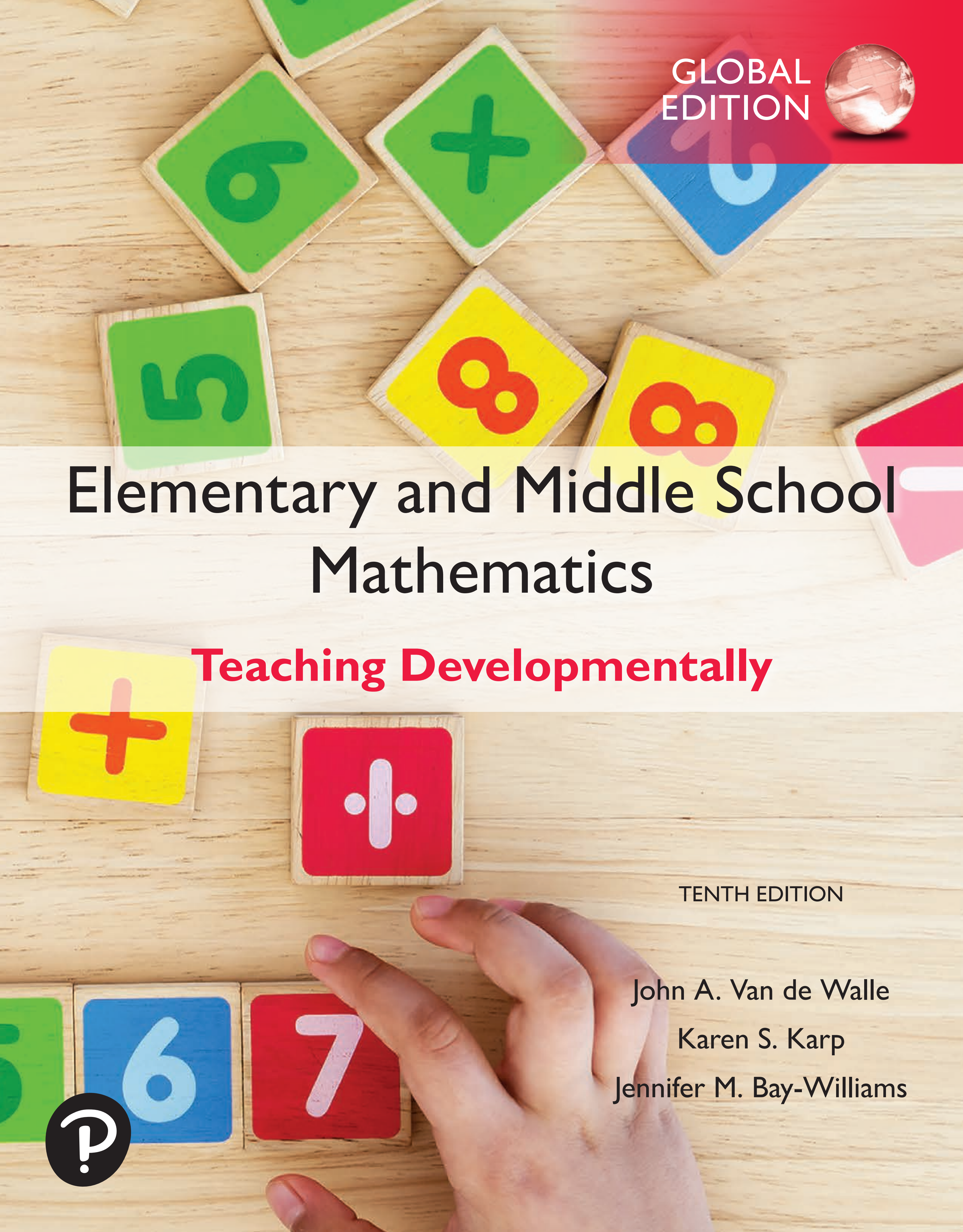

GLOBAL EDITION

Elementary and Middle School Mathematics

Teaching Developmentally

TENTH EDITION

John A. Van de Walle
Karen S. Karp
Jennifer M. Bay-Williams

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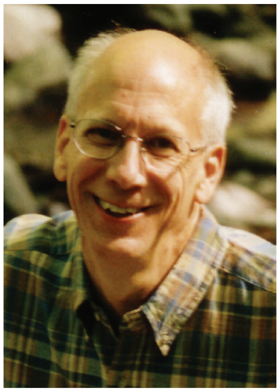
Authorized adaptation from the United States edition, entitled *Elementary and Middle School Mathematics: Teaching Developmentally*, 10th Edition, ISBN 978-0-13-480208-4 by John A. Van de Walle, Karen S. Karp, and Jennifer M. Bay-Williams, published by Pearson Education © 2020.

British Library Cataloguing-in-Publication Data
A catalogue record for this book is available from the British Library

ISBN 10: 1-292-33139-9
ISBN 13: 978-1-292-33139-3
eBook ISBN 13: 978-1-292-33144-7

Typeset in Janson Text LT Pro by SPi Global

About the Authors



John A. Van de Walle

The late John A. Van de Walle was a professor emeritus at Virginia Commonwealth University. He was a leader in mathematics education who regularly gave professional development workshops for K–8 teachers in the United States and Canada focused on mathematics instruction that engaged students in mathematical reasoning and problem solving. He visited and taught in many classrooms and worked with teachers to implement student-centered mathematics lessons. He co-authored the *Scott Foresman-Addison Wesley Mathematics K–6* series and contributed to the original Pearson School mathematics program *enVisionMATH*. Additionally, John was very active in the National Council of Teachers of Mathematics (NCTM), writing book chapters and journal articles, serving on the board of directors, chairing the educational materials committee, and speaking at national and regional meetings.

Karen S. Karp

Karen S. Karp is a visiting professor at Johns Hopkins University (Maryland). Previously, she was a professor of mathematics education at the University of Louisville for more than twenty years. Prior to entering the field of teacher education, she was an elementary school teacher in New York. She is the coauthor of *Developing Essential Understanding of Addition and Subtraction for Teaching Mathematics in PreK–Grade 2*, *Discovering Lessons for the Common Core State Standards in Grades K–5* and *Putting Essential Understanding of Addition and Subtraction into Practice PreK–Grade 2*. She is a former member of the board of directors for the National Council of Teachers of Mathematics (NCTM) and a former president of the Association of Mathematics Teacher Educators. She continues to work in classrooms to support teachers in ways to instruct students with disabilities.

Jennifer M. Bay-Williams

Dr. Jennifer Bay-Williams is a professor at the University of Louisville. She has written many articles and books around K–12 mathematics education, including the three book series related to this book—*Teaching Student Centered Mathematics*—and various other books, including *Everything You Need for Mathematics Coaching*, *Developing Essential Understanding of Addition and Subtraction in Prekindergarten–Grade 2*, *On the Money* (financial literacy book series), and *Math and Literature: Grades 6–8*. Jennifer taught elementary, middle, and high school in Missouri and in Peru, and continues to learn and work in K–8 classrooms in rural and urban settings. Jennifer has been a member of the NCTM board of directors, AMTE secretary, president, and lead writer for *Standards for the Preparation of Teachers of Mathematics* (AMTE, 2017), and is currently on the TODOS: Mathematics for All Board of Directors.

About the Contributors

Jonathan Wray is the technology contributor to *Elementary and Middle School Mathematics, Teaching Developmentally* (sixth–tenth editions). He is the acting coordinator of secondary mathematics curricular programs in the Howard County public school system. He has served as the president of the Association of Maryland Mathematics Teacher Educators (AMMTE) and the Maryland Council of Teachers of Mathematics (MCTM) and currently is manager of the Elementary Mathematics Specialists and Teacher Leaders (ems&tl) Project. Jon also served on the NCTM Board of Directors (2012–2015). He has been recognized for his expertise in infusing technology in mathematics teaching and was named an outstanding technology leader in education by the Maryland Society for Educational Technology (MSET). He was a primary and intermediate grades classroom teacher, gifted and talented resource teacher, elementary mathematics specialist, curriculum and assessment developer, grant project manager, and educational consultant.

Elizabeth Todd Brown is the assessment and media contributor to *Elementary and Middle School Mathematics, Teaching Developmentally* (tenth edition). She is a professor emeritus from the University of Louisville. She was also an elementary and middle school teacher in Iowa, Missouri, and Kentucky and continues to volunteer weekly at a local elementary school in Louisville. Todd coauthored the book, *Feisty Females*, and has written articles on the importance of early mathematics learning. She served as president of the local NCTM affiliate Greater Louisville Council of the Teachers of Mathematics and was the chair of the volunteers for the NCTM Regional in Louisville in 1998 and 2013. Todd was the 1998 Kentucky Presidential Awardee for Excellence in Teaching Mathematics and Science.

Brief Contents

Contents

The fundamental core of effective teaching of mathematics combines an understanding of how students learn, how to promote that learning by teaching through problem solving, and how to plan for and assess that learning daily. That is the focus of these first six chapters, providing discussion, examples, and activities that develop the core ideas of learning, teaching, planning, and assessment for each and every student.

PART II Teaching Student-Centered Mathematics

Each of these chapters *applies* the core ideas of Part I to the content taught in K–8 mathematics. Clear discussions are provided for how to teach the topic, what a learning progression for that topic might be, and what worthwhile tasks look like. Hundreds of problem-based, engaging tasks and activities are provided to show how the concepts can be developed with students. These chapters are designed to help you develop pedagogical strategies now, and serve as a resource and reference for your teaching now and in the future.

Preface

All students can learn mathematics with understanding. It is through the teacher's actions that every student can have this experience. We believe that teachers must create a classroom environment in which students are given opportunities to solve problems and work together, using their ideas and strategies, to solve them. Effective mathematics instruction involves posing tasks that engage students in the mathematics they are expected to learn. Then, by allowing students to interact with and productively struggle with *their own mathematical ideas* and *their own strategies*, they will learn to see the connections among mathematical topics and the real world. Students value mathematics and feel empowered to use it.

Creating a classroom in which students design solution pathways, engage in productive struggle, and connect one mathematical idea to another, is complex. Questions arise, such as, "How do I get students to wrestle with problems if they just want me to show them how to do it? What kinds of tasks lend themselves to this type of engagement? Where can I learn the mathematics content I need to be able to teach in this way?" With these and other questions firmly in mind, we have several objectives in the tenth edition of this textbook:

1. Illustrate what it means to teach mathematics using a problem-based approach.
2. Serve as a go-to reference for all of the mathematics content suggested for grades preK–8 as recommended in the Common Core State Standards (NGA Center & CCSSO, 2010) and in standards used in other states, and for the research-based strategies that illustrate how students best learn this content.
3. Present a practical resource of robust, problem-based activities and tasks that can engage students in the use of significant mathematical concepts and skills.
4. Focus attention on student thinking, including the ways students might reason about numbers, and possible challenges and misconceptions they might have.

We are hopeful that you will find that this book is a valuable resource for teaching and learning mathematics!

New to this Edition

The following are highlights of the most significant changes in the tenth edition.

Common Challenges and Misconceptions

Every chapter in Part II offers at least one table that summarizes common challenges students encounter in learning that topic (Chapter 15, Fraction Operations has three). The table includes the challenge, provides an example of what that might look like in either a sample of student work or a statement, and then offers some brief ideas of what you might do to help. Knowing common student challenges and misconceptions is a critical part of planning and can greatly influence how a lesson is structured and what problems you use. The research from many sources has been merged into these practical references.

Routines

More and more classrooms are using innovative lesson designs and short discussion routines to help students develop number sense, flexibility, and the mathematical practices. In Chapter 4, we have added se new sections on: 3-Act Tasks, Number Talks, and Worked Examples. For example, worked examples are mentioned in some of the tables identifying student challenges, because there is research to suggest that analyzing worked examples is effective in helping students learn.

Mathematical Modeling

Since the ninth edition, there has been significant national dialogue about the importance of mathematical modeling and what this might look like across the grades. The *Guidelines for Assessment & Instruction in Mathematical Modeling Education* (GAIMME) Report (COMAP & SIAM, 2016) provides excellent guidance. Therefore, the section in Chapter 13 on mathematical modeling was completely rewritten to reflect the GAIMME report, as well as to showcase a number of excellent books and articles that have emerged recently.

Infusion of Technology

You may notice that Chapter 7 (Technology) from the previous edition is gone. Readers and reviewers have commented that this chapter is not needed in part because using technology is much more commonly understood and used, and in part because it makes far more sense to talk about technology *as it relates to the mathematics.* We have heard you and we have integrated technology discussions, tools, and ideas throughout the book.

MyLab Education

Digital learning and assessment resources have been expanded significantly via MyLab Education. The following resources have been designed to help you develop the pedagogical knowledge *and* content knowledge needed to be a successful teacher of mathematics:

- **Video examples:** Embedded throughout all chapters, these examples allow you to see key concepts in action through authentic classroom video, as well as clips of children solving math problems. Additional videos feature your authors and other experts introducing and briefly explaining strategies for teaching important topics.
- **Self-checks:** Designed for self-study, these multiple-choice items are tied to each chapter learning outcome, and help you assess how well you have mastered the concepts covered in the reading. These exercises are self-grading and provide a rationale for the correct answer. Similar questions are available in the book. Answers to the questions in the book are given at the end of the Self Check section.
- **Application exercises:** Video and scenario-based exercises appear throughout the chapters and provide an opportunity for you to apply what you have learned to real classroom situations. There are also ten exercises on *observing and responding to student thinking* that include video clips of children talking through and solving problems on a whiteboard app; accompanying questions ask you to analyze and child's reasoning, identify any misconceptions, and explain any actions or prompts you might use as the teacher to guide the student's learning. Expert feedback is provided after submitting your response.
- **Math practice:** Located at the end of most content chapters, these sets of questions provide an opportunity to practice or refresh your own mathematics skills through solving exercises associated with the content from that chapter. These questions are also self-grading.
- **Blackline masters, activity pages, and expanded lessons:** These documents are linked throughout each chapter and make it easy for instructors and students to download and print classroom-ready handouts that can be used in a methods class or school settings.

Major Changes to Specific Chapters

Every chapter in the tenth edition has been revised to reflect the most current research, standards, and exemplars. This is evident in the approximately *300 new references* in the tenth edition! This represents our ongoing commitment to synthesize and present the most current evidence of effective mathematics teaching. Here we share changes to what we consider the most significant (and that have not already been mentioned above).

Teaching Mathematics in the 21st Century (Chapter 1)

The new Association of Mathematics Teacher Educators (AMTE) Standards for Preparing Teachers of Mathematics (AMTE, 2017) are described in Chapter 1. We added a section on how to create a whole school agreement with a cohesive mathematics message.

Exploring What It Means to Know and Do Mathematics (Chapter 2)

Chapter 2 was revised in several significant ways, including revisions to the exemplar tasks (one in each content domain) to each have a common format, and to each have a stronger focus on multiple strategies. The discussions on theory were condensed, and making connections between theory and teaching were revised to be more succinct and explicit.

Teaching through Problem Solving (Chapter 3)

The NCTM Teaching Practices (2014) have been integrated into Chapter 3. A completely revamped section, now titled Developing Procedural Fluency, focuses on the importance of connecting conceptual and procedural knowledge, and includes a new list of ways to adapt drill-related tasks to emphasize understanding and connections (Boaler, 2016). Talk moves in the Discourse section have been revised to include eight talk moves (Chapin, O'Conner, & Anderson, 2013).

Teaching through Problem Solving (Chapter 4)

Beyond the new routines section (described above), the families section was heavily revised and the lesson plan steps condensed and formatted for easier readability.

Teaching Mathematics Equitably to All Students (Chapter 6)

We expanded our emphasis on using an asset-based approach, focusing on students' strengths rather than deficits. We emphasize a focus on using students' prior knowledge and experiences to drive instructional decisions. There is also a revamping of the section on gifted and talented students including attention to an excellence gap (students who may be overlooked).

Basic Facts (Chapter 9)

Recent research (e.g., Baroody et al., 2016) has uncovered a new and effective addition reasoning strategy—Use 10, which has been added to this chapter, along with new visuals and insights on teaching subtraction facts effectively.

Developing Strategies for Multiplication and Division (Chapter 12)

In new updates in this chapter, there are expanded examinations of the written records of computing multiplication and division problems including lattice multiplication, open arrays, and partial quotients. There is also a new section of the use of the break apart or decomposition strategy for division. A conversation about the selection of numbers for computational estimation problems is also shared.

Algebraic Thinking, Equations, and Functions (Chapter 13)

In addition to the new section on mathematical modeling, there are several new ideas and strategies for supporting algebraic thinking, including adapting the hundreds chart to explore patterns and options for creating tables with more structure to help students notice relationships.

Developing Fraction Concepts (Chapters 14)

Fraction concepts has an expanded focus on the fundamental ideas of sharing and iterating. This chapter also has been reorganized, has more contexts for comparing fractions, and more attention to student challenges in understanding fractions.

Ratios, Proportions, and Proportional Reasoning (Chapter 17)

The sections on additive and multiplicative reasoning have been significantly revised, including a new discussion on social justice mathematics. Additionally, significantly more literature connections are provided in this chapter and new activities.

Developing Concepts of Data Analysis (Chapter 20)

This chapter had numerous enhancements and changes! In addition to four new figures and completely updated technology options, the discussion of variability is woven throughout the chapter (including more attention to measures that are resistant to outlier), and sections on boxplots, histograms, and bivariate data were expanded and revised (see new subsection on bivariate categorical data).

An Introduction to *Teaching Developmentally*

If you look at the table of contents, you will see that the chapters are separated into two distinct sections. The first section consists of six chapters and covers important ideas that cross the boundaries of specific areas of content. The second section, consisting of 16 chapters, offers teaching suggestions and activities for every major mathematics topic in the preK–8 curriculum. Chapters in Part I offer perspectives on the challenging task of helping students learn mathematics. Having a feel for the discipline of mathematics—that is, to know what it means to "do mathematics"—is critical to learning how to teach mathematics well. In addition, understanding constructivist and sociocultural perspectives on learning mathematics and how they are applied to teaching through problem solving provides a foundation and rationale for how to teach and assess preK–8 students. You will be teaching diverse students including students who are English learners, are gifted, or have disabilities. In this text, you will learn how to apply instructional strategies in ways that support and challenge *all* learners. Formative assessment strategies and strategies for diverse learners are addressed in specific chapters in Part I (Chapters 5, and 6, respectively), and throughout Part II chapters.

Each chapter of Part II focuses on one of the major content areas in preK–8 mathematics curriculum. It begins with identifying the big ideas for that content, and provides guidance on how students best learn that content through many problem-based activities to engage them in understanding mathematics, as well as considering what challenges they may encounter and how you might help them.

Hundreds of tasks and activities are embedded in the text. Take out pencil and paper, or use technology, and try the problems, thinking about how you might solve them *and* how students at the intended grades might solve them. This is one way to actively engage in *your learning* about *students learning* mathematics. In so doing, this book will increase your own understanding of mathematics, the students you teach, and how to teach them effectively.

Some Special Features of This Text

By flipping through the book, you will notice many section headings, a large number of figures, and various special features. All are designed to make the book more useful as a long-term resource. Here are a few things to look for.

CHAPTER 14

Developing Fraction Concepts

LEARNER OUTCOMES

After reading this chapter and engaging in the embedded activities and reflections, you should be able to:

- **14.1** Describe and give examples for fractions constructs and fraction models.
- **14.2** Explain foundational concepts of fractional parts, including iteration and partitioning.
- **14.3** Illustrate the concept of equivalence across fraction models.
- **14.4** Describe strategies for comparing fractions and ways to teach this topic conceptually.

Fractions are one of the most important topics students need to understand to be successful in algebra and beyond, yet it is an area in which U.S. students, as well as students in many countries, struggle (OECD, 2014). National Assessment of Educational Progress (NAEP) results have consistently shown that students have a weak understanding of fraction concepts (Sowder & Wearne, 2006; Wearne & Kouba, 2000). This lack of understanding is then translated into difficulties with fraction computation, decimal and percent concepts, and algebra (Bailey, Hoard, Nugent, & Geary, 2012; Booth & Newton, 2012; Brown & Quinn, 2007; National Mathematics Advisory Panel, 2008; Siegler, Fazio, Bailey, & Zhou, 2013). Therefore, it is critical that you teach fractions well, present fractions as interesting and important, and commit to helping students understand the meaning of fractions.

BIG IDEAS

- Fractions can and should be represented across different interpretations (e.g., part-whole and division) and different models: area (e.g., $\frac{1}{3}$ of a garden), length (e.g., $\frac{3}{4}$ of an inch), and set (e.g., $\frac{1}{2}$ of the marbles).
- Fractions are equal shares of a whole or a unit. Therefore, equal sharing activities (e.g., 2 sandwiches shared with 4 friends) build on whole-number knowledge to introduce fractional quantities.
- Partitioning and iterating are strategies students can use to understand the meaning of fractions. Partitioning can be thought of as splitting the whole equally (e.g., splitting a whole into fourths), and iterating can be thought of as making a copy of each piece and counting them (e.g., one-fourth, two-fourths, etc.).
- Equivalent fractions are ways of describing the same amount by using different-sized fractional parts.
- Fractions can be compared by reasoning about the relative size of the fractions. Estimation and reasoning are important in teaching understanding of fractions.

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◀ Learning Outcomes

To help readers know what they should expect to learn, each chapter begins with learning outcomes. Self-checks are numbered to cover and thus align with each learning outcome.

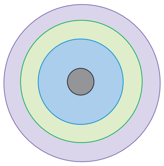
◀ Big Ideas

Much of the research and literature espousing a student-centered approach suggests that teachers plan their instruction around big ideas rather than isolated skills or concepts. At the beginning of each chapter in Part II, you will find a list of the big mathematical ideas associated with the chapter. Teachers find these lists helpful to quickly envision the mathematics they are to teach.

Activity 21.9 CCSS-M: 7.G.B.4; 7.SP.C.6; 7.SP.C.7b

Chance of Hitting the Target?

Project a target such as the one illustrated here with concentric circles having radii of 2 inches, 6 inches, 8 inches, and 10 inches, each region shaded a different color. Ask students to determine the fraction and percent of each colored region in the circle.



Ask students to discuss what the probability for landing on the center (assuming all throws land on the circle and are thrown randomly). Ask students to discuss why data may or may not match the percent of the area that is covered (e.g., people with good aim will be able to hit the smaller areas more often). Then, have students propose what point values they would assign to each region. Students may assign values in various ways. For example, they may think the skinny outer circle is harder to land on and give it more points than other sections, even though the area of that region may be more. Allow them time to share their reasoning and to critique others' ways of assigning points.

Activities

The numerous activities found in every chapter of Part II have always been rated by readers as one of the most valuable parts of the book. Some activity ideas are described directly in the text and in the illustrations. Others are presented in the numbered Activity boxes. Every activity is a problem-based task (as described in Chapter 3) and is designed to engage students in doing mathematics.

Adaptations for Students with Special Needs and English Learners

Chapter 6 provides detailed background and strategies for how to support students with special needs and English learners (ELs). But, many adaptations are specific to a activity or task. Therefore, Part II chapters offer adaptations and instructions within activities (look for the icon) that can meet the needs of students with special needs and ELs.

Activity 9.2 CCSS-M: 1.OA.A.1; 1.OA.C.6; 2.OA.B.2

How Many Feet in the Bed?

