

$$2x - 1 = 2x - 4, \quad x - \frac{1}{2} = x - 2$$

61. Group Activity Determine whether the two equations are equivalent.

(a) $3x = 6x + 9, \quad x = 2x + 9$

(b) $6x + 2 = 4x + 10, \quad 3x + 1 = 2x + 5$

62. Group Activity Determine whether the two equations are equivalent.

(a) $3x + 2 = 5x - 7, \quad -2x + 2 = -7$

(b) $2x + 5 = x - 7, \quad 2x = x - 7$

Standardized Test Questions

63. True or False $-6 > -2$. Justify your answer.

64. True or False $2 \le \frac{6}{3}$. Justify your answer.

In Exercises 65–68, you may use a graphing calculator to solve these problems.

65. Multiple Choice Which of the following equations is equivalent to the equation $3x + 5 = 2x + 1$?

(A) $3x = 2x$ (B) $3x = 2x + 4$

(C) $\frac{3}{2}x + \frac{5}{2} = x + 1$ (D) $3x + 6 = 2x$

(E) $3x = 2x - 4$

66. Multiple Choice Which of the following inequalities is equivalent to the inequality $-3x < 6$?

(A) $3x < -6$ (B) $x < 10$

(C) $x > -2$ (D) $x > 2$

(E) $x > 3$

67. Multiple Choice Which of the following is the solution to the equation $x(x + 1) = 0$?

(A) $x = 0$ or $x = -1$ (B) $x = 0$ or $x = 1$

(C) Only $x = -1$ (D) Only $x = 0$

(E) Only $x = 1$

68. Multiple Choice Which of the following represents an equation equivalent to the equation

$$\frac{2x}{3} + \frac{1}{2} = \frac{x}{4} - \frac{1}{3}$$

that is cleared of fractions?

(A) $2x + 1 = x - 1$ (B) $8x + 6 = 3x - 4$

(C) $4x + 3 = \frac{3}{2}x - 2$ (D) $4x + 3 = 3x - 4$

(E) $4x + 6 = 3x - 4$

Explorations

69. Testing Inequalities on a Calculator

(a) The calculator we use indicates that the statement $2 < 3$ is true by returning the value 1 (for true) when $2 < 3$ is entered. Try it with your calculator.

(b) The calculator we use indicates that the statement $2 < 1$ is false by returning the value 0 (for false) when $2 < 1$ is entered. Try it with your calculator.

(c) Use your calculator to test which of these two numbers is larger: $799/800$, $800/801$.

(d) Use your calculator to test which of these two numbers is larger: $-102/101$, $-103/102$.

(e) If your calculator returns 0 when you enter $2x + 1 < 4$, what can you conclude about the value stored in x?

Extending the Ideas

70. Perimeter of a Rectangle The formula for the perimeter P of a rectangle is

$$P = 2(L + W).$$

Solve this equation for W.

71. Area of a Trapezoid The formula for the area A of a trapezoid is

$$A = \frac{1}{2}h(b_1 + b_2).$$

Solve this equation for b_1.

72. Volume of a Sphere

The formula for the volume V of a sphere is

$$V = \frac{4}{3}\pi r^3.$$

Solve this equation for r.

73. Celsius and Fahrenheit The formula for Celsius temperature in terms of Fahrenheit temperature is

$$C = \frac{5}{9}(F - 32).$$

Solve the equation for F.

P.4 Lines in the Plane

What you'll learn about

- Slope of a Line
- Point-Slope Form Equation of a Line
- Slope-Intercept Form Equation of a Line
- Graphing Linear Equations in Two Variables
- Parallel and Perpendicular Lines
- Applying Linear Equations in Two Variables

... and why

Linear equations are used extensively in applications involving business and behavioral science.

Slope of a Line

The slope of a nonvertical line is the vertical change divided by the horizontal change between any two points on the line. For the points (x_1, y_1) and (x_2, y_2), the vertical change is $\Delta y = y_2 - y_1$, and the horizontal change is $\Delta x = x_2 - x_1$. (Δy is read "delta" y.) See Figure P.21.

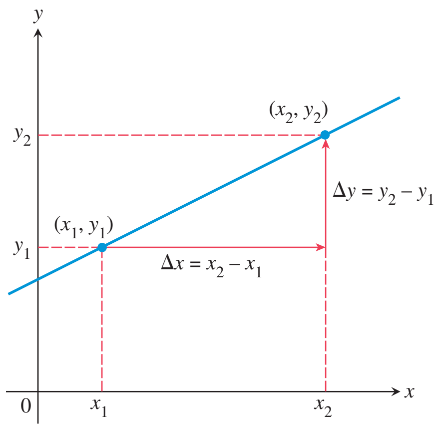


Figure P.21 The slope of a nonvertical line can be found from the coordinates of any two points on the line.

DEFINITION Slope of a Line

The **slope** of a nonvertical line through the points (x_1, y_1) and (x_2, y_2) is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

If the line is vertical, then $x_1 = x_2$ and the slope is undefined.

EXAMPLE 1 Finding the Slope of a Line

Find the slope of the line through the two points. Sketch a graph of the line.

(a) $(-1, 2)$ and $(4, -2)$

(b) $(1, 1)$ and $(3, 4)$

SOLUTION

(a) The two points are $(x_1, y_1) = (-1, 2)$ and $(x_2, y_2) = (4, -2)$. Thus,

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(-2) - 2}{4 - (-1)} = -\frac{4}{5} = -0.8.$$

(b) The two points are $(x_1, y_1) = (1, 1)$ and $(x_2, y_2) = (3, 4)$. Thus,

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 1}{3 - 1} = \frac{3}{2} = 1.5.$$

The graphs of these two lines are shown in Figure P.22.

Now try Exercise 3.

Slope Formula

The slope does not depend on the order of the points. We could use $(x_1, y_1) = (4, -2)$ and $(x_2, y_2) = (-1, 2)$ in Example 1a. Check it out.

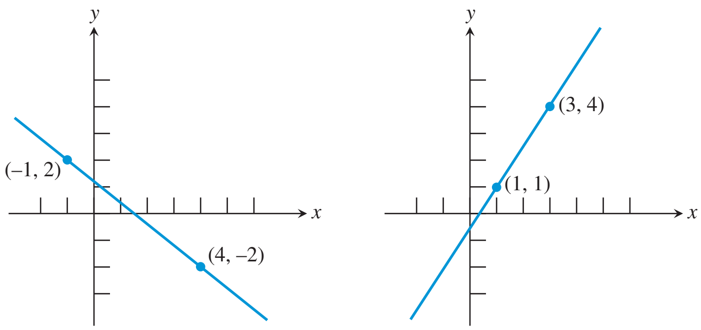


Figure P.22 The graphs of the two lines in Example 1.

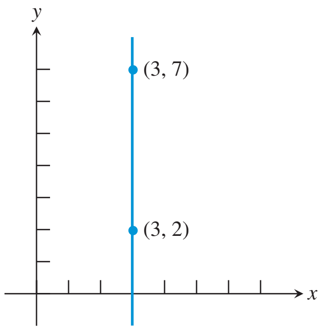


Figure P.23 Applying the slope formula to this vertical line gives $m = 5/0$, which is not defined. Thus, the slope of a vertical line is undefined.

Figure P.23 shows a vertical line through the points $(3, 2)$ and $(3, 7)$. If we try to calculate its slope using the slope formula $(y_2 - y_1)/(x_2 - x_1)$, we get zero in the denominator. So, it makes sense to say that a vertical line does not have a slope, or that its slope is undefined.

Point-Slope Form Equation of a Line

If we know the coordinates of one point on a line and the slope of the line, then we can find an equation for that line. For example, the line in Figure P.24 passes through the point (x_1, y_1) and has slope m. If (x, y) is any other point on this line, the definition of the slope yields the equation

$$m = \frac{y - y_1}{x - x_1} \quad \text{or} \quad y - y_1 = m(x - x_1).$$

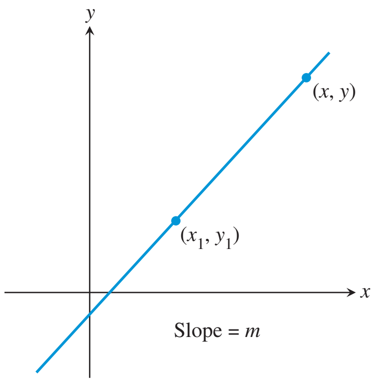


Figure P.24 The line through (x_1, y_1) with slope m.

An equation written this way is in the *point-slope form*.

DEFINITION Point-Slope Form of an Equation of a Line

The **point-slope form** of an equation of a line that passes through the point (x_1, y_1) and has slope m is

$$y - y_1 = m(x - x_1).$$

EXAMPLE 2 Using the Point-Slope Form

Use the point-slope form to find an equation of the line that passes through the point $(-3, -4)$ and has slope 2.

SOLUTION We substitute $x_1 = -3$, $y_1 = -4$, and $m = 2$ into the point-slope form, and simplify the resulting equation.

$$\begin{aligned} y - y_1 &= m(x - x_1) && \text{Point-slope form} \\ y - (-4) &= 2(x - (-3)) && x_1 = -3,\ y_1 = -4,\ m = 2 \\ y + 4 &= 2(x + 3) && \text{Simplify.} \end{aligned}$$

For graphing purposes, this equation can be written as $y = 2(x + 3) - 4$ or as $y = 2x + 2$.

Now try Exercise 11.

y-Intercept

The b in $y = mx + b$ is often referred to as "the y-intercept" instead of "the y-coordinate of the y-intercept."

Slope-Intercept Form Equation of a Line

The **y-intercept** of a nonvertical line is the point where the line intersects the y-axis. If we know the y-intercept and the slope of the line, we can apply the point-slope form to find an equation of the line.

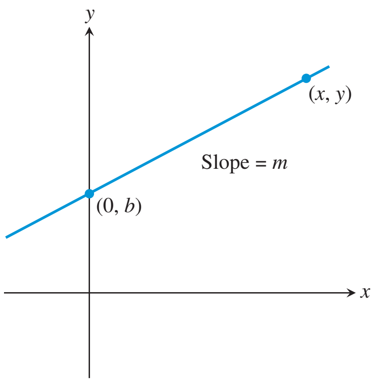


Figure P.25 The line with slope m and y-intercept b.

Alternative Solution

You could solve Example 3 using the point-slope form:

$$\begin{aligned} y - 6 &= 3(x - (-1)) \\ y &= 3(x + 1) + 6 \\ y &= 3x + 3 + 6 \\ y &= 3x + 9 \end{aligned}$$

Figure P.25 shows a line with slope m and y-intercept $(0, b)$, or b for short. A point-slope form equation for this line is $y - b = m(x - 0)$. By rewriting this equation, we obtain the form known as the *slope-intercept form*.

DEFINITION Slope-Intercept Form of an Equation of a Line

The **slope-intercept form** of an equation of a line with slope m and y-intercept $(0, b)$ is

$$y = mx + b.$$

EXAMPLE 3 Using the Slope-Intercept Form

Using the slope-intercept form, write an equation of the line with slope 3 that passes through the point $(-1, 6)$.

SOLUTION

$$\begin{aligned} y &= mx + b && \text{Slope-intercept form} \\ y &= 3x + b && m = 3 \\ 6 &= 3(-1) + b && y = 6 \text{ when } x = -1 \\ b &= 9 \end{aligned}$$

The slope-intercept form of the equation is $y = 3x + 9$. **Now try Exercise 21.**

We should not use the phrase "*the* equation of a line" because each line has many equations. Every line has an equation that can be written in the form $Ax + By + C = 0$ where A and B are not both zero. This form is the **general form** for an equation of a line.

If $B \neq 0$, the general form can be changed to the slope-intercept form as follows:

$$\begin{aligned} Ax + By + C &= 0 \\ By &= -Ax - C \\ y &= \underbrace{-\frac{A}{B}}_{\text{slope}}x + \underbrace{\left(-\frac{C}{B}\right)}_{y\text{-intercept}} \end{aligned}$$

Forms of Equations of Lines

General form:	$Ax + By + C = 0$, A and B not both zero
Slope-intercept form:	$y = mx + b$
Point-slope form:	$y - y_1 = m(x - x_1)$
Vertical line:	$x = a$
Horizontal line:	$y = b$

Graphing Linear Equations in Two Variables

A **linear equation in x and y** is one that can be written in the form

$$Ax + By = C,$$

where A and B are not both zero. Rewriting the equation as $Ax + By - C = 0$, we see that it is closely related to the general form. If $B = 0$, the line is vertical, and if $A = 0$, the line is horizontal.

The **graph** of an equation in x and y consists of all pairs (x, y) that are solutions of the equation. For example, $(1, 2)$ is a **solution** of the equation $2x + 3y = 8$ because substituting $x = 1$ and $y = 2$ into the equation leads to the true statement $8 = 8$. The pairs $(-2, 4)$ and $(2, 4/3)$ are also solutions.

Because the graph of a linear equation in x and y is a straight line, to draw the graph we can find two solutions and then connect them with a straight line. If a line is neither horizontal nor vertical, then two easy points to find are its x-intercept and y-intercept. The **x-intercept** is the point $(a, 0)$ where the graph intersects the x-axis. Set $y = 0$ and solve for x to find the x-intercept. To find the y-intercept, set $x = 0$ and solve for y.

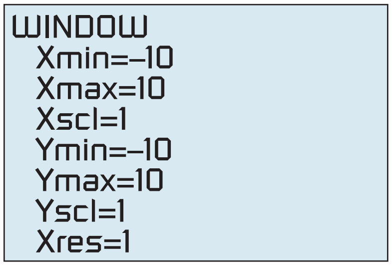


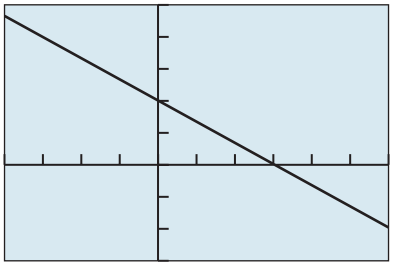
Figure P.26 The window dimensions for the *standard window*. The notation "$[-10, 10]$ by $[-10, 10]$" is used to represent window dimensions like these.

Graphing with a Graphing Utility

To draw a graph of an equation using a grapher:

1. Rewrite the equation in the form $y =$ (an expression in x).
2. Enter the expression into the grapher.
3. Select an appropriate *viewing window*. (See Figures P.26 and P.27, Example 4, and the margin note.)
4. Press the "graph" key.

A graphing utility, or *grapher*, computes y-values for a select set of x-values between Xmin and Xmax and plots the corresponding (x, y) points.



$[-4, 6]$ by $[-3, 5]$

Figure P.27 The graph of $2x + 3y = 6$. Notice that the points $(0, 2)$ (y-intercept) and $(3, 0)$ (x-intercept) lie on the graph and are solutions of the equation. (Example 4)

EXAMPLE 4 Using a Graphing Utility

Draw the graph of $2x + 3y = 6$.

SOLUTION First we solve for y.

$$2x + 3y = 6$$

$$3y = -2x + 6 \qquad \text{Solve for } y.$$

$$y = -\frac{2}{3}x + 2 \qquad \text{Divide by 3.}$$

Figure P.27 shows the graph of $y = -(2/3)x + 2$, or equivalently, the graph of the linear equation $2x + 3y = 6$ in the $[-4, 6]$ by $[-3, 5]$ viewing window.

Now try Exercise 27.

Viewing Window

The **viewing window** $[-4, 6]$ by $[-3, 5]$ in Example 4 and Figure P.27 means $-4 \le x \le 6$ and $-3 \le y \le 5$.

Parallel and Perpendicular Lines

EXPLORATION 1 Investigating Graphs of Linear Equations

1. What do the graphs of $y = mx + b$ and $y = mx + c$, $b \ne c$, have in common? How are they different?
2. Graph $y = 2x$ and $y = -(1/2)x$ in a *square viewing window*. (See margin note.) On the grapher we use, the "decimal window" is square. Estimate the angle between the two lines.
3. Repeat part 2 for $y = mx$ and $y = -(1/m)x$ with $m = 1, 3, 4,$ and 5.

Square Viewing Window

A **square viewing window** on a grapher is one in which angles appear to be true. For example, the line $y = x$ will appear to make a 45° angle with the positive x-axis. Furthermore, a distance of 1 on the x- and y-axes will appear to be the same.

Parallel lines and perpendicular lines were involved in Exploration 1. Using a grapher to decide whether lines are parallel or perpendicular is risky. Here is an algebraic test to determine whether two lines are parallel or perpendicular.

Parallel and Perpendicular Lines

1. Two nonvertical lines are parallel if and only if their slopes are equal. Any two distinct vertical lines are parallel.
2. Two nonvertical lines are perpendicular if and only if their slopes m_1 and m_2 are opposite reciprocals, that is, if and only if

$$m_1 = -\frac{1}{m_2}.$$

A vertical line is perpendicular to a horizontal line, and vice versa.

EXAMPLE 5 Finding an Equation of a Parallel Line

Find an equation of the line through $P(1, -2)$ that is parallel to the line l with equation $3x - 2y = 1$.

SOLUTION We find the slope of l by writing its equation in slope-intercept form.

$$\begin{aligned} 3x - 2y &= 1 && \text{Equation for } l \\ -2y &= -3x + 1 && \text{Subtract } 3x. \\ y &= \frac{3}{2}x - \frac{1}{2} && \text{Divide by } -2. \end{aligned}$$

The slope of l is $3/2$.

The line whose equation we seek has slope $3/2$ and contains the point $(x_1, y_1) = (1, -2)$. Thus, the point-slope form equation for the line we seek is

$$y + 2 = \frac{3}{2}(x - 1),$$

which also can be written as

$$y = \frac{3}{2}x - \frac{7}{2} \quad \text{or} \quad 3x - 2y = 7.$$

Now try Exercise 41(a).

EXAMPLE 6 Finding an Equation of a Perpendicular Line

Find an equation of the line through $P(2, -3)$ that is perpendicular to the line l with equation $4x + y = 3$. Support the result with a grapher.

SOLUTION We find the slope of l by writing its equation in slope-intercept form.

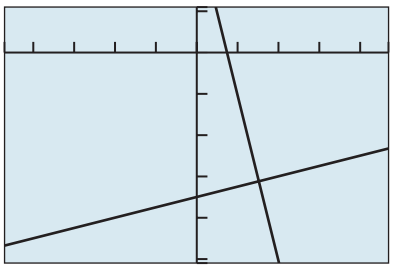
$$\begin{aligned} 4x + y &= 3 && \text{Equation for } l \\ y &= -4x + 3 && \text{Subtract } 4x. \end{aligned}$$

The slope of l is -4.

The line whose equation we seek has slope $-1/(-4) = 1/4$ and passes through the point $(x_1, y_1) = (2, -3)$. We use the point-slope form, then simplify the equation:

$$\begin{aligned} y - (-3) &= \frac{1}{4}(x - 2) \\ y + 3 &= \frac{1}{4}x - \frac{1}{2} && \text{Distributive property} \\ y &= \frac{1}{4}x - \frac{7}{2} \end{aligned}$$

Figure P.28 shows the graphs of the two equations in a square viewing window and suggests that the graphs are indeed perpendicular. **Now try Exercise 43(b).**



$[-4.7, 4.7]$ by $[-5.1, 1.1]$

Figure P.28 The graphs of $y = -4x + 3$ and $y = (1/4)x - 7/2$ in this square viewing window appear to intersect at a right angle. (Example 6)

Applying Linear Equations in Two Variables

Linear equations and their graphs occur frequently in applications. Algebraic solutions to these application problems often require finding an equation of a line and solving a linear equation in one variable. Grapher techniques complement algebraic ones.

EXAMPLE 7 Finding the Depreciation of Real Estate

Camelot Apartments purchased a \$50,000 building. For tax purposes, its value depreciates \$2000 per year over a 25-year period.

(a) Write a linear equation giving the value y of the building in terms of the years x after the purchase.

(b) In how many years will the value of the building be \$24,500?

SOLUTION

(a) We need to determine the value of m and b so that $y = mx + b$, where $0 \le x \le 25$. We know that $y = 50{,}000$ when $x = 0$, so the line has y-intercept $(0, 50{,}000)$ and $b = 50{,}000$. One year after purchase $(x = 1)$, the value of the building is $50{,}000 - 2000 = 48{,}000$. So when $x = 1$, $y = 48{,}000$. Therefore,

$$\begin{aligned} y &= mx + b \\ 48{,}000 &= m \cdot 1 + 50{,}000 && y = 48{,}000 \text{ when } x = 1. \\ -2000 &= m && \text{Subtract } 50{,}000. \end{aligned}$$

The value y of the building x years after its purchase is

$$y = -2000x + 50{,}000.$$

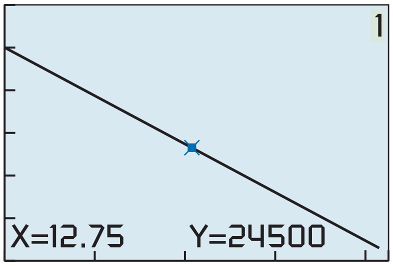
(b) We need to find the value of x when $y = 24{,}500$. So, we substitute 24,500 for y in the equation $y = -2000x + 50{,}000$.

$$\begin{aligned} 24{,}500 &= -2000x + 50{,}000 && \text{Set } y = 24{,}500. \\ -25{,}500 &= -2000x && \text{Subtract } 50{,}000. \\ 12.75 &= x && \text{Divide by } -2000. \end{aligned}$$

The value of the building will be \$24,500 precisely 12.75 years, or 12 years 9 months, after the building was purchased by Camelot Apartments.

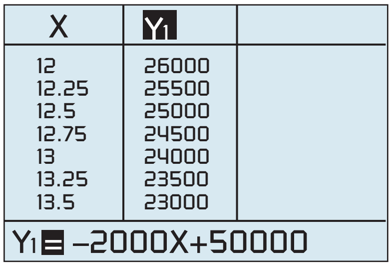
We can support our algebraic work both graphically and numerically. The trace coordinates in Figure P.29a show graphically that $(12.75, 24{,}500)$ is a solution of $y = -2000x + 50{,}000$. This means that $y = 24{,}500$ when $x = 12.75$. Figure P.29b is a table of values for $y = -2000x + 50{,}000$ for a few values of x. The fourth line of the table shows numerically that $y = 24{,}500$ when $x = 12.75$.

Now try Exercise 45.



[0, 23.5] by [0, 60000]

(a)



X	Y1	
12	26000	
12.25	25500	
12.5	25000	
12.75	24500	
13	24000	
13.25	23500	
13.5	23000	

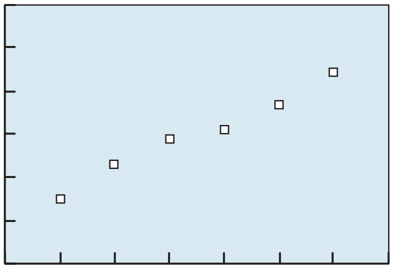
Y1 = −2000X+50000

(b)

Figure P.29 A (a) graph and (b) table of values for $y = -2000x + 50{,}000$. (Example 7)

Figure P.30 on page 58 shows Americans' income from 2010 through 2015 in trillions of dollars and a corresponding scatter plot of these data. In Example 8, we model the data in Figure P.30 with a linear equation.

Time (Years)	Income (trillions of $)
2010	12.5
2011	13.3
2012	13.9
2013	14.1
2014	14.7
2015	15.4



[2009, 2016] by [11, 17]

Figure P.30 Americans' personal income. (Example 8)

Source: U.S. Census Bureau, The World Almanac and Book of Facts 2017.

EXAMPLE 8 Finding a Linear Model for Americans' Personal Income over Time

Americans' total personal income in trillions of dollars is given in Figure P.30.

(a) Write a linear equation for Americans' income y in terms of the year x using the points $(2010, 12.5)$ and $(2015, 15.4)$.

(b) Use the equation in (a) to estimate Americans' income in 2012.

(c) Use the equation in (a) to predict Americans' income in 2020.

(d) Superimpose a graph of the linear equation in (a) on a scatter plot of the data like the one shown in Figure P.30.

SOLUTION

(a) Let $y = mx + b$. The slope of the line through $(2010, 12.5)$ and $(2015, 15.4)$ is

$$m = \frac{15.4 - 12.5}{2010 - 2005} = \frac{2.9}{5} = 0.58.$$

Using this slope and the point (2010, 12.5) yields

$$y = 0.58(x - 2010) + 12.5.$$

(b) To estimate Americans' total personal income for the year 2012, we let $x = 2012$ in the equation found in part (a):

$$\begin{aligned} y &= 0.58(2012 - 2010) + 12.5 \\ &= 0.58 \cdot 2 + 12.5 \\ &= 1.16 + 12.5 \\ &= 13.66 \end{aligned}$$

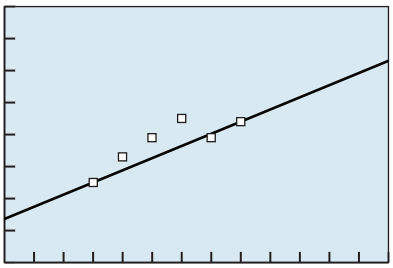
This value of roughly \$13.7 trillion underestimates the actual total income of \$13.9 trillion.

(c) To predict Americans' total personal income for 2020, we let $x = 2020$:

$$\begin{aligned} y &= 0.58(2020 - 2010) + 12.5 \\ &= 0.58 \cdot 10 + 12.5 \\ &= 5.8 + 12.5 \\ &= 18.3 \end{aligned}$$

Our linear model projects that the total personal income of all Americans in 2020 should be about \$18.3 trillion.

(d) Figure P.31 shows the scatter plot as well as the graph of the line we used to make our estimates and predictions. **Now try Exercise 51.**



[2002, 2015] by [8, 16]

Figure P.31 Linear model for Americans' personal income. (Example 8)

The Speed of Light

Whether light traveled instantaneously or actually took some time was an open question for thousands of years. Galileo Galilei (1564–1642) was one of the first to approximate the speed of light. Jean Bernard Léon Foucault (1819–1868) established a modern estimate for light's speed. The speed of light played an important role in Einstein's development of special relativity and continues to be important in physics and astronomy. In 2013, whether the speed of light in a vacuum is truly a universal constant was called into question.

CHAPTER OPENER Problem (from page 25)

Problem: Assume that the speed of light is about 299,800 km/sec (186,300 mi/sec).

(a) If light travels from the Sun to Earth in 8.32 min, approximate the distance between Earth and the Sun.

(b) If the distance from the Moon to Earth is roughly 384,400 km, approximate the time required for light to travel from the Moon to Earth.

(c) If light travels on average from the Sun to Pluto in about 5 hr 28 min, approximate the average distance between the Sun and Pluto.

Solution: We use the linear equation $d = r \cdot t$ (distance = rate $\times$ time) and the given rate $r = 299{,}800$ km/sec.

(a) Because $t = 8.32$ min $= 499.2$ sec,

$$d = r \cdot t = 299{,}800 \text{ km/sec} \times 499.2 \text{ sec} \approx 150{,}000{,}000 \text{ km}.$$

The distance from the Sun to Earth is about 150 million kilometers (93 million miles).

(b) Because $d = 384{,}400$ km,

$$t = \frac{d}{r} = \frac{384{,}400 \text{ km}}{299{,}800 \text{ km/sec}} \approx 1.282 \text{ sec}.$$

The time it takes light to travel from the Moon to Earth is about 1.282 sec.

(c) Because $t = 5$ hr 28 min $= 328$ min $= 19{,}680$ sec,

$$d = r \cdot t = 299{,}800 \text{ km/sec} \times 19{,}680 \text{ sec} = 5{,}900{,}064{,}000 \text{ km}.$$

The average distance from the Sun to Pluto is about 5.9×10^9 km.

QUICK REVIEW P.4

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator*.

In Exercises 1–4, solve for x.

1. $-75x + 25 = 200$

2. $400 - 50x = 150$

3. $3(1 - 2x) + 4(2x - 5) = 7$

4. $2(7x + 1) = 5(1 - 3x)$

In Exercises 5–8, solve for y.

5. $2x - 5y = 21$

6. $\frac{1}{3}x + \frac{1}{4}y = 2$

7. $2x + y = 17 + 2(x - 2y)$

8. $x^2 + y = 3x - 2y$

In Exercises 9 and 10, simplify the fraction.

9. $\dfrac{9 - 5}{-2 - (-8)}$

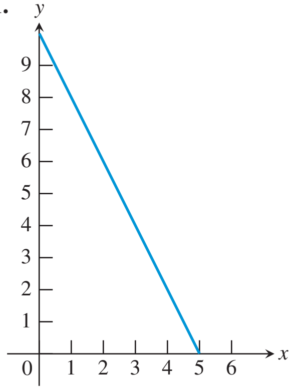
10. $\dfrac{-4 - 6}{-14 - (-2)}$

SECTION P.4 Exercises

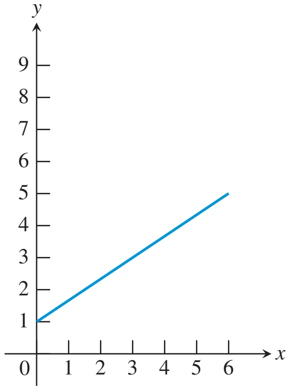
Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator*.

In Exercises 1 and 2, estimate the slope of the line.

1.



2.



In Exercises 3–6, find the slope of the line through the pair of points.

3. $(-3, 5)$ and $(4, 9)$

4. $(-2, 1)$ and $(5, -3)$

5. $(-2, -5)$ and $(-1, 3)$

6. $(5, -3)$ and $(-4, 12)$

In Exercises 7–10, find the value of x or y so that the line through the pair of points has the given slope.

	Points	Slope
7.	$(x, 3)$ and $(5, 9)$	$m = 2$
8.	$(-2, 3)$ and $(4, y)$	$m = -3$
9.	$(-3, -5)$ and $(4, y)$	$m = 3$
10.	$(-8, -2)$ and $(x, 2)$	$m = 1/2$

In Exercises 11–14, find a point-slope form equation for the line through the point with given slope.

	Point	Slope		Point	Slope
11.	$(1, 4)$	$m = 2$	**12.**	$(-4, 3)$	$m = -2/3$
13.	$(5, -4)$	$m = -2$	**14.**	$(-3, 4)$	$m = 3$

In Exercises 15–20, find a general form equation for the line through the pair of points.

15. $(-7, -2)$ and $(1, 6)$

16. $(-3, -8)$ and $(4, -1)$

17. $(1, -3)$ and $(5, -3)$

18. $(-1, -5)$ and $(-4, -2)$

19. $(-1, 2)$ and $(2, 5)$

20. $(4, -1)$ and $(4, 5)$

In Exercises 21–26, find a slope-intercept form equation for the line.

21. The line through $(0, 5)$ with slope $m = -3$

22. The line through $(1, 2)$ with slope $m = 1/2$

23. The line through the points $(-4, 5)$ and $(4, 3)$

24. The line through the points $(4, 2)$ and $(-3, 1)$

25. The line $2x + 5y = 12$

26. The line $7x - 12y = 96$

In Exercises 27–30, graph the linear equation on a grapher. Choose a viewing window that shows the line intersecting both the x- and y-axes.

27. $8x + y = 49$

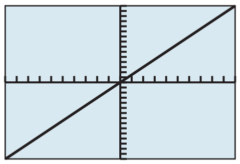
28. $2x + y = 35$

29. $123x + 7y = 429$

30. $2100x + 12y = 3540$

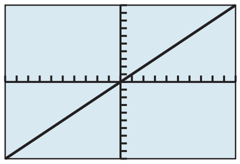
In Exercises 31 and 32, the line contains the origin and the point in the upper right corner of the grapher screen.

31. Writing to Learn Which line shown here has the greater slope? Explain.



$[-10, 10]$ by $[-15, 15]$

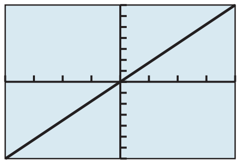
(a)



$[-10, 10]$ by $[-10, 10]$

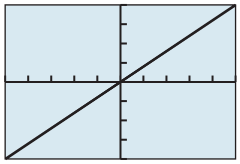
(b)

32. Writing to Learn Which line shown here has the greater slope? Explain.



$[-20, 20]$ by $[-35, 35]$

(a)



$[-5, 5]$ by $[-20, 20]$

(b)

In Exercises 33–36, find the value of x and the value of y for which $(x, 14)$ and $(18, y)$ are points on the graph.

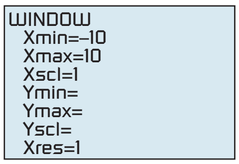
33. $y = 0.5x + 12$

34. $y = -2x + 18$

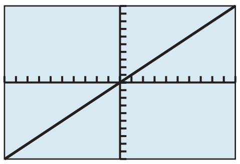
35. $3x + 4y = 26$

36. $3x - 2y = 14$

In Exercises 37–40, find the values for Ymin, Ymax, and Yscl that will make the graph of the line appear in the viewing window as shown here.



(a)



(b)

37. $y = 3x$

38. $y = 5x$

39. $y = \frac{2}{3}x$

40. $y = \frac{5}{4}x$

In Exercises 41–44, (a) find an equation for the line passing through the point and parallel to the given line, and (b) find an equation for the line passing through the point and perpendicular to the given line. Support your work graphically.

	Point	Line
41.	$(1, 2)$	$y = 3x - 2$
42.	$(-2, 3)$	$y = -2x + 4$
43.	$(3, 1)$	$2x + 3y = 12$
44.	$(6, 1)$	$3x - 5y = 15$

45. Real Estate Appreciation Bob Michaels purchased a house 8 years ago for \$42,000. This year it was appraised at \$67,500.

(a) A linear equation $V = mt + b$, $0 \le t \le 15$, represents the value V of the house for 15 years after it was purchased. Determine m and b.

(b) Graph the equation and trace to estimate in how many years after purchase this house will be worth \$72,500.

(c) Write and solve an equation algebraically to determine how many years after purchase this house will be worth \$74,000.

(d) Determine how many years after purchase this house will be worth \$80,250.

46. Investment Planning Mary Ellen plans to invest \$18,000, putting part of the money x into a savings account that pays 5% annually and the rest into an account that pays 8% annually.

(a) What are the possible values of x in this situation?

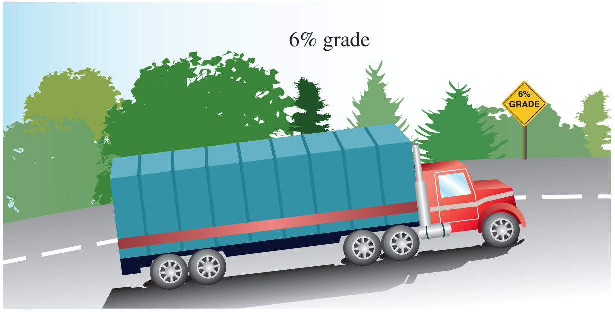
(b) If Mary Ellen invests x dollars at 5%, write an equation that describes the total interest I received from both accounts at the end of one year.

(c) Graph and trace to estimate how much Mary Ellen invested at 5% if she earned \$1020 in total interest at the end of the first year.

(d) Use your grapher to generate a table of values for I to find out how much Mary Ellen should invest at 5% to earn \$1185 in total interest in one year.

47. Navigation A commercial jet airplane climbs at takeoff with slope $m = 3/8$. How far in the horizontal direction will the airplane fly to reach an altitude of 12,000 ft above the takeoff point?

48. Grade of a Highway Interstate 70 west of Denver, Colorado, has a section posted as a 6% grade. This means that for a horizontal change of 100 ft there is a 6-ft vertical change.



(a) Find the slope of this section of the highway.

(b) On a highway with a 6% grade what is the horizontal distance required to climb 250 ft?

(c) A sign along the highway says 6% grade for the next 7 mi. Estimate how many feet of vertical change there are along those next 7 mi. (There are 5280 ft in 1 mile.)

49. Writing to Learn Building Specifications Asphalt shingles do not meet code specifications on a roof that has less than a 4-12 pitch. A 4-12 pitch means there are 4 ft of vertical change in 12 ft of horizontal change. A certain roof has slope $m = 3/8$. Could asphalt shingles be used on that roof? Explain.

50. Revisiting Example 8 Use the linear equation found in Example 8 to estimate Americans' income in 2010, 2013, 2015 displayed in Figure P.30.

51. Americans' Spending The (x, y) table shows total personal consumption expenditures (y) in the United States in trillions of dollars for selected years (x). (*Source: U.S. Bureau of Economic Analysis, The World Almanac and Book of Facts 2017.*)

x	1990	1995	2000	2005	2010	2015
y	3.8	5.0	6.8	8.8	10.2	12.3

(a) Write a linear equation for Americans' spending (y) in terms of the year (x), using the points (1990, 3.8) and (2010, 10.2).

(b) Use the equation in (a) to estimate Americans' expenditures in 2005.

(c) Use the equation in (a) to predict Americans' expenditures in 2020.