ENGLISH LEARNERS

STUDENTS *with* SPECIAL NEEDS

Read *How Many Feet in the Bed?* by Diane Johnston Hamm. On the second time through the book, ask students how many more feet are in the bed when a new person gets in. Ask students to record the equation (e.g., $6 + 2$) and tell how many. Two less can be considered as family members get out of the bed. Find opportunities to make the connection between counting on and adding using a number line. For ELs, be sure that they know what the phrases "two more" and "two less" mean (and clarify the meaning of *foot,* which is also used for measuring). Acting out with students in the classroom can be a great illustration for both ELs and students with disabilities.

FORMATIVE ASSESSMENT Notes. To assess understanding of division algorithms, call on different students to explain individual steps using the appropriate terminology that connects to the concept of division. Use an Observation Checklist to record students' responses, indicating how well they understand the algorithm. For students who are having difficulty, you may want to conduct a short diagnostic interview to explore their level of understanding in more detail. Begin by having the student complete $115 \div 9$ and ask them to talk about what they are thinking as they carry out specific steps in the process. If there is difficulty explaining, have the student use base-ten materials to directly model the problem and attempt to link the actions to the procedure. Then ask them to discuss verbally the connections between what was done with the models and what was written symbolically. ■

Formative Assessment Notes

Assessment is an integral process within instruction. Similarly, it makes sense to think about what to be listening for (assessing) as you read about different areas of content development. Throughout the content chapters, there are formative assessment notes with brief descriptions of ways to assess the topic in that section. Reading these assessment notes as you read the text can help you understand how best to assist students who struggle.

Technology Notes

Infusing technological tools is important in learning mathematics. We have infused technology notes throughout Part II. A technology icon is used to identify places within the text or activity where a technology idea or resource is discussed. Descriptions include open-source (free) software, applets, and other Web-based resources, as well as ideas for calculator use.

TECHNOLOGY Note. An amazing computer tool for drawing two-dimensional views of block buildings is the Isometric Drawing Tool, available at the NCTM Illuminations website. Using mouse clicks students can draw either whole cubes, faces, or just lines. The drawings, however, are actually "buildings" and can be viewed as three-dimensional objects that when rotated can be seen from any vantage point. Prepared investigations lead students through the features of the tool. ■

Standards for Mathematical Practice

MP2. Reason abstractly and quantitatively.

Standards for Mathematical Practice Margin Notes

Connections to the eight Standards of Mathematical Practice from the *Common Core State Standards* are highlighted in the margins. The location of the note indicates an example of the identified practice in the nearby text.

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RESOURCES FOR CHAPTER 14

LITERATURE CONNECTIONS

The Doorbell Rang
Hutchins (1986)
Often used to investigate whole-number operations of multiplication and division, this book is also an excellent early introduction to fractions. The story is a simple tale of two children preparing to share a plate of 12 cookies. Just as they have figured out how to share the cookies, the doorbell rings and more children arrive. You can change the number of children to create a sharing situation that requires fractions (e.g., 8 children).

The Man Who Counted: A Collection of Mathematical Adventures
Tahan (1993)
This book contains a story, "Beasts of Burden," about a wise mathematician, Beremiz, and the narrator, who are traveling together on one camel. They are asked by three brothers to solve an argument: *Their father has left them 35 camels to divide among them: half to one brother, one-third to another, and one-ninth to the third brother*. The story is an excellent context for fractional parts of sets (and adding fractions). Changing the number of camels to 36 or 34, does not solve the challenge because the sum of $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{9}$ will never be one whole, no matter how many camels are involved. See Bresser (1995) for three days of activities with this book.

Apple Fractions
Pallotta (2002)
This book offers interesting facts about apples while introducing fractions as fair shares (of apples, a healthier option than books that focus on chocolate and cookies!). In addition, the words for fractions are used and connected to fraction symbols, making it a good connection for fractions in grades 1–3.

RECOMMENDED READINGS

Articles

Clarke, D. M., Roche, A., & Mitchell, A. (2008). Ten practical tips for making fractions come alive and make sense. *Mathematics Teaching in the Middle School, 13*(7), 373–380.

Ten excellent tips for teaching fractions are discussed and favorite activities are shared. An excellent overview of teaching fractions.

Lewis, R. M., Gibbons, L. K., Kazemi, E., & Lind T. (2015). Unwrapping students ideas about fractions. *Teaching Children Mathematics, 22*(3), 158–168.

This excellent read provides a how-to for implementing sharing tasks, including sequencing of tasks, questions to pose, and formatives assessment tool to monitor student understanding.

Freeman, D. W., & Jorgensen, T. A. (2015). Moving beyond brownies and pizzas. *Teaching Children Mathematics, 21*(7), 412–420.

This article describes student thinking as they compare fractions. In the more4U pages, they offer excellent sets of tasks with a range of contexts, each set focusing on a different reasoning strategy for comparing fractions.

Books

Lamon, S. (2012). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies.* New York, NY: Taylor & Francis Group.

As the title implies, this book has a wealth of information to help with better understanding fractions and teaching fractions well. Many rich tasks and student work are provided throughout.

McNamara, J., & Shaughnessy, M. M. (2010). *Beyond pizzas and pies: 10 essential strategies for supporting fraction sense (grades 3–5)*. Sausalito, CA: Math Solutions Publications.

This book has it all—classroom vignettes, discussion of research on teaching fractions, and many activities, including student work.

Websites

Rational Number Project (http://www.cehd.umn.edu/ci/rationalnumberproject/rnp1-09.html).

This project offers excellent lessons and other materials for teaching fraction concepts effectively.

End of Chapter Resources

The end of each chapter there are *Resources*, which include "Literature Connections" (found in all Part II chapters) and "Recommended Readings."

Literature Connections. Here you will find examples of great children's literature for launching into the mathematics concepts in the chapter just read. For each title suggested, there is a brief description of how the mathematics concepts in the chapter can be connected to the story. These literature-based mathematics activities will help you engage students in interesting contexts for doing mathematics.

Recommended Readings. In this section, you will find an annotated list of articles and books to augment the information found in the chapter. These recommendations include NCTM articles and books, and other professional resources designed for the classroom teacher.

Supplements for Instructors

Qualified college adopters can contact their Pearson sales representatives for information on ordering any of the supplements described below. The following instructor supplements are all posted and available for download at www.pearsonglobaleditions.com/Van de Walle:

- **Instructor's resource manual:** The Instructor's Resource Manual for the tenth edition includes a wealth of resources designed to help instructors teach the course, including chapter notes, activity suggestions, and suggested assessment and test questions.
- **Electronic test bank:** An electronic test bank (TB) contains hundreds of challenging questions as multiple-choice or short-answer questions. Instructors can choose from these questions and create their own customized exams.
- **PowerPoint™ presentation:** Ideal for instructors to use for lecture presentations or student handouts, the PowerPoint presentation provides ready-to-use graphics and text images tied to the individual chapters and content development of the text.

Acknowledgments

Many talented people have contributed to the success of this book and we are deeply grateful to all those who have assisted over the years. Without the success of the first edition, there would certainly not have been a second, much less ten editions. The following people worked closely with John on the first edition, and he was sincerely indebted to Warren Crown, John Dossey, Bob Gilbert, and Steven Willoughby, who gave time and great care in offering detailed comments on the original manuscript.

In preparing this tenth edition, we have received thoughtful input from the following mathematics teacher educators who offered comments on the ninth edition. Each reviewer challenged us to think through important issues. Many specific suggestions have found their way into this book, and their feedback helped us focus on important ideas. Thank you to Jessica Cohen, Western Washington University; Shea Mosely Culpepper, University of Houston; Shirley Dissler, High Point University; Cynthia Gautreau, California State University in Fullerton; Kevin LoPresto, Radford University; Ryan Nivens, East Tennessee State University; Adrienne Redmond-Sanogo, Oklahoma State University; and Douglas Roebuck, Ball State University. We are indebted to you for your dedicated and professional insight.

Additionally, we are very grateful for the ideas and reviews as we developed the tenth edition. Graham Fletcher, mathematics specialist, Atlanta, Georgia, provided strong support in the development of high leverage routines, including, of course, the new section on 3-Act Tasks. Susan Peters, mathematics teacher educator, University of Louisville, provided critical feedback and helpful ideas for developing concepts of data analysis. Their input resulted in significant improvements to those chapters. We continue to seek suggestions from teachers who use this book so please email us at teachingdevelopmentally@gmail.com with any advice, ideas, or insights you would like to share.

We are extremely grateful to our Pearson team of editors! Each of them has worked hard to turn our ideas (and yours) into a reality. And that is why we have been able to continue to evolve this book in a way to make it accessible online and via hard copy. Drew Bennett, our editor, has helped us define the direction of this edition and make important decisions that would make the book a better product for pre-service and in-service teachers. Our development editor, Kim Norbuta, has been supportive and positive, keeping us on target, even with the tightest of deadlines. Our content producer Yagnesh Jani was always available with the missing resources and answers we needed. Finally, we are very grateful to Jason Hammond and his editing team at SPi-Global, who carefully and conscientiously assisted in preparing this edition for publication. It has been a pleasure to interact with each of them and they have given us peace of mind to have knowledgeable, strong support.

We would each like to thank our families for their many contributions and support. On behalf of John, we thank his wife, Sharon, who was John's biggest supporter and a sounding board as he wrote the first six editions of this book. We also recognize his daughters, Bridget (a fifth-grade teacher in Chesterfield County, Virginia) and Gretchen (an Associate Professor of psychology at Rutgers University–Newark). They were John's first students, and he tested many ideas that are in this book by their sides. We can't forget those who called John "Math Grandpa": his granddaughters, Maggie, Aidan, and Grace.

From Karen Karp: I would like to express thanks to my husband, Bob Ronau, who as a mathematics educator graciously helped me think about decisions while offering insights and encouragement. In addition, I thank my children, Matthew, Tammy, Joshua, Misty, Matt, Christine, Jeffrey, and Pamela for their kind support and inspiration. I also am grateful for my wonderful grandchildren, Jessica, Zane, Madeline, Jack and Emma, who have helped deepen my understanding about how children think.

From Jennifer Bay-Williams: I would like to begin by saying thank you to the many mathematics teachers and teacher educators whose presentations at conferences, blogs, tweets, articles and classroom lessons have challenged and inspired me. I am forever grateful to my husband, Mitch Williams, whose background in English/Language Arts and great listening skills have been an amazing support. Finally, thank you to my children, MacKenna (14 years) and Nicolas (11 years), along with their peers and teachers, who continue to help me think more deeply about mathematics teaching and learning.

Global Edition Acknowledgments

This Global Edition is the result of the individuals who have contributed their insights, reviews, and suggestions to this project. We are deeply grateful for these collaborations:

Chunlian Jiang, University of Macau
Terry Tin-Yau Wong, The University of Hong Kong

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CHAPTER

1

Teaching Mathematics in the 21st Century

LEARNER OUTCOMES

After reading this chapter and engaging in the embedded activities and reflections, you should be able to:

1.1 Summarize the factors that influence the effective teaching of mathematics.

1.2 Describe the importance of content standards, process standards and standards of mathematical practice.

1.3 Explore the qualities needed to learn and grow as a professional teacher of mathematics.

Some of you will soon find yourself in front of a class of students; others of you may already be teaching. What general ideas will guide the way you will teach mathematics as you grow in the teaching profession? This book will help you become comfortable with the mathematics content of the preK–8 curriculum. You will also learn about research-based strategies that help students come to know mathematics and be confident in their ability to do mathematics. These two things—your knowledge of mathematics and how students learn mathematics—are the most important tools you can acquire to be successful.

Becoming an Effective Teacher of Mathematics

As part of your personal desire to build successful learners of mathematics, you might recognize the challenge that mathematics is sometimes seen as the subject that people love to hate. At social events of all kinds—even at parent–teacher conferences—other adults may respond to the fact that you are a teacher of mathematics with comments such as "I could never do math," or "I can't calculate the tip at a restaurant–I just hope they include suggestions for tips at the bottom of my receipt." Instead of dismissing or ignoring these disclosures, consider what positive action you can take. Would people confide that they don't read and hadn't read a book in years—not likely. Families' and teachers' attitudes toward mathematics may enhance or detract from students' ability to do math. It is important for you and for students' families to know that mathematics ability is not inherited—anyone can learn mathematics. Moreover, learning mathematics is an essential life skill (OECD, 2016). So, you need to find ways of countering negative statements about mathematics, especially if they are declared in the presence of students. Point out that it is a myth that only some people can be successful in learning mathematics. Only in that way can the chain of passing apprehension from family member to child, or in rare cases from teacher to student, be broken. There is much joy to be had in solving mathematical problems, and it is essential that you model an excitement for learning and nurture a passion for mathematics in your students.

Ultimately, your students need to think of themselves as mathematicians in the same way as they think of themselves as readers. As students interact with our increasingly mathematical and technological world, they need to construct, modify, communicate or integrate new information in many forms. Solving novel problems and approaching new situations with a mathematical perspective should come as naturally as using reading to comprehend facts, insights, or news. Particularly because this century is a quantitative one (Hacker, 2016), we must prepare students to interpret the language and power of numeracy. Hacker states that "decimals and ratios are now as crucial as nouns and verbs" (p. 2). So, for your students' sake, consider how important mathematics is to interpreting and successfully surviving in our complex economy and in our changing environment. Learning mathematics opens up a world of important ideas to students.

The goal of this book is to help you understand the mathematics methods that will make you an effective teacher. We also base this book on the collective wisdom of an organization of mathematics educators and mathematicians who developed a set of standards for what knowledge, skills and dispositions are important in cultivating a well-prepared beginning teacher of mathematics (Association of Mathematics Teacher Educators, 2017). This book infuses those standards for developing elementary and middle school teachers of mathematics using the suggestions of what best supports teacher candidates in methods courses. Because the authors of this book were also engaged in the creation and writing of the *Standards for Preparing Teachers of Mathematics*, the book is aligned with the AMTE standards. As you dig into the information in the chapters ahead, your vision of what is possible for all students and your confidence to explore and teach mathematics will grow.

A Changing World

In *The World Is Flat* (2007), Thomas Friedman discusses how globalization has created the need for people to have skills that are long lasting and will survive the ever-changing landscape of available jobs. He names categories of workers who regardless of the shifting terrain of job options—will always be successful in finding employment. One of these "untouchable" categories is—math lover. Friedman emphasizes that in a world that is digitized and surrounded by algorithms, math lovers will always have career opportunities and choices. Yet, there is a skills gap of qualified people as science, technology, engineering, and mathematics (STEM) jobs take more than twice as long to fill as other jobs in the marketplace (Rothwell, 2014).

Now every teacher of mathematics has the job to prepare students with career skills while developing a "love of math" in students. Lynn Arthur Steen, a well-known mathematician and educator, stated, "As information becomes ever more quantitative and as society relies increasingly on computers and the data they produce, an innumerate citizen today is as vulnerable as the illiterate peasant of Gutenberg's time" (1997, p. xv). So, as you see there are an array of powerful reasons why children will benefit from the study of mathematics and the instructional approaches you will learn in this book. Your students need to acquire the mental tools to make sense of mathematics—in some cases for mathematical applications that might not yet be known! This knowledge serves as a lens for interpreting the world.

Our changing world influences what should be taught in preK–8 mathematics classrooms as there is a relationship between early mathematics performance and success in middle school (Bailey, Siegler, & Geary, 2014) and high school mathematics (Watts, Duncan, Siegler, & Davis-Kean, 2014). As we prepare preK–8 students for jobs that possibly do not currently exist, we can predict that there will be few jobs where just knowing simple computation is enough to be successful. We can also predict that many jobs will require interpreting complex data, designing algorithms to make predictions, and using multiple strategies to approach new problems.

As you prepare to help students learn mathematics for the future, you will need some perspective on the forces that effect change in the mathematics classroom. This chapter addresses the leadership that you, the teacher, will develop as you shape the mathematics experience for your students. Your beliefs about what it means to know and do mathematics and about how students make sense of mathematics will affect how you approach instruction and the understandings and skills your students take from the classroom. The enthusiasm you demonstrate about mathematical ideas will translate into your students' interest in this amazing and beautiful discipline.

Factors to Consider

Over the years, there have been significant reforms in mathematics education that reflect the technological and informational needs of our society, research on how students learn mathematics, the importance of providing opportunities to learn for all students, and ideas on how and what to teach from an international perspective. Just as we would not expect doctors to be using the exact same techniques and medicines that were prevalent when you were a child, teachers' methods are evolving and transforming via a powerful collection of expert knowledge about how the mind functions and how to design effective instruction (Wiggins, 2013).

There are several significant factors in this transformation. One factor is the public or political pressure for change in mathematics education due largely to information about student performance in national and international studies. These large-scale comparisons of student performance continue to make headlines, provoke public opinion, and pressure legislatures to call for tougher standards backed by testing. This research is important because international and national assessments provide strong evidence that mathematics teaching *must* change if students are to be competitive in the global market and able to understand the complex issues they will confront as responsible citizens of the world (Green, 2014).

National Assessment of Education Progress (NAEP). Since the 1960s, the United States regularly gathers data on how fourth-, eighth-, and twelfth-grade students are doing in mathematics on the NAEP (https://nces.ed.gov/nationsreportcard). These data provide a tool for policy makers and educators to measure the overall improvement of U.S. students over time in what is called the "Nation's Report Card." NAEP uses four achievement levels: below basic, basic, proficient, and advanced, with proficient and advanced representing substantial grade-level achievement. The criterion-referenced test is designed to reflect the current curriculum but keeps a few stable items for purposes of long-term comparison. In the most recent NAEP mathematics assessment in 2015, less than half of all U.S. students in grades 4 and 8 performed at the desirable levels of proficient and advanced (40 percent in fourth grade and 33 percent in eighth grade) (National Center for Education Statistics, 2015). Despite encouraging gains in the NAEP scores over the last 30 years due to important shifts in instructional practices (particularly at the elementary level) (Kloosterman, Rutledge, & Kenney, 2009b), students' performance in 2015 still reveals disappointing levels of competency. For the first time in 25 years the number of students performing at proficient and advanced dropped two points at fourth grade and three points at eighth grade (Toppo, 2015). We still have work to do!

Trends in International Mathematics and Science Study (TIMSS). In 2015, 49 nations participated in the third International Mathematics and Science Study (https://timssandpirls.bc.edu), the largest international comparative study of students' mathematics and science achievement—given regularly since 1995. Data are gathered in grades 4, and 8 from a randomly selected group of students resulting in a sample of more than 600,000 with approximately 20,000 of the students from the United States. The results revealed that U.S. students performed above the international average of the TIMSS countries at both the fourth grade and the eighth grade but were outperformed at the fourth-grade level by education systems in Singapore, Hong Kong, Republic of Korea, Chinese Taipei, Japan, Northern Ireland, Russian Federation, Norway, Ireland, England, Belgium, Kazakhstan, and Portugal and outperformed at the eighth-grade level by education systems in Singapore, Republic of Korea, Chinese Taipei, Hong Kong, Japan, Russian Federation, Kazakhstan, Canada, and Ireland. These data provide valuable benchmarks that allow the United States to reflect on our teaching practices and our overall competitiveness in preparing students for a global economy. If you've heard people talk about how mathematics is taught in Singapore—these rankings are why. But these data do not suggest that we should use the curriculum from other higher performing countries as there are many variables to consider. However we can learn a common theme from these examples: a teaching focus in these nations that emphasizes conceptual understanding and procedural fluency.Both of which are critically important to the long-term growth of problem solving skills (OECD, 2016; Rittle-Johnson, Schenider, & Star, 2015). In fact, teaching in the high-achieving countries more closely resembles the long-standing recommendations of the National Council of Teachers of Mathematics, the major professional organization for mathematics teachers, discussed next.

National Council of Teachers of Mathematics (NCTM). One transformative factor in the teaching of mathematics is the leadership of the National Council of Teachers of Mathematics (NCTM). The NCTM, with more than 60,000 members, is the world's largest mathematics education organization. This group holds an influential role in the support of teachers and an emphasis on what is best for learners. Their guidance in the creation and dissemination of standards for curriculum, assessment, and teaching led the way for other disciplines to create standards and for the eventual creation of the CCSS-M. For an array of resources, including the web-based Illuminations component which consists of a set of exciting instructional experiences for your students, visit the NCTM website (www.nctm.org).

The Movement toward Shared Standards

We share the history of the standards here so you have a sense of how mathematics instruction has changed over time and how external factors and emerging research play a role in that process. These important ideas are all connected to your future as a teacher of elementary or middle school mathematics.

The momentum for reform in mathematics education began in earnest in the early 1980s. The main impetus was a response to a need for more problem solving as well as the research of developmental psychologists who identified how students can best learn mathematics. Then in 1989, NCTM published the first set of standards for a subject area in the *Curriculum and Evaluation Standards for School Mathematics*. Many believe that no other document has had such an enormous effect on school mathematics or on any other area of the curriculum.

NCTM followed in 1991 with a set of standards for teaching that articulated a vision of teaching mathematics for all students, not just a few. In 1995, NCTM added to the collection the *Assessment Standards for School Mathematics*, which focused on the importance of integrating assessment with instruction and indicated the key role that assessment plays in implementing change (see Chapter 5). In 2000, NCTM released *Principles and Standards for School Mathematics* as an update of its original standards document. Combined, these documents prompted a revolutionary reform movement in mathematics education throughout the world.

As these documents influenced teacher practice, ongoing debate about the mathematics curriculum continued with many arguing that instead of hurrying through numerous topics every year, the curriculum needed to address content more deeply. Guidance was needed in deciding what mathematics content should be taught at each grade level and, in 2006, NCTM released *Curriculum Focal Points*, a little publication with a big message—the mathematics taught at each grade level needs to be focused, provide depth, and explicitly show connections. The goal of the Focal Points was to support a coherent curriculum and give clarity to teachers and students as to what should be taught at each grade. The resulting sequence of key concepts provided a "structural fiber" that helped students understand mathematics (Dossey, McCrone, & Halvorsen, 2016, p. 18).

In 2010, the National Governors Association (NGA) Center for Best Practices and Council of Chief State School Officers (CCSSO) presented the *Common Core State Standards*, which are grade-level specific standards which incorporate ideas from *Curriculum*

Focal Points as well as international curriculum documents. A large majority of U.S. states adopted these as their standards and other states were stimulated to create new standards of their own. In less than 25 years, the standards movement transformed the country from having little to no coherent vision on what mathematics should be taught and when, to a more widely shared idea of what students should know and be able to do at each grade level.

In the following sections, we discuss three significant components of the standards that are critical to your work as a highly effective teacher of mathematics.

Mathematics Content Standards

As noted earlier, the dialogue on improving mathematics teaching and learning extends beyond mathematics educators. Policymakers and elected officials considered previous NCTM standards documents, international assessments, and research on the best way to prepare students to be "college and career ready." The National Governors Association Center for Best Practices and the Council of Chief State School Officers (CCSSO) collaborated with other professional groups and entities to develop shared expectations for K–12 students across states, a focused set of mathematics content standards and practices, and efficiency of material and assessment development (Porter, McMaken, Hwang, & Yang, 2011). As a result, they developed *Common Core State Standards for Mathematics* (CCSS-M) which can be downloaded at http://www.corestandards.org/math.

The CCSS-M articulates an overview of *critical areas* of mathematics content that are expectations for each grade from K–8 to provide a coherent curriculum built around big mathematical ideas. These larger groups of related standards are called *domains*, and there are eleven that relate to grades K–8 (see Figure 1.1). At this time, approximately 37 states, Washington, D.C., four territories, and Department of Defense Schools have adopted the CCSS-M. A few states chose not to adopt the standards from the start, some created their own versions, and others are still deciding their level of participation or reevaluating their own standards compared to CCSS-M. This change represents the largest shift of mathematics content in the United States in more than 100 years.

MyLab Education **Video Example 1.1**

Watch this video (https://www.youtube.com/watch?v=5pBOnvzC_Yw&list=PLD7F4C7DE7CB3D2E6&index=15) by one of the authors of the CCSS-M to hear more about the shifts made in these standards.

The *Common Core State Standards* were developed with strong consideration given to the research on what is known about the development of students' understanding of mathematics over time (Cobb & Jackson, 2011). The selection of topics at particular grades reflects not only rigorous mathematics, but also what is known from research and practice about learning progressions which are sometimes referred to as *learning trajectories* (Clements & Sarama, 2014; Confrey, Maloney, & Corley, 2014). These progressions can help teachers know the sequence of what came before a particular concept as well as what to expect next as students reach key points

<table>
<tr><th>Kindergarten</th><th>Grade 1</th><th>Grade 2</th><th>Grade 3</th><th>Grade 4</th><th>Grade 5</th><th>Grade 6</th><th>Grade 7</th><th>Grade 8</th></tr>
<tr><td>Counting and Cardinality</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td colspan="6">Operations and Algebraic Thinking</td><td colspan="3">Expressions and Equations</td></tr>
<tr><td colspan="6">Number and Operations in Base-Ten</td><td colspan="3">The Number System</td></tr>
<tr><td colspan="6">Measurement and Data</td><td colspan="3">Statistics and Probability</td></tr>
<tr><td colspan="9">Geometry</td></tr>
<tr><td></td><td></td><td></td><td colspan="3">Number and Operations—Fractions</td><td colspan="2">Ratios and Proportional Relationships</td><td>Functions</td></tr>
</table>

FIGURE 1.1 *Common Core State Standards* domains by grade level.

along a pathway to desired learning targets (Corcoran, Mosher, & Rogat, 2009). Although these paths are not identical for all students, they can inform the order of instructional experiences which will support movement toward understanding and application of mathematics concepts. There is a website for the "Progressions Documents for the Common Core Math Standards" (http://ime.math.arizona.edu/progressions) where progressions for the domains in the Common Core State Standards can be found.

Although you may have heard people suggest that they are not in favor of the *Common Core State Standards*, many of those comments reflect people's concern with the testing that is associated with the standards, not the content standards or the mathematical practices which are described next.

The Process Standards and Standards for Mathematical Practice

To prepare students for college and career readiness and a lifetime of enjoying mathematical ideas, there are additional standards that emphasize the important processes in doing mathematics. The process standards refer to the mathematical methods and strategies which preK–12 students acquire to enhance their use of mathematical content knowledge. NCTM developed these standards as part of the *Principles and Standards* document (2000) and stated that the process standards should not be regarded as separate content or strands in the mathematics curriculum, rather, they are central and integral components of all mathematics learning and teaching. The five process standards and ways you can develop these elements in your students can be found in Table 1.1. Members of NCTM have free access to the *Principles and Standards* and nonmembers can sign up for 120 days of free access to the full document on the NCTM website (www.nctm.org) under the tab *Standards and Focal Points*.

The *Common Core State Standards* also go beyond specifying mathematics content expectations to also include Standards for Mathematical Practice. These are "'processes and proficiencies'

TABLE 1.1 THE FIVE PROCESS STANDARDS FROM *PRINCIPLES AND STANDARDS FOR SCHOOL MATHEMATICS*

Process Standard	How Can You Develop These Processes in Your Students?
Problem Solving	• Start instruction with a problem to solve—as problem solving is the vehicle for developing mathematical ideas. • Select meaningful mathematical tasks. • Set problems in a situation to which students can relate. • Use a variety of strategies to solve problems. • Have students self-assess their understanding of the problem and their strategy use.
Reasoning and Proof	• Have students consider evidence of why something is true or not. • Create opportunities for students to evaluate conjectures—do they hold true? • Encourage students to use logical reasoning to see if something always works or their answers make sense. • Demonstrate a variety of ways for students to justify their thinking through finding examples and counterexamples to use in a logical argument.
Communication	• Invite students to talk about, write about, describe, and explain their mathematical ideas as a way to examine their thinking. • Give students opportunities to share ideas so that others can understand and actively discuss their reasoning. • Share examples of student work, so students can compare and assess others' thinking. • Present precise mathematical language and notation so that the word usage and definitions can act as a foundation for students' future learning.
Connections	• Emphasize how mathematical ideas explicitly connect to students' prior mathematical knowledge and future learning. • Assist students in developing the relationships between the mathematics being learned and real-world contexts and in other subject areas.
Representation	• Encourage students to use multiple representations to explore relationships and communicate their thinking. • Create opportunities for students to move from one representation of a mathematical concept or idea to another to add depth of understanding. • Provide problems where students can use mathematical models to clarify or represent a situation.

Source: Adapted with permission from NCTM (National Council of Teachers of Mathematics). (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

with longstanding importance in mathematics education" (NGA Center & CCSSO, 2010, p. 6) that are based on the underlying frameworks of the NCTM process standards (Koestler, Felton, Bieda, & Otten, 2013). Teachers must develop these mathematical practices in each and every student (NGA Center & CCSSO, 2010, pp. 7–8) as described briefly in Table 1.2 to help them reach proficiency. A more detailed description of the Standards for Mathematical Practice can be found in Appendix A and you may find versions of these practices that spell out explanations and examples by individual grade level either through your state documents or on the web or in publications such as the Koestler, Felton, Bieda, and Otten (2013) book described in the resources section at the end of the chapter.

Regardless of the standards used in your state it is your job to support the parents and families of your students to educate them about the research behind the standards used. Incorporate your classroom website, newsletters, back to school night, family math events to share examples of how concepts are being built in very purposeful ways—even if they differ from the "way the parents were taught" when they were in school (Walkowiak, 2015). There is a nonprofit website called YouCubed (www.youcubed.org) that offers a parent section where videos and resources are available to help you support parents' understanding of these ideas and approaches.

TABLE 1.2 THE STANDARDS FOR MATHEMATICAL PRACTICE FROM THE CCSS-M

Mathematical Practice	K–8 Students Should Be Able to:
Make sense of problems and persevere in solving them.	• Explain what the problem is asking. • Describe possible approaches to a solution. • Consider similar problems to gain insights. • Use concrete objects or drawings to think about and solve problems. • Monitor and evaluate their progress and change strategies if needed. • Check their answers using a different method. • Try again with another approach if one attempt is not successful or when they feel "stuck."
Reason abstractly and quantitatively.	• Explain the relationship between quantities in problem situations. • Represent situations using symbols (e.g., writing expressions or equations). • Create representations that fit the word problem. • Use flexibly the different properties of operations and objects.
Construct viable arguments and critique the reasoning of others.	• Understand and use assumptions, definitions, and previous results to explain or justify solutions. • Make conjectures by building a logical set of statements. • Analyze situations and use examples and counterexamples. • Explain their thinking and justify conclusions in ways that are understandable to teachers and peers. • Compare two possible arguments for strengths and weaknesses to enhance the final argument.
Model with mathematics.	• Apply mathematics to solve problems in everyday life. • Make assumptions and approximations to simplify a problem. • Identify important quantities and use tools or representations to connect their relationships. • Reflect on the reasonableness of their answer based on the context of the problem.
Use appropriate tools strategically.	• Consider a variety of tools, choose the most appropriate tool, and use the tool correctly (e.g., manipulative, ruler, technology) to support their problem solving. • Use estimation to detect possible errors and establish a reasonable range of answers. • Use technology to help visualize, explore, and compare information.
Attend to precision.	• Communicate precisely using clear definitions and appropriate mathematical language. • State accurately the meanings of symbols. • Specify appropriate units of measure and labels of axes. • Use a level of precision suitable for the problem context.
Look for and make use of structure.	• Identify and explain mathematical patterns or structures. • Shift viewpoints and see things as single objects or as comprised of multiple objects or see expressions in many equivalent forms. • Explain why and when properties of operations are true in a particular context.
Look for and express regularity in repeated reasoning.	• Notice if patterns in calculations are repeated and use that information to solve other problems. • Use and justify the use of general methods or shortcuts by identifying generalizations. • Self-assess as they work to see whether a strategy makes sense, checking for reasonableness prior to finalizing their answer.

Source: Based on Council of Chief State School Officers. (2010). *Common Core State Standards*.

TABLE 1.3 THE SIX GUIDING PRINCIPLES FROM THE *PRINCIPLES TO ACTIONS*

Guiding Principle	Suggestions for Classroom Actions That Align with the Principles
Teaching and learning	• Select focused mathematics goals. • Use meaningful instructional tasks that develop reasoning, sense making, and problem-solving strategies. • Present and encourage a variety of mathematical representations that connect the same ideas or concepts. • Facilitate student discussions and conversations about important mathematical ideas. • Ask essential questions that are planned to be a catalyst for deeper levels of thinking. • Use a strong foundation of conceptual understanding as a foundation for procedural fluency. • Encourage productive struggle—as it is a way to deepen understanding and move toward student application of their learning. • Generate ways for students to provide evidence of their thinking through discussions, illustrations, and written responses.
Access and equity	• Establish high expectations for all students. • Provide supports targeted to student needs (equity not equality). • Provide instructional opportunities for students to demonstrate their competence in different ways—creating tasks with easy entry points for students who struggle and extension options for those who finish quickly. • Identify obstacles to students' success and find ways to bridge or eliminate those barriers. • Develop all students' confidence that they can do mathematics. • Enhance the learning of all by celebrating students' diversity.
Curriculum	• Build connections across mathematics topics to capitalize on broad themes and big ideas. • Look for both horizontal and vertical alignment to build coherence. • Avoid thinking of a curriculum as a checklist or disconnected set of daily lessons.
Tools and technology	• Include an array of technological tools and manipulatives to support the exploration of mathematical concepts, structures, and relationships. • Think beyond computation when considering the integration of technology. • Explore connections to how technology use for problem solving links to career readiness.
Assessment	• Incorporate a continuous assessment plan to follow how students are performing and how instruction can be modified and thereby improved. • Move beyond test results that just look at overall increases and decreases to pinpoint specific student needs. • Consider the use of multiple assessments to capture a variety of student performance. • Encourage students to self-assess sometimes by evaluating the work of others to enhance their own performance. • Teach students how to check their work.
Professionalism	• Develop a long-term plan for building your expertise. • Build collaborations that will enhance the work of the group of collaborators as you enhance the performance of the students in the school. • Take advantage of all coaching, mentoring and professional development opportunities and be a life-long learner. • Structure in time to reflect on and analyze your instructional practices.

How to Effectively Teach the Standards

NCTM also developed a publication that capitalizes on the timing of the adoption of the new standards across many states to explore the specific learning conditions, school structures, and teaching practices which will be important for a high-quality education for all students. The book *Principles to Actions* (2014) uses detailed classroom stories and student work samples to illustrate the careful, reflective work required of effective teachers of mathematics through 6 guiding principles (see Table 1.3 and Appendix B). A series of presentations (webcasts), led by the authors of the publication, explore several of the guiding principles and are available on the *Principles to Actions* portion of NCTM's website (www.nctm.org).

Pause & Reflect

Take a moment now to select one or two of the six guiding principles that seem especially significant to you and are areas in which you wish to develop more expertise. Why do you think these are the most important to your teaching?

MyLab Education **Application Exercise 1.1: Helping Teachers: Coherence and Focus** Click the link to access this exercise, then watch the video and answer the accompanying questions.

An Invitation to Learn and Grow

Think back to when you were a student in preK–8 classrooms. What are your remembrances of learning mathematics? Here are some thoughts from in-service and pre-service teachers of whom we asked the same question. Which description do you resonate with?

> I was really good at math in lower elementary grades, but because I never understood why math works, it made it very difficult to embrace the concepts as I moved into higher grades. I started believing I wasn't good at math so I didn't get too upset when my grades reflected that. *Kathryn*
>
> As a student, I always felt lost during mathematics instruction. It was as if everyone around me had a magic key or code that I missed out on getting. *Tracy*
>
> I remember math being very challenging, intimidating, and capable of making me literally sick to my stomach. Math was a bunch of rules and formulas I was expected to memorize, but not to understand. *Mary Rebekah*
>
> I consider myself to be really good at math and I enjoy mathematics-related activities, but I often wonder if I would have been GREAT at math and had a completely different career if I cared about math as much as I do now. Sometimes I feel robbed. *April*
>
> Math went from engaging, interactive instruction that I excelled at and loved, to lecture-style instruction that I struggled with. I could not seek outside help, even though I tried, because the teacher's way was so different from the way of the people trying to help me. I went from getting all As to getting low Bs and Cs without knowing how the change happened. *Janelle*
>
> Math class was full of elimination games where students were pitted against each other to see who could answer a math fact the fastest. Because I have a good memory I did well, but I hated every moment. It was such a nerve-wracking experience and for the longest time that is what I thought math was. *Lawrence*
>
> Math was never a problem because it was logical, everything made sense. *Tova*

As you can see these memories run the gamut with an array of emotions and experiences. The question now becomes, what do you hope your former students will say as they think back to your mathematics instruction? The challenge is to get each and every student to learn mathematics with understanding and enthusiasm. Would you relish hearing your students, 15 years after leaving your classroom, state that you encouraged them to be mathematically minded, curious about solving new problems, self-motivated, able to critically think about both correct and incorrect strategies, and that you nurtured them to be risk takers willing to try and persevere on challenging tasks? What will your legacy be?

The mathematics education described in this book may not be the same as the mathematics content and the mathematics teaching you experienced in grades K–8. As a practicing or prospective teacher facing the challenge of teaching mathematics from a problem-solving approach, this book may require you to confront some of your personal beliefs—beliefs about what it means to *do mathematics*, how one goes about *learning mathematics*, how to *teach mathematics*, and what it means to *assess mathematics*. Success in mathematics isn't merely about speed or the notion that there is "one right answer." Thinking and talking about mathematics as a means to sense making is a strategy that will serve us well in becoming a society where all citizens are confident in their ability to do math.

Becoming a Teacher of Mathematics

This book and this course of study are critical to your professional teaching career. The mathematics education course you are taking now as a pre-service teacher or the professional development you are experiencing as an in-service teacher will be the foundation for much of the mathematics instruction you do in your classroom for the next decade. The authors of this book take that seriously, as we know you do. Remember like a workout to benefit fully from this book you can't just go and watch others exercise (or do mathematics), you must participate with enthusiasm, energy, and effort. You bring many strengths to the teaching of mathematics including your willingness to try new things, a fresh perspective on how technology can be integrated into instruction and your stance as a lifelong learner. Therefore, this section lists and describes the characteristics, habits of thought, skills, and dispositions you will continue to cultivate to reach success as an effective teacher of mathematics.

Knowledge of Mathematics. You will need to have a profound, flexible, and adaptive knowledge of mathematics content (Ma, 1999). This statement is not intended to scare you if you feel that mathematics is not your strong suit, but it is meant to help you prepare for a serious semester of learning about mathematics and how to teach it. You cannot teach what you do not know. Additionally, the "school effects" for mathematics are great, meaning that unlike other subject areas, where students have frequent interactions with their family or others outside of school on topics such as reading books, exploring nature, or discussing current events, in the area of mathematics what we do in school is often "it" for many students. An absence of in-school opportunities to gain mathematics knowledge can result in students forever being economically disadvantaged and without the potential for social mobility (OECD, 2016). This is not merely about time spent but instead on how that time is used; the quality of instruction. These findings add to the gravity of your responsibility, because a student's learning for the year in mathematics will likely come from you. If you are not sure of a fractional concept or other aspect of mathematics content knowledge, now is the time to make changes in your depth of understanding and flexibility with mathematical ideas to best prepare for your role as an instructional leader. You will need to be able to "translate confusion into understanding" (Green, 2014, p. 105). You don't want to work on the brink of your knowledge base—instead you need to soak up the knowledge so you will feel more confident and can speak with added passion and enthusiasm. This book and your professor or instructor will help you in that process.

Persistence. You need the ability to stave off frustration and demonstrate persistence. Dweck (2007) has described the brain as similar to a muscle—one that can be strengthened with a good workout! People are not just "wired" for learning mathematics they must perform hard work and persevere to understand new ideas. As you move through this book and work the problems yourself, you will learn methods and strategies that will help you anticipate the barriers to students' learning and identify strategies to get them past these stumbling blocks. It is likely that what works for you as a learner will work for your students. As you conduct these mental "workouts," if you ponder, struggle, talk about your thinking, and reflect on how these new ideas fit or don't fit with your prior knowledge, then you will enhance your repertoire as a teacher. Remember as you model these characteristics for your students, they too will value perseverance more than speed. In fact, Einstein did not describe himself as intelligent—instead he suggested he was just someone who continued to work on problems longer than others. Creating opportunities for your students to productively struggle is part of the learning process (Stigler & Hiebert, 2009; Warshauer, 2015).

Positive Disposition. Prepare yourself by developing a positive attitude toward the subject of mathematics. Research shows that teachers with positive attitudes teach math in more successful ways that result in their students liking math more (Karp, 1991) and performing at higher levels (Palardy & Rumberger, 2008). If in your heart you say, "I never liked math," that mindset will be evident in your instruction (Maloney, Gunderson, Ramirez, Levin, & Beilock, 2014). The good news is that research shows that changing attitudes toward mathematics is relatively easy (Tobias, 1995) and that attitude changes are long-lasting (Dweck, 2006). Additionally, math methods courses have been found to be effective in increasing positive attitudes, more so than student teaching experiences (Jong & Hodges, 2015). Also, teachers who studied key concepts in math methods classes were more effective in and attentive to planning lessons on those big ideas—even years after taking the course (Morris & Hiebert, 2017). By expanding your knowledge of the subject and trying new ways to approach problems you can learn to enjoy doing activities in class and presenting mathematics instruction in schools. Not only can you acquire a positive attitude toward mathematics, as a professional educator it is essential that you do.

To explore your students' attitudes toward mathematics consider using this interview protocol. Here you can explore how the classroom environment may affect their attitudes.

MyLab Education **Teacher Resource: Interview Protocol**

Readiness for Change. Demonstrate a readiness for change, even for change so radical that it may cause you disequilibrium. You may find that what is familiar will become unfamiliar and, conversely, what is unfamiliar will become familiar. For example, you may have always referred to "reducing fractions" as the process of changing $\frac{2}{4}$ to $\frac{1}{2}$, but this language is misleading as the fractions are not getting smaller. Such terminology can lead to mistaken connections. Did the reduced fraction go on a diet? A careful look will point out that *reducing* is not the term to use; rather, you are writing an equivalent fraction that is *simplified* or in *lowest terms*. Even though you have used the language *reducing* for years, you need to become familiar with more precise language such as "simplifying fractions."

On the other hand, what is unfamiliar will become more comfortable. It may feel uncomfortable for you to be asking students, "Did anyone solve it differently?" especially if you are worried that you might not understand their approach. Yet this question is essential to effective teaching. As you bravely use this strategy, it will become comfortable (and you will learn new strategies!).

Another potentially difficult shift in practice is toward an emphasis on concepts as well as procedures. What happens in a procedure-focused classroom when a student doesn't understand division of fractions? A teacher with only procedural knowledge is often left to repeat the procedure louder and slower, "Just change the division sign to multiplication, flip over the second fraction, and multiply." We know the use of a memorized approach doesn't work well if we want students to fully understand the process of dividing fractions, so let's consider an example using $3\frac{1}{2} \div \frac{1}{2} =$ ________. You might start by relating this division problem to prior knowledge of a whole number division problem such as $25 \div 5 =$ ________. A corresponding story problem might be, "How many orders of 5 pizzas are there in a group of 25 pizzas?" Then ask students to put words around the fraction division problem, such as "You plan to serve each guest $\frac{1}{2}$ a pizza. If you have $3\frac{1}{2}$ pizzas, how many guests can you serve?" Yes, there are seven halves in $3\frac{1}{2}$ and therefore 7 guests can be served. Are you surprised that you can do this division of fractions problem in your head?

To respond to students' challenges, uncertainties, and frustrations you may need to unlearn and relearn mathematical concepts, developing comprehensive conceptual understanding and a variety of representations along the way. Supporting your mathematics content knowledge on solid, well-supported terrain is your best hope of making a lasting difference in your students' learning of mathematics—so be ready for change. What you already understand will provide you with many "Aha" moments as you read this book and connect new information to your current mathematics knowledge.

Willingness to Be a Team Player. Your school must work as a unit where all teachers are supporting children not just for the one grade they teach but in a coherent manner across the grades. When this idea of a team is implemented, then teachers agree to use the same language, symbols, models and notation to give students a familiar thread that ties the concepts and procedures together year after year. This established understanding is called a Whole School Agreement (Karp, Bush, & Dougherty, 2016) and your eager collaboration is essential in making this approach work well.

Lifelong Learning, Make Time to Be Self-Aware and Reflective. As Leinwand wrote, "If you don't feel inadequate, you're probably not doing the job" (2007, p. 583). No matter whether you are a preservice teacher or an experienced teacher, there is always more to learn about the content and methodology of teaching mathematics. The ability to examine oneself for areas that need improvement or to reflect on successes and challenges is critical for growth and development. The best teachers are always trying to improve their practice through the reading latest article, reading the newest book, attending the most recent conference, or signing up for the next series of professional development opportunities. These teachers don't say, "Oh, that's what I am already doing"; instead, they identify and celebrate each new insight. Highly effective teachers never "finish" learning nor exhaust the number of new mental connections that they make and, as a result, they never see teaching as stale or stagnant. An ancient Chinese proverb states, "The best time to plant a tree is twenty years ago; the second best time is today." Explore this self-reflection chart on professional growth to list your strengths and indicate areas for continued growth.

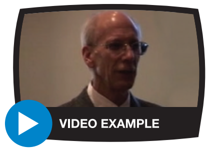
Think back to the quotations from the teachers at the beginning of this section. Again, what memories will you create for your students?

As you begin this adventure let's be reminded of what John Van de Walle said with every new edition, "Enjoy the journey!"

MyLab Education **Teacher Resource: Professional Growth Chart**

MyLab Education **Video Example 1.2**

Watch this video of John Van de Walle as he speaks to teachers about the challenges of teaching in ways focused on a problem-solving approach.



RESOURCES FOR CHAPTER 1

RECOMMENDED READINGS

Articles

Buckheister, K., Jackson, C., & Taylor, C. (2015). An inside track: Fostering mathematical practices. *Teaching Children Mathematics*, *22*(1), 28–35.

The authors share a game used with early elementary students and through teacher-student dialogue they describe how the several mathematical practices can be developed.

Hoffman, L., & Brahier, D. (2008). Improving the planning and teaching of mathematics by reflecting on research. *Mathematics Teaching in the Middle School*, *13*(7), 412–417.

This article addresses how teachers' beliefs influence their mathematics instruction. The authors discuss how to pose higher-level problems, ask thought-provoking questions, face students' frustration, and use students' mistakes to enhance their understanding of concepts.

Books

Bush, S., & Karp, K. (2015) *Discovering lessons for the Common Core State Standards in grades K–5*. Reston, VA: NCTM.