(d) Superimpose a graph of the linear equation in (a) on a scatter plot of the data.

52. U.S. Imports from Mexico The (x, y) table shows total U.S. imports from Mexico (y) in billions of dollars for selected years (x). (*Source: U.S. Census Bureau, The World Almanac and Book of Facts 2017.*)

x	2005	2010	2012	2013	2014	2015
y	170.1	230.0	277.6	280.6	295.7	296.4

(a) Use the pairs (2005, 170.1) and (2015, 296.4) to write a linear equation for x and y.

(b) Superimpose the graph of the linear equation in (a) on a scatter plot of the data.

(c) Use the equation in (a) to predict the total U.S. imports from Mexico in 2020.

53. World Population Table P.7 shows the midyear worldwide human population for selected years.

Table P.7 World Population

Year	Population (millions)
1980	4453
1990	5282
2000	6085
2010	6972
2013	7130

Source: U.S. Census Bureau.

(a) Let $x = 0$ represent 1980, $x = 1$ represent 1981, and so forth. Draw a scatter plot of the data.

(b) Use the 1980 and 2010 data to write a linear equation for the population y (in millions) in terms of the year x. Superimpose the graph of the linear equation on the scatter plot of the data.

(c) Use the graph in (b) to predict the midyear world population in 2020.

54. U.S. Exports to Canada Table P.8 shows the total exports from the United States to Canada in billions of dollars for selected years.

Table P.8 U.S. Exports to Canada

Time (years)	U.S. Exports (billions of \$)
2005	211.9
2010	249.3
2012	292.7
2013	300.8
2014	312.8
2015	280.6

Source: U.S. Census Bureau, The World Almanac and Book of Facts 2017.

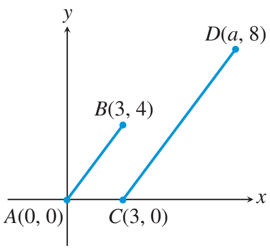
(a) Draw a scatter plot of the data.

(b) Use the 2005 and 2015 data to write a linear equation for the U.S. exports y in terms of the year x. Superimpose the graph of the equation on the scatter plot.

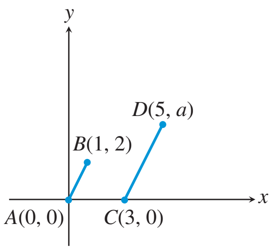
(c) Use the equation in (b) to predict the U.S. exports to Canada for 2020.

In Exercises 55 and 56, determine a so that the line segments AB and CD are parallel.

55.

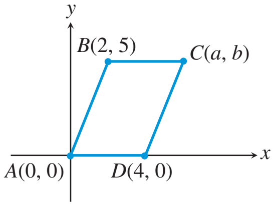


56.

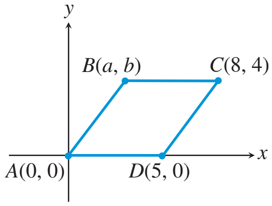


In Exercises 57 and 58, determine a and b so that figure $ABCD$ is a parallelogram.

57.



58.



59. Writing to Learn Perpendicular Lines

(a) Is it possible for two lines with positive slopes to be perpendicular? Explain.

(b) Is it possible for two lines with negative slopes to be perpendicular? Explain.

60. Group Activity Parallel and Perpendicular Lines

(a) Assume that $c \neq d$ and a and b are not both zero. Show that $ax + by = c$ and $ax + by = d$ are parallel lines. Explain why the restrictions on a, b, c, and d are necessary.

(b) Assume that a and b are not both zero. Show that $ax + by = c$ and $bx - ay = d$ are perpendicular lines. Explain why the restrictions on a and b are necessary.

Standardized Test Questions

61. True or False The slope of a vertical line is zero. Justify your answer.

62. True or False The graph of any equation of the form $ax + by = c$, where a and b are not both zero, is always a line. Justify your answer.

In Exercises 63–66, you may use a graphing calculator to solve these problems.

63. Multiple Choice Which of the following is an equation of the line through the point $(-2, 3)$ with slope 4?

(A) $y - 3 = 4(x + 2)$ (B) $y + 3 = 4(x - 2)$

(C) $x - 3 = 4(y + 2)$ (D) $x + 3 = 4(y - 2)$

(E) $y + 2 = 4(x - 3)$

64. Multiple Choice Which of the following is an equation of the line with slope 3 and y-intercept -2?

(A) $y = 3x + 2$ (B) $y = 3x - 2$

(C) $y = -2x + 3$ (D) $x = 3y - 2$

(E) $x = 3y + 2$

65. Multiple Choice Which of the following lines is perpendicular to the line $y = -2x + 5$?

(A) $y = 2x + 1$ (B) $y = -2x - \frac{1}{5}$

(C) $y = -\frac{1}{2}x + \frac{1}{3}$ (D) $y = -\frac{1}{2}x + 3$

(E) $y = \frac{1}{2}x - 3$

66. Multiple Choice Which of the following is the slope of the line through the two points $(-2, 1)$ and $(1, -4)$?

(A) $-\frac{3}{5}$ (B) $\frac{3}{5}$

(C) $-\frac{5}{3}$ (D) $\frac{5}{3}$

(E) -3

Explorations

67. Exploring the Graph of $\frac{x}{a} + \frac{y}{b} = c$, $a \neq 0$, $b \neq 0$

Let $c = 1$.

(a) Draw the graph for $a = 3, b = -2$.

(b) Draw the graph for $a = -2, b = -3$.

(c) Draw the graph for $a = 5, b = 3$.

(d) Use your graphs in (a), (b), (c) to conjecture what a and b represent when $c = 1$. Prove your conjecture.

(e) Repeat (a)–(d) for $c = 2$. What do a and b represent when $c = 2$?

(f) If $c = -1$, what do a and b represent?

68. Investigating Graphs of Linear Equations

(a) Graph $y = mx$ for $m = -3, -2, -1, 1, 2, 3$ in the window $[-8, 8]$ by $[-5, 5]$. What do these graphs have in common? How are they different?

(b) If $m > 0$, what do the graphs of $y = mx$ and $y = -mx$ have in common? How are they different?

(c) Graph $y = 0.3x + b$ for $b = -3, -2, -1, 0, 1, 2, 3$ in $[-8, 8]$ by $[-5, 5]$. What do these graphs have in common? How are they different?

Extending the Ideas

69. Connecting Algebra and Geometry Show that if the midpoints of consecutive sides of any quadrilateral (see figure) are connected, the result is a parallelogram.

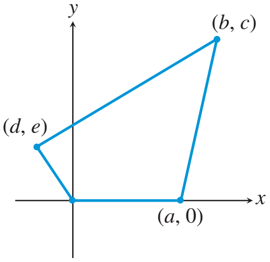


Figure for Exercise 69

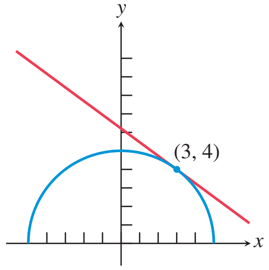
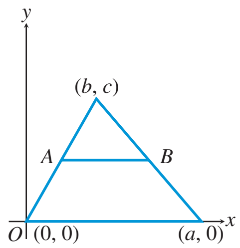


Figure for Exercise 70

70. Connecting Algebra and Geometry Consider the semicircle of radius 5 centered at $(0, 0)$ as shown in the figure. Find an equation of the line tangent to the semicircle at the point $(3, 4)$. (*Hint:* A line tangent to a circle is perpendicular to the radius at the point of tangency.)

71. Connecting Algebra and Geometry Show that in any triangle (see figure), the line segment joining the midpoints of two sides is parallel to the third side and is half as long.



P.5 Solving Equations Graphically, Numerically, and Algebraically

What you'll learn about

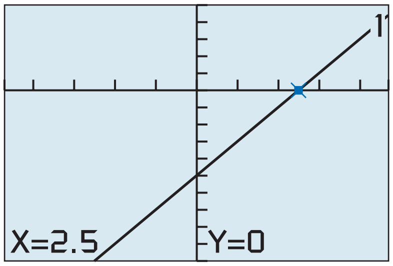
- Solving Equations Graphically
- Solving Quadratic Equations
- Approximating Solutions of Equations Graphically
- Approximating Solutions of Equations Numerically Using Tables
- Solving Equations by Finding Intersections

... and why

These are the basic techniques used to solve equations in this text.

Solving Equations Graphically

The graph of the equation $y = 2x - 5$ (in x and y) can be used to solve the equation $2x - 5 = 0$ (in x). Using the techniques of Section P.3, we can show algebraically that $x = 5/2$ is the solution of $2x - 5 = 0$. Therefore, the ordered pair $(5/2, 0)$ is a solution of $y = 2x - 5$. Figure P.32 supports this reasoning graphically. It shows that the x-intercept of the graph of the line $y = 2x - 5$ is the point $(5/2, 0)$.



$[-4.7, 4.7]$ by $[-10, 5]$

Figure P.32 Using the TRACE feature of a grapher, we see that $(2.5, 0)$ is an x-intercept of the graph of $y = 2x - 5$ and, therefore, $x = 2.5$ is a solution of the equation $2x - 5 = 0$.

The graphical method for solving an equation in x by using the graph of the associated equation in x and y is surprisingly general.

EXAMPLE 1 Solving by Finding *x*-Intercepts

Solve the equation $2x^2 - 3x - 2 = 0$ graphically. Confirm algebraically.

SOLUTION

Solve Graphically Find the x-intercepts of the graph of $y = 2x^2 - 3x - 2$ (Figure P.33). We use TRACE to see that $(-0.5, 0)$ and $(2, 0)$ are x-intercepts of this graph. Thus, the solutions of this equation are $x = -0.5$ and $x = 2$. Answers obtained graphically are really approximations, although in general they are very good approximations.

Confirm Algebraically In this case, we can use factoring to find exact values.

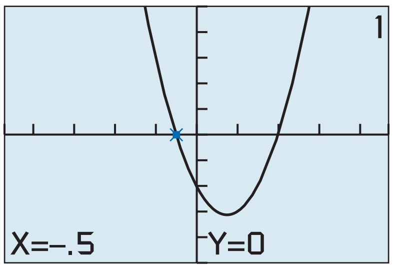
$$2x^2 - 3x - 2 = 0$$
$$(2x + 1)(x - 2) = 0 \quad \text{Factor.}$$

Therefore,

$$2x + 1 = 0 \quad \text{or} \quad x - 2 = 0$$
$$x = -1/2 \quad \text{or} \quad x = 2$$

So, $x = -1/2$ and $x = 2$ are the exact solutions of the original equation.

Now try Exercise 1.



$[-4.7, 4.7]$ by $[-5, 5]$

Figure P.33 It appears that $(-0.5, 0)$ and $(2, 0)$ are x-intercepts of the graph of $y = 2x^2 - 3x - 2$. (Example 1)

The "therefore" step of the algebraic confirmation in Example 1 used the following important property.

Zero Factor Property

Let a and b be real numbers.

$$\text{If } ab = 0, \text{ then } a = 0 \text{ or } b = 0.$$

Solving Quadratic Equations

Linear equations $(ax + b = 0)$ and *quadratic equations* are two members of the family of *polynomial equations*, which will be studied in detail in Chapter 2.

DEFINITION Quadratic Equation in x

A **quadratic equation in x** is one that can be written in the form

$$ax^2 + bx + c = 0,$$

where a, b, and c are real numbers and $a \neq 0$.

We review some of the basic algebraic techniques for solving quadratic equations. One algebraic technique that we have already used in Example 1 is *factoring*.

Quadratic equations of the form $(ax + h)^2 = k$ are fairly easy to solve as illustrated in Example 2.

Finding Square Roots

If $t^2 = k > 0$, then $t = \sqrt{k}$ or $t = -\sqrt{k}$.

EXAMPLE 2 Solving by Extracting Square Roots

Solve $(2x - 1)^2 = 9$ algebraically.

SOLUTION

$$(2x - 1)^2 = 9$$

$$2x - 1 = \pm 3 \qquad \text{Extract square roots.}$$

$$2x = 4 \quad \text{or} \quad 2x = -2 \qquad \text{Add 1.}$$

$$x = 2 \quad \text{or} \quad x = -1 \qquad \text{Divide by 2.}$$

Now try Exercise 9.

The technique of Example 2 is more general than you might think because every quadratic equation can be written in the form $(x + h)^2 = k$. The procedure we need to accomplish this is *completing the square*.

Completing the Square

To solve $x^2 + bx = c$ by **completing the square**, add $(b/2)^2$ to both sides of the equation and factor the left side of the new equation.

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = c + \left(\frac{b}{2}\right)^2$$

$$\left(x + \frac{b}{2}\right)^2 = c + \frac{b^2}{4}$$

In general, to solve a quadratic equation by completing the square, first we divide both sides of the equation by the coefficient of x^2, and then we complete the square as illustrated in Example 3.

EXAMPLE 3 Solving by Completing the Square

Solve $4x^2 - 20x + 17 = 0$ by completing the square.

SOLUTION

$$4x^2 - 20x + 17 = 0$$

$$x^2 - 5x + \frac{17}{4} = 0 \qquad \text{Divide by 4.}$$

$$x^2 - 5x = -\frac{17}{4} \qquad \text{Subtract } \frac{17}{4}.$$

We now complete the square of the left side by adding $(b/2)^2$ to both sides.

$$x^2 - 5x + \left(-\frac{5}{2}\right)^2 = -\frac{17}{4} + \left(-\frac{5}{2}\right)^2 \qquad \text{Add } \left(-\frac{5}{2}\right)^2.$$

$$\left(x - \frac{5}{2}\right)^2 = 2 \qquad \text{Factor and simplify.}$$

$$x - \frac{5}{2} = \pm\sqrt{2} \qquad \text{Extract square roots.}$$

$$x = \frac{5}{2} \pm \sqrt{2}$$

$$x = \frac{5}{2} + \sqrt{2} \approx 3.91 \text{ or } x = \frac{5}{2} - \sqrt{2} \approx 1.09$$

Now try Exercise 13.

The procedure of Example 3 can be used to solve the general quadratic equation $ax^2 + bx + c = 0$, producing the following formula. (See Exercise 68.)

Quadratic Formula

The solutions of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are given by the **quadratic formula**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

EXAMPLE 4 Using the Quadratic Formula

Solve the equation $3x^2 - 6x = 5$.

SOLUTION First we subtract 5 from both sides of the equation to put it in the form $ax^2 + bx + c = 0$: $3x^2 - 6x - 5 = 0$. Notice that $a = 3$, $b = -6$, and $c = -5$.

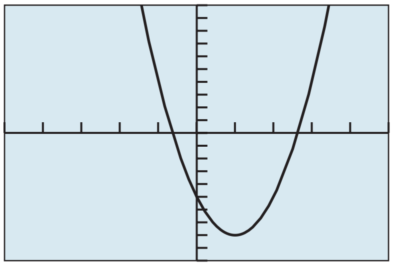
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{Quadratic formula}$$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(3)(-5)}}{2(3)} \qquad a = 3, b = -6, c = -5$$

$$x = \frac{6 \pm \sqrt{96}}{6} \qquad \text{Simplify.}$$

$$x = \frac{6 + \sqrt{96}}{6} \approx 2.63 \quad \text{or} \quad x = \frac{6 - \sqrt{96}}{6} \approx -0.63$$

The graph of $y = 3x^2 - 6x - 5$ in Figure P.34 supports our conclusion that the x-intercepts are approximately -0.63 and 2.63. **Now try Exercise 19.**



$[-5, 5]$ by $[-10, 10]$

Figure P.34 The graph of $y = 3x^2 - 6x - 5$. (Example 4)

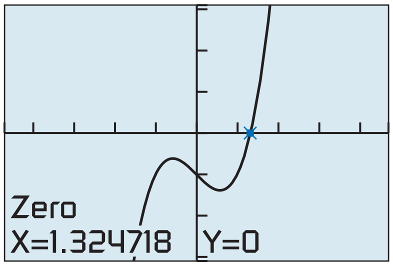
Solving Quadratic Equations Algebraically

There are four basic ways to solve quadratic equations algebraically.

1. **Factoring** (see Example 1)
2. **Extracting Square Roots** (see Example 2)
3. **Completing the Square** (see Example 3)
4. **Using the Quadratic Formula** (see Example 4)

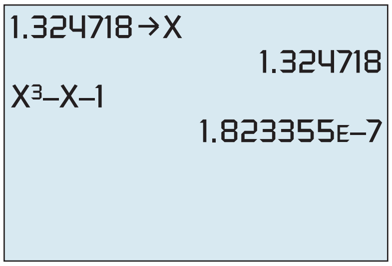
Approximating Solutions of Equations Graphically

A solution of the equation $x^3 - x - 1 = 0$ is a value of x that makes the value of $y = x^3 - x - 1$ equal to zero. Example 5 illustrates a built-in procedure on graphing calculators to find such values of x.



$[-4.7, 4.7]$ by $[-3.1, 3.1]$

(a)



(b)

Figure P.35 The graph of $y = x^3 - x - 1$. (a) shows that $(1.324718, 0)$ is an approximation to the x-intercept of the graph. (b) supports this conclusion. (Example 5)

EXAMPLE 5 Solving Graphically

Solve the equation $x^3 - x - 1 = 0$ graphically.

SOLUTION Figure P.35a suggests that $x \approx 1.324718$ is the solution we seek. Figure P.35b provides numerical support that $x = 1.324718$ is a close approximation to the solution because, when $x = 1.324718$, $x^3 - x - 1 \approx 1.82 \times 10^{-7}$, which is nearly zero.

Now try Exercise 31.

When solving equations graphically, we usually get approximate solutions and not exact solutions. We will use the following agreement about accuracy in this textbook.

Agreement About Approximate Solutions

For applications, round to a value that is reasonable for the context of the problem. For all other problems, round to two decimal places unless directed otherwise.

With this accuracy agreement, we would report the approximate solution found in Example 5 as 1.32.

Approximating Solutions of Equations Numerically Using Tables

The table feature on graphing calculators provides a numerical *zoom-in procedure* that we can use to find accurate solutions of equations. We illustrate this procedure in Example 6 with the same equation used in Example 5.

EXAMPLE 6 Solving Using Tables

Solve the equation $x^3 - x - 1 = 0$ using grapher tables.

SOLUTION From Figure P.35a, we know that the solution we seek is between $x = 1$ and $x = 2$. Figure P.36a sets the starting point of the table $(\text{TblStart} = 1)$ at $x = 1$ and the input increments in the table $(\Delta\text{Tbl} = 0.1)$ at 0.1. Figure P.36b shows that the zero of $x^3 - x - 1$ lies between $x = 1.3$ and $x = 1.4$.

(continued)

The next two steps in this process are shown in Figure P.37.

From Figure P.37a, we can read that the zero is between $x = 1.32$ and $x = 1.33$; from Figure P.37b, we can read that the zero is between $x = 1.324$ and $x = 1.325$. Thus, we conclude that the zero is approximately 1.32, according to our accuracy agreement.
Now try Exercise 37.

TABLE SETUP
TblStart=1
ΔTbl=.1
Indpnt: **Auto** Ask
Depend: **Auto** Ask

(a)

X	Y1	
1	−1	
1.1	−.769	
1.2	−.472	
1.3	−.103	
1.4	.344	
1.5	.875	
1.6	1.496	

Y1 = X3−X−1

(b)

Figure P.36 (a) gives the setup that produces the table in (b). (Example 6)

X	Y1	
1.3	−.103	
1.31	−.0619	
1.32	−.02	
1.33	.02264	
1.34	.0661	
1.35	.11038	
1.36	.15546	

Y1 = X3−X−1

(a)

X	Y1	
1.32	−.02	
1.321	−.0158	
1.322	−.0116	
1.323	−.0073	
1.324	−.0031	
1.325	.0012	
1.326	.00547	

Y1 = X3−X−1

(b)

Figure P.37 In (a) TblStart $= 1.3$ and ΔTbl $= 0.01$, and in (b) TblStart $= 1.32$ and ΔTbl $= 0.001$. (Example 6)

EXPLORATION 1 Finding Real Zeros of Equations

Consider the equation $4x^2 - 12x + 7 = 0$.

1. Use a graph to show that this equation has two real solutions, one between 0 and 1 and the other between 2 and 3.
2. Use the numerical zoom-in procedure illustrated in Example 6 to find each zero accurate to two decimal places.
3. Use the built-in zero finder (see Example 5) to find the two solutions. Then round them to two decimal places.
4. If you are familiar with the graphical zoom-in process, use it to find each solution accurate to two decimal places.
5. Compare the numbers obtained in parts 2, 3, and 4.
6. Support the results obtained in parts 2, 3, and 4 numerically.
7. Use the numerical zoom-in procedure illustrated in Example 6 to find each zero accurate to six decimal places. Compare with the answer found in part 3 with the zero finder.

Solving Equations by Finding Intersections

Sometimes we can solve an equation graphically by finding the *points of intersection* of two graphs. A point (a, b) is a **point of intersection** of two graphs if it lies on both graphs.

We illustrate this procedure with the absolute value equation in Example 7.

EXAMPLE 7 Solving by Finding Intersections

Solve the equation $|2x - 1| = 6$.

SOLUTION Figure P.38 suggests that the V-shaped graph of $y = |2x - 1|$ intersects the graph of the horizontal line $y = 6$ twice. We can use TRACE or the intersection feature of our grapher to see that the two points of intersection have coordinates $(-2.5, 6)$ and $(3.5, 6)$. This means that the original equation has two solutions: -2.5 and 3.5.

We can use algebra to find the exact solutions. The only two real numbers with absolute value 6 are 6 itself and -6. So, if $|2x - 1| = 6$, then

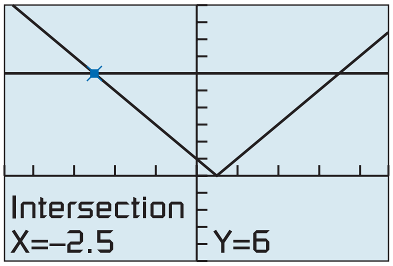
$$2x - 1 = 6 \quad \text{or} \quad 2x - 1 = -6$$

$$x = \frac{7}{2} = 3.5 \quad \text{or} \quad x = -\frac{5}{2} = -2.5$$

Now try Exercise 39.

Tech Tip

In Example 7, graph using $y_1 = |2x - 1|$ and $y_2 = 6$. Figure P.38 indicates that both y_1 and y_2 equal 6 when $x = -2.5$. So, $|2x - 1| = 6$ when $x = -2.5$. This is only one of the two solutions. The other can be found similarly.



$[-4.7, 4.7]$ by $[-5, 10]$

Figure P.38 The graphs of $y = |2x - 1|$ and $y = 6$ intersect at $(-2.5, 6)$ and $(3.5, 6)$. (Example 7)

QUICK REVIEW P.5

In Exercises 1–4, expand the product.

1. $(3x - 4)^2$ **2.** $(2x + 3)^2$

3. $(2x + 1)(3x - 5)$ **4.** $(3y - 1)(5y + 4)$

In Exercises 5–8, factor the expression.

5. $25x^2 - 20x + 4$ **6.** $15x^3 - 22x^2 + 8x$

7. $3x^3 + x^2 - 15x - 5$ **8.** $y^4 - 13y^2 + 36$

In Exercises 9 and 10, combine the fractions and reduce the resulting fraction to lowest terms.

9. $\dfrac{x}{2x + 1} - \dfrac{2}{x + 3}$

10. $\dfrac{x + 1}{x^2 - 5x + 6} - \dfrac{3x + 11}{x^2 - x - 6}$

SECTION P.5 Exercises

In Exercises 1–6, solve the equation graphically by finding x-intercepts. Confirm by using factoring to solve the equation.

1. $x^2 - x - 20 = 0$ **2.** $2x^2 + 5x - 3 = 0$

3. $4x^2 - 8x + 3 = 0$ **4.** $x^2 - 8x = -15$

5. $x(3x - 7) = 6$ **6.** $x(3x + 11) = 20$

In Exercises 7–12, solve the equation by extracting square roots.

7. $4x^2 = 25$ **8.** $2(x - 5)^2 = 17$

9. $3(x + 4)^2 = 8$ **10.** $4(u + 1)^2 = 18$

11. $2y^2 - 8 = 6 - 2y^2$ **12.** $(2x + 3)^2 = 169$

In Exercises 13–18, solve the equation by completing the square.

13. $x^2 + 6x = 7$

14. $x^2 + 5x - 9 = 0$

15. $x^2 - 7x + \frac{5}{4} = 0$

16. $4 - 6x = x^2$

17. $2x^2 - 7x + 9 = (x - 3)(x + 1) + 3x$

18. $3x^2 - 6x - 7 = x^2 + 3x - x(x + 1) + 3$

In Exercises 19–24, solve the equation using the quadratic formula.

19. $x^2 + 8x - 2 = 0$

20. $2x^2 - 3x + 1 = 0$

21. $3x + 4 = x^2$

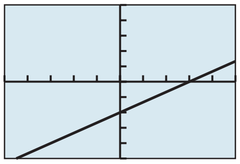
22. $x^2 - 5 = \sqrt{3}x$

23. $x(x + 5) = 12$

24. $x^2 - 2x + 6 = 2x^2 - 6x - 26$

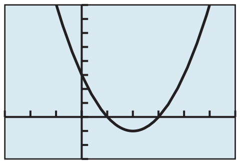
In Exercises 25–28, estimate any x- and y-intercepts that are shown in the graph.

25.



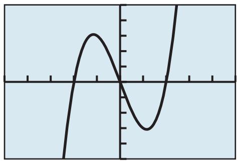
$[-5, 5]$ by $[-5, 5]$

26.



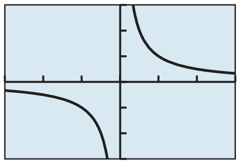
$[-3, 6]$ by $[-3, 8]$

27.



$[-5, 5]$ by $[-5, 5]$

28.



$[-3, 3]$ by $[-3, 3]$

In Exercises 29–34, solve the equation graphically by finding x-intercepts.

29. $x^2 + x - 1 = 0$

30. $4x^2 + 20x + 23 = 0$

31. $x^3 + x^2 + 2x - 3 = 0$

32. $x^3 - 4x + 2 = 0$

33. $x^2 + 4 = 4x$

34. $x^2 + 2x = -2$

In Exercises 35 and 36, the table permits you to estimate a zero of an expression. State the expression and give the zero as accurately as can be read from the table.

35.

X	Y1
.4	–.04
.41	–.0119
.42	.0164
.43	.0449
.44	.0736
.45	.1025
.46	.1316

Y1 = X²+2X–1

36.

X	Y1
–1.735	–.0177
–1.734	–.0117
–1.733	–.0057
–1.732	3E–4
–1.731	.0063
–1.73	.01228
–1.729	.01826

Y1 = X³–3X

In Exercises 37 and 38, use tables to find the indicated number of solutions of the equation accurate to two decimal places.

37. Two solutions of $x^2 - x - 1 = 0$

38. One solution of $-x^3 + x + 1 = 0$

In Exercises 39–44, solve the equation graphically by finding intersections. Confirm your answer algebraically.

39. $|t - 8| = 2$

40. $|x + 1| = 4$

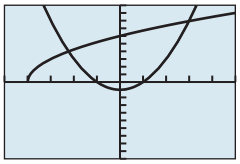
41. $|2x + 5| = 7$

42. $|3 - 5x| = 4$

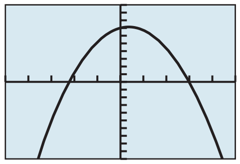
43. $|2x - 3| = x^2$

44. $|x + 1| = 2x - 3$

45. Interpreting Graphs The two figures below can be used to solve the equation $3\sqrt{x + 4} = x^2 - 1$ graphically.



$[-5, 5]$ by $[-10, 10]$
(a)



$[-5, 5]$ by $[-10, 10]$
(b)

(a) Figure (a) illustrates the intersection method for solving the given equation. Identify the two equations that are graphed.

(b) Figure (b) illustrates the x-intercept method for solving the given equation. Identify the equation that is graphed.

(c) **Writing to Learn** How are the intersection points in Figure (a) related to the x-intercepts in Figure (b)?

46. Writing to Learn Revisiting Example 6 Explain why all real numbers x that satisfy $1.324 < x < 1.325$ round to 1.32.

In Exercises 47–56, use a method of your choice to solve the equation.

47. $x^2 + x - 2 = 0$

48. $x^2 - 3x = 12 - 3(x - 2)$

49. $|2x - 1| = 5$

50. $x + 2 - 2\sqrt{x + 3} = 0$

51. $x^3 + 4x^2 - 3x - 2 = 0$

52. $x^3 - 4x + 2 = 0$

53. $|x^2 + 4x - 1| = 7$

54. $|x + 5| = |x - 3|$

55. $|0.5x + 3| = x^2 - 4$

56. $\sqrt{x + 7} = -x^2 + 5$

57. Group Activity Discriminant of a Quadratic The radicand $b^2 - 4ac$ in the quadratic formula is called the **discriminant** of the quadratic polynomial $ax^2 + bx + c$ because its value distinguishes the nature of the zeros of the polynomial.

(a) **Writing to Learn** If $b^2 - 4ac > 0$, what can you say about the zeros of the quadratic polynomial $ax^2 + bx + c$? Explain your answer.

(b) **Writing to Learn** If $b^2 - 4ac = 0$, what can you say about the zeros of the quadratic polynomial $ax^2 + bx + c$? Explain your answer.

(c) **Writing to Learn** If $b^2 - 4ac < 0$, what can you say about the zeros of the quadratic polynomial $ax^2 + bx + c$? Explain your answer.

58. Group Activity Discriminant of a Quadratic Use the information learned in Exercise 57 to create a quadratic polynomial with the following numbers of real zeros. Support your answers graphically.

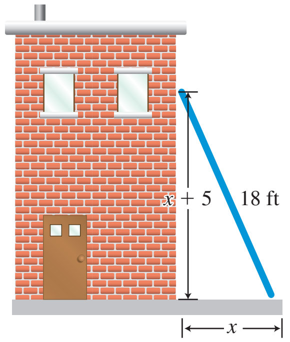
(a) Two real zeros

(b) Exactly one real zero

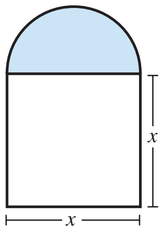
(c) No real zeros

59. Size of a Soccer Field Several of the World Cup '94 soccer matches were played in Stanford University's stadium in Menlo Park, California. The field is 30 yd longer than it is wide, and the area of the field is 8800 yd^2. What are the dimensions of this soccer field?

60. Height of a Ladder John's paint crew knows from experience that its 18-ft ladder is particularly stable when the distance from the ground to the top of the ladder is 5 ft more than the distance from the building to the base of the ladder as shown in the figure. In this position, how far up the building does the ladder reach?



61. Finding the Dimensions of a Norman Window A Norman window has the shape of a square with a semicircle mounted on it. Find the width of the window if the total area of the square and the semicircle is to be 200 ft^2.



Standardized Test Questions

62. True or False If 2 is an x-intercept of the graph of $y = ax^2 + bx + c$, then 2 is a solution of the equation $ax^2 + bx + c = 0$. Justify your answer.

63. True or False If $2x^2 = 18$, then x must be equal to 3. Justify your answer.

In Exercises 64–67, you may use a graphing calculator to solve these problems.

64. Multiple Choice Which of the following are the solutions of the equation $x(x - 3) = 0$?

(A) Only $x = 3$ (B) Only $x = -3$

(C) $x = 0$ and $x = -3$ (D) $x = 0$ and $x = 3$

(E) There are no solutions.

65. Multiple Choice Which value of k makes $x^2 - 5x + k$ a perfect square?

(A) $-\frac{5}{2}$ (B) $\left(-\frac{5}{2}\right)^2$ (C) $(-5)^2$

(D) $\left(-\frac{2}{5}\right)^2$ (E) -6

66. Multiple Choice Which of the following are the solutions of the equation $2x^2 - 3x - 1 = 0$?

(A) $\frac{3}{4} \pm \sqrt{17}$ (B) $\frac{3 \pm \sqrt{17}}{4}$ (C) $\frac{3 \pm \sqrt{17}}{2}$

(D) $\frac{-3 \pm \sqrt{17}}{4}$ (E) $\frac{3 \pm 1}{4}$

67. Multiple Choice Which of the following are the solutions of the equation $|x - 1| = -3$?

(A) Only $x = 4$ (B) Only $x = -2$

(C) Only $x = 2$ (D) $x = 4$ and $x = -2$

(E) There are no solutions.

Explorations

68. Deriving the Quadratic Formula Follow these steps to use completing the square to solve $ax^2 + bx + c = 0$, $a \neq 0$.

(a) Subtract c from both sides of the original equation and divide both sides of the resulting equation by a to obtain

$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

(b) Add the square of one-half of the coefficient of x in (a) to both sides and simplify to obtain

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

(c) Extract square roots in (b) and solve for x to obtain the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Extending the Ideas

69. Finding Number of Solutions Consider the equation $|x^2 - 4| = c$.

(a) Find a value of c for which this equation has four solutions. (There are many such values.)

(b) Find a value of c for which this equation has three solutions. (There is only one such value.)

(c) Find a value of c for which this equation has two solutions. (There are many such values.)

(d) Find a value of c for which this equation has no solutions. (There are many such values.)

(e) **Writing to Learn** Are there any other possible numbers of solutions of this equation? Explain.

70. Sums and Products of Solutions of $ax^2 + bx + c = 0, a \neq 0$ Suppose that $b^2 - 4ac > 0$.

(a) Show that the sum of the two solutions of this equation is $-b/a$.

(b) Show that the product of the two solutions of this equation is c/a.

71. Exercise 70 Continued The equation $2x^2 + bx + c = 0$ has two solutions x_1 and x_2. If $x_1 + x_2 = 5$ and $x_1 \cdot x_2 = 3$, find the two solutions.

P.6 Complex Numbers

What you'll learn about

- Complex Numbers
- Operations with Complex Numbers
- Complex Conjugates and Division
- Complex Solutions of Quadratic Equations

... and why

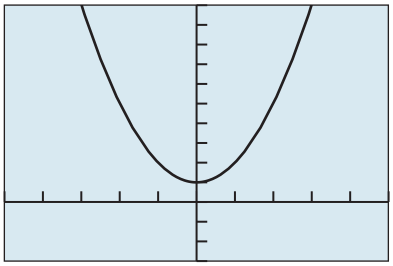
The zeros of polynomials are complex numbers.

Complex Numbers

Figure P.39 shows that the function $f(x) = x^2 + 1$ has no real zeros, so $x^2 + 1 = 0$ has no real-number solutions. To remedy this situation, mathematicians in the 17th century extended the definition of $\sqrt{a}$ to include negative real numbers a. First the number $i = \sqrt{-1}$ is defined as a solution of the equation $i^2 + 1 = 0$ and is the **imaginary unit**. Then for any negative real number a, we define $\sqrt{a} = \sqrt{|a|} \cdot i$.

The extended system of numbers, called the *complex numbers*, consists of all real numbers and sums of real numbers and real number multiples of i. The following are all examples of complex numbers:

$$-6, \quad 5i, \quad \sqrt{5}, \quad -7i, \quad \frac{5}{2}i + \frac{2}{3}, \quad -2 + 3i, \quad 5 - 3i, \quad 1 + \sqrt{3}i$$



[−5, 5] by [−3, 10]

Figure P.39 The graph of $f(x) = x^2 + 1$ has no x-intercepts.

DEFINITION Complex Number

A **complex number** is any number that can be written in the form

$$a + bi,$$

where a and b are real numbers. The real number a is the **real part**, the real number b is the **imaginary part**, and $a + bi$ is the **standard form**.

A real number a is the complex number $a + 0i$, so *all real numbers are also complex numbers*. If $a = 0$ and $b \neq 0$, then $a + bi$ becomes bi, and is an **imaginary number**. For instance, $5i$ and $-7i$ are imaginary numbers.

Two complex numbers are **equal** if and only if their real and imaginary parts are equal. For example,

$$x + yi = 2 + 5i \quad \text{if and only if} \quad x = 2 \text{ and } y = 5.$$

Historical Note

René Descartes (1596–1650) coined the term *imaginary* in a time when negative solutions to equations were considered *false*. Carl Friedrich Gauss (1777–1855) gave us the term *complex number* and the symbol i for $\sqrt{-1}$. Today practical applications of complex numbers abound.

Operations with Complex Numbers

Adding complex numbers is done by adding their real and imaginary parts separately. Subtracting complex numbers is also done using the same parts.

DEFINITION Addition and Subtraction of Complex Numbers

If $a + bi$ and $c + di$ are two complex numbers, then

Sum: $(a + bi) + (c + di) = (a + c) + (b + d)i,$

Difference: $(a + bi) - (c + di) = (a - c) + (b - d)i.$

EXAMPLE 1 Adding and Subtracting Complex Numbers

(a) $(7 - 3i) + (4 + 5i) = (7 + 4) + (-3 + 5)i = 11 + 2i$

(b) $(2 - i) - (8 + 3i) = (2 - 8) + (-1 - 3)i = -6 - 4i$

Now try Exercise 3.