Bush, S., & Karp, K. (2014) *Discovering lessons for the Common Core State Standards in grades 6–8*. Reston, VA: NCTM.

These two books align lessons in articles from the past 15 years of NCTM journals with the Common Core State Standards and the Standards for Mathematical Practices. The books provide a way to see how the standards play out in instructional tasks for classroom use.

Koestler, C. Felton, M., Bieda, K. & Otten, S. (2013). *Connecting the NCTM process standards and the CCSSM practices*. Reston, VA: NCTM.

Expanding on the brief description of the Standards for Mathematical Practice in the CCSSM document, this book integrates those processes into the content with specific examples.

SELF CHECK

Similar questions are available on the MyLab Education.

Before attempting the Self-Check questions, refer to the learning outcomes (LOs) listed at the beginning of the chapter.

LO 1.1 **1.** Many factors influence how mathematics is taught in a school system and which mathematics is covered. What are some of the most influential factors?

A. The presence of a new classroom textbook and a new teacher
B. The size and wealth of the school system
C. The age of the students and their geographic location
D. National and international testing results

LO 1.1 **2.** There have been significant reforms in mathematics that reflect the technological and informational needs of our society. Which statement is a factor to consider in change?

A. STEM jobs take twice as long to fill as other jobs
B. Developing a "love of math" in students
C. Prepare preK-12 students for jobs that do not currently exist
D. U.S. student performance in national and international studies

LO 1.1 **3.** The National Council of Teachers of Mathematics has been transformative and influential in the changes in the teaching of mathematics. Three of the statements below are true about NCTM's contributions. Identify the one that is *not* true.

A. Creation and dissemination of curriculum standards
B. Illuminations instructional resources
C. Creation and dissemination of assessment standards
D. Trends in International Mathematics and Science Study

LO 1.2 **4.** The six *Principles and Standards for School Mathematics* articulate high-quality mathematics education. Which of the following statements represents the equity principle?

A. Coherence speaks to the importance of building instruction around big ideas.
B. Calculators and computers should be seen as essential tools for doing and learning mathematics.
C. Mathematics today requires not only computation skills, but also the ability to think and reason.
D. The message of high expectations for all is intertwined with every other principle.

LO 1.2 **5.** The Common Core State Standards divide the content expectations for students into large groupings called what?

A. Factors
B. Units of study
C. Domains
D. Lesson plans

LO 1.2 **6.** A process standard refers to the mathematical processes that preK–12 students acquire and the mathematical knowledge they use. Identify the process standard that highlights how mathematical concepts relate to real-world and other subjects.

A. Connections
B. Problem solving
C. Reasoning and proof
D. Representation
E. Communication

LO 1.2 **7.** Which one of the following Standards for Mathematical Practice asks students to analyze situations by breaking them into cases and can recognize and use counter examples?

A. Express regularity in repeated reasoning
B. Reason abstractly and quantitatively
C. Construct viable arguments and critique reasoning of others
D. Attend to precision

LO 1.3 **8.** Although all are important, which of the following teacher characteristics is most essential to demonstrate in order to help students persevere, think to try other strategies, and check their answers to problems?

A. Lifelong learning
B. Persistence
C. Positive attitude
D. Reflective disposition

LO 1.3 **9.** Which of the following statements is the best definition for the term "school effects" as it pertains to mathematics education?

A. Some schools have better learning environments for mathematics than others.
B. For many students, school is the only place where they get to experience mathematics.
C. Schools have no effect on students' learning of mathematics, and students' learning is completely determined by how they interact with mathematics outside of the classroom.
D. Schools affect students' learning of mathematics by creating the daily schedule for when various content areas are taught.

LO 1.3 **10.** Three of the following statements are ways for teachers to maintain a positive disposition about mathematics. Which of the following would *not* contribute to that disposition?

A. Enjoy doing activities in class
B. Trying new ways to approach problems
C. Expanding your knowledge of the subject
D. In your heart you say, "I never liked math"

Answers: LO 1.1 - 1. D. 2. D. 3. D.; LO 1.2 - 4. D. 5. C. 6. A. 7. C.; LO 1.3 - 8. B. 9. B. 10. D.

CHAPTER

2

Exploring What It Means to Know and Do Mathematics

LEARNER OUTCOMES

After reading this chapter and engaging in the embedded activities and reflections, you should be able to:

2.1 Investigate what it means to do mathematics.

2.2 Describe essential components of mathematical proficiency, including the importance of a relational understanding.

2.3 Connect learning theories to effective teaching practices.

This chapter explains how to help students learn mathematics. To get at how to help students learn, however, we must first consider what is important *to* learn. Let's look at a poorly understood topic, division of fractions, as an opening example. If a student has learned this topic well, what will they know and what should they be able to do? The answer is more than being able to successfully implement a procedure (e.g., commonly called the "invert and multiply" procedure). There is much more to know and understand about division of fractions:

- What does $3 \div \frac{1}{4}$ mean conceptually?
- What situation might this expression represent?
- Will the result be greater than or less than 3 and why?
- What strategies can we employ to solve this problem?
- What illustration or manipulative might illustrate this problem?
- How does this expression relate to subtraction? To multiplication?

All of these questions can be answered by a student who fully understands a topic such as division of fractions. We must help students reach this level of procedural fluency and conceptual understanding.

This chapter focuses on the "what" and "how" of teaching mathematics. First, *what* does doing mathematics look like (be ready to experience this yourself through four great tasks!) and what is important to know about mathematics? Second, *how* do we help students develop a strong understanding of mathematics?

What Does It Mean to Do Mathematics?

Doing mathematics means generating strategies for solving a problem, applying that strategy, and checking to see whether an answer makes sense. Doing mathematics means demonstrating mathematical process (see Table 1.1), which in the CCSS-M are effectively described in eight Mathematical Practices (see Table 1.2 and Appendix A). Doing mathematics begins with establishing goals for students that reflect these practices and then posing worthwhile tasks that open up the opportunity for such processes to develop (NCTM, 2014).

Goals for Students

One way to determine if your goals for students focus on doing mathematics is to consider the verbs in lesson and unit plans. Objectives or instructions that ask students to listen, copy, memorize, drill, and compute are lower-level thinking tasks and do not adequately prepare students for the real act of doing mathematics. In contrast, the following verbs engage students in doing higher-level mathematics:

analyze	design	justify
apply	develop	model
compare	explain	predict
connect	explore	represent
construct	formulate	solve
critique	generalize	use
describe	investigate	verify

These verbs may look familiar to you, as they are on the higher level of Bloom's (revised) Taxonomy (Anderson & Krathwohl, 2001) (see Figure 2.1).

In observing a third-grade classroom where the teacher focused on doing mathematics (i.e., focusing on the process standards), researchers found that students began to (1) connect to previous material, (2) respond with information beyond the required response, and (3) conjecture or predict (Fillingim & Barlow, 2010). When students engage in mathematical processes and practices a daily basis, they receive an empowering message: "You are capable of making sense of this—you are capable of doing mathematics!"

An Invitation to Do Mathematics

As noted above, after selecting goals that focus on mathematical processes and practices, the next step is to pose worthwhile tasks. The purpose of this section is to provide *you* with opportunities to engage with such tasks—four different problems across the mathematical strands and across K–8. None requires mathematics beyond elementary school mathematics, but they do require higher-level thinking. For each, stop and solve first. Then read the "Explore" section. Continue to engage with the task. Then, you will be doing mathematics and seeing how others may think about the problem differently (or the same). Have fun!

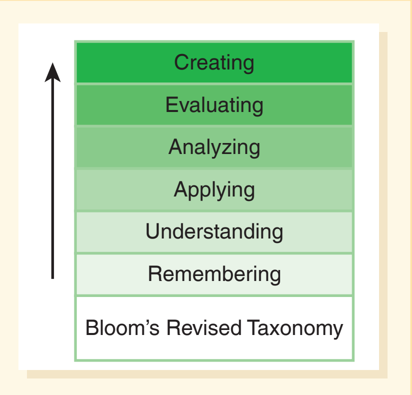


FIGURE 2.1 Bloom's (Revised) Taxonomy (Anderson & Krathwohl, 2001).

Source: Anderson, L.W., & Krathwohl, D. R. (Eds.). (2001). *A taxonomy for learning, teaching, and assessing: A revision of Bloom's Taxonomy of educational objectives: Complete Edition.* New York, NY: Addison Wesley, Longman.

Task 1. Pattern Search: Start and Jump Numbers.

Begin with a number (start) and add (jump) a fixed amount. For example, start with 3 and jump by 5s. Use the Start and Jump Numbers Activity Page or write the list on a piece of paper. Examine the list and record as many patterns as you see.

MyLab Education Activity Page: Start and Jump Numbers

Explore. Here are some questions to guide your pattern search:

- Do you see at least one alternating pattern?
- Have you noticed an odd/even pattern? Why is this pattern true?
- What do you notice about the numbers in the tens place?
- Do the patterns change when the numbers are greater than 100?

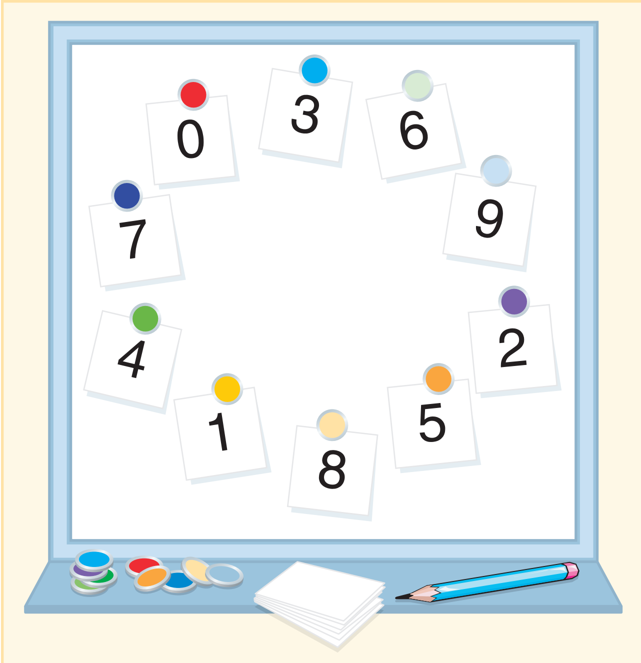


FIGURE 2.2 For jumps of 3, this cycle of digits will occur in the ones place. The start number determines where the cycle begins.

Have you thought about what happens to your patterns after the numbers are more than 100, for example 113? One way is as 1 hundred, 1 ten, and 3 ones. But, of course, it could also be "eleventy-three," where the tens digit has gone from 9 to 10 to 11. How do these different perspectives affect your patterns? What would happen after 999?

Extend. Sometimes when you have discovered some patterns in mathematics, it is a good idea to make some changes and see how the changes affect the patterns. What changes might you make in this problem? Your changes may be even more interesting than the following suggestions. But here are some ideas:

- Change the start number but keep the jump number equal to 5. What is the same and what is different?
- Keep the same start number and explore with different jump numbers.
- What patterns do different jump numbers make? For example, when the jump number was 5, the ones-digit pattern repeated every two numbers—it had a "pattern length" of 2. But when the jump number is 3, the length of the ones-digit pattern is 10! Do other jump numbers create different pattern lengths?
- For a jump number of 3, how does the ones-digit pattern relate to the circle of numbers in Figure 2.2? Are there similar circles of numbers for other jump numbers?
- Using the circle of numbers for 3, find the pattern for jumps of multiples of 3, that is, jumps of 6, 9, or 12.

Calculators facilitate exploration of this problem. Using calculators make the list generation accessible for young children who can't skip count yet, and they make it possible to readily explore bigger jump numbers like 25 or 36. Most simple calculators have an automatic constant feature that will add the same number successively. For example, if you press 3 + 5 = and then keep pressing =, the calculator will keep counting by fives from the previous answer. Consider demonstrating this with an online calculator (http://calculator-1.com) or app for the white board so the class can observe and discuss the counting.

Task 2. Analyzing a Situation: Two Machines, One Job.

Ron's recycle shop started when Ron bought a used paper-shredding machine. Business was good, so Ron bought a new shredding machine. The old machine could shred a truckload of paper in 4 hours. The new machine could shred the same truckload in only 2 hours. How long will it take to shred a truckload of paper if Ron runs both shredders at the same time?

MyLab Education **Application Exercise 2.1: Observing and Responding to Student Thinking** Click the link to access this exercise, then watch the video and answer the accompanying questions.

Use the Two Machines, One Job Activity Page to record your solution to this problem. Do not read on until you have an answer or are stuck. Can you check that you are correct? Can you approach the problem using a picture?

MyLab Education **Activity Page: Two Machines, One Job**

Explore. This task is more challenging than the last one, though you might be surprised at how it can be solved logically. Here are some things to consider:

- Have you tried to predict approximately how much time you think it should take the two machines? For example, will it be closer to 1 hour or closer to 4 hours? Estimating can sometimes lead to a new insight.
- What type of illustration might help solve this problem? For example, could you draw a rectangle or a line segment to stand for the truckload of paper?
- Is there a manipulative (chips, plastic cubes, pennies) you might use to make a collection that represents the truckload?

Strategies. Hopefully you have solved this problem in some way that makes sense to you—there are lots of ways to solve this particular task! Understanding other people's strategies can develop our own understanding. And, teachers are always in a position where they must try to figure out how their students are thinking about a problem. The following is one explanation for solving the problem, using strips (based on Schifter & Fosnot, 1993):

> "This rectangle [see Figure 2.3] stands for the whole truckload. In 1 hour, the new machine will do half of this." The rectangle is divided in half. "In 1 hour, the old machine could shred $\frac{1}{4}$ of the paper." The rectangle is divided accordingly. "So in 1 hour, the two machines have done $\frac{3}{4}$ of the truck, and there is $\frac{1}{4}$ left. What is left is one-third as much as what they already did, so it should take the two machines one-third as long to do that part as it took to do the first part. One-third of an hour is 20 minutes. That means it takes 1 hour and 20 minutes to do it all.

As with the teachers in these examples, it is important to decide whether your solution is correct through justifying why you did what you did; this reflects real problem solving rather than checking with an answer key. After you have justified that you have solved the problem in a correct manner, try to find other ways that students might solve the problem—in considering multiple ways, you are making mathematical connections.

Task 3. Generalizing Relationships: One Up, One Down. This problem has two parts, addition and multiplication. For your use, or to use with students, download One Up, One Down: Addition Activity Page or One Up, One Down: Multiplication Activity Page.

MyLab Education	**Activity Page: One Up, One Down: Addition**
MyLab Education	**Activity Page: One Up, One Down: Multiplication**

Addition. When you add 7 + 7, you get 14. When you make the first number 1 more and the second number 1 less, you get the same answer:

$$\begin{array}{ccccc} \uparrow & & \downarrow & & \\ 7 & + & 7 & = & 14 \\ 8 & + & 6 & = & 14 \end{array}$$

Does it work for 5 + 5? For what other problems is this pattern true?
What else do you notice about this pattern? Why is this pattern true?

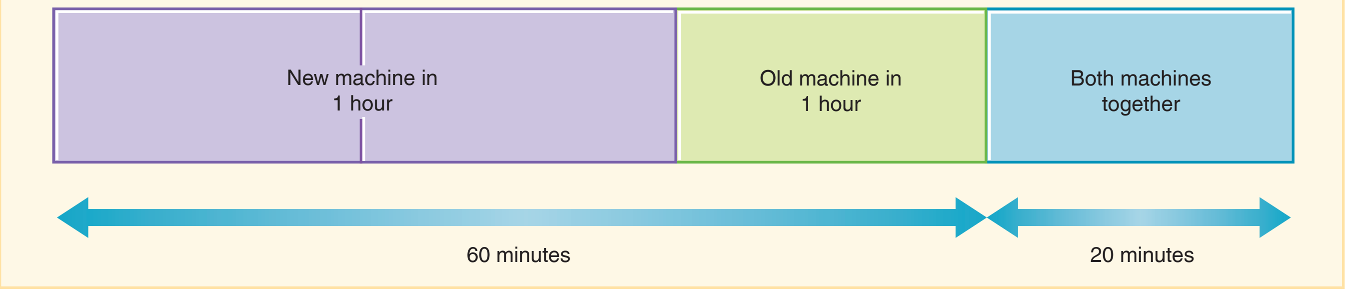


FIGURE 2.3 Cora's solution to the paper-shredding problem.

Multiplication. **When you multiply 7 × 7 you get 49. When you make the first number 1 more and the second number 1 less, you get one less:**

$$\begin{array}{ccccc} \uparrow & & \downarrow & & \\ 7 & \times & 7 & = & 49 \\ 8 & \times & 6 & = & 48 \end{array}$$

Does this work for 5 × 5? For what other problems is this pattern true?
What else do you notice about this pattern? Why is this pattern true?

Explore. If you explored both of these, you may have noticed that there are many more patterns or generalizations in the addition situation than in the multiplication situation. Consider:

- What manipulative or picture might illustrate the patterns?
- How is the pattern altered if the sums/products begin as two consecutive numbers (e.g., 8 × 7)? If they differed by 2 or by 3?
- What if you instead go "Two up, two down"? (e.g., 7 + 7 to 9 + 5 OR 7 × 7 to 9 × 5)?
- What if both factors increase (i.e., one up, one up)?
- What manipulative or picture might illustrate why the patterns work?
- In what ways is the addition situation similar to and different than the multiplication situation?

Strategies. Let's look at the multiplication pattern using illustrations. To compare the before and after products, draw rectangles (or arrays) with a length and height of each of the factors (see Figure 2.4[a]), then draw the new rectangle (e.g., 8-by-6-unit rectangle). You may prefer to think of multiplication as equal sets. For example, using stacks of chips, 7 × 7 is seven stacks with seven chips in each stack (set) (see Figure 2.4[b]). The expression 8 × 6 is represented by eight stacks of six (though six stacks of eight is a possible interpretation). See how the stacks for each expression compare.

Have you made some mathematical connections and conjectures in exploring this problem? In doing so you have hopefully felt a sense of accomplishment and excitement—one of the benefits of *doing* mathematics.

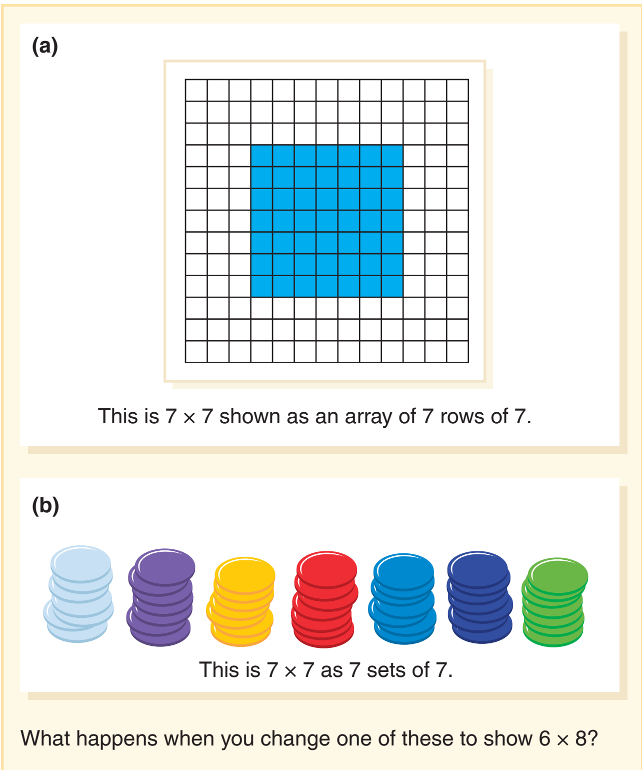


FIGURE 2.4 Two physical ways to think about multiplication that might help in the exploration.

Task 4. Experimenting and Justifying: The Best Chance of Purple.

Samuel, Susan, and Sandu are playing a game. The first one to spin "purple" wins! Purple means that one spin lands on red and one spin lands on blue (see Figure 2.5). Each person chooses to spin each spinner once or one of the spinners twice. Samuel chooses to spin spinner A twice; Susan chooses to spin spinner B twice; and Sandu chooses to spin each spinner once. Who has the best chance of purple? (based on Lappan & Even, 1989).

Think about the problem and what you know. Experiment. Use the Best Chance of Purple Activity Page to explore this problem.

MyLab Education **Activity Page: Best Chance of Purple**

Explore. A good strategy for learning is to first explore a problem concretely, then analyze it abstractly. This is particularly helpful in situations involving chance or probability. Use a paper clip with the spinners on your Activity Page, or use a virtual spinner. Consider the following as you explore:

- Who is most likely to win and why?
- For Sandu's turn (spinner A, then spinner B), would it matter if he spun B first and then A? Why or why not?
- How might you change one spinner so that Susan has the best chance at purple?

Strategies. Just like the earlier tasks, there are multiple strategies for approaching this task.

Strategy 1: Tree Diagrams. On spinner A, the four colors each have the same chance of coming up. You could make a tree diagram for A with four branches and all the branches would have the same chance (see Figure 2.6). Spinner B has different-sized sections, leading to the following questions:

- What is the relationship between the blue region and each of the others?
- How could you make a tree diagram for B with each branch having the same chance?
- How can you add to the diagram for spinner A so that it represents spinning A twice in succession?
- Which branches on your diagram represent getting purple?
- How could you make tree diagrams for each player's choices?
- How do the tree diagrams relate to the spinners?

Tree diagrams are only one way to approach this. If the strategy makes sense to you, stop reading and solve the problem. If tree diagrams do not seem like a strategy you want to use, read on.

Strategy 2: Grids. Partition squares to represent all the possible outcomes for spinner A and spinner B. Although there are many ways to divide a square into four equal parts, if you use lines going all in the same direction, you can make comparisons of all the outcomes of one event (one whole square) with the outcomes of another event (drawn on a different square). For two independent events, you can then create lines going the other direction for the second event. Samuel's two spins are represented in Figure 2.7(a). If these two squares are overlapped, you can visually see that two parts (two-sixteenths) are "blue on red" or "red on blue." Susan's probability can be determined by layering the squares in Figure 2.7(b); and Sandu's from layering one square from Figure 2.7(a) with one from Figure 2.7(b).

Why are there four parts for spinner A and 6 parts for spinner B? How is the grid strategy alike and different from the tree diagram? One strategy may make more sense to you, and one may make more sense to another. Hearing other students' explanations and reasoning for both strategies are important in building a strong understanding of mathematics.

Interesting mathematics problems such as the four presented here are plentiful. The Math Forum, for example, has

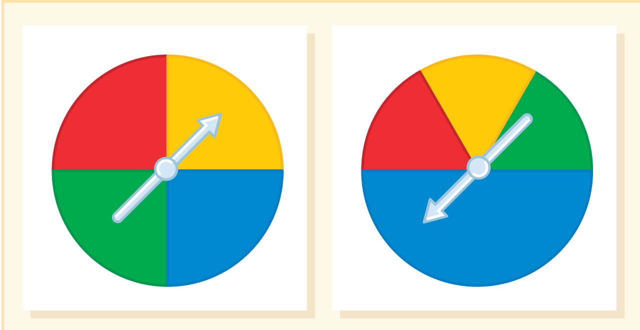


FIGURE 2.5 You may spin A twice, B twice, or A then B. Which choice gives you the best chance of spinning a red and a blue (purple)?

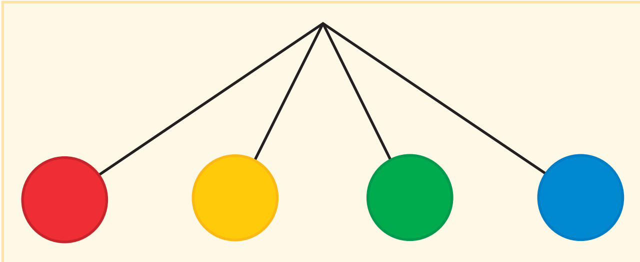


FIGURE 2.6 A tree diagram for spinner A in Figure 2.5.

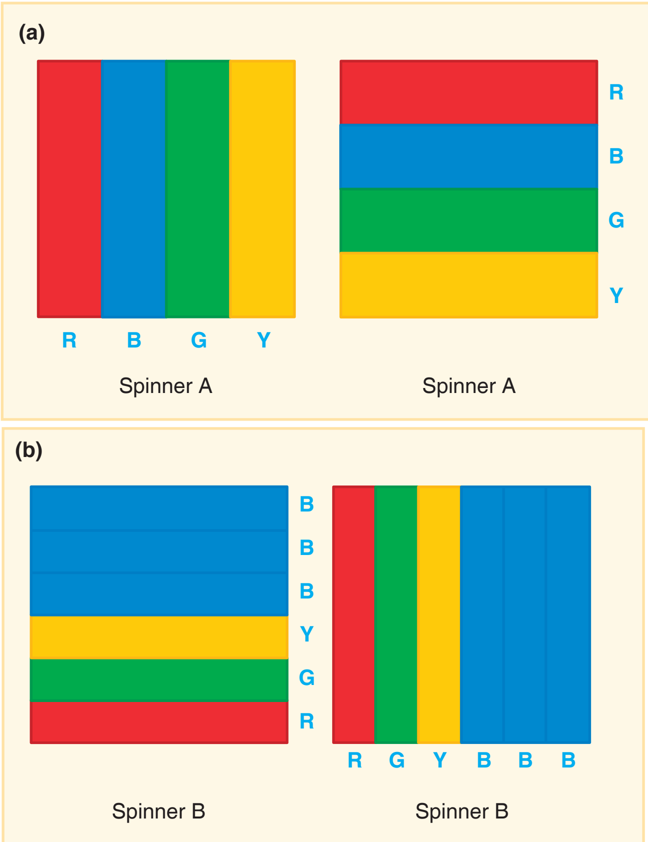


FIGURE 2.7 Grids can illustrate the chance of spinning purple with two spins.

a large collection of classic problems along with discussion, solutions, and extensions. NCTM teacher journals include monthly problems, and readers submit student solutions for these tasks, which appear in an issue a few months later.

Where Are the Answers?

Did you notice that no answers were shared for these four rich tasks? You may be wondering if your answer is correct, or if there are other answers. But, one aspect of being mathematically proficient is to rely on one's own justification and reasoning to determine if an answer is correct. Consider the message students receive when the textbook or the teacher is the source of whether an answer is correct: "Your job is to find the answers that the teacher already has." In the real world of problem solving and doing mathematics, there are no answer books. A person must be able to make sure they have used an appropriate strategy and reached a reasonable conclusion— we hope you feel this is the case for the tasks you solved in this section.

What Does It Mean to Know Mathematics?

In setting learning objectives for students, we often ask, "What will students know and be able to do?" The previous section focused on what students need to be able to do; this section focused on what students need to know. *Procedural knowledge* refers to *how* to complete an algorithm or procedure. *Conceptual knowledge* refers to *connected* knowledge: "mental connections among mathematical facts, procedures, and ideas" (Hiebert and Grouws 2007, p. 380). Both procedural and conceptual knowledge are foundational to procedural fluency and conceptual understanding, discussed later.

As an example, consider what is important for a student to know about a fraction such as $\frac{6}{8}$? At what point do they know enough that they can claim they "understand" fractions? It is more complicated than it might first appear. Here is a partial list of what they might know or be able to do:

- Read the fraction.
- Identify the 6 and 8 as the numerator and denominator, respectively.
- Recognize it is equivalent to $\frac{3}{4}$.
- Know it is more than $\frac{1}{2}$ (recognize relative size).
- Draw a region that is shaded in a way to show $\frac{6}{8}$.
- Locate $\frac{6}{8}$ on a number line.
- Illustrate $\frac{6}{8}$ of a set of 48 pennies or counters.
- Know that there are infinitely many equivalencies to $\frac{6}{8}$.
- Recognize $\frac{6}{8}$ as a rationale number.
- Realize $\frac{6}{8}$ might also be describing a ratio (girls to boys, for example).
- Be able to represent $\frac{6}{8}$ as a decimal fraction.

A number of items on this list refer to procedural knowledge (e.g., being able to find an equivalent fraction) and others refer to conceptual knowledge (e.g., recognizing $\frac{6}{8}$ is greater than $\frac{1}{2}$ by analyzing the relationship between the numerator and denominator). A student may know that $\frac{6}{8}$ can be simplified to $\frac{3}{4}$ but not recognize that $\frac{3}{4}$ and $\frac{6}{8}$ are equivalent (having procedural knowledge without conceptual knowledge). A student may be able to find one fraction between $\frac{1}{2}$ and $\frac{6}{8}$, but not be able to find others, meaning that while they have some procedural knowledge to find a common denominator, they do not have enough conceptual knowledge to recognize they could also change to denominators of sixteenths to find more in-between fractions (and recognize there are infinitely many options).

Understanding is hard to define, but it can be explained as a measure of the quality and quantity of connections that an idea has with existing ideas. The extent that a student understands why an algorithm works or understands relationships is their depth of understanding.

Relational Understanding

Understanding exists along a continuum from an *instrumental understanding*—doing something without understanding (see Figure 2.8) to a *relational understanding*—knowing what to do and why. These two terms were introduced by Richard Skemp in 1978 and continue to be a useful way to think about the depth of a student's mathematical understanding.

In the $\frac{6}{8}$ example, a student who only knows a procedure for simplifying $\frac{6}{8}$ to $\frac{3}{4}$ has an understanding near the instrumental end of the continuum, while a student who can draw diagrams, give examples, and find numerous equivalencies, has an understanding toward the relational end of the continuum. Here we briefly share two (interrelated) important ways to nurture a relational understanding.

Explore with Tools. A *tool* is any object, picture, or drawing that can be used to explore a concept. *Manipulatives* are physical objects that students and teachers can use to illustrate and discover mathematical concepts, whether made specifically for mathematics (e.g., connecting cubes) or for other purposes (e.g., buttons). Choices for manipulatives (including virtual manipulatives) abound—from common objects such as lima beans to commercially produced materials such as Pattern Blocks. Figure 2.9 shows six examples, each representing a different concept, just to give a glimpse (Part II of this book is full of more options). More and more of these interactives and others (e.g., algebra tiles, geometric solids, number lines, adjustable spinners) are available in a digital format, for example, on NCTM's Illuminations website (http://illuminations.nctm.org) and on the National Library of Virtual Manipulatives (NLVM) website (http://nlvm.usu.edu).

A tool does not "illustrate" a concept. The tool is used to visualize a mathematical concept and only your mind can impose the mathematical relationship on the object (Suh, 2007b; Thompson, 1994). As noted in *Task 4: Experimenting and Justifying: The Best Chance of Purple*, manipulatives can be a testing ground for emerging ideas. They are more concrete and provide insights into new and abstract relationships. Consider each of the concepts and the corresponding model in Figure 2.9. Try to separate the physical tool from the relationship that you must impose on the tool in order to *see* the concept.

The examples in Figure 2.9 are models that can show the following concepts:

a. The concept of "6" is a relationship between sets that can be matched to the words *one*, *two*, *three*, *four*, *five*, or *six*. Changing a set of counters by adding one changes the relationship. The difference between the set of 6 and the set of 7 is the relationship "one more than."

b. The concept of "measure of length" is a comparison. The length measure of an object is a comparison relationship of the length of the object to the length of the unit.

c. The concept of "rectangle" includes both spatial and length relationships. The opposite sides are of equal length and parallel and the adjacent sides meet at right angles.

d. The concept of "hundred" is not in the larger square but in the relationship of that square to the strip ("ten") and to the little square ("one").

e. "Chance" is a relationship between the frequency of an event happening compared with all possible outcomes. The spinner can be used to create relative frequencies. These can be predicted by observing relationships of sections of the spinner.

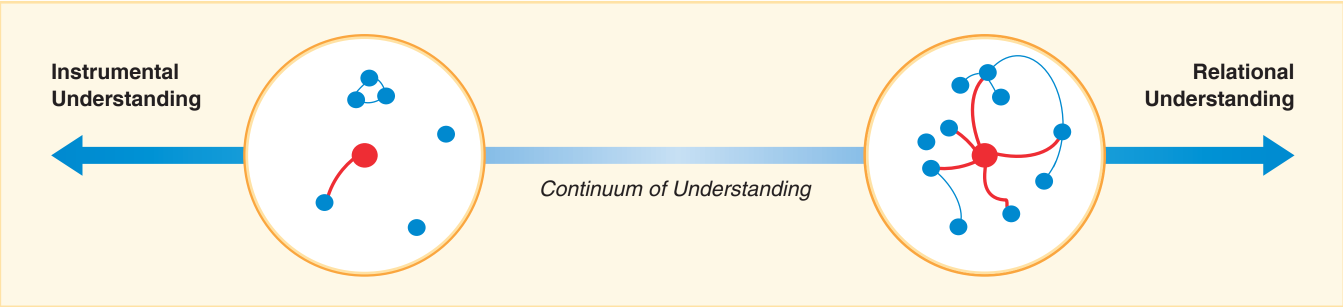


FIGURE 2.8 Understanding is a measure of the quality and quantity of connections that a new idea has with existing ideas. The greater the number of connections to a network of ideas, the better the understanding.

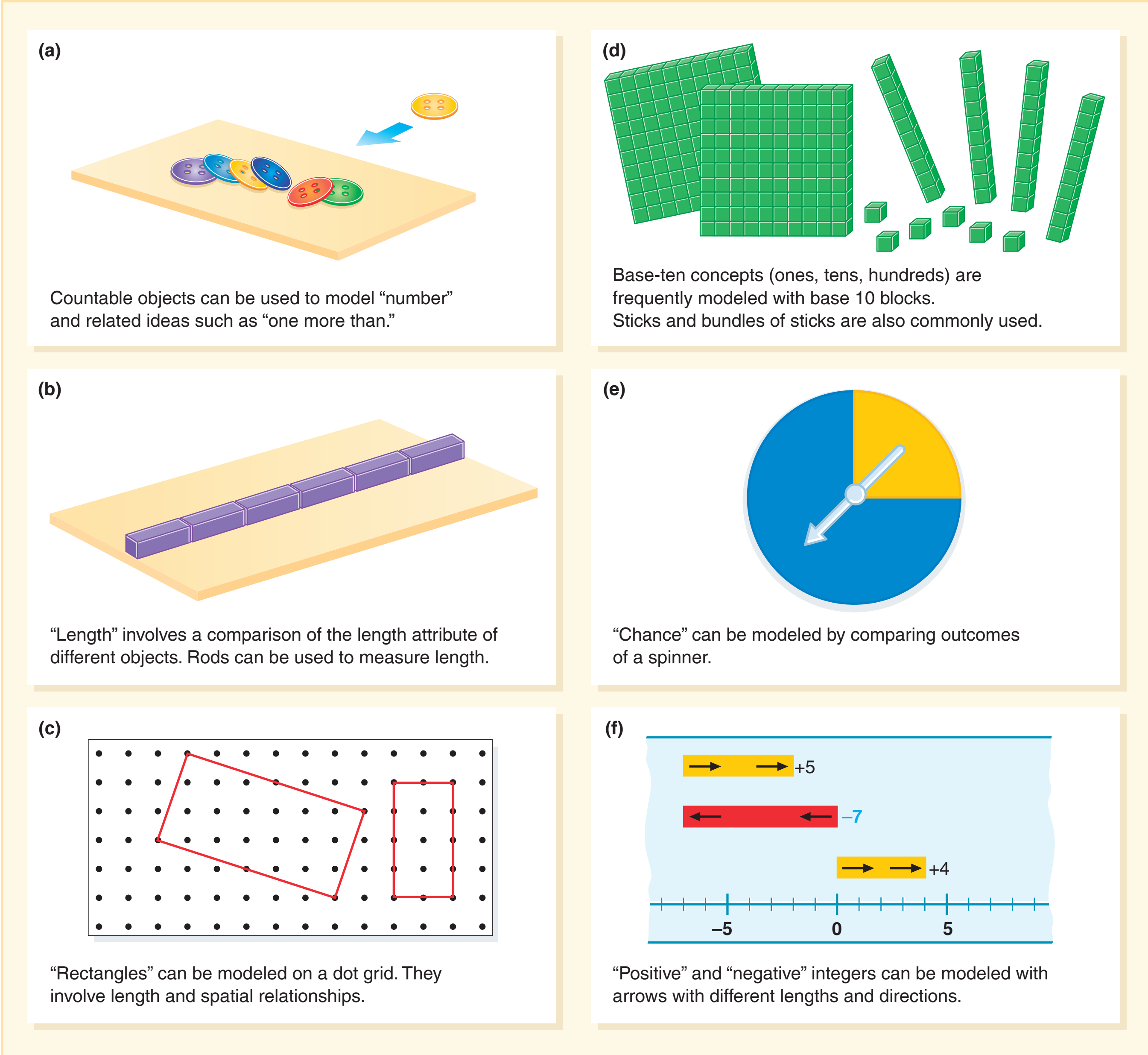


FIGURE 2.9 Examples of tools to illustrate mathematics concepts.

f. The concept of a "negative integer" is based on the relationships of "magnitude" and "is the opposite of." Negative quantities exist only in relation to positive quantities. Arrows on the number line model the opposite of relationship in terms of direction and size or magnitude relationship in terms of length.

A variety of tools (including calculators) should be accessible for students to select and use appropriately as they engage in doing mathematics.

While tools can be used to support relational understanding, they can be used ineffectively and not accomplish this goal. One of the most widespread misuses of tools occurs when the teacher tells students, "Do as I do." There is a natural temptation to get out the materials and show students how to use them to "show" the concept. It is just as possible to move blocks around mindlessly as it is to "invert and multiply" mindlessly. Neither promotes thinking or aids in the development of concepts (Ball, 1992; Clements & Battista, 1990; Stein & Bovalino, 2001). On the other extreme, it is ineffective to provide no focus or purpose for using the tools. This will result in nonproductive and unsystematic investigation (Stein & Bovalino, 2001). The goal is to set up tasks with the tools so that students notice important mathematical relationships that can be discussed, connecting the concrete representations to abstract concepts.

Connect Representations. In order for students to build connections among ideas, different representations must be included in instruction, and opportunities must be provided for students to make connections among the representations. Figure 2.10 illustrates a Web of Representations showing ways to demonstrate understanding. Students who have difficulty translating a concept from one representation to another also have difficulty solving problems and understanding computations (Lesh, Cramer, Doerr, Post, & Zawojewski, 2003). Therefore, strengthening students' ability to move between and among these representations improves their understanding. For example, give students the Translation Task Activity Page to complete for a topic they are learning. You can fill out one box and ask them to insert the other representations, or you can invite a group to work on all four representations for a given topic (e.g., multiplication of whole numbers).

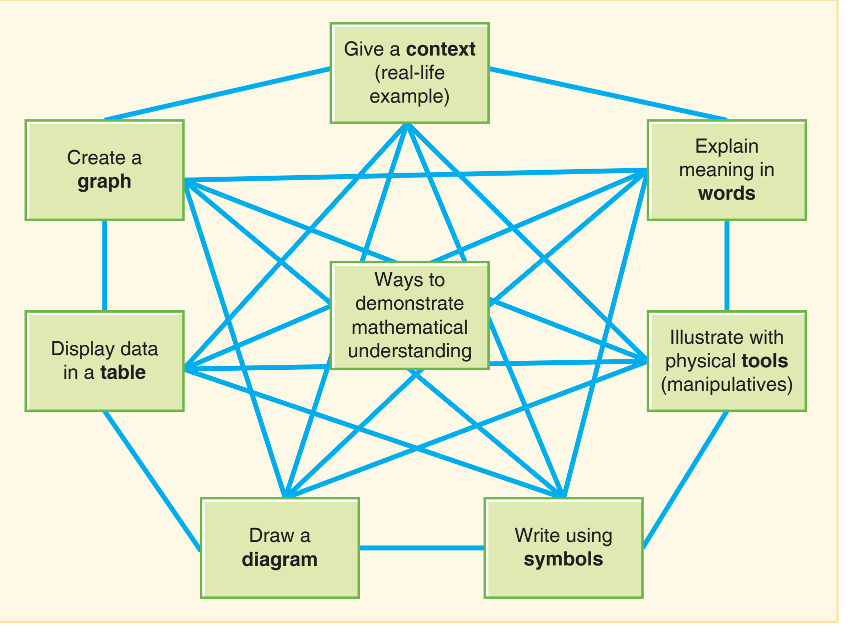


FIGURE 2.10 Web of Representations. Translations between and within each representation of a mathematical idea can help students build a relation understanding of a mathematical concept.

MyLab Education Activity Page: Translation Task

Mathematical Proficiency

This chapter started with an invitation to do mathematics, engaging you in the mathematical processes or practices (See Tables 1.1, 1.2, and Appendix A). Students who are able to demonstrate these practices are mathematically proficient. In other words, proficiency isn't only knowing the list of content from your grade, it is being able to demonstrate the practices *as it applies to that content*. The mathematical practices are based on research on how students learn as described in the National Research Council (NRC) report *Adding It Up* (NRC, 2001), which can be read for free at https://www.nap.edu/read/9822. Figure 2.11 illustrates these interrelated and interwoven strands.

Conceptual Understanding. Conceptual understanding is a flexible web of connections and relationships within and between ideas, interpretations and images of mathematical concepts—a relational understanding. Consider the web of associations for ratio as shown in Figure 2.12. Students with a conceptual understanding will connect what they know about division and numbers to make sense of scaling, unit prices, and so on. Note how much is involved in having a relational understanding of ratio.

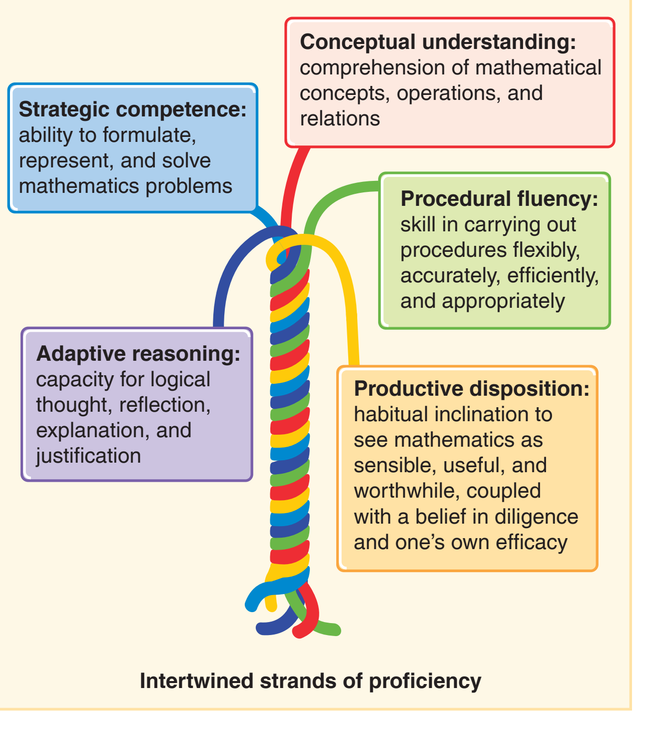


FIGURE 2.11 *Adding It Up* describes five strands of mathematical proficiency.

Source: National Research Council. (2001). *Adding It Up: Helping Children Learn Mathematics*, p. 5. Reprinted with permission from the National Academy of Sciences, courtesy of the National Academies Press, Washington, DC.

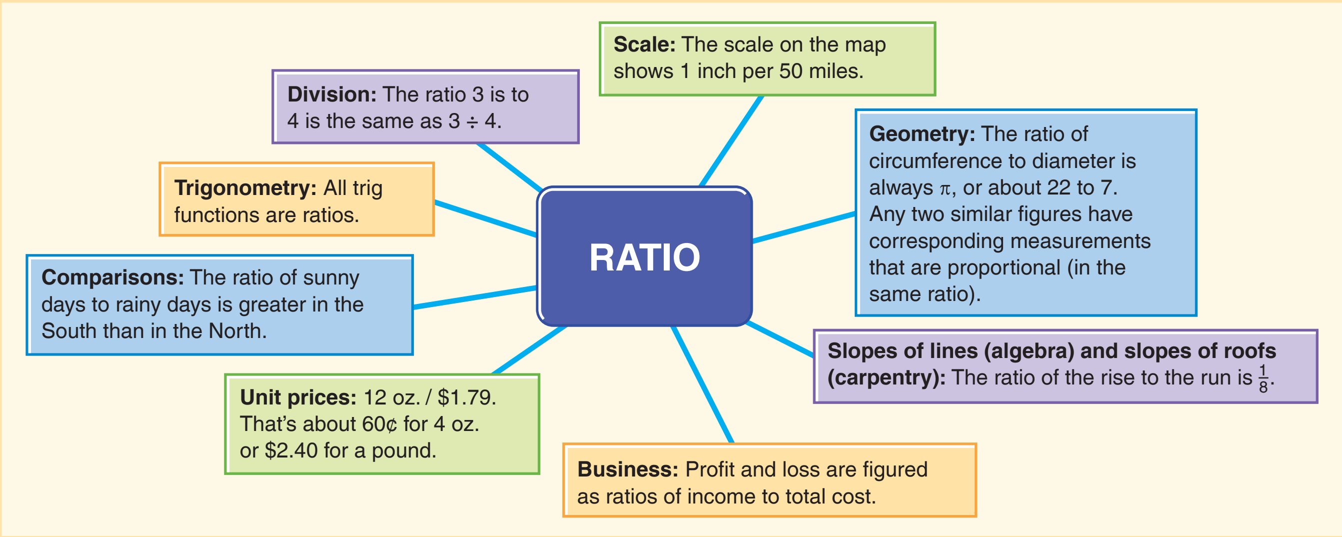


FIGURE 2.12 Potential web of ideas that could contribute to the understanding of "ratio."

Conceptual understanding includes the network of representations and interpretations of a concepts through the use of pictures, manipulatives, tables, graphs, words, and so on (see Figure 2.10). An illustration for ratios across these representations is provided in Figure 2.13.

Procedural Fluency. Procedural fluency is sometimes confused with being able to do standard algorithms correctly and quickly, but it is much bigger than that. Look at the four descriptors of procedural fluency in Figure 2.11. Recall that procedural knowledge is a foundation (knowing how to do an algorithm). Also important is procedural understanding (knowing why an algorithm works). But, procedural fluency is more than procedural understanding. Fluency includes four components: efficiency, accuracy, flexibility and appropriate strategy selection. Let's look at the problem 37+28. Figure 2.14 illustrates four approaches. Which ones represent a student who is has procedural fluency with two-digit addition? A fluent student does not automatically stack the numbers and apply the standard algorithm (though they know how to do this); the fluent student looks at the problem and considers which strategy will be *efficient* given the numbers in the problem and *selects an appropriate strategy*. In this case, a student might move two from the 37 to the 28 to adapt the equation to 35+30 (If this reminds you of One Up, One Down, good for you!). Or a student might add 2 onto 28, add 37+30, and then take 2 away from the answer (see Figure 2.14[c]). Given a different problem, like 54+37, the student might opt for a different strategy, such as adding the tens (80) and the ones (11) and combing them (91), showing *flexibility* in which strategies they use.