The **additive identity** for the complex numbers is $0 = 0 + 0i$. The **additive inverse** of $a + bi$ is $-(a + bi) = -a - bi$ because

$$(a + bi) + (-a - bi) = 0 + 0i = 0.$$

Many of the properties of real numbers also hold for complex numbers. These include

- *Commutative* properties of addition and multiplication,
- *Associative* properties of addition and multiplication, and
- *Distributive* properties of multiplication over addition and subtraction.

Using these properties and the fact that $i^2 = -1$, complex numbers can be multiplied by treating them as algebraic expressions.

EXAMPLE 2 Multiplying Complex Numbers

$$\begin{aligned}(2 + 3i)\cdot(5 - i) &= 2(5 - i) + 3i(5 - i) && \text{Distributive property}\\ &= 10 - 2i + 15i - 3i^2 && \text{Distributive property}\\ &= 10 + 13i - 3(-1) && i^2 = -1\\ &= 13 + 13i\end{aligned}$$

Now try Exercise 9.

We can generalize Example 2 as follows:

$$\begin{aligned}(a + bi)(c + di) &= ac + adi + bci + bdi^2\\ &= (ac - bd) + (ad + bc)i\end{aligned}$$

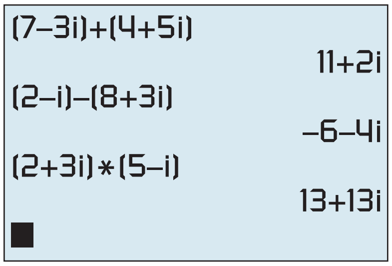


Figure P.40 Complex number operations on a grapher. (Examples 1 and 2)

Many graphers can perform basic calculations on complex numbers. Figure P.40 shows how the operations of Examples 1 and 2 look on some graphers.

We compute positive integer powers of complex numbers using the commutative, associative, and distributive properties of complex numbers and the key fact that $i^2 = -1$.

EXAMPLE 3 Raising a Complex Number to a Power

If $z = \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$, find z^2 and z^3.

SOLUTION

$$\begin{aligned}z^2 &= \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\\ &= \frac{1}{4} + \frac{\sqrt{3}}{4}i + \frac{\sqrt{3}}{4}i + \frac{3}{4}i^2\\ &= \frac{1}{4} + \frac{2\sqrt{3}}{4}i + \frac{3}{4}(-1)\\ &= -\frac{1}{2} + \frac{\sqrt{3}}{2}i\end{aligned}$$

$$\begin{aligned}z^3 = z^2\cdot z &= \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\\ &= -\frac{1}{4} - \frac{\sqrt{3}}{4}i + \frac{\sqrt{3}}{4}i + \frac{3}{4}i^2\\ &= -\frac{1}{4} + 0i + \frac{3}{4}(-1)\\ &= -1\end{aligned}$$

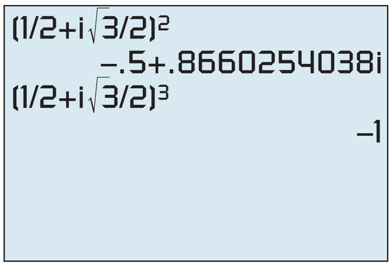


Figure P.41 The square and cube of a complex number. (Example 3)

Figure P.41 supports these results numerically. **Now try Exercise 27.**

Example 3 demonstrates that $1/2 + (\sqrt{3}/2)i$ is a cube root of -1 and a solution of $x^3 + 1 = 0$. In Section 2.5, we will explore complex zeros of polynomial functions in depth. Section 6.6 delves into the geometry of complex numbers.

Complex Conjugates and Division

The product of the complex numbers $a + bi$ and $a - bi$ is a positive real number:

$$(a + bi) \cdot (a - bi) = a^2 - (bi)^2 = a^2 + b^2$$

We introduce the following definition, which arises from this special relationship.

DEFINITION Complex Conjugate

The **complex conjugate** of the complex number $z = a + bi$ is

$$\bar{z} = \overline{a + bi} = a - bi.$$

The **multiplicative identity** for the complex numbers is $1 = 1 + 0i$. The **multiplicative inverse**, or **reciprocal**, of $z = a + bi$ is

$$z^{-1} = \frac{1}{z} = \frac{1}{a + bi} = \frac{1}{a + bi} \cdot \frac{a - bi}{a - bi} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i.$$

In general, a quotient of two complex numbers, written in fraction form, can be simplified as we just simplified $1/z$—by multiplying the numerator and denominator of the fraction by the complex conjugate of the denominator.

EXAMPLE 4 Dividing Complex Numbers

Write the complex number in standard form.

(a) $\dfrac{2}{3 - i}$ **(b)** $\dfrac{5 + i}{2 - 3i}$

SOLUTION Multiply the numerator and denominator by the complex conjugate of the denominator.

(a)
$$\frac{2}{3 - i} = \frac{2}{3 - i} \cdot \frac{3 + i}{3 + i} = \frac{6 + 2i}{3^2 + 1^2} = \frac{6}{10} + \frac{2}{10} i = \frac{3}{5} + \frac{1}{5} i$$

(b)
$$\frac{5 + i}{2 - 3i} = \frac{5 + i}{2 - 3i} \cdot \frac{2 + 3i}{2 + 3i} = \frac{10 + 15i + 2i + 3i^2}{2^2 + 3^2} = \frac{7 + 17i}{13} = \frac{7}{13} + \frac{17}{13} i$$

Now try Exercise 33.

Complex Solutions of Quadratic Equations

Recall that the solutions of the quadratic equation $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$, are given by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The radicand $b^2 - 4ac$ is the **discriminant** and tells us whether the solutions are real numbers. In particular, if $b^2 - 4ac < 0$, the solutions involve the square root of a negative number and thus lead to complex-number solutions. In all, there are three cases, which we now summarize:

Discriminant of a Quadratic Equation

For a quadratic equation $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$,

- If $b^2 - 4ac > 0$, there are two distinct real solutions.
- If $b^2 - 4ac = 0$, there is one (repeated) real solution.
- If $b^2 - 4ac < 0$, there is a complex conjugate pair of solutions.

EXAMPLE 5 Solving a Quadratic Equation

Solve $x^2 + x + 1 = 0$.

SOLUTION

Solve Algebraically Using the quadratic formula with $a = b = c = 1$, we obtain

$$x = \frac{-(1) \pm \sqrt{(1)^2 - 4(1)(1)}}{2(1)} = \frac{-(1) \pm \sqrt{-3}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i.$$

So the solutions are $-1/2 + (\sqrt{3}/2)i$ and $-1/2 - (\sqrt{3}/2)i$, a complex conjugate pair.

Confirm Numerically Substituting $-1/2 + (\sqrt{3}/2)i$ into the original equation, we obtain

$$\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^2 + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + 1$$
$$= \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + 1 = 0.$$

By a similar computation we can confirm the second solution.

Now try Exercise 41.

QUICK REVIEW P.6

In Exercises 1–4, add or subtract, and simplify.

1. $(2x + 3) + (-x + 6)$ **2.** $(3y - x) + (2x - y)$

3. $(2a + 4d) - (a + 2d)$ **4.** $(6z - 1) - (z + 3)$

In Exercises 5–10, multiply and simplify.

5. $(x - 3)(x + 2)$

6. $(2x - 1)(x + 3)$

7. $(x - \sqrt{2})(x + \sqrt{2})$

8. $(x + 2\sqrt{3})(x - 2\sqrt{3})$

9. $[x - (1 + \sqrt{2})][x - (1 - \sqrt{2})]$

10. $[x - (2 + \sqrt{3})][x - (2 - \sqrt{3})]$

SECTION P.6 Exercises

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator*.

In Exercises 1–8, write the sum or difference in the standard form $a + bi$ without using a calculator.

1. $(2 - 3i) + (6 + 5i)$

2. $(2 - 3i) + (3 - 4i)$

3. $(7 - 3i) + (6 - i)$ **4.** $(2 + i) - (9i - 3)$

5. $(2 - i) + (3 - \sqrt{-3})$

6. $(\sqrt{5} - 3i) + (-2 + \sqrt{-9})$

7. $(i^2 + 3) - (7 + i^3)$

8. $(\sqrt{7} + i^2) - (6 - \sqrt{-81})$

In Exercises 9–16, write the product in standard form without using a calculator.

9. $(2 + 3i)(2 - i)$

10. $(2 - i)(1 + 3i)$

11. $(1 - 4i)(3 - 2i)$

12. $(5i - 3)(2i + 1)$

13. $(7i - 3)(2 + 6i)$

14. $(\sqrt{-4} + i)(6 - 5i)$

15. $(-3 - 4i)(1 + 2i)$

16. $(\sqrt{-2} + 2i)(6 + 5i)$

In Exercises 17–20, write the expression in the form bi, where b is a real number.

17. $\sqrt{-16}$

18. $\sqrt{-25}$

19. $\sqrt{-3}$

20. $\sqrt{-5}$

In Exercises 21–24, find the real numbers x and y that make the equation true.

21. $2 + 3i = x + yi$

22. $3 + yi = x - 7i$

23. $(5 - 2i) - 7 = x - (3 + yi)$

24. $(x + 6i) = (3 - i) + (4 - 2yi)$

In Exercises 25–28, write the complex number in standard form.

25. $(3 + 2i)^2$

26. $(1 - i)^3$

27. $\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^4$

28. $\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)^3$

In Exercises 29–32, find the product of the complex number and its conjugate.

29. $2 - 3i$

30. $5 - 6i$

31. $-3 + 4i$

32. $-1 - \sqrt{2}i$

In Exercises 33–40, write the expression in standard form without using a calculator.

33. $\frac{1}{2 + i}$

34. $\frac{i}{2 - i}$

35. $\frac{2 + i}{2 - i}$

36. $\frac{2 + i}{3i}$

37. $\frac{(2 + i)^2(-i)}{1 + i}$

38. $\frac{(2 - i)(1 + 2i)}{5 + 2i}$

39. $\frac{(1 - i)(2 - i)}{1 - 2i}$

40. $\frac{(1 - \sqrt{2}i)(1 + i)}{(1 + \sqrt{2}i)}$

In Exercises 41–44, solve the equation.

41. $x^2 + 2x + 5 = 0$

42. $3x^2 + x + 2 = 0$

43. $4x^2 - 6x + 5 = x + 1$

44. $x^2 + x + 11 = 5x - 8$

Standardized Test Questions

45. True or False There are no complex numbers z satisfying $z = -\bar{z}$. Justify your answer.

46. True or False For the complex number i, $i + i^2 + i^3 + i^4 = 0$. Justify your answer.

In Exercises 47–50, solve the problem without using a calculator.

47. Multiple Choice Which of the following is the standard form for the product $(2 + 3i)(2 - 3i)$?

(A) $-5 + 12i$ (B) $4 - 9i$ (C) $13 - 3i$

(D) -5 (E) $13 + 0i$

48. Multiple Choice Which of the following is the standard form for the quotient $\frac{1}{i}$?

(A) 1 (B) -1 (C) i (D) $-1/i$ (E) $0 - i$

49. Multiple Choice Assume that $2 - 3i$ is a solution of $ax^2 + bx + c = 0$, where a, b, and c are real numbers. Which of the following is also a solution of the equation?

(A) $2 + 3i$ (B) $-2 - 3i$ (C) $-2 + 3i$

(D) $3 + 2i$ (E) $\frac{1}{2 - 3i}$

50. Multiple Choice Which of the following is the standard form for the power $(1 - i)^3$?

(A) $-4i$ (B) $-2 + 2i$ (C) $-2 - 2i$

(D) $2 + 2i$ (E) $2 - 2i$

Explorations

51. Group Activity The Powers of *i*

(a) Simplify the complex numbers $i, i^2, \ldots, i^8$ by evaluating each one.

(b) Simplify the complex numbers $i^{-1}, i^{-2}, \ldots, i^{-8}$ by evaluating each one.

(c) Evaluate i^0.

(d) **Writing to Learn** Discuss your results from (a)–(c) with the members of your group, and write a summary statement about the integer powers of i.

52. Writing to Learn Describe the nature of the graph of $f(x) = ax^2 + bx + c$ when a, b, and c are real numbers and the equation $ax^2 + bx + c = 0$ has nonreal complex solutions.

Extending the Ideas

53. Prove that the difference between a complex number and its conjugate is a complex number whose real part is 0.

54. Prove that the product of a complex number and its complex conjugate is a complex number whose imaginary part is zero.

55. Prove that the complex conjugate of a product of two complex numbers is the product of their complex conjugates.

56. Prove that the complex conjugate of a sum of two complex numbers is the sum of their complex conjugates.

57. Writing to Learn Explain why $-i$ is a solution of $x^2 - ix + 2 = 0$ but i is not.

P.7 Solving Inequalities Algebraically and Graphically

What you'll learn about

- Solving Absolute Value Inequalities
- Solving Quadratic Inequalities
- Approximating Solutions to Inequalities
- Projectile Motion

... and why

These techniques are involved in using a graphing utility to solve inequalities in this text.

Solving Absolute Value Inequalities

We now extend the methods for solving inequalities introduced in Section P.3. We start with two algebraic relationships used to solve absolute value inequalities.

Solving Absolute Value Inequalities

Let u be an algebraic expression in x, and let a be a positive real number.

1. If $|u| < a$, then u is in the interval $(-a, a)$. That is,

$$|u| < a \quad \text{if and only if} \quad -a < u < a.$$

2. If $|u| > a$, then u is in the interval $(-\infty, -a)$ or (a, ∞). That is,

$$|u| > a \quad \text{if and only if} \quad u < -a \text{ or } u > a.$$

The inequalities $<$ and $>$ can be replaced with $\leq$ and $\geq$, respectively. See Figure P.42.

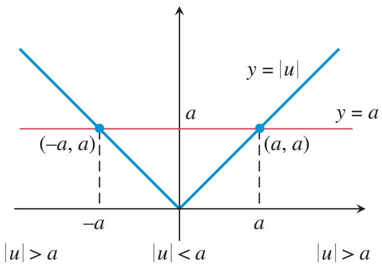


Figure P.42 The solution of $|u| < a$ is represented by the portion of the number line for which the graph of $y = |u|$ is below the graph of $y = a$. The solution of $|u| > a$ is represented by the portion of the number line where the graph of $y = |u|$ is above the graph of $y = a$.

EXAMPLE 1 Solving an Absolute Value Inequality

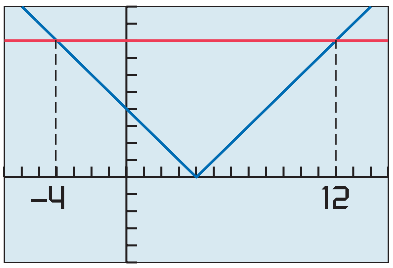
Solve $|x - 4| < 8$.

SOLUTION

$$\begin{aligned} |x - 4| &< 8 && \text{Original inequality} \\ -8 < x - 4 &< 8 && \text{Equivalent double inequality} \\ -4 < x &< 12 && \text{Add 4.} \end{aligned}$$

As an interval the solution is $(-4, 12)$.

Figure P.43 shows that points on the graph of $y = |x - 4|$ are below the points on the graph of $y = 8$ for values of x between -4 and 12. So, $|x - 4|$ is less than 8 when $-4 < x < 12$. **Now try Exercise 3.**



$[-7, 15]$ by $[-5, 10]$

Figure P.43 The graphs of $y = |x - 4|$ and $y = 8$. (Example 1)

EXAMPLE 2 Solving Another Absolute Value Inequality

Solve $|3x - 2| \geq 5$.

SOLUTION The solution of this one absolute value inequality consists of the combined solutions of the following two inequalities:

$$\begin{aligned} 3x - 2 \leq -5 \quad &\text{or} \quad 3x - 2 \geq 5 \\ 3x \leq -3 \quad &\text{or} \quad 3x \geq 7 && \text{Add 2.} \\ x \leq -1 \quad &\text{or} \quad x \geq \frac{7}{3} && \text{Divide by 3.} \end{aligned}$$

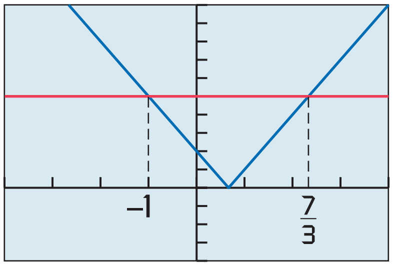
(continued)

Union of Two Sets

The **union of two sets A and B**, denoted by $A \cup B$, is the set of all objects that belong to A or B or both.

The solution consists of all numbers that are in either one of the two intervals $(-\infty, -1]$ or $[7/3, \infty)$, which may be written as $(-\infty, -1] \cup [7/3, \infty)$. The notation "$\cup$" is read as "union." (See the margin note.)

Figure P.44 shows that points on the graph of $y = |3x - 2|$ are above or on the points on the graph of $y = 5$ for values of x to the left of and including -1 and to the right of and including $7/3$, supporting the algebraic solution. **Now try Exercise 7.**



$[-4, 4]$ by $[-4, 10]$

Figure P.44 The graphs of $y = |3x - 2|$ and $y = 5$. (Example 2)

Solving Quadratic Inequalities

To solve a quadratic inequality such as $x^2 - x - 12 > 0$, we begin by solving the corresponding quadratic equation $x^2 - x - 12 = 0$. Then we determine the values of x for which the graph of $y = x^2 - x - 12$ lies above the x-axis.

EXAMPLE 3 Solving a Quadratic Inequality

Solve $x^2 - x - 12 > 0$.

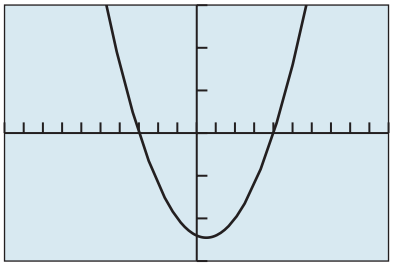
SOLUTION First we solve the corresponding equation $x^2 - x - 12 = 0$.

$$\begin{aligned} x^2 - x - 12 &= 0 \\ (x - 4)(x + 3) &= 0 && \text{Factor.} \\ x - 4 = 0 \quad \text{or} \quad x + 3 &= 0 && ab = 0 \Rightarrow a = 0 \text{ or } b = 0 \\ x = 4 \quad \text{or} \quad x &= -3 && \text{Solve for } x. \end{aligned}$$

The solutions of the corresponding quadratic equation are -3 and 4, and they are not solutions of the original inequality because $0 > 0$ is false. Figure P.45 shows that the points on the graph of $y = x^2 - x - 12$ are above the x-axis for values of x to the left of -3 and to the right of 4.