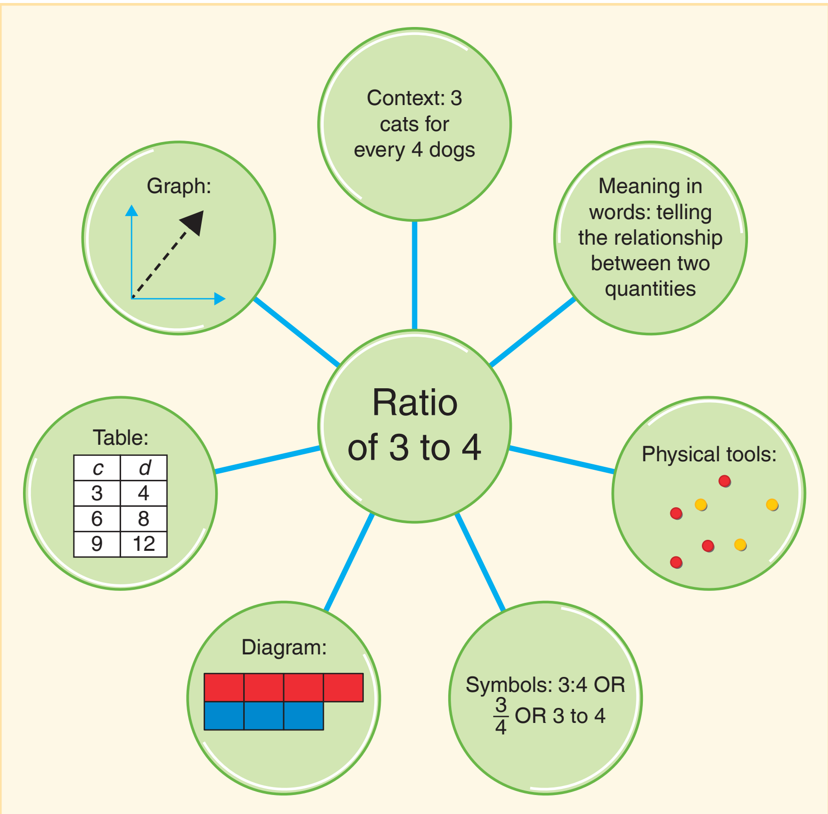


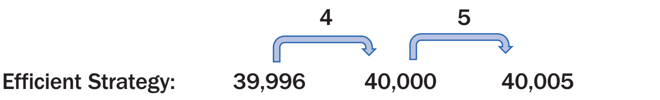
FIGURE 2.13 Multiple representations for ratio of 3 to 4.

MyLab Education

Video Example 2.1

Watch Jennifer Bay-Williams discuss procedural fluency as it relates to multi-digit addition and subtraction.

Ironically, procedural fluency is often mistaken for learning standard algorithms and being able to do them quickly and accurately. An overemphasis on standard algorithms can actually interfere with the development of fluency. Think about the following problem: 40,005 − 39,996 = ________. Applying the standard algorithm involves regrouping across zeros, a tedious, inefficient, and prone-to-error method. Noticing that the numbers are close together, and therefore lend to a counting up strategy, efficiently leads to a result of 9. Number lines are an important representation in building fluency:



Procedures support concepts (and vice versa) and both conceptual and procedural knowledge contributed to student development of procedural flexibility (Schneider, Rittle-Johnson, & Star, 2011). Both conceptual understanding and procedural fluency are crucial to becoming mathematically proficient (Baroody, Feil, & Johnson, 2007; Bransford, Brown, & Cocking, 2000).

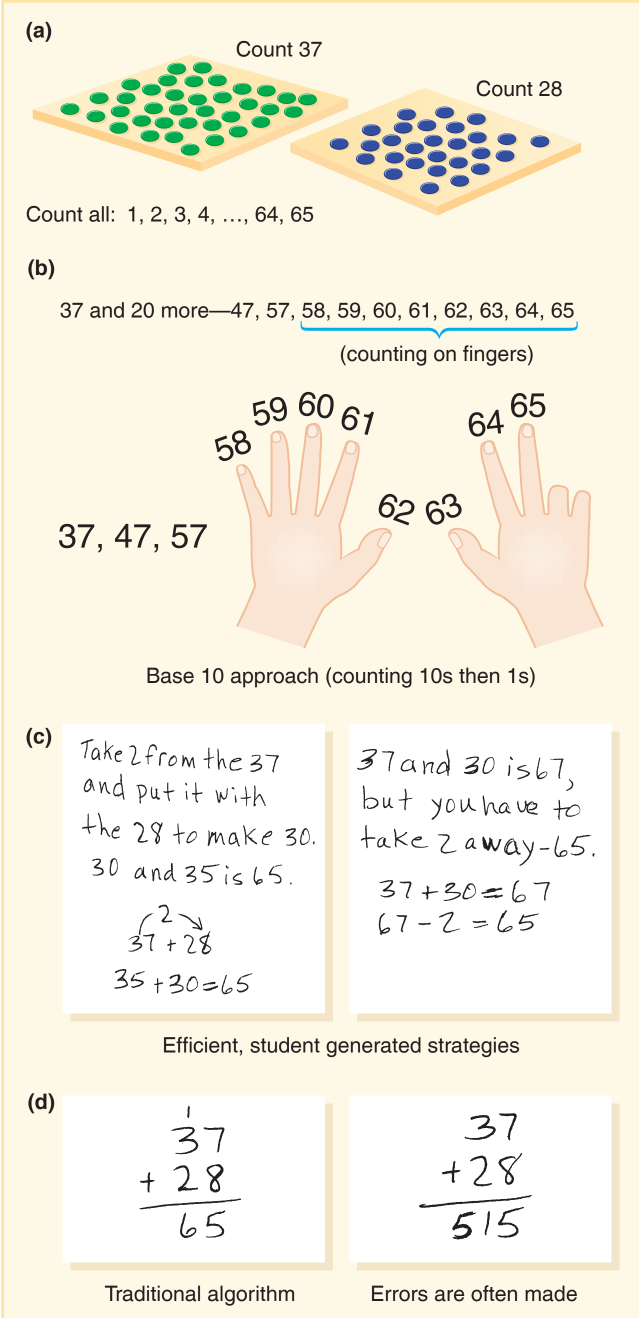


FIGURE 2.14 A range of levels of procedural fluency for 37 + 28.

Productive Disposition. As the Mathematical Practices and the Strands of Proficiency describe, being proficient at mathematics is not just what you know, but how you go about solving problems. Mathematical practices need to be interwoven with conceptual development (Kobiela & Lehrer, 2015). When teachers are intentional and explicit about mathematical practices, students' participation and confidence improves—they develop a productive disposition (Wilburne, Wildmann, Morret, & Stipanovic, 2014). For which of the questions below might a student with a productive disposition typically say "yes"?

- When you read a problem you don't know how to solve, do you think, "Great, something challenging. I can solve this."?
- Do you consider several possible approaches before diving in to solve?
- As you work, do you draw a picture or use a manipulative?
- Do you recognize a wrong path and try something else?
- When you finish a problem, do you wonder whether it is right? If there are other answers?
- Do you have a way of convincing yourself or a peer that your answer is correct?

And, if students say yes to these questions, which mathematical practices are they accessing? Procedural fluency and conceptual understanding are supported by and support a productive disposition.

How Do Students Learn Mathematics?

Now that you have had the chance to experience doing mathematics, you may have a series of questions: Why take the time to solve these problems—isn't it better to do a lot of shorter practice problems? Can students solve such challenging tasks? In other words, how does "doing mathematics" relate to learning mathematics? The answer lies in learning theory and research on how people learn.

Learning theories such as constructivism and sociocultural theory have influenced the way in which mathematics is taught. Your teaching practices will be influenced by how you believe people learn, which may be informed by one of these learning theories and from your own pragmatic experiences. It is important for you to attend to your own beliefs and how they relate to your teaching practice (Davis & Sumara, 2012). Here we briefly describe two theories that are important to understanding how students learn mathematics. These theories are not competing and are compatible (Norton & D'Ambrosio, 2008).

Constructivism

Constructivism is rooted in Jean Piaget's work, developed in the 1930s and translated to English in the 1950s. At the heart of constructivism is the notion that learners are not blank slates but rather creators or constructors of their own learning (Cobb, 1988; Fosnot, 1996; von Glasersfeld, 1996). Integrated *networks*, or *cognitive schemas*, are formed by constructing knowledge and they are used to build new knowledge. Through *reflective thought*—the effort to connect existing ideas to new information—people modify their existing schemas to incorporate new ideas (Fosnot, 1996). All people construct or give meaning to things they perceive or think about. As you read these words you are giving meaning to them. Whether listening passively to a lecture or actively engaging in synthesizing findings in a project, your brain is applying prior knowledge in your existing schemas to make sense of the new information.

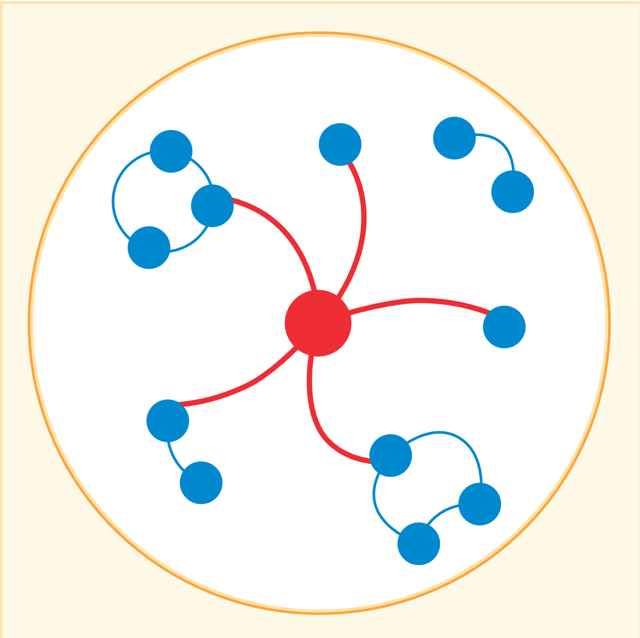


FIGURE 2.15 We use the ideas we already have (blue dots) to construct a new idea (red dot), in the process developing a network of connections between ideas. The more ideas used and the more connections made, the better we understand.

To connect to the metaphor of building construction, the *tools* we use to build understanding are our existing ideas and knowledge. The *materials* we use are things we see, hear, or touch, as well as our own thoughts and ideas. In Figure 2.15, blue and red dots are used as symbols for ideas. Consider the picture to be a small section of our cognitive makeup. The blue dots represent existing ideas. The lines joining the ideas represent our logical connections or relationships that have developed between and among ideas. The red dot is an emerging idea, one that is being constructed. Whatever existing ideas, blue dots, are used in the construction will be connected to the new idea, red dot, because those were the ideas that gave meaning to it. If a potentially relevant idea, blue dot, is not accessed by the learner when learning a new concept, red dot, then that potential connection will not be made.

Sociocultural Theory

In the 1920s and 1930s, Lev Vygotsky, a Russian psychologist, began developing what is now called sociocultural theory. Like constructivism, this theory assumes active meaning-seeking on the part of the learner. An important aspect of sociocultural theory is that the way in which information is internalized, or learned, depends on whether it was within a learner's zone of proximal development (ZPD) (Vygotsky, 1978). Simply put, the ZPD refers to a range of knowledge that may be out of reach for a person to learn on his or her own, but is accessible if the learner has support from peers or more knowledgeable others. In a true mathematical community of learners there is something of a common ZPD that emerges across learners as well as the individual ZPDs of each person (Cobb, 1994; Goos, 2004). Social interaction is essential for learning to occur. And, a community of learners is affected not only by culture the teacher creates, but by the broader social and historical culture of the members of the classroom (Forman, 2003).

Implications for Teaching Mathematics

Learning theories are not teaching strategies—theory *informs* teaching. This section outlines teaching strategies that are informed by constructivist and sociocultural perspectives. You will see these strategies revisited in more detail in Chapters 3 and 4, where a problem-based model for instruction is discussed, and in Part II of this book, where you learn how to apply these ideas to specific areas of mathematics.

Importantly, if these strategies are grounded in how people learn, it means *all* people learn this way—students with special needs, English language learners, students who struggle, and students who are gifted. Too often, when teachers make adaptations and modifications for particular learners, they trade in strategies that align with learning theories and research for methods that seem "easier" for students. These strategies, however, provide fewer opportunities for students to connect ideas and build knowledge—thereby impeding, not supporting, learning.

Build New Knowledge from Prior Knowledge. If you are teaching a new concept, like division, it must be developed using what students already know about equal subtraction and sharing. Consider the following task and how you might introduce it to third graders and if you are grounding your work in constructivist or sociocultural learning theories.

Goodies Toy Store is creating bags with 3 squishy balls in each. If they have 24 squishy balls, how many bags will they be able to make?

You will plan for students to access their prior knowledge, use cultural tools, and build new knowledge. You might ask students to use manipulatives or to draw pictures to solve this problem. As they work, they might have different ways of thinking about the problem (e.g., skip counting up by 3s, or skip counting down by 3s). These ideas become part of a classroom discussion, connecting what they know about equal subtraction and addition, and connecting that to multiplication and division.

Interestingly, this practice of connecting ideas is not only grounded in learning theory, but has been established through research studies. Making mathematical relationships explicit is connected with improving student conceptual understanding (Hiebert & Grouws, 2007). The teacher's role in making mathematical relationships explicit is to be sure that students are making the connections that are implied in a task. For example, asking students to relate today's topic to one they investigated last week, or by asking "How is Laila's strategy like Marco's strategy?" are ways to be "explicit" about mathematical relationships.

Provide Opportunities to Communicate about Mathematics. Learning is enhanced when the learner is engaged with others working on the same ideas. The interaction in such a classroom allows students to engage in reflective thinking and to internalize concepts that may be out of reach without the interaction and input from peers and their teacher. In discussions with peers, students will be adapting and expanding on their existing networks of concepts.

MyLab Education **Video Example 2.2**

Watch this video of Cathy's classroom problem solving task and how the students are communicating their process and solutions.

Create Opportunities for Reflective Thought. Classrooms need to provide structures and supports to help students make sense of mathematics in light of what they know. For a new idea to be interconnected in a rich web of interrelated ideas, children must be mentally engaged. They must find the relevant ideas they possess and bring them to bear on the development of the new idea. In terms of the dots in Figure 2.15 we want to activate every blue dot students have that is related to the new red dot we want them to learn. It is through student thinking, talking, and writing, that we can help them reflect on how mathematical ideas are connected to each other.

Encourage Multiple Strategies. Encourage students to use strategies that make sense to them. The student whose work is presented in Figure 2.16 may not understand the algorithm she used. If instead she were asked to use her own approach to find the difference, she might be able to get to a correct solution and build on her understanding of place value and subtraction.

Even learning a basic fact, like 7 × 8, can have better results if a teacher promotes multiple strategies. Imagine a class where students discuss and share ways to figure out the product. One student might think of 5 eights (40) and then 2 more eights (16) to equal 56. Another may have learned 7 × 7 (49) and add on 7 more to get 56. Still another might think "8 sevens" and take half of the sevens (4 × 7) to get 28 and double 28 to get 56. A class discussion sharing these ideas brings to the fore a wide range of useful mathematical "dots" relating addition and multiplication concepts.

Engage Students in Productive Struggle. Have you ever just wanted to think through something yourself without being interrupted or told how to do it? Yet, how often in mathematics class does this happen? As soon as a student pauses in solving a problem the teacher steps in to show or explain. While this may initially get the student to an answer faster, it does not help the student learn mathematics—engaging in productive struggle is what helps students learn mathematics. As Piaget describes, learners are going to experience disequilibrium in developing new ideas. Let students know this disequilibrium is part of the process.

Productive struggle is critical to developing conceptual understanding (Hiebert & Grouws, 2007). Notice the importance of both words in "productive struggle." Students must have the tools and prior knowledge to solve a problem, and not be given a problem that is out of reach, or they will struggle without being productive; yet students should not be given tasks that are straightforward and easy or they will not be struggling with mathematical ideas. When students, even very young students, know that struggle is expected as part of the process of doing mathematics, they embrace the struggle and feel success when they reach a solution (Carter, 2008). We must redefine what it means to "help" students. Rather than showing students how to do something, we must employ strategies, like asking probing questions, that keep students engaged in productive struggle.

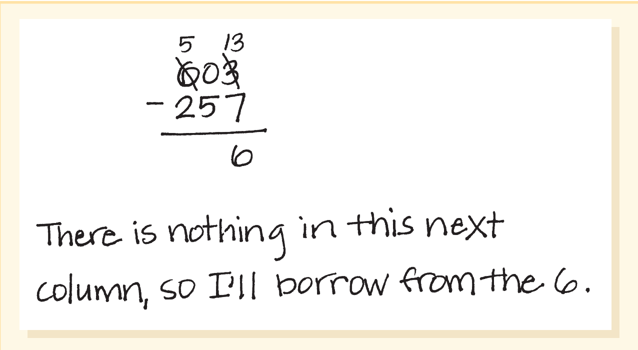


FIGURE 2.16 This student's work indicates that she has a misconception about place value and regrouping.

Treat Errors as Opportunities for Learning. When students make errors, it can mean a misapplication of their prior knowledge in a new situation. Remember that from a constructivist perspective, the mind is sifting through what it knows in order to find useful approaches for the new situation. Students rarely give random responses, so their errors are insight into limited or misconceptions they might have. For example, students comparing decimals may incorrectly apply "rules" of whole numbers, such as "the more digits, the bigger the number" (Martinie, 2014). Often one student's misconception is shared by others in the class and discussing the problem publicly can help other students understand (Hoffman, Breyfogle, & Dressler, 2009). You can introduce errors and ask students to imagine what might have led to that answer (Rathouz, 2011). This public negotiation of meaning allows students to construct deeper meaning for the mathematics they are learning.

Scaffold New Content. The practice of *scaffolding*, often associated with sociocultural theory, is based on the idea that a task otherwise outside of a student's ZPD can become accessible if it is carefully structured. For concepts completely new to students, the learning requires more structure or assistance, including the use of tools (e.g., manipulatives) or more assistance from peers. As students become more comfortable with the content, the scaffolds are removed and the student becomes more independent. Scaffolding can provide support for those students who may not have a robust collection of "blue dots."

Honor Diversity. Finally, and importantly, these theories emphasize that each learner is unique, with a different collection of prior knowledge and cultural experiences. Since new knowledge is built on existing knowledge and experience, effective teaching incorporates and builds on what the students bring to the classroom, honoring those experiences. Thus, lesson contexts are selected based on students' interests, knowledge and experiences. Classroom culture influences the individual learning of your students. Support a range of approaches and strategies for doing mathematics. Each students' ideas should be valued and included in classroom discussions of the mathematics. (See also the discussion of Culturally Responsive Mathematics Instruction in Chapter 6.)

MyLab Education **Application Exercise 2.2:**
How Do Students Learn Mathematics? Click the link to access this exercise, then watch the video and answer the accompanying questions.

Connecting the Dots

It seems appropriate to close this chapter by connecting some dots, especially because the ideas represented here are the foundation for the approach to each topic in the content chapters. This chapter began with discussing what *doing* mathematics is and challenging you to do some mathematics. Each of these tasks offered opportunities to make connections between mathematics concepts—connecting the blue dots.

Second, you read about what is important to know about mathematics—that having relational knowledge (knowledge in which blue dots are well connected) requires conceptual and procedural understanding as well as other proficiencies. The problems that you solved in the first section emphasized concepts and procedures while placing you in a position to use strategic competence, adaptive reasoning, and a productive disposition.

Finally, you read how learning theory—the importance of having opportunities to connect the dots—connects to mathematics learning. The best learning opportunities, according to constructivism and sociocultural theories, are those that engage learners in using their own knowledge and experience to solve problems through social interactions and reflection. This is what you were asked to do in the four tasks. Did you learn something new about mathematics? Did you connect an idea that you had not previously connected?

This chapter focused on connecting the dots between theory and practice—building a case that your teaching must focus on opportunities for students to develop their own networks of blue dots. As you plan and design instruction, you should constantly reflect on how to elicit prior knowledge by designing tasks that reflect the social and cultural backgrounds of students, to challenge students to think critically and creatively, and to include a comprehensive treatment of mathematics.

RESOURCES FOR CHAPTER 2

RECOMMENDED READINGS

Articles

Carter, S. (2008). Disequilibrium & questioning in the primary classroom: Establishing routines that help students learn. *Teaching Children Mathematics*, *15*(3), 134–137.

This is a wonderful teacher's story of how she infused the constructivist notion of disequilibrium and the related idea of productive struggle to support learning in her first-grade class.

Whitacre, I., Schoen, R. C., Champagne, Z., & Goddard, A. (2016). Relational thinking: What's the difference? *Teaching Children Mathematics*, *23*(5), 303–309.

These authors illustrate the importance of relational thinking sharing interesting results from students related to subtraction, and offering strategies for developing procedural flexibility.

Wilburne, J. M., Wildmann, T., Morret, M., & Stipanovic, J. (2014). Classroom strategies to make sense and persevere. *Mathematics Teaching in the Middle School*, *20*(3), 144–151.

Four strategies are shared that not only help build perseverance, but also develop the other mathematical practices and ultimately develop students' productive dispositions.

Books

Boaler, J. (2016). *Mathematical mindsets: Unleashing student potential through creative math, inspiring messaging, and innovative teaching*. San Francisco, CA: Jossey Bass.

Full of excellent tasks and strategies for engaging students, this very popular book is a great resource for K–12 teachers. Research and examples build a strong case for engaging students in doing mathematics to help them become mathematically proficient.

Mason, J., Burton, L., & Stacey, K. (2010). *Thinking mathematically* (2nd ed.). Harlow, England: Pearson Education.

This classic book is about doing mathematics. There are excellent problems to explore along the way, with strategy suggestions. It is an engaging book that will help you learn more about your own problem solving and become a better teacher of mathematics.

SELF CHECK

Similar questions are available on the MyLab Education.

Before attempting the Self-Check questions, refer to the learning outcomes (LOs) listed at the beginning of the chapter.

LO 2.1 **1.** Based on the discussion in this chapter, which of the following is the most effective way to pose a task to students so that they have the experience of "doing mathematics"?

A. Teach the skill that is needed to solve the problem first, then give this problem as an extension.
B. Have students take the problem home and solve it, then bring it back to school to discuss.
C. Share the problem, ask the students to explore, stop them to see how they are doing, and let them keep going.
D. Give an example of how to do a problem that is similar to the one you have selected, but slightly easier, so the students know what to do.

LO 2.1 **2.** Using a "doing mathematics" verb in a lesson or unit lesson plans guides students to higher level thinking tasks. Identify the statement below that would invite students to think at a higher level.

A. Compare the problem solutions and determine what problem-solving strategy was used and why.
B. Complete the drill and practice sheet in the allotted time.
C. Listen to the directions and copy the problem to solve.
D. Memorize the procedure for solving for n and be prepared to tell a friend.

LO 2.1 **3.** Why is it good practice for teachers to solve the problems they are asking their students to do?

A. To have the right key for students to use in checking their answer.
B. To use the one operation they want students to use with the problem.
C. To be aware of the varied ways the problem could be solved and still make sense.
D. To identify the one strategy they want their students to use to solve the problem.

LO 2.2 **4.** Procedural knowledge refers to:
- **A.** knowing how to complete an algorithm or procedure.
- **B.** knowing how to approach a new problem.
- **C.** connecting facts and ideas for a procedure.
- **D.** being able to describe underlying meaning.

LO 2.2 **5.** The extent to which a student understands why an algorithm works or connects relationships between concepts and procedures refers to:
- **A.** connected knowledge.
- **B.** procedural knowledge.
- **C.** depth of understanding.
- **D.** problem solving.

LO 2.2 **6.** Identify the statement below that would be an example of instrumental understanding.
- **A.** Using manipulatives to show other equivalent fractions
- **B.** Being able to draw diagrams of how 6/8 = 3/4
- **C.** Giving a real-life example of how 6/8 relates to 3/4
- **D.** Knowing the procedure for simplifying 6/8 to 3/4

LO 2.2 **7.** Mathematically proficient students demonstrate practices that show their knowledge of the content, and are able to apply that knowledge, showcasing the fact that this requires conceptual understanding. Identify the statement that represents conceptual understanding.
- **A.** Considers several possible approaches before trying to solve a problem
- **B.** Connects knowledge of division to rates, and uses that to figure out unit prices
- **C.** Efficient, accurate and flexible strategy selection
- **D.** Uses an algorithm to regroup across zeros to solve 40,005 − 39,996 =

LO 2.3 **8.** Constructivism and sociocultural theories have implications for teaching. Which of the following teaching strategies would be "weak" in terms of helping students learn based on these theories?
- **A.** Having students sort the facts that they know and then work on the facts that they do not know by trying to memorize them.
- **B.** Introducing multiplication by reading a children's book about arrays, such as *100 Hungry Ants*.
- **C.** Showing students two different samples of student work in which the answers were different and discussing publicly which one is correct (or are both correct).
- **D.** Illustrating how to fill a ten frame and then asking students to share how many counters they see and how they see it.

LO 2.3 **9.** Learning theories are not teaching strategies. They are about how people learn. What strategy described below refers to scaffold new content?
- **A.** Introduce errors and invite students to think what would have led to that error.
- **B.** Emphasize that each learner is unique.
- **C.** Asking probing questions that keep student engaged in the process.
- **D.** Include the use of tools (manipulatives) and more assistance from peers.

LO 2.3 **10.** Classrooms that provide structures and supports to help students make sense of mathematics in light of what they know are employing which one of the following strategies:
- **A.** Opportunities to communicate about mathematics
- **B.** Encourage multiple strategies
- **C.** Opportunities for reflective thought
- **D.** Use manipulatives to solve problems

LO 2.3 **11.** Which of the following statements about dots is *true*?
- **A.** Red dots (new ideas) are best added to a person's network of concepts by being connected to blue dots (existing ideas).
- **B.** Teachers already have complete webs of blue dots, which enables them to show students how those dots are connected.
- **C.** Some students will not have blue dots to connect to red dots, so those blue dots will have to be developed through instruction.
- **D.** Learning theory describes various characteristics of learning, providing clear direction on how to teach mathematics.

Answers: LO 2.1 - 1. C. 2. A. 3. C.; LO 2.2 - 4. A. 5. C. 6. D. 7. B.; LO 2.3 - 8. A. 9. D. 10. C. 11. A.

CHAPTER

3

Teaching through Problem Solving

LEARNER OUTCOMES

After reading this chapter and engaging in the embedded activities and reflections, you should be able to:

3.1 Contrast and describe approaches to problem solving.

3.2 Describe teaching practices that support student learning for all students

3.3 Critique mathematical tasks to determine if they promote problem solving and procedural fluency.

3.4 Explain ways to engage students in classroom discourse.

Imagine yourself in a mathematics classroom. What are students working on? What are they talking about? If that classroom embodies the ideas in Chapter 2 of doing mathematics, then you will see students working on a task carefully selected by the teacher that allows them to add to their prior knowledge, connect mathematical ideas, and learn important conceptual and procedural knowledge related to the topic. *Principles to Actions* (NCTM, 2014) includes reasoning and problem solving as one of eight mathematics teaching practices, explaining that effective teachers engage students in solving and talking about tasks that can be solved in different ways by different students. The first of the eight mathematical practices in CCSS-M explains that mathematically proficient students are able to make sense of a situation, select solution paths, consider alternative strategies, and monitor their progress—problem solve (NGA Center & CCSSO, 2010). In this chapter, we focus on how to teach through problem solving, including how to select worthwhile tasks and facilitate student engagement (e.g., talking and writing) with those tasks.

Problem Solving

The world in which we live and work has changed, and will continue to change, dramatically. In particular the mathematics we need for careers and for personal finance is completely different now than it was 25 years or 50 years ago. Yet, mathematics lessons in many classrooms may look the same as a generation or two generations ago. What do you remember about the types of problems or tasks that you were expected to do in school? Were you given instructions and then asked to follow them to get to a correct solution? Were you given encouragement to solve the problem differently? To come up with your own way to solve the problem? To consider if a particular strategy was efficient? Were you asked to determine when a particular strategy was useful (and when it was not)?

Skills needed in the 21st-century workplace are less about being able to compute and more about being able to design solution strategies. Priorities for students today include critical thinking, communication, collaboration, and creativity, as well as being able to use technology (Partnership for 21st Century Skills, n.d.). Students engaged in these inquiry-based or problem-based practices are encouraged to ask: Why? What would happen if? What is another way? How does this way compare to that way? Will this always work? Inquiry is a disposition of openness, curiosity, and wonder (Clifford & Marinucci, 2008). This disposition comes naturally to young students—our goal as teachers is to nurture it, not squelch it.

Too often mathematics teaching still follows the pattern of the teacher showing one way to do a skill, and students practicing that skill using the same procedure. Unfortunately, this approach to mathematics teaching has not been successful for many students and it does not prepare students for their 21st-century lives. Here are a few shortcomings of a teach-by-telling approach:

- It communicates that there is only one way to solve the problem, misrepresenting the field of mathematics and disempowering students who naturally may want to try to do it their own way.
- It positions the student as a passive learner, dependent on the teacher to present ideas, rather than as an independent thinker with the capability and responsibility for solving the problem.
- It assumes that all students have the necessary prior knowledge to understand the teacher's explanations—which is rarely, if ever, the case.
- It decreases the likelihood a student will attempt a novel problem without explicit instructions on how to solve it. But that's what doing mathematics is—figuring out an approach to solve the problem at hand.

Are you thinking that showing students is the helpful, preventing students from struggling while also saving time? If so, you are in good company. But, there is strong research showing that engaging students in productive struggle (i.e., allowing time and opportunity for them to grapple with a task and figure out mathematical strategies) leads to increased conceptual understanding (Hiebert & Grouws, 2007; NCTM, 2014). To be effective in preparing students to do mathematics, you have to consider alternatives to teaching by telling, and find ways to engage students in productive struggle (while not frustrating them).

Schroeder and Lester (1989) describe three approaches to problem solving that are used in classrooms. The distinctions between these ways are important because, as you will read, only attending to one of these approaches, will not lead to your students becoming mathematically proficient. Spoiler alert: the approaches are sequenced with the most important coming last!

Teaching *for* Problem Solving

Teaching *for* problem solving starts with learning the abstract concept and then moving to solving problems as a way to apply the learned skills (explain-practice-apply). For example, students learn the algorithm for adding fractions and, once that is mastered, solve story problems that involve adding fractions. This is the most common approach, and is often the way textbooks are written (practice first, then solve story problems). The major shortcoming of this approach is that students learn very early in school that the stories they encounter are going to be solved using the skill they just learned. Therefore, there is no point in reading the story to see what is happening and what needs solved, the numbers can be lifted out and the skill applied. This habit has resulted in students having difficulties with story problems, multistep problems, and solving high-level tasks. In other words, the pattern of explain-practice-apply works against preparing students to *do mathematics*. Yet, solving application problems after a skill is learned *is* important. The key is to be sure that the application problem is complex enough that reading and making sense of the situation is necessary to solving it.

Teaching *about* Problem Solving

Students need guidance on *how* to problem solve. This includes the process of problem solving and learning strategies that can help in solving problems—for example, "draw a picture."

Four-Step Problem Solving Process. George Pólya, a famous mathematician, wrote a classic book, *How to Solve It* (1945), which outlined four steps for problem solving. These steps for problem solving continue to be widely used today. For example, they are reflected in the first mathematical practice, "Making sense of problems and persevere in solving them" (NGA Center & CCSSO, 2010, p. 6). The four steps are summarized here:

1. *Understand the problem.* First, you must figure out what the problem is about and identify what question or problem is being posed.
2. *Devise a plan.* Next, you think about how to solve the problem. Will you want to write an equation? Will you want to model the problem with a manipulative? (See the next section, "Problem-Solving Strategies.")
3. *Carry out the plan.* This step is the implementation of your selected plan.
4. *Look back.* This final step, arguably the most important as well as most often skipped, is when you determine whether your answer from step 3 answers the problem as originally understood in step 1. Does your answer make sense? If not, loop back to step 2 and select a different strategy to solve the problem, or loop back to step 3 to fix something within your strategy.

The beauty of Pólya's framework is its generalizability; it can and should be applied to many different types of problems, from simple computational exercises to authentic and worthwhile multistep problems. Explicitly teaching these four steps to students can improve their ability to think mathematically.

Problem-Solving Strategies. Strategies for solving problems are identifiable methods for approaching a task. These strategies are "habits of mind" associated with thinking mathematically (Levasseur & Cuoco, 2003; Mark, Cuoco, Glodenberg, & Sword, 2010). Students select or design a strategy as they devise a plan (Pólya's step 2). The following labeled strategies are commonly encountered in grades K–8, though not all of them are used at every grade level.

- *Visualize.* Seeing is not only believing—it is also a means for understanding! Using manipulatives, acting it out, drawing a picture, or using dynamic software are ways to help represent, understand, and communicate mathematical concepts.
- *Look for patterns.* Searching for patterns, including regularity and repetition in everyday, spatial, symbolic, or imaginary contexts, is an important entry point into thinking mathematically. Patterns in number and operations play a huge role in helping students learn and master basic skills starting at the earliest levels and continuing into middle and high school.
- *Predict and check for reasonableness.* This is sometimes called "Guess and Check," but students are predicting more than they are guessing. This is not as easy as it may sound, as it involves making a strategic attempt, reflecting, and adjusting if necessary. The quantitative analysis (the answer is too small or too big) supports student sense making and is a bridge to algebra (Guerrero, 2010).
- *Formulate conjectures and justify claims.* As students interpret a problem, making conjectures and then testing them can help students solve the problem and deepen their understanding of the mathematical relationships. This reasoning is central to doing mathematics (Lannin, Ellis, & Elliott, 2011).
- *Create a list, table, or chart.* Systematically accounting for possible outcomes in a situation can provide insights into its solution. Students may make an organized list, a table or t-chart, or chart information on a graph. The list, table, or chart is used to search for patterns in order to solve the problem.
- *Simplify or change the problem.* Simplifying the quantities in a problem can make a situation easier to understand and analyze. This can lead to insights that can be applied to the original, more complex quantities in a problem. One way to simplify the problem is to test specific examples. The results of testing examples can provide insights into the structure of the task.
- *Write an equation.* Using or inventing symbols, numbers, notations, and equations are compact ways of modeling a situation. Writing an equation can provide insights into the structure of the problem and be used in solving the task.

Mathematical problem solving is founded in curiosity. The ways of thinking described here become the tools by which students can enter into unfamiliar and novel tasks. These strategies are not distinct, but interrelated. For example, creating a list is a way of looking for patterns. When students employ one of these strategies, it should be identified, highlighted, and discussed. Labeling a strategy provides a useful means for students to talk about their methods, which can help students make connections between and among strategies and representations. Over time, these mathematical ways of thinking will become habits.

It is important not to "proceduralize" problem solving. In other words, don't take the problem solving out of problem solving by telling students the strategy they should pick and how to do it. Instead, pose a problem that lends itself to different strategies, and ask students to approach the problem in a way that makes the most sense to them,

MyLab Education **Video Example 3.1**

Watch the third-grade teacher in this video describe a problem-solving approach.

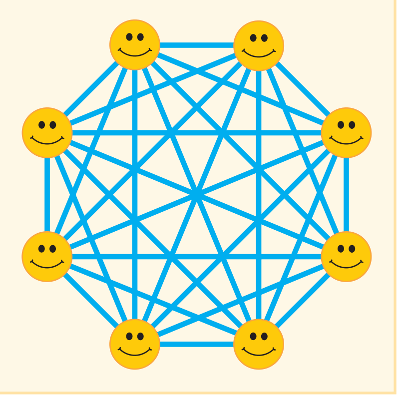
The classic handshake problem (see also the Handshake Problem Activity Page) is an example of a task that lends to many strategies:

Eight friends met for a skating party. Each friend shook hands once with everyone else. How many handshakes occurred?

MyLab Education **Activity Page: Handshakes**

Without suggesting any strategy, ask students to explore this problem, design a solution strategy, implement it, and be ready to share. The following are common solution strategies:

Visualize by acting it out or drawing a picture:



Create a smaller problem and record in a table:

Number of friends	2	3	4	5	6	7	8
Number of handshakes	1	3	6	10	15	21	28

Write an equation:

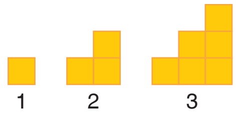
$$7 + 6 + 5 + 4 + 3 + 2 + 1 + 0 = 28$$

Note: The first friend can shake 7 hands, the next friend only has 6 hands to shake (she already shook hands with the first friend), and so on down to 0 handshakes for the last person.

During the sharing of results, you can help students understand the strategies other students used and see connections among the strategies. Additionally, you can highlight a particular strategy so that more students are able to use that strategy on a future task.

Similar problems can be posed to support the development of these problem solving strategies for example:

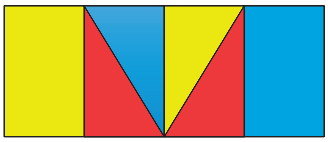
1. **If six softball teams play each other once in a round-robin tournament, how many games will be needed?**
2. **How many blocks are needed for the 10th staircase?**



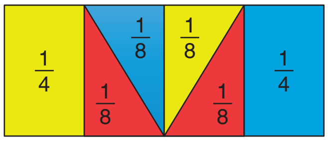
Teaching *through* Problem Solving

This approach means that students learn mathematics through inquiry. They explore real contexts, problems, situations, and models, and from those explorations they learn mathematics. So, teaching *through* problem solving might be described as upside down from teaching *for* problem solving—with the problem or task presented at the beginning of a lesson and related knowledge or skills emerging from exploring the problem. Consider the following task, given to students who have not learned the algorithm for adding fractions with different denominators:

What fraction of this flag is blue?



By partitioning, students might label each section with their fraction of the whole, labeling the blue parts as illustrated here:



As students work, they recognize that they need equal-sized pieces in order to combine. They change $\frac{1}{4}$ to $\frac{2}{8}$ (either by partitioning or knowing the fraction equivalency) and then add the blue pieces to get $\frac{3}{8}$. After students have solved the task, the teacher convenes the class to highlight important mathematical ideas—in this case that you can only combine same-sized pieces. The teacher asks students to share their ideas and asks questions that help build and connect conceptual *and* procedural knowledge. With more tasks like this one, students learn the procedure for adding fractions with different denominators (and understand it!). Notice that mathematical ideas are the outcomes of the problem solving experience rather than elements taught before problem solving (Hiebert et al., 1996, 1997). Importantly, the process of solving problems is completely interwoven with the learning; children are *learning* mathematics by *doing* mathematics and by doing mathematics they are learning mathematics (Cai, 2010).

Teaching through problem solving acknowledges what we now know about what it means to learn and do mathematics (see Chapter 2). Our understanding is always changing, incomplete, situated in context, and interconnected. What we learn becomes part of our expanding and evolving network of ideas—a network without endpoints. What we learn through problem solving and inquiry can change what we thought we knew before, and can be the basis for asking new questions that can lead to new learning (Thomas & Brown, 2011). Teaching through problem solving positions students to engage in inquiry and has a positive impact on students

(Boaler & Sengupta-Irving, 2016; Cobb, Gresalfi, & Hodge, 2009; Langer-Osuna, 2011). In these more effective classrooms, students:

- Ask questions
- Determine solution paths
- Use mathematical tools
- Make conjectures
- Seek out patterns
- Communicate findings
- Make connections to other content
- Make generalizations
- Reflect on results

Hopefully this sounds familiar. This list reflects the Standards for Mathematical Practice (CCSS-M) and reflects the strands of mathematical proficiency.

Teaching Practices for Teaching through Problem Solving

Classrooms where students are learning through problem solving do not happen by accident—they happen because the teacher uses practices and establishes expectations that encourage risk taking, reasoning, the generation and sharing of ideas, and so forth. Teaching *through* problem solving involves more than just tweaking a few things; it is a paradigm shift from traditional mathematics teaching where students repeat what the teacher demonstrates. At first glance, it may seem that the teacher's role is less demanding because the students are doing the mathematics, but the teacher's role is actually more demanding in such classrooms. Table 3.1 lists eight research-informed teaching practices from NCTM's (2014) *Principles to Actions*, along with the teacher actions that relate to that practice. All eight are addressed throughout this book and all are important to teaching mathematics *through* problem solving. In this chapter, we address tasks, building procedural fluency, and orchestrating discourse (three of the teaching practices).

Ensuring Success for Every Student

The NCTM Teaching Practices were designed to address issues related to access and equity—in other words, they are intended to ensure that all students have access to learning important mathematics. Teaching through problem solving provides opportunities for all students to become mathematically proficient (Boaler, 2008; Diversity in Mathematics Education, 2007; Silver & Stein, 1996). Teaching through problem solving:

- *Focuses students' attention on ideas and sense making.* When solving problems, students are necessarily reflecting on the concepts inherent in the problems. Emerging concepts are more likely to be integrated with existing ones, thereby improving understanding. This approach honors the different knowledge students bring to the classroom.
- *Develops mathematical practices and processes.* By definition, teaching through problem solving positions students to be the "doers," and as they are doing the mathematics, they are developing mathematical practices that are essential to becoming mathematically proficient.
- *Develops student confidence and identities.* As students engage in learning through problem solving, they begin to identify themselves as doers of mathematics (Boaler, 2008, Cobb, Gresalfi, & Hodge, 2009; Leatham & Hill, 2010). When students' peers and teachers listen to and respect their ideas, it impacts students' emerging mathematical identities (Aguirre, Mayfield-Ingram, & Martin, 2013) and later pursuit of careers that lead to higher socioeconomic status (Boaler & Selling, 2017).

TABLE 3.1 EIGHT MATHEMATICAL TEACHING PRACTICES THAT SUPPORT STUDENT LEARNING.

Teaching Practice	To Enact the Mathematics Teaching Practice, a Teacher:
1. Establish mathematics goals to focus learning	• Articulates clear learning goals that identify the mathematics students will learn in a lesson or lessons. • Identifies how the learning goals relate to a mathematics learning progression. • Helps students understand how the work they are doing relates to the learning goals. • Uses the articulated goals to inform instructional decisions involved in planning and implementation.

2. Implement tasks that promote reasoning and problem solving	• Selects tasks that:
	  • Have maximum potential to build and extend students' current mathematical understanding.
	  • Have multiple entry points.
	  • Require a high level of cognitive demand.
	• Supports students to make sense of and solve tasks using multiple strategies and representations, without doing the thinking for the students.

3. Use and connect mathematical representations	• Supports students to use and make connections between various representations. • Introduces representations when appropriate. • Expects students to use various representations to support their reasoning and explanations. • Allows students to choose which representations to use in their work. • Helps students attend to the essential features of a mathematical idea represented in a variety of ways.
4. Facilitate meaningful mathematical discourse	• Facilitates productive discussions among students by focusing on reasoning and justification. • Strategically selects and sequences students' strategies for whole class discussion. • Makes explicit connections between students' strategies and ideas.

5. Pose purposeful questions	• Asks questions that
	  • Probe students' thinking and that require explanation and justification.
	  • Build on students' ideas and avoids funneling (i.e., directing to one right answer or idea).
	  • Make students' ideas and the mathematics more visible so learners can examine the ideas more closely.
	• Provides appropriate amounts of wait time to allow students to organize their thoughts.

6. Build procedural fluency from conceptual understanding	• Encourages students to make sense of, use, and explain their own reasoning and strategies to solve tasks. • Makes explicit connections between strategies produced by students and conventional strategies and procedures.
7. Support productive struggle in learning mathematics	• Helps students see mistakes, misconceptions, naïve conceptions, and struggles as opportunities for learning. • Anticipates potential difficulties and prepares questions that will help scaffold and support students' thinking. • Allows students time to struggle with problems. • Praises students for their efforts and perseverance in problem solving.
8. Elicit and use evidence of student thinking	• Decides what will count as evidence of students' understanding. • Gathers evidence of students' understanding at key points during lesson. • Interprets students' thinking to gauge understanding and progress toward learning goals. • Decides during the lesson how to respond to students to probe, scaffold, and extend their thinking. • Uses evidence of students' learning to guide subsequent instruction.

Source: Based on Principles to Actions: Ensuring Mathematical Success For All (NCTM), © 2014.

- *Builds on students' strengths.* Because good problems have multiple paths to the solution, students can apply strategies that they understand and that lend to their learning preferences. Students may solve 42−26 by applying various mental strategies, using a manipulative such as base-ten blocks, by counting forward (or backward) on a hundreds chart, or by applying an algorithm. Furthermore, students expand on these ideas and grow in their understanding as they hear and reflect on the solution strategies of others.
- *Allows for extensions and elaborations.* Extensions and "what if" questions can motivate and challenge all students, as well as provide enrichment for advanced learners or quick finishers. For example, students can generate their own questions related to the problem they solved.
- *Engages students so that there are fewer discipline problems.* Many discipline issues in a classroom are the result of boredom, not understanding directions or an algorithm, or simply finding little relevance in the task. Most students like to be challenged and enjoy being permitted to solve problems in ways that make sense to them, giving them less reason to act out or cause trouble.

- *Provides formative assessment data.* As students discuss ideas, draw pictures or use manipulatives, defend their solutions and evaluate those of others, and write reports or explanations, they provide the teacher with a steady stream of valuable information. These products provide rich evidence of how students are solving problems, what misconceptions they might have, and how they are connecting and applying new concepts. With a better understanding of what students know, a teacher can plan more effectively and accommodate each student's learning needs.
- *Invites creativity* Students enjoy the creative process of problem solving, searching for patterns, and showing how they figured something out. Teachers find it exciting to see the surprising and inventive ways students think. Teachers know more about their students and appreciate the diversity within their classrooms when they focus on problem solving.

When students have confidence, show perseverance, and enjoy mathematics, it makes sense that they will achieve at a higher level and want to continue learning about mathematics, opening many doors to them in the future.