The solution of the original inequality is $(-\infty, -3) \cup (4, \infty)$.

Now try Exercise 11.



$[-10, 10]$ by $[-15, 15]$

Figure P.45 The graph of $y = x^2 - x - 12$ appears to cross the x-axis at $x = -3$ and $x = 4$. (Example 3)

In Example 4, the quadratic inequality involves the symbol $\le$. In this case, the solutions of the corresponding quadratic equation are also solutions of the inequality.

EXAMPLE 4 Solving Another Quadratic Inequality

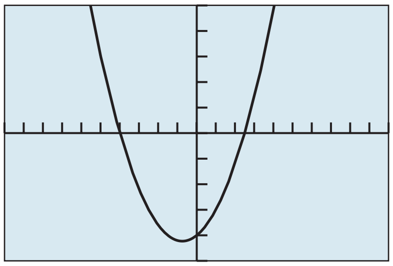
Solve $2x^2 + 3x \le 20$.

SOLUTION First we subtract 20 from both sides of the inequality to obtain $2x^2 + 3x - 20 \le 0$. Next, we solve the corresponding quadratic equation $2x^2 + 3x - 20 = 0$.

$$\begin{aligned} 2x^2 + 3x - 20 &= 0 \\ (x + 4)(2x - 5) &= 0 && \text{Factor.} \\ x + 4 = 0 \quad \text{or} \quad 2x - 5 &= 0 && ab = 0 \Rightarrow a = 0 \text{ or } b = 0 \\ x = -4 \quad \text{or} \quad x &= \frac{5}{2} && \text{Solve for } x. \end{aligned}$$

The solutions of the corresponding quadratic equation are -4 and $5/2 = 2.5$. You can check that they are also solutions of the inequality.

Figure P.46 shows that the points on the graph of $y = 2x^2 + 3x - 20$ are below the x-axis for values of x between -4 and 2.5. The solution of the original inequality is $[-4, 2.5]$. We use square brackets because the endpoints -4 and 2.5 are solutions of the inequality. **Now try Exercise 9.**



$[-10, 10]$ by $[-25, 25]$

Figure P.46 The graph of $y = 2x^2 + 3x - 20$ appears to be below the x-axis for $-4 < x < 2.5$. (Example 4)

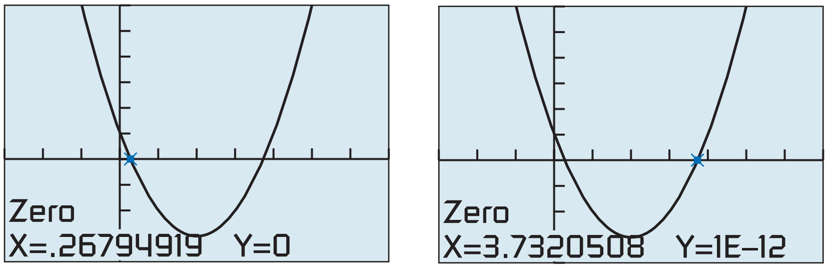
In Examples 3 and 4 the corresponding quadratic equations could be factored. When they cannot be factored, we approximate the real zeros of the quadratic equation if it has any. Then we use our accuracy agreement from Section P.5 to write the endpoints of any intervals accurate to two decimal places, as illustrated in Example 5.

EXAMPLE 5 Solving a Quadratic Inequality Graphically

Solve $x^2 - 4x + 1 \geq 0$ graphically.

SOLUTION We can use the graph of $y = x^2 - 4x + 1$ in Figure P.47 to determine that the solutions of the equation $x^2 - 4x + 1 = 0$ are about 0.27 and 3.73. Thus, the solution of the original inequality is roughly $(-\infty, 0.27] \cup [3.73, \infty)$. We use square brackets because the zeros of the quadratic equation are solutions of the inequality even though we only have approximations to their values.

Now try Exercise 21.



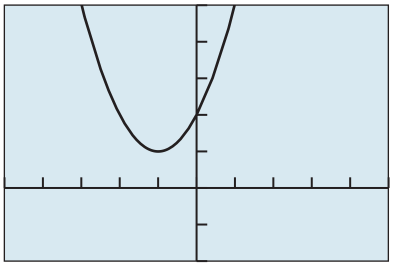
$[-3, 7]$ by $[-4, 6]$

(a)

$[-3, 7]$ by $[-4, 6]$

(b)

Figure P.47 This figure suggests that $y = x^2 - 4x + 1$ is approximately zero for $x \approx 0.27$ and $x \approx 3.73$. (Example 5)



$[-5, 5]$ by $[-2, 5]$

Figure P.48 The values of $y = x^2 + 2x + 2$ are never negative. (Example 6)

EXAMPLE 6 Showing That a Quadratic Inequality Has No Solution

Solve $x^2 + 2x + 2 < 0$.

SOLUTION Figure P.48 shows that the graph of $y = x^2 + 2x + 2$ lies above the x-axis for all values for x. Thus, the inequality $x^2 + 2x + 2 < 0$ has *no* solution.

Now try Exercise 25.

Figure P.48 also shows that the solution of the inequality $x^2 + 2x + 2 > 0$ is the set of all real numbers or, in interval notation, $(-\infty, \infty)$. A quadratic inequality can also have exactly one solution (see Exercise 31).

Approximating Solutions to Inequalities

To solve the inequality in Example 7, we approximate the zeros of the corresponding graph. Then we determine the values of x for which the corresponding graph is above or on the x-axis.

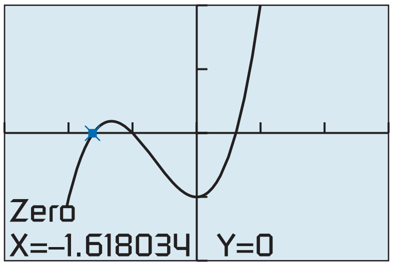
EXAMPLE 7 Solving a Cubic Inequality

Solve $x^3 + 2x^2 - 1 \geq 0$ graphically.

SOLUTION We can use the graph of $y = x^3 + 2x^2 - 1$ in Figure P.49 to show that the solutions of the corresponding equation $x^3 + 2x^2 - 1 = 0$ are approximately -1.62, -1, and 0.62. The points on the graph of $y = x^3 + 2x^2 - 1$ are above the x-axis for values of x between -1.62 and -1, and for values of x to the right of 0.62.

The solution of the inequality is $[-1.62, -1] \cup [0.62, \infty)$. We use square brackets because the zeros of $y = x^3 + 2x^2 - 1$ are solutions of the inequality.

Now try Exercise 27.



$[-3, 3]$ by $[-2, 2]$

Figure P.49 The graph of $y = x^3 + 2x^2 - 1$ appears to be above the x-axis between the two negative x-intercepts and to the right of the positive x-intercept. (Example 7)

Projectile Motion

The movement of an object that is propelled vertically, and then subject only to the force of gravity, is an example of **projectile motion**.

Projectile Motion

Suppose an object is launched vertically from a point s_0 feet above the ground with an initial velocity of v_0 feet per second. The vertical position s (in feet) of the object t seconds after it is launched is

$$s = -16t^2 + v_0 t + s_0.$$

EXAMPLE 8 Finding the Height of a Projectile

A projectile is launched straight up from ground level with an initial velocity of 288 ft/sec.

(a) When will the projectile's height above ground be 1152 ft?

(b) When will the projectile's height above ground be at least 1152 ft?

SOLUTION Here $s_0 = 0$ and $v_0 = 288$. So, the projectile's height is $s = -16t^2 + 288t$.

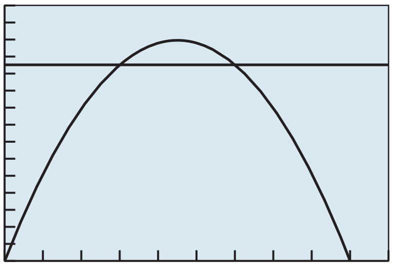
(a) We need to determine when $s = 1152$.

$$\begin{aligned} s &= -16t^2 + 288t \\ 1152 &= -16t^2 + 288t && \text{Substitute } s = 1152. \\ 16t^2 - 288t + 1152 &= 0 && \text{Add } 16t^2 - 288t. \\ t^2 - 18t + 72 &= 0 && \text{Divide by 16.} \\ (t - 6)(t - 12) &= 0 && \text{Factor.} \\ t = 6 \quad \text{or} \quad t &= 12 && \text{Solve for } t. \end{aligned}$$

The projectile is 1152 ft above ground twice; the first time at $t = 6$ sec on the way up, and the second time at $t = 12$ sec on the way down (Figure P.50).

(b) The projectile will be at least 1152 ft above ground when $s \geq 1152$. We can see from Figure P.50 together with the algebraic work in (a) that the solution is $[6, 12]$. This means that the projectile is at least 1152 ft above ground for times between $t = 6$ sec and $t = 12$ sec, including 6 and 12 sec.

In Exercise 32 we ask you to use algebra to solve the inequality $s = -16t^2 + 288t \geq 1152$. **Now try Exercise 33.**



$[0, 20]$ by $[0, 1500]$

Figure P.50 The graphs of $s = -16t^2 + 288t$ and $s = 1152$. We know from Example 8a that the two graphs intersect at $(6, 1152)$ and $(12, 1152)$.

QUICK REVIEW P.7

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator*.

In Exercises 1–3, solve for x.

1. $-7 < 2x - 3 < 7$

2. $5x - 2 \geq 7x + 4$

3. $|x + 2| = 3$

In Exercises 4–6, factor the expression completely.

4. $4x^2 - 9$

5. $x^3 - 4x$

6. $9x^2 - 16y^2$

In Exercises 7 and 8, reduce the fraction to lowest terms.

7. $\dfrac{z^2 - 25}{z^2 - 5z}$

8. $\dfrac{x^2 + 2x - 35}{x^2 - 10x + 25}$

In Exercises 9 and 10, add the fractions and simplify.

9. $\dfrac{x}{x - 1} + \dfrac{x + 1}{3x - 4}$

10. $\dfrac{2x - 1}{x^2 - x - 2} + \dfrac{x - 3}{x^2 - 3x + 2}$

SECTION P.7 Exercises

In Exercises 1–8, solve the inequality algebraically. Write the solution in interval notation and draw its number line graph.

1. $|x + 4| \geq 5$

2. $|2x - 1| > 3.6$

3. $|x - 3| < 2$

4. $|x + 3| \leq 5$

5. $|4 - 3x| - 2 < 4$

6. $|3 - 2x| + 2 > 5$

7. $\left|\dfrac{x + 2}{3}\right| \geq 3$

8. $\left|\dfrac{x - 5}{4}\right| \leq 6$

In Exercises 9–16, solve the inequality. Use algebra to solve the corresponding equation.

9. $2x^2 + 17x + 21 \leq 0$

10. $6x^2 - 13x + 6 \geq 0$

11. $2x^2 + 7x > 15$

12. $4x^2 + 2 < 9x$

13. $2 - 5x - 3x^2 < 0$

14. $21 + 4x - x^2 > 0$

15. $x^3 - x \geq 0$

16. $x^3 - x^2 - 30x \leq 0$

In Exercises 17–26, solve the inequality graphically.

17. $x^2 - 4x < 1$

18. $12x^2 - 25x + 12 \geq 0$

19. $6x^2 - 5x - 4 > 0$

20. $4x^2 - 1 \leq 0$

21. $9x^2 + 12x - 1 \geq 0$

22. $4x^2 - 12x + 7 < 0$

23. $4x^2 + 1 > 4x$

24. $x^2 + 9 \leq 6x$

25. $x^2 - 8x + 16 < 0$

26. $9x^2 + 12x + 4 \geq 0$

In Exercises 27–30, solve the cubic inequality graphically.

27. $3x^3 - 12x + 2 \geq 0$

28. $8x - 2x^3 - 1 < 0$

29. $2x^3 + 2x > 5$

30. $4 \leq 2x^3 + 8x$

31. Group Activity Give an example of a quadratic inequality with the indicated solution.

(a) All real numbers

(b) No solution

(c) Exactly one solution

(d) $[-2, 5]$

(e) $(-\infty, -1) \cup (4, \infty)$

(f) $(-\infty, 0] \cup [4, \infty)$

32. Revisiting Example 8 Solve $-16t^2 + 288t \geq 1152$ algebraically. Compare your answer with the result obtained in part (b) of Example 8.

33. Projectile Motion A projectile is launched straight up from ground level with an initial velocity of 256 ft/sec.

(a) When will the projectile's height above ground be 768 ft?

(b) When will the projectile's height above ground be at least 768 ft?

(c) When will the projectile's height above ground be less than or equal to 768 ft?

34. Projectile Motion A projectile is launched straight up from ground level with an initial velocity of 272 ft/sec.

(a) When will the projectile's height above ground be 960 ft?

(b) When will the projectile's height above ground be more than 960 ft?

(c) When will the projectile's height above ground be less than or equal to 960 ft?

35. Writing to Learn Explain the role of equation solving in the process of solving an inequality. Give an example.

36. Travel Planning Barb wants to drive to a city 105 mi from her home in no more than 2 h. What is the lowest average speed she must maintain on the drive?

37. Connecting Algebra and Geometry Consider the collection of all rectangles that have length 2 in. less than twice their width.

(a) Find the possible widths (in inches) of these rectangles if their perimeters are less than 200 in.

(b) Find the possible widths (in inches) of these rectangles if their areas are less than or equal to 1200 in.^{2}

38. Boyle's Law For a certain gas, $P = 400/V$, where P is pressure and V is volume. If $20 \leq V \leq 40$, what is the corresponding range for P?

39. Cash-Flow Planning A company has current assets (cash, property, inventory, and accounts receivable) of \$200,000 and current liabilities (taxes, loans, and accounts payable) of \$50,000. How much can it borrow if it wants its ratio of assets to liabilities to be no less than 2? Assume the amount borrowed is added to both current assets and current liabilities.

Standardized Test Questions

40. True or False The absolute value inequality $|x - a| < b$, where a and b are real numbers, always has at least one solution. Justify your answer.

41. True or False Every real number is a solution of the absolute value inequality $|x - a| \geq 0$, where a is a real number. Justify your answer.

In Exercises 42–45, solve these problems without using a calculator.

42. Multiple Choice Which of the following is the solution to $|x - 2| < 3$?

(A) $x = -1$ or $x = 5$ (B) $[-1, 5)$
(C) $[-1, 5]$ (D) $(-\infty, -1) \cup (5, \infty)$
(E) $(-1, 5)$

43. Multiple Choice Which of the following is the solution to $x^2 - 2x + 2 \geq 0$?

(A) $[0, 2]$ (B) $(-\infty, 0) \cup (2, \infty)$
(C) $(-\infty, 0] \cup [2, \infty)$ (D) All real numbers
(E) There is no solution.

44. Multiple Choice Which of the following is the solution to $x^2 > x$?

(A) $(-\infty, 0) \cup (1, \infty)$ (B) $(-\infty, 0] \cup [1, \infty)$
(C) $(1, \infty)$ (D) $(0, \infty)$
(E) There is no solution.

45. Multiple Choice Which of the following is the solution to $x^2 \leq 1$?

(A) $(-\infty, 1]$ (B) $(-1, 1)$
(C) $[1, \infty)$ (D) $[-1, 1]$
(E) There is no solution.

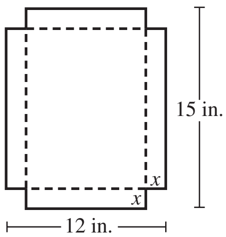
Explorations

46. Constructing a Box with No Top An open box is formed by cutting squares from the corners of a regular piece of cardboard (see figure) and folding up the flaps.

(a) What size corner squares should be cut to yield a box with a volume of 125 in.^{3}?

(b) What size corner squares should be cut to yield a box with a volume more than 125 in.^{3}?

(c) What size corner squares should be cut to yield a box with a volume of at most 125 in.^{3}?



Extending the Ideas

In Exercises 47 and 48, use a combination of algebraic and graphical techniques to solve the inequalities.

47. $|2x^2 + 7x - 15| < 10$ **48.** $|2x^2 + 3x - 20| \geq 10$

CHAPTER P Key Ideas

Properties, Theorems, and Formulas

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CHAPTER P Review Exercises

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator*.

The collection of exercises marked in red could be used as a chapter test.

In Exercises 1 and 2, find the endpoints and state whether the interval is bounded or unbounded.

1. $[0, 5]$ **2.** $(2, \infty)$

3. Distributive Property Use the distributive property to write the expanded form of $2(x^2 - x)$.

4. Distributive Property Use the distributive property to write the factored form of $2x^3 + 4x^2$.

In Exercises 5 and 6, simplify the expression. Assume that denominators are not zero.

5. $\dfrac{(uv^2)^3}{v^2u^3}$

6. $(3x^2y^3)^{-2}$

In Exercises 7 and 8, write the number in scientific notation.

7. The mean distance from Pluto to the Sun is about 3,680,000,000 mi (miles).

8. The diameter of a red blood corpuscle is about 0.000007 m (meters).

In Exercises 9 and 10, write the number in decimal form.

9. Our solar system is about 5×10^9 years old.

10. The mass of an electron is about 9.1094×10^{-28} g (grams).

11. Student Loan Debt Table P.9 shows the growing student loan debt in the United States over the period 2010–2014. Without using a calculator, write the debt for each year in scientific notation.

Table P.9 U.S. Student Loan Debt

Time (years)	Student Loan Debt (billions of $)
2010	803.5
2011	866.3
2012	959.9
2013	1071.0
2014	1155.5

Source: The World Almanac and Book of Facts 2017.

(a) Student loan debt in 2010

(b) Student loan debt in 2011

(c) Student loan debt in 2012

(d) Student loan debt in 2013

(e) Student loan debt in 2014

12. Decimal Form Find the decimal form for $-5/11$. State whether it repeats or terminates.

In Exercises 13 and 14, find **(a)** the distance between the points and **(b)** the midpoint of the line segment determined by the points.

13. -5 and 14

14. $(-4, 3)$ and $(5, -1)$

In Exercises 15 and 16, show that the figure determined by the points is the indicated type.

15. Right triangle: $(-2, 1)$, $(3, 11)$, $(7, 9)$

16. Equilateral triangle: $(0, 1)$, $(4, 1)$, $\left(2, 1 - 2\sqrt{3}\right)$

In Exercises 17 and 18, find the standard form equation for the circle.