Tasks That Promote Problem Solving

In order to create classroom experiences for students where they develop the mathematical practices, you must select tasks that promote problem solving. Such a worthwhile task may take on many different forms. It might be clearly defined or open-ended; it may involve problem solving or problem posing; it may include words or be purely symbols; it may take only a few minutes to solve or may take weeks to investigate; it may be real-life or abstract. Also, a task can be problematic initially, but then become routine as a student's knowledge and experience grows, or be problematic for some students, but not for others, based on their previous experiences. For a task to lend to problem-solving, in other words be *problematic*, it must pose a question for which (1) there is no prescribed rules or methods to solve and (2) there is not a perception that there is one "correct" solution method (Hiebert et al., 1997).

Here is a task for you to try (see also the Missing Numbers Activity Page):

$10 + \blacksquare = 4 + (3 + \blacktriangle)$.

Find a number to replace the square and a number to replace the triangle so that the equation is true.

Find more pairs of numbers that will make the equation true.

What do you notice about the numbers for any correct solution?

MyLab Education **Activity Page: Missing Numbers**

Is this a worthwhile task? It does not have a prescribed approach and there are numerous ways to approach the problem—so it meets the first criteria of being problematic. This task has other features that also make it worthwhile, which include: high-level cognitive demand, multiple entry and exit points, and relevant contexts.

High-Level Cognitive Demand

Tasks that promote problem solving are cognitively demanding, meaning they involve high-level thinking. Low cognitive demand tasks (also called *routine problems* or *lower-level tasks*) involve stating facts, following known procedures (computation), and solving routine problems. On Bloom's Taxonomy, they are at the remembering level–the lowest level.

MyLab Education **Video Example 3.2**

As described in How Thinking Works TED Talk (https://www.youtube.com/watch?v=dUqRTWCdXt4), schools must focus on teaching students to *think* (rather than follow instructions and remember facts).

VIDEO EXAMPLE

High-level cognitive demand tasks, on the other hand, involve understanding, analyzing information and applying it, and evaluating strategies,, as indicated by the other levels of Bloom's Taxonomy (see Chapter 2, Figure 2.1). Table 3.2 provides descriptors for the low-level and high-level cognitive demand descriptors (Smith & Stein, 1998). The Missing Numbers task above and the tasks in Chapter 2 each involve high-level cognitive demand. And, the task does not need to have a context or take days to solve to have a high level of cognitive demand—but it must provide students an opportunity to reason and make sense of the mathematics (NCTM, 2014).

Multiple Entry and Exit Points

Because your students will likely have a wide range of experiences in mathematics, it is important to use problems that have *multiple entry points*, meaning that the task can be approached in a variety of ways and has varying degrees of challenge within it. Having multiple entry points can accommodate the diversity of learners in your classroom because students are encouraged to use a variety of strategies that are supported by their prior experiences. Having a choice of strategies can lower the anxiety of students, particularly English learners (Murrey, 2008). Students are encouraged to engage with the task in a way that makes sense to them, rather than trying to recall or replicate a procedure shown to them.

Tasks should also have *multiple exit points*, or various ways to express solutions that reveal a range of mathematical sophistication and have the potential to generate new questions. As

TABLE 3.2 LEVELS OF COGNITIVE DEMAND

Low-Level Cognitive Demand	High-Level Cognitive Demand
Memorization Tasks • Involve either producing previously learned facts, rules, formulas, or definitions or memorizing • Are routine—involving exact reproduction of previously learned procedure • Have no connection to related concepts	**Procedures with Connections Tasks** • Focus students' attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas • Suggest general procedures that have close connections to underlying conceptual ideas • Are usually represented in multiple ways (e.g., visuals, manipulatives, symbols, problem situations) • Require that students engage with the conceptual ideas that underlie the procedures in order to successfully complete the task
Procedures without Connections Tasks • Specifically call for use of the procedure • Require little cognitive demand, with little ambiguity about what needs to be done and how to do it • Have no connection to related concepts • Are focused on producing correct answers rather than developing mathematical understanding • Require no explanations, or explanations focus solely on the procedure that was used	**Doing Mathematics Tasks** • Require complex and nonalgorithmic thinking (i.e., nonroutine—there is not a predictable, known approach) • Require students to explore and understand the nature of mathematical concepts, processes, or relationships • Demand self-monitoring or self-regulation of one's own cognitive processes • Require students to access relevant knowledge in working through the task • Require students to analyze the task and actively examine task constraints that may limit possible solution strategies and solutions • Require considerable cognitive effort

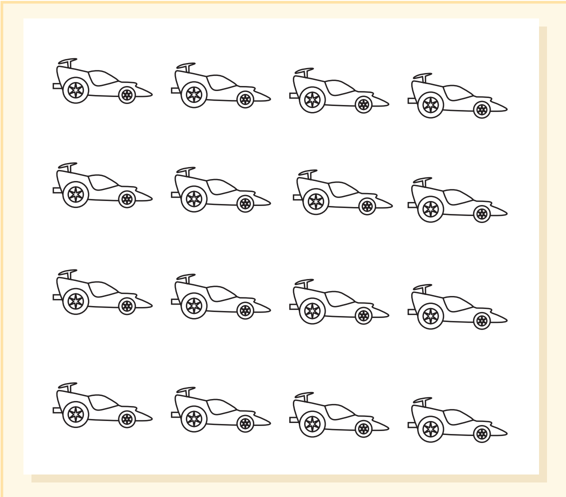
Source: Adapted from Smith, M. S., & Stein, M. K. (1998). Selecting and creating mathematical tasks: From research to practice. *Mathematics Teaching in the Middle School, 3*(5), 344–350.

students employ their multiple approaches and varied displays of their solutions (e.g., various pictures, demonstrations with manipulatives, acting out a problem, using a table or a graph, etc.), opportunities emerge for students to defend their strategy, critique the reasoning of others, learn new approaches, all the while thinking at a high level about mathematics.

Consider the opportunities for multiple entry and exit points in the following kindergarten or first-grade tasks.

TASK 1: [The teacher places a bowl of objects (e.g., toy cars) on the table.] Do we have enough [toy cars] for everyone in the class?

TASK 2: [The teacher gives each student a page with pictures of cars copied in rows (see Do We Have Enough? Activity Page). Do we have enough cars for everyone in the class?



MyLab Education **Activity Page: Do We Have Enough?**

At first glance, the first task might seem the more engaging because it includes actual manipulatives. But, in the particular case, having toy cars (or a cube representing a car) available, might lead to one (low-level) strategy: passing the cars out to see if there are enough for each child. The second task is more problematic. Students might count the cars. As they count, you can observe their thinking: Do they start at the top and count across the rows? Or do they haphazardly count and miss or double-count? Do they count by ones? By twos? By fours? Instead of counting cars, students might count their friends in the class first. Or, students might "assign" a car to each friend, writing names on each car. Counting the cars is just one aspect of the task; students must also decide how that number compares to the number of students in the class. Does a student just know that the number of cars is greater or less than the number of students? Do they represent each child in the class with a counter, and match a counter with a pictured car? Do they look for the two numbers on a hundreds chart or number line to compare? Because the second task is more problematic, it offers many more options for how to begin the task (entry) and how to show a solution for the task (exit), making it a worthwhile task for the class.

Figure 3.1(a) provides a high level task that has multiple entry and exit points.

Pause & Reflect

Before studying the solutions in Figure 3.1(b), read the problem, select a strategy and solve it. ●

Figure 3.1(b) illustrates a range of solutions. Student (b) used percentages as a way to compare; student (d) found simplified fractions to compare quantities; student (g) used part-part

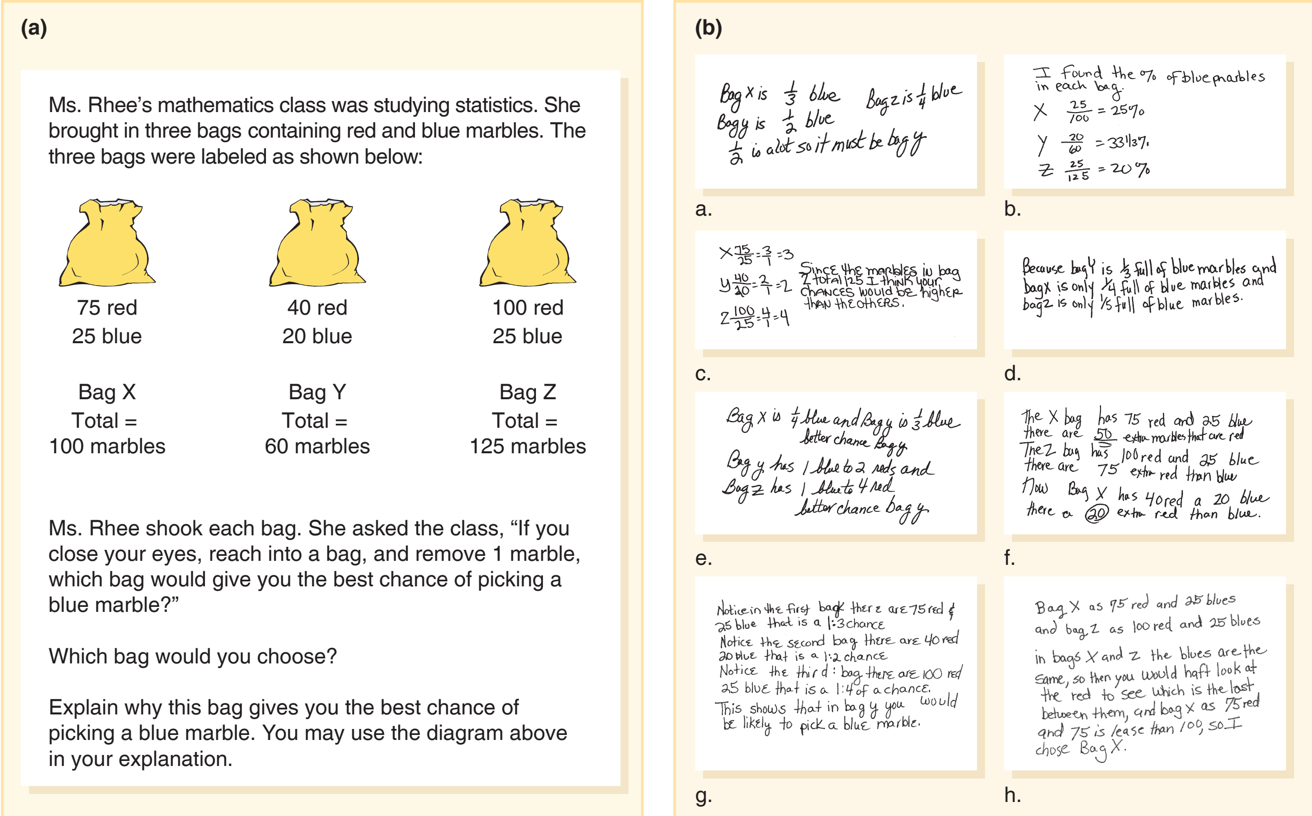


FIGURE 3.1 A task with multiple entry and exit points, as illustrated by the range of student solutions.

Source: Based on Smith, M. S., Bill, V., & Hughes, E. K. (2008). Thinking Through a Lesson: Successfully Implementing High-Level Tasks. *Mathematics Teaching in the Middle School, 14*(3), 132–138.

ratios to reason about the quantities. In addition, several solutions reveal student misconceptions. Notice that in solution (a), the student has recorded part-part ratios, but then is comparing the values as though they are fractions (part-whole) and in solution (f) the student is comparing differences, rather than attending to the multiplicative relationships. During a classroom discussion, the teacher's role is to ensure that the strategies are strategically shared (perhaps sharing some less advanced strategies first or related strategies together). In doing this, students can clear up misconceptions, make connections among the strategies and among mathematical ideas (i.e., ratio, fractions, percents, and probability), and thereby advance their understanding of mathematics (Smith, Bill, & Hughes, 2008).

Relevant Contexts

Certainly one of the most powerful features of a worthwhile task is that the problem that begins the lesson can get students excited about learning mathematics. Compare the following introductory tasks on multiplying 2 two-digit numbers. Which one do you think would more exciting to fourth-grade students?

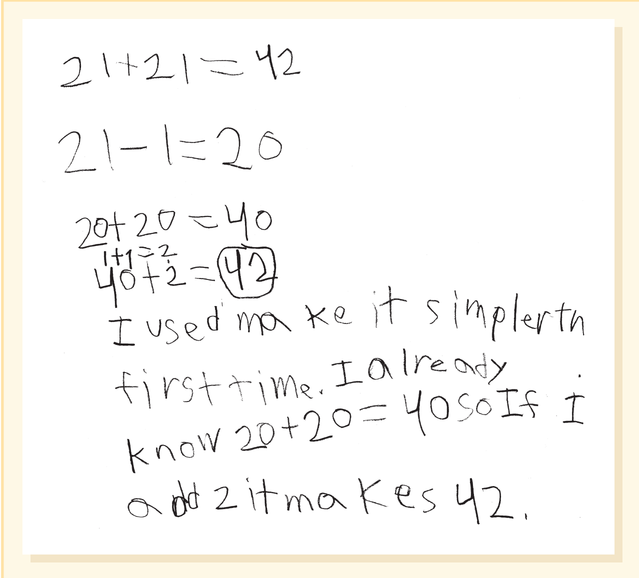
> *Classroom A:* "Today we are going to use grid paper to show the sub-products when we multiply 2 two-digit numbers."
>
> *Classroom B:* "The school is planning a fall festival and class is selling water. The principal said we have 14 cases of bottled water in the storage closet. I looked in the storage closet and could see that a case had 7 rows, with 5 bottles in each row. How can we use the information about one case to figure out how many bottles of water we have?" If we sell each one for $2.00, how much money might we make?.

Contexts must reflect the cultures and interests of the students in your classroom (this is a critical component of Culturally Responsive Mathematics Instruction in Chapter 5). Using everyday situations can increase student participation, increase student use of different problem strategies, and help students develop a productive disposition (Tomaz & David, 2015). Here we share two strategies for incorporating relevant contexts—using literature and connecting to other subjects.

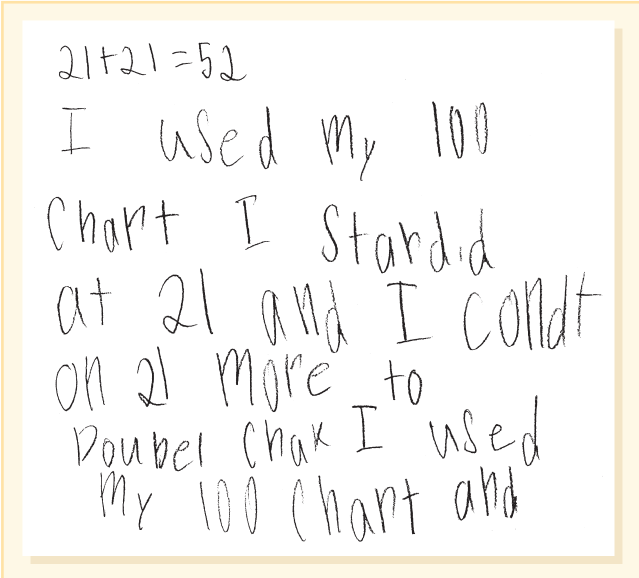
Use Literature. Children's literature and adolescent literature are rich sources of problems. Picture books, poems, media, and chapter books can be used to create high cognitive demand tasks with multiple entry points. An example of literature lending itself to mathematical problems is the very popular children's picture book *Two of Everything* (Hong, 1993). In this magical Chinese folktale, a couple finds a pot that doubles whatever is put into it. (Imagine where the story goes when Mrs. Haktak falls in the pot!). Students can explore the following problem: How many students would be in our class if our whole class fell in the Magic Pot? Figure 3.2 illustrates different ways that students in second grade approached the problem (multiple entry points) and different ways they explained and illustrated how they figured it out (multiple exit points). Notice that the student using the hundreds chart is incorrect. The teacher will need to follow up to determine whether this was a copy error or a misconception.

In *Harry Potter and the Sorcerer's Stone* (Rowling, 1998), various lessons can be built on the description of Hagrid as twice as tall and five times as wide as the average man. Students in grades 2–3 can cut strips of paper that are as tall as they are and as wide as their shoulders are (you can cut strips from cash register rolls). Then they can figure out how big Hagrid would be if he were twice as tall and five times as wide as they are. In grades 4–5, students can create a table that shows each student's height and width and look for a pattern (it turns out to be about 3 to 1). Then they can figure out Hagrid's height and width and see whether they keep the same ratio (it is 5 to 2). In grades 6–8, students can create a scatter plot of their widths and heights and see where Hagrid's data would be plotted on the graph. Measurement, number, and algebra content are all embedded in this example. Whether students are 5 or 13, literature resonates with their experiences and imaginations, making them more enthusiastic about solving the related mathematics problems and more likely to learn and to see mathematics as a useful tool for exploring the world.

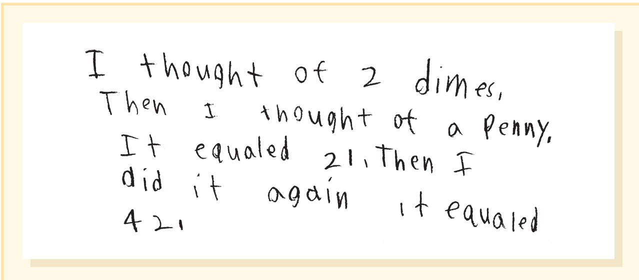
Nonfiction literature (picture books, chapter books, newspapers, magazines, and the web) have the added benefit of students learning about the *real* world around them. There are books of lists (e.g., Scholastic Book of Lists (Buckley & Stremme, 2006)) and world record books, for example, which provide many great contexts for exploring the world, and comparing world data to your class (see Bay-Williams & Martinie, 2009; Petersen, 2004; Sheffield



Robbie adds tens and ones to solve.



Kylee uses a hundreds chart and counts on.



Benjamin uses the context of money to combine.

FIGURE 3.2 Second-grade students use different problem-solving strategies to figure out how many students there would be if their class of 21 were doubled.

& Gallagher, 2004 for collections of mathematics lessons using nonfiction). Dates lend to creating number lines. For example, the dates for the seven wonders of the ancient world can be used to explore negative numbers.

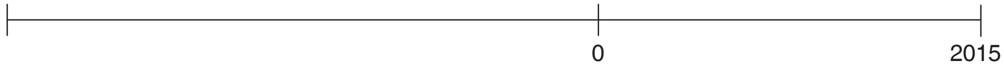
Where on the number line do these wonders of the ancient world go? Can you find the other wonders and place them on the number line?

Hanging Gardens of Babylon: About 600 B.C.

Great Pyramids: About 2500 B.C.

Statue of Zeus: 435 B.C.

Lighthouse of Alexandria: About 250 B.C.



In Part II, each end-of-chapter resource section includes "Literature Connections," quick descriptions of picture books, poetry, and novels that can be used to explore the mathematics of that chapter. Literature ideas are also found in articles in journal articles (e.g., *Teaching Children Mathematics*) and teacher books (e.g., *Math & Literature Series*, *Using Children's Literature to Teach Problem Solving in Math* (White, 2014)).

Connect to Other Disciplines. Interdisciplinary lessons help students see connections among the courses/topics they are studying, which often feel completely separate to them. Elementary teachers can pull ideas from the topics being taught in social studies, science, and language arts; likewise, middle school teachers can link to these subjects as they collaborate with grade-level colleagues. Other familiar contexts such as art, sports, and pop culture are also worthwhile contexts.

For example, in kindergarten, students can link their study of natural systems in science to mathematics by sorting leaves based on color, smooth or jagged edges, feel of the leaf, and shape. Students learn about rules for sorting and can use Venn diagrams to keep track of their sorts. They can observe and analyze what is common and different in leaves from different trees. Older students can find the perimeter and area of various types of leaves and learn about why these perimeters and areas differ. Like with literature, there are high quality print and online resources. For example AIMS (Activities in Mathematics and Science) has numerous books that integrate mathematics and science. In *Looking at Lines* (AIMS, 2005), middle school students hang paper clips from a handmade balance to learn about linear equations (mathematics) and force and motion (science).

The social studies curriculum is rich with opportunities to do mathematics. Timelines of historic events are excellent opportunities for students to work on the relative sizes of numbers and to make better sense of history. Students can explore the areas and populations of various countries, provinces, or states and compare the population densities, while in social studies they can talk about how life differs between regions with 200 people living in a square mile and regions with 5 people per square mile.

Evaluating and Adapting Tasks

Throughout this book, in student textbooks, on the Internet, at workshops you attend, and in articles you read, you will find suggestions for tasks that *someone* believes are effective for teaching a particular mathematics concept. Yet a large quantity of what is readily available falls short when measured against the standards of being (1) high level and (2) promoting problem solving. This is particularly true with the plethora of worksheets that pop up on web searches and sites for teachers. Beware of the low-level cognitive demand tasks cloaked in clever artwork—they may look fun, but if the mathematics is not problematic, those graphics are not going to help your students think at a high level. Make sure it is the mathematics itself that is clever and engaging.

Task Evaluation and Selection Guide	
Task Potential	Try it and ask . . . ❑ What is problematic about the task? ❑ Is the mathematics interesting? ❑ What mathematical goals does the task address (and are they aligned to what you are seeking)? ❑ What strategies might students use? ❑ What key concepts and/or misconceptions might this task elicit?
Problem-Solving Strategies	Will the task elicit more than one problem-solving strategy? Which strategies are possible? ❑ Visualize ❑ Look for patterns ❑ Predict and check for reasonableness ❑ Formulate conjectures and justify claims ❑ Create a list, table, or chart ❑ Simplify or change the problem ❑ Write an equation
Features	To what extent does the task have these key features? ❑ High cognitive demand ❑ Multiple entry and exit points ❑ Relevant contexts
Assessment	In what ways does the task provide opportunities for you to gain insights into student understanding through? ❑ Using tools or models to represent mathematics ❑ Student reflection, justification, and explanation ❑ Multiple ways to demonstrate understanding

FIGURE 3.3 Use these reflective questions in selecting tasks that promote problem solving.

An intentional selection process can help with selecting high quality tasks (Barlow, 2010; Breyfogle & Williams, 2008–2009). The Task Evaluation and Selection Guide in Figure 3.3 provides reflective questions to help you evaluate whether a task you are considering has the maximum potential to help your children learn relevant mathematics. These questions are meant to help you consider to what extent the task meets these criteria, so all boxes do not need to be checked off. A task could rate very high on the number of problem solving strategies, but miss the mark in terms of being relevant for students, hence you decide to trade out the context for something more interesting. Or, the task is complete with the features and problem solving strategies, but it does not match your mathematical goals for the lesson. You may choose to alter the task to focus on the mathematics you have selected or save it for when it is a better match.

Certainly, teaching the operations has often been devoid of the features described in this chapter. You may be wondering how you will be able to teach all the operations by teaching through problem solving. Boaler (2016) offers ideas for adapting tasks are particularly applicable to the operations, but applies to all content areas.

1. *Allow multiple ways:* Explicitly ask students to use multiple methods, strategies, and representations to solve.
2. *Make it an exploration:* Change the task so there is more to it than a single computation. For example, rather than ask kindergartners to add 3+5, ask them to find numbers they can add to get the number 8.

3. *Postpone teaching a standard method:* Begin with students' intuition about how to solve a problem type before learning about conventional methods.
4. *