17. Center $(0, 0)$, radius 2

18. Center $(5, -3)$, radius 4

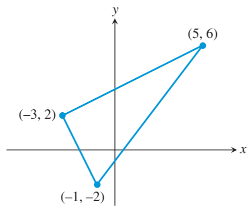
In Exercises 19 and 20, find the center and radius of the circle.

19. $(x + 5)^2 + (y + 4)^2 = 9$

20. $x^2 + y^2 = 1$

21. (a) Find the lengths of the sides of the triangle in the figure.

(b) **Writing to Learn** Prove that the triangle is a right triangle.



22. Distance and Absolute Value Use absolute value notation to write the statement that the distance between z and -3 is less than or equal to 1.

23. Finding a Line Segment with Given Midpoint Let $(3, 5)$ be the midpoint of the line segment with endpoints $(-1, 1)$ and (a, b). Determine a and b.

24. Finding Slope Find the slope of the line through the points $(-1, -2)$ and $(4, -5)$.

25. Point-Slope Form Find an equation in point-slope form for the line through the point $(2, -1)$ with slope $m = -2/3$.

26. Find an equation of the line through the points $(-5, 4)$ and $(2, -5)$ in the general form $Ax + By + C = 0$.

In Exercises 27–32, find an equation in slope-intercept form for the line.

27. The line through $(3, -2)$ with slope $m = 4/5$

28. The line through the points $(-1, -4)$ and $(3, 2)$

29. The line through $(-2, 4)$ with slope $m = 0$

30. The line $3x - 4y = 7$

31. The line through $(2, -3)$ and parallel to the line $2x + 5y = 3$

32. The line through $(2, -3)$ and perpendicular to the line $2x + 5y = 3$

33. SAT Math Scores Table P.10 shows the average SAT math scores in the United States for selected years.

Table P.10 Average SAT Math Scores

Year	Annual Average Score
2008	514
2009	514
2010	515
2011	514
2012	514
2013	514
2014	513
2015	511
2016	508

Source: The College Board, The World Almanac and Book of Facts 2017.

(a) Draw a scatter plot of the data.

(b) Use the 2008 and 2016 data to write a linear equation for the average SAT math score y in terms of the year x. Superimpose the graph of the equation on the scatter plot.

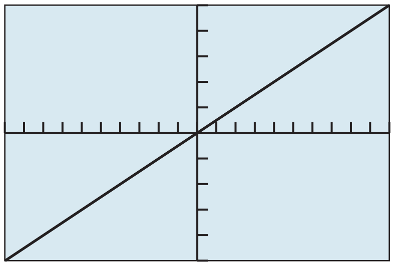
(c) Use the equation in part (b) to predict the average SAT math score for 2020.

(d) **Writing to Learn** Do you think the prediction in part (c) is valid? Explain. (Check it if possible.)

34. Consider the point $(-6, 3)$ and Line L: $4x - 3y = 5$. Write an equation **(a)** for the line passing through this point and parallel to L, and **(b)** for the line passing through this point and perpendicular to L. Support your work graphically.

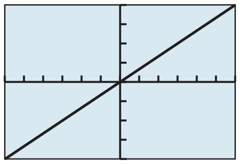
In Exercises 35 and 36, assume that each graph contains the origin and the point in the upper right-hand corner of the viewing window.

35. Find the slope of the line in the figure.



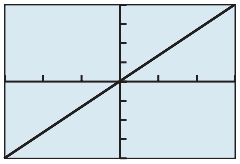
$[-10, 10]$ by $[-25, 25]$

36. Writing to Learn Which line has the greater slope? Explain.



$[-6, 6]$ by $[-4, 4]$

(a)



$[-15, 15]$ by $[-12, 12]$

(b)

In Exercises 37–52, solve the equation algebraically without using a calculator.

37. $3x - 4 = 6x + 5$

38. $\dfrac{x-2}{3} + \dfrac{x+5}{2} = \dfrac{1}{3}$

39. $2(5 - 2y) - 3(1 - y) = y + 1$

40. $3(3x - 1)^2 = 21$

41. $x^2 - 4x - 3 = 0$

42. $16x^2 - 24x + 7 = 0$

43. $6x^2 + 7x = 3$

44. $2x^2 + 8x = 0$

45. $x(2x + 5) = 4(x + 7)$

46. $|4x + 1| = 3$

47. $4x^2 - 20x + 25 = 0$

48. $-9x^2 + 12x - 4 = 0$

49. $x^2 = 3x$

50. $4x^2 - 4x + 2 = 0$

51. $x^2 - 6x + 13 = 0$

52. $x^2 - 2x + 4 = 0$

53. Completing the Square Use completing the square to solve the equation $2x^2 - 3x - 1 = 0$.

54. Quadratic Formula Use the quadratic formula to solve the equation $3x^2 + 4x - 1 = 0$.

In Exercises 55–58, solve the equation graphically.

55. $3x^3 - 19x^2 - 14x = 0$

56. $x^3 + 2x^2 - 4x - 8 = 0$

57. $x^3 - 2x^2 - 2 = 0$

58. $|2x - 1| = 4 - x^2$

In Exercises 59 and 60, solve the inequality and draw a number line graph of the solution.

59. $-2 < x + 4 \le 7$

60. $5x + 1 \ge 2x - 4$

In Exercises 61–72, solve the inequality.

61. $\dfrac{3x - 5}{4} \le -1$

62. $|2x - 5| < 7$

63. $|3x + 4| \ge 2$

64. $4x^2 + 3x > 10$

65. $2x^2 - 2x - 1 > 0$

66. $9x^2 - 12x - 1 \le 0$

67. $x^3 - 9x \le 3$

68. $4x^3 - 9x + 2 > 0$

69. $\left|\dfrac{x + 7}{5}\right| > 2$

70. $2x^2 + 3x - 35 < 0$

71. $4x^2 + 12x + 9 \ge 0$

72. $x^2 - 6x + 9 < 0$

In Exercises 73–80, perform the indicated operation, and write the result in the standard form $a + bi$ without using a calculator.

73. $(3 - 2i) + (-2 + 5i)$

74. $(5 - 7i) - (3 - 2i)$

75. $(1 + 2i)(3 - 2i)$

76. $(1 + i)^3$

77. $(1 + 2i)^2(1 - 2i)^2$

78. i^{29}

79. $\sqrt{-16}$

80. $\dfrac{2 + 3i}{1 - 5i}$

81. Projectile Motion A projectile is launched straight up from ground level with an initial velocity of 320 ft/sec.

(a) When will the projectile's height above ground be 1538 ft?

(b) When will the projectile's height above ground be at most 1538 ft?

(c) When will the projectile's height above ground be greater than or equal to 1538 ft?

82. Navigation A commercial jet airplane climbs at takeoff with slope $m = 4/9$. How far in the horizontal direction will the airplane fly to reach an altitude of 20,000 ft above the takeoff point?

83. Connecting Algebra and Geometry Consider the collection of all rectangles that have length 1 cm more than three times their width w.

(a) Find the possible widths (in cm) of these rectangles if their perimeters are less than or equal to 150 cm.

(b) Find the possible widths (in cm) of these rectangles if their areas are greater than 1500 cm^2.

CHAPTER 1

Functions and Graphs

One of the central principles of economics is that the value of money is not constant; it is a function of time. Since fortunes are made and lost by people attempting to predict the future value of money, much attention is paid to quantitative measures like the Consumer Price Index, a basic measure of inflation in various sectors of the economy. For a look at how the Consumer Price Index for housing has behaved over time, see page 166.

Chapter 1 Overview

In this chapter we begin the study of functions that will continue throughout the text. Your previous courses have introduced you to some basic functions. These functions can be visualized using a graphing calculator, and their properties can be described using the notation and terminology that will be introduced in this chapter. A familiarity with this terminology will serve you well in later chapters when we explore properties of functions in greater depth.

1.1 Modeling and Equation Solving

What you'll learn about

- Numerical Models
- Algebraic Models
- Graphical Models
- The Zero Factor Property
- Problem Solving
- Grapher Failure and Hidden Behavior
- A Word About Proof

... and why

Numerical, algebraic, and graphical models provide different methods to visualize, analyze, and understand data.

Numerical Models

Scientists and engineers have always used mathematics to model the real world and thereby to unravel its mysteries. A **mathematical model** is a mathematical structure that approximates phenomena for the purpose of studying or predicting their behavior. Thanks to advances in computer technology, the process of devising mathematical models is now a rich field of study itself, **mathematical modeling**.

In this text we will be concerned primarily with three types of mathematical models: *numerical models*, *algebraic models*, and *graphical models*. Each type of model gives insight into real-world problems, but the best insights are often gained by switching from one kind of model to another. Developing the ability to do that will be one of the goals of this course.

Perhaps the most basic kind of mathematical model is the **numerical model**, in which numbers (or *data*) are analyzed to gain insights into phenomena. A numerical model can be as simple as the major league baseball standings or as complicated as the network of interrelated numbers that measure the global economy.

Table 1.1 Minimum Hourly Wage in France

Year	Minimum Hourly Wage (MHW)	Purchasing Power in 2020 Dollars
1960	2.9	3.0
1965	3.0	3.2
1970	4.3	4.5
1975	6.0	6.3
1980	6.9	7.2
1985	8.0	8.4
1990	8.3	8.7
1995	8.8	9.2
2000	9.4	9.9
2005	10.6	11.2
2010	11.2	11.7
2015	11.5	12.0
2020	11.6	12.1

Source: https://stats.oecd.org.

EXAMPLE 1 Tracking the Minimum Wage

Table 1.1 shows the growth of the minimum hourly wage (MHW) in France from 1960 through 2020 (at the 2020 exchange rate). The table also shows the MHW adjusted to the purchasing power of 2020 dollars.

(a) In what five-year period did the actual MHW increase the most?

(b) In what year did a worker earning the MHW seem to enjoy the greatest purchasing power?

(c) A worker on minimum wage in 1975 was earning twice as much as a worker on minimum wage in 1965, and yet there was great pressure to raise the minimum wage again. Why?

SOLUTION

(a) In the period 1970 to 1975 it increased by \$1.70. Notice that the minimum wage never goes down, so we can tell that there were no other increases of this magnitude even though the table does not give data from every year.

(b) In 2020.

(c) Although the MHW increased from \$3.00 to \$6.00 in that period, the ratio of purchasing power with the actual MHW dropped from 1.07 to 1.05. This is one way inflation can affect the economy. **Now try Exercise 11.**

Working with large numerical models is standard operating procedure in business and industry, where computers are relied upon to provide fast and accurate data processing.

EXAMPLE 2 Gender of Personnel Serving in the U.K. Regular Armed Forces

Table 1.2 shows the fluctuation in the number of the U.K. regular armed forces from 2012 to 2021. Is the proportion of female personnel increasing over time?

SOLUTION The numbers in each column go up and down depending on the needs of the military at the time. It does look like the *number* of female personnel has been decreasing, but it is difficult to discern from Table 1.2 whether the *proportion* of female personnel is decreasing. What we need is another column of numbers showing the ratio of female personnel to the total number of personnel each year.

We could compute all the ratios separately, but it is easier to do this kind of repetitive calculation with a single command on a computer spreadsheet. You can also do this on a graphing calculator by manipulating lists (see Exercise 19). Table 1.3 shows what percentage of the total number of personnel each year are females. With these data to extend our numerical model, it is clear that the proportion of female personnel increased steadily between 2012 and 2021.

Table 1.2 Strength of U.K. Regular Armed Forces (thousands)

Year	Total	Male	Female
2012	175.94	158.88	17.06
2013	166.46	150.16	16.31
2014	156.63	140.89	15.74
2015	152.15	136.75	15.40
2016	150.25	134.97	15.28
2017	147.52	132.30	15.23
2018	144.90	129.64	15.26
2019	144.65	129.00	15.65
2020	146.33	130.22	16.11
2021	149.54	132.84	16.71

Source: U.K. Government Services and Information, U.K. armed forces biannual diversity statistics: 1 October 2021.

Table 1.3 Percentage of Personnel in the U.K. Regular Armed Forces Who Are Female

Year	Percentage
2012	9.7
2013	9.8
2014	10.0
2015	10.1
2016	10.2
2017	10.3
2018	10.5
2019	10.8
2020	11.0
2021	11.2

Source: U.K. Government Services and Information, U.K. armed forces biannual diversity statistics: 1 October 2021.

Now try Exercise 19.

Algebraic Models

An **algebraic model** uses formulas to relate variable quantities associated with the phenomena being studied. The added power of an algebraic model over a numerical model is that it can be used to generate numerical values of unknown quantities by relating them to known quantities.

EXAMPLE 3 Comparing Pizzas

A pizzeria sells a rectangular 18 in. by 24 in. pizza for the same price as its large round pizza (24-in. diameter). If both pizzas are of the same thickness, which option gives the most pizza for the money?

SOLUTION We need to compare the *areas* of the pizzas. Fortunately, geometry has provided algebraic models that allow us to compute the areas from the given information.

For the rectangular pizza:

$$Area = l \times w = 18 \times 24 = 432 \text{ in.}^2$$

For the circular pizza:

$$Area = \pi r^2 = \pi\left(\frac{24}{2}\right)^2 = 144\pi \approx 452.4 \text{ in.}^2$$

The round pizza is larger and therefore gives more for the money.

Now try Exercise 21.

The algebraic models in Example 3 come from geometry, but you have probably encountered algebraic models from many other sources in your algebra and science courses.

EXPLORATION 1 Designing an Algebraic Model

A department store is having a sale in which everything is discounted 30% off the marked price. The discount is taken at the sales counter, and then a national sales tax of 5% and a local sales tax of 0.8% are added on.

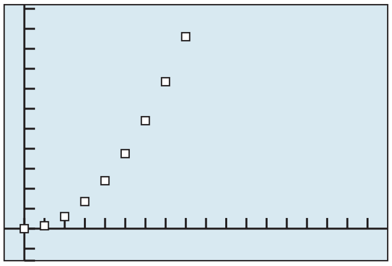
1. The discount price d is related to the marked price m by the formula $d = km$, where k is a certain constant. What is k?
2. The actual sale price s is related to the discount price d by the formula $s = d + td$, where t is a constant related to the combined total sales tax. What is t?
3. Using the answers from steps 1 and 2 you can find a constant p that relates s directly to m by the formula $s = pm$. What is p?
4. If you have only €50, can you afford to buy a pair of trousers marked €69.99?
5. If you have a debit card but are determined to spend no more than €100, what is the maximum total value of your marked purchases before you present them at the sales counter?

The ability to generate numbers from formulas makes an algebraic model far more useful as a predictor of behavior than a numerical model. Indeed, one optimistic goal of scientists and mathematicians when modeling phenomena is to fit an algebraic model to numerical data and then (even more optimistically) to analyze why it works. Not all models can be used to make accurate predictions. For example, nobody has ever devised a successful formula for predicting the ups and downs of the stock market as a function of time, although that does not stop investors from trying.

If numerical data do behave reasonably enough to suggest that an algebraic model might be found, it is often helpful to look at a picture first. That brings us to graphical models.

Graphical Models

A **graphical model** is a visible representation of a numerical model or an algebraic model that gives insight into the relationships between variable quantities. Learning to interpret and use graphs is a major goal of this text.



[−1, 18] by [−8, 56]

Figure 1.1 A scatter plot of the data from a Galileo gravity experiment. (Example 4)

EXAMPLE 4 Visualizing Galileo's Gravity Experiments

Galileo Galilei (1564–1642) spent a good deal of time rolling balls down inclined planes, carefully recording the distance they traveled as a function of elapsed time. His experiments are commonly repeated in physics classes today, so it is easy to reproduce a typical table of Galilean data.

Elapsed time (sec)	0	1	2	3	4	5	6	7	8
Distance traveled (in.)	0	0.75	3	6.75	12	18.75	27	36.75	48

What graphical model fits the data? Can you find an algebraic model that fits?

SOLUTION A scatter plot of the data is shown in Figure 1.1.

Galileo's experience with quadratic functions suggested to him that this figure was a parabola with its vertex at the origin; he therefore modeled the effect of gravity as a quadratic function:

$$d = kt^2.$$

Because the ordered pair $(1, 0.75)$ must satisfy the equation, it follows that $k = 0.75$, yielding the equation

$$d = 0.75t^2.$$

You can verify numerically that this algebraic model correctly predicts the rest of the data points. We will have much more to say about parabolas in Chapter 2.

Now try Exercise 23.

This insight led Galileo to discover several basic laws of motion that would eventually be named after Isaac Newton. Although Galileo had found the algebraic model to describe the path of the ball, it would take Newton's calculus to explain why that model worked.

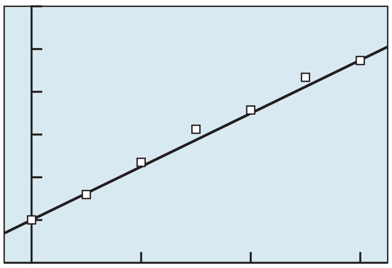
EXAMPLE 5 Fitting a Curve to Data

Table 1.4 shows the average life expectancy for persons born in Singapore in each given year. The data are drawn from census years between 1960 and 2020.

Table 1.4 Years of Life Expected at Birth in Singapore, 1960–2020

Years After 1960	0	10	20	30	40	50	60
Life Expectancy	62.9	65.8	72.1	75.3	78.0	81.7	83.9

Source: https://singstat.gov.sg.



[−5, 65] by [60, 90]

Figure 1.2 The line with equation $y = 0.35x + 62.9$ is a good model for the life expectancy data in Table 1.4. (Example 5).

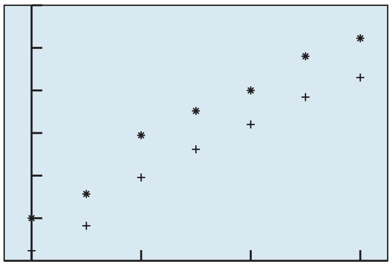
Model the trend graphically and use the graphical model to suggest an algebraic model.

SOLUTION A scatter plot of the data is shown in Figure 1.2. Since the points show a linear pattern, a linear equation is an appropriate algebraic model. We will describe in Chapter 2 how a statistician would find the best line to fit the data, but we can get a pretty good fit for now by putting the line through $(0, 62.9)$ and $(60, 83.9)$.

The slope is $(83.9 - 62.9)/(60 - 0) = 0.35$ and the y-intercept is 62.9.

You can observe that the line $y = 0.35x + 62.9$ does a very nice job of modeling the data.

Now try Exercises 13 and 15.



[−5, 65] by [60, 90]

Figure 1.3 A scatter plot showing the life expectancy of Singaporean residents (the circles represent the females, while the crosses represent the males) for the same years as shown in Figure 1.2. The high point in 1980 is even more apparent.

EXPLORATION 2 Interpreting the Model

The parabola in Example 4 arose from a law of physics that governs falling objects, which should inspire more confidence than the linear model in Example 5. We can repeat Galileo's experiment many times with differently sloped ramps, with different units of measurement, and even on different planets, and a quadratic model will fit it every time. The purpose of this Exploration is to think more deeply about the linear model for the life expectancy data.

1. The linear model we found will not continue to predict life expectancy indefinitely. Why must it eventually fail?
2. Do you think that our linear model should give an accurate estimate for the life expectancy of a person born in Singapore in 2030? Why or why not?
3. The linear model is such a good fit that it actually calls our attention to the unusually high point for females in 1980. Statisticians might look for a reason for why the gap in life expectancy between females and males increased or decreased in later years for both genders. Can you think of one? As a hint, consider the scatter plot in Figure 1.3, which shows the life expectancy for *females* and *males* in Singapore for the same period.

There are other ways of graphing numerical data that are particularly useful for statistical studies. We will treat some of them in Chapter 10. The scatter plot will be our choice of data graph for the time being, because it provides the closest connection to graphs of functions in the Cartesian plane.

The Zero Factor Property

The main reason for studying algebra through the ages has been to solve equations. We develop algebraic models for phenomena so that we can solve problems, and the solutions to the problems usually come down to finding solutions of algebraic equations.

If we are fortunate enough to be solving an equation in a single variable, we might proceed as in the following example.

EXAMPLE 6 Solving an Equation Algebraically

Find all real numbers x for which $6x^3 = 11x^2 + 10x$.

SOLUTION We begin by changing the form of the equation to $6x^3 - 11x^2 - 10x = 0$.

We can then solve this equation algebraically by factoring:

$$\begin{aligned} 6x^3 - 11x^2 - 10x &= 0 \\ x(6x^2 - 11x - 10) &= 0 \\ x(2x - 5)(3x + 2) &= 0 \end{aligned}$$

$$x = 0 \quad \text{or} \quad 2x - 5 = 0 \quad \text{or} \quad 3x + 2 = 0$$

$$x = 0 \quad \text{or} \quad x = \frac{5}{2} \quad \text{or} \quad x = -\frac{2}{3}$$

Now try Exercise 31.

In Example 6, we used the important Zero Factor Property of real numbers.

Zero Factor Property

A product of real numbers is zero if and only if at least one of the factors in the product is zero.

It is this property that algebra students use to solve equations in which an expression is set equal to zero. Modern problem solvers are fortunate to have an alternative way to find such solutions.

If we graph the expression, then the x-intercepts of the graph of the expression will be the values for which the expression equals 0.

EXAMPLE 7 Solving an Equation: Comparing Methods

Solve the equation $x^2 = 10 - 4x$.

SOLUTION

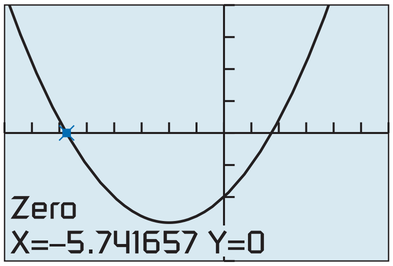
Solve Algebraically The given equation is equivalent to $x^2 + 4x - 10 = 0$. This quadratic equation has irrational solutions that can be found by the quadratic formula.

$$x = \frac{-4 + \sqrt{16 + 40}}{2} \approx 1.7416574$$

and

$$x = \frac{-4 - \sqrt{16 + 40}}{2} \approx -5.7416574$$

Although the decimal answers are certainly accurate enough for all practical purposes, it is important to note that only the expressions found by the quadratic formula give the *exact* real number answers. The tidiness of exact answers is a worthy mathematical goal. Realistically, however, exact answers are often impossible to obtain, even with the most sophisticated mathematical tools.



[−8, 6] by [−20, 20]

Figure 1.4 The graph of $y = x^2 + 4x - 10$. (Example 7)

Solve Graphically We first find an equivalent equation with 0 on the right-hand side: $x^2 + 4x - 10 = 0$. We next graph the equation $y = x^2 + 4x - 10$, as shown in Figure 1.4.

We then use the grapher to locate the x-intercepts of the graph:

$$x \approx 1.7416574 \text{ and } x \approx -5.7416574$$

Now try Exercise 35.

Solving Equations with Technology

Example 7 shows one method of solving an equation with technology. Some graphers could also solve the equation in Example 7 by finding the *intersection* of the graphs of $y = x^2$ and $y = 10 - 4x$. Some graphers have built-in equation solvers. Each method has its advantages and disadvantages, but we recommend the "finding the x-intercepts" technique for now, because it most closely parallels the classical algebraic techniques for finding roots of equations and makes the connection between the algebraic and graphical models easier to follow and appreciate.

We used the graphing utility of the calculator to **solve graphically** in Example 7. Most calculators also have solvers that would enable us to **solve numerically** for the same decimal approximations without considering the graph. Some calculators have computer algebra systems that will solve numerically to produce exact answers in certain cases. In this text we will distinguish between these two technological methods and the traditional pencil-and-paper methods used to **solve algebraically**.

Every method of solving an equation usually comes down to finding where an expression equals zero. If we use $f(x)$ to denote an algebraic expression in the variable x, the connections are as follows:

Fundamental Connection

If a is a real number that solves the equation $f(x) = 0$, then these three statements are equivalent:

1. The number a is a **root** (or **solution**) of the **equation** $f(x) = 0$.
2. The number a is a **zero** of $y = f(x)$.
3. The number a is an ***x*-intercept** of the **graph** of $y = f(x)$. (Sometimes the point $(a, 0)$ is referred to as an x-intercept.)

Problem Solving

George Pólya (1887–1985) is called the father of modern problem solving, not only because he was good at it (as he certainly was) but also because he published the most famous analysis of the problem-solving process: *How to Solve It: A New Aspect of Mathematical Method*. His "four steps" are well known to most mathematicians:

Pólya's Four Problem-Solving Steps

1. Understand the problem.
2. Devise a plan.
3. Carry out the plan.
4. Look back.

The increased emphasis on problem solving in recent years has been accompanied by a more prescriptive approach to modeling. The *Guidelines for Assessment and Instruction in Mathematical Modeling Education* (GAIMME) report recommends the following approach, echoing Pólya's four steps:

Mathematical Modeling (GAIMME)

1. Identify the problem.
2. Make assumptions and identify variables.
3. Carry out the mathematics.
4. Analyze and assess the solution.
5. Iterate. (In actual practice, the final step would be to implement the model.)

The American Statistical Association's *Guidelines for Assessment and Instruction in Statistics Education* (GAISE) *Report* recommends the following approach for modeling that is based on data analysis and statistical inference:

Statistical Problem Solving (GAISE)

1. Formulate questions.
2. Collect data.
3. Analyze the data.
4. Interpret the results.

The problem-solving process that we recommend you use throughout this course will be the following version of Pólya's four steps.

Problem-Solving Process

Step 1—Understand the problem.

- Read the problem as stated, several times if necessary.
- Be sure you understand the meaning of each term used.
- Restate the problem in your own words. Discuss the problem with others if you can.
- Identify clearly the information that you need to solve the problem.
- Find the information you need from the given data.

Step 2—Develop a mathematical model of the problem.

- Draw a picture to visualize the problem situation. It usually helps.
- Introduce a variable to represent the quantity you seek. (In some cases there may be more than one.)
- Use the statement of the problem to find an equation or inequality that relates the variables you seek to quantities that you know.

Step 3—Solve the mathematical model and support or confirm the solution.

- **Solve algebraically** using traditional algebraic methods, and **support graphically or support numerically** using a graphing utility.
- **Solve graphically or numerically** using a graphing utility and **confirm algebraically** using traditional algebraic methods.
- **Solve graphically or numerically** because there is no other way possible.

Step 4—Interpret the solution in the problem setting.

- Translate your mathematical result into the problem setting and decide whether the result makes sense.

EXAMPLE 8 Applying the Problem-Solving Process

The engineers at an auto manufacturer pay students \$0.08 per mile plus \$25 per day to road test their new vehicles.

(a) How much did the auto manufacturer pay Sally to drive 440 mi in one day?

(b) John earned \$93 test-driving a new car in one day. How far did he drive?

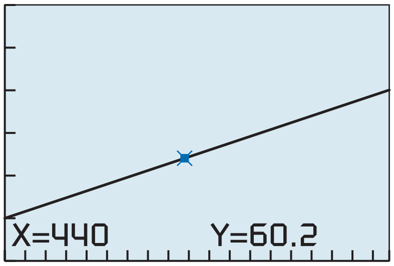
SOLUTION

Model A picture of a car or of Sally or John would not be helpful, so we go directly to designing the model. Both John and Sally earned \$25 for one day, plus \$0.08 per mile. Multiply dollars/mile by miles to get dollars.

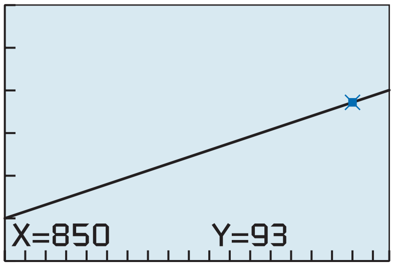
So if p represents the pay for driving x miles in one day, our algebraic model is

$$p = 25 + 0.08x.$$

(continued)



[0, 940] by [0, 150]
(a)



[0, 940] by [0, 150]
(b)

Figure 1.5 Graphical support for the algebraic solutions in Example 8.

Solve Algebraically

(a) To get Sally's pay we let $x = 440$ and solve for p:

$$\begin{aligned} p &= 25 + 0.08(440) \\ &= 60.20 \end{aligned}$$

(b) To get John's mileage we let $p = 93$ and solve for x:

$$\begin{aligned} 93 &= 25 + 0.08x \\ 68 &= 0.08x \\ x &= \frac{68}{0.08} \\ x &= 850 \end{aligned}$$

Support Graphically Figure 1.5a shows that the point $(440, 60.20)$ is on the graph of $y = 25 + 0.08x$, supporting our answer to (a). Figure 1.5b shows that the point $(850, 93)$ is on the graph of $y = 25 + 0.08x$, supporting our answer to (b). (We could also have **supported** our answer **numerically** by simply substituting in for each x and confirming the value of p.)

Interpret Sally earned \$60.20 for driving 440 mi in one day. John drove 850 mi in one day to earn \$93.00. **Now try Exercise 47.**

It is not really necessary to *show* written support as part of an algebraic solution, but it is good practice to support answers wherever possible, simply to reduce the chance for error. We will often show written support of our solutions in this book in order to highlight the connections among the algebraic, graphical, and numerical models.

Grapher Failure and Hidden Behavior

Although the graphs produced by computers and graphing calculators are wonderful tools for understanding algebraic models and their behavior, it is important to keep in mind that machines have limitations. Occasionally they can produce graphical models that misrepresent the phenomena we wish to study, a problem we call **grapher failure**. Sometimes the viewing window will be too large, obscuring details of the graph, which we call **hidden behavior**. We will give an example of each just to illustrate what can happen, but rest assured that these difficulties rarely occur with graphical models that arise from real-world problems.

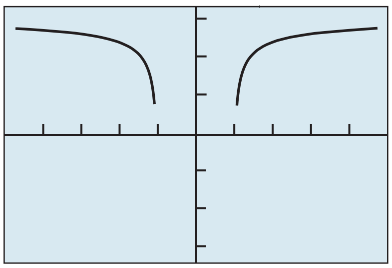
Technology Note

One way to get the table in Figure 1.6b is to use the "Ask" feature of your graphing calculator and enter each x-value separately.

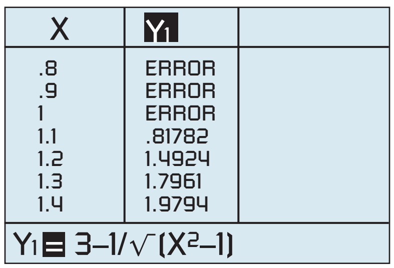
EXAMPLE 9 Seeing Grapher Failure

Look at the graph of $y = 3 - \dfrac{1}{\sqrt{x^2 - 1}}$ in the ZDecimal window on a graphing calculator. Are there any x-intercepts?

SOLUTION The graph is shown in Figure 1.6a.



$[-4.7, 4.7]$ by $[-3.1, 3.1]$
(a)



(b)

Figure 1.6 (a) A graph with no apparent intercepts. (b) The function $y = 3 - 1/\sqrt{x^2 - 1}$ is undefined when $|x| \le 1$.

The graph seems to have no x-intercepts, yet we can find some by solving the equation $0 = 3 - 1/\sqrt{x^2 - 1}$ algebraically:

$$\begin{aligned} 0 &= 3 - 1/\sqrt{x^2 - 1} \\ 1/\sqrt{x^2 - 1} &= 3 \\ \sqrt{x^2 - 1} &= 1/3 \\ x^2 - 1 &= 1/9 \\ x^2 &= 10/9 \\ x &= \pm\sqrt{10/9} \approx \pm 1.054 \end{aligned}$$

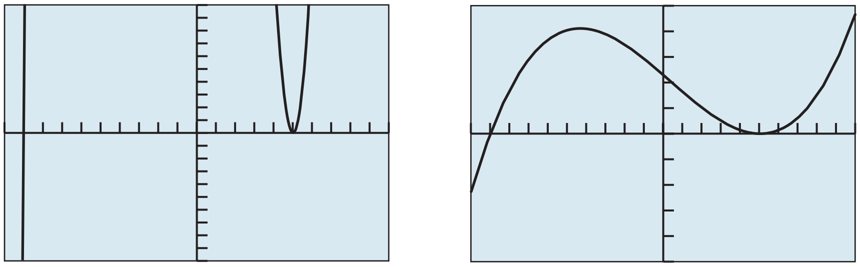
There should be x-intercepts at about ± 1.054. What went wrong?

The answer is a simple form of grapher failure. As the table shows, the function is undefined for the sampled x-values until $x = 1.1$, at which point the graph "turns on," beginning with the pixel at $(1.1, 0.81782)$ and continuing from there to the right. Similarly, the graph coming from the left "turns off" at $x = -1$, before it gets to the x-axis. The x-intercepts might well appear in other windows, but for this particular function in this particular window, the behavior we expect to see is not there. **Now try Exercise 49.**

EXAMPLE 10 Not Seeing Hidden Behavior

Solve graphically: $x^3 - 1.1x^2 - 65.4x + 229.5 = 0.$

SOLUTION Figure 1.7a shows the graph in the standard $[-10, 10]$ by $[-10, 10]$ window, an inadequate choice because too much of the graph is off the screen. Our horizontal dimensions look fine, so we adjust our vertical dimensions to $[-500, 500]$, which yields the graph in Figure 1.7b.



[−10, 10] by [−10, 10]

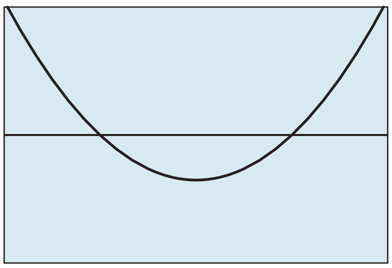
(a)

[−10, 10] by [−500, 500]

(b)

Figure 1.7 The graph of $y = x^3 - 1.1x^2 - 65.4x + 229.5$ in two viewing windows. (Example 10)

We use the grapher to locate an x-intercept near -9 (which we find to be -9) and then an x-intercept near 5 (which we find to be 5). The graph leads us to believe that we have finished. However, if we zoom in closer to observe the behavior near $x = 5$, the graph tells a new story (Figure 1.8).



[4.95, 5.15] by [−0.1, 0.1]

Figure 1.8 A closer look at the graph of $y = x^3 - 1.1x^2 - 65.4x + 229.5$. (Example 10)

In this graph we see that there are actually *two* x-intercepts near 5 (which we find to be 5 and 5.1). There are therefore three roots (or zeros) of the equation $x^3 - 1.1x^2 - 65.4x + 229.5 = 0$: $x = -9$, $x = 5$, and $x = 5.1$. **Now try Exercise 51.**

You might wonder if there could be still *more* hidden x-intercepts in Example 10! We will learn in Chapter 2 how the *Fundamental Theorem of Algebra* guarantees that there are not.

A Word About Proof

While Example 10 is still fresh in our minds, let us point out a subtle, but very important, consideration about our solution.

We *solved graphically* to find two solutions, then eventually three solutions, to the given equation. Although we did not show the steps, it is easy to *confirm numerically* that the three numbers found are actually solutions by substituting them into the equation. But the problem asked us to find *all* solutions. Although we could explore that equation graphically in a hundred more viewing windows and never find another solution, our failure to find them would not *prove* that they are not out there somewhere. That is why the Fundamental Theorem of Algebra is so important. It tells us that there can be at most three real solutions to *any* cubic equation, so we know for a fact that there are no more.

Exploration is encouraged throughout this text because it is how mathematical progress is made. Mathematicians are never satisfied, however, until they have *proved* their results. We will show you proofs in later chapters, and we will ask you to produce proofs occasionally in the exercises. That will be a time for you to set the technology aside, get out a pencil, and show in a logical sequence of algebraic steps that something is undeniably and universally true. This process is called **deductive reasoning**.

EXAMPLE 11 Proving a Peculiar Number Fact

Prove that 6 is a factor of $n^3 - n$ for every positive integer n.

SOLUTION You can explore this expression for various values of n on your calculator. Table 1.5 shows it for the first 12 values of n.

Table 1.5 The First 12 Values of $n^3 - n$

n	1	2	3	4	5	6	7	8	9	10	11	12
$n^3 - n$	0	6	24	60	120	210	336	504	720	990	1320	1716

All of these numbers are divisible by 6, but that does not prove that they will continue to be divisible by 6 for all values of n. In fact, a table with a billion values, all divisible by 6, would not constitute a proof. Here is a proof:

Let n be *any* positive integer.

- We can factor $n^3 - n$ as the product of three numbers: $(n - 1)(n)(n + 1)$.
- The factorization shows that $n^3 - n$ is always the product of three consecutive integers.
- Every set of three consecutive integers must contain a multiple of 3.
- Since 3 divides a factor of $n^3 - n$, it follows that 3 is a factor of $n^3 - n$ itself.
- Every set of three consecutive integers must contain a multiple of 2.
- Since 2 divides a factor of $n^3 - n$, it follows that 2 is a factor of $n^3 - n$ itself.
- Since both 2 and 3 are factors of $n^3 - n$, we know that 6 is a factor of $n^3 - n$.

End of proof!

Now try Exercise 53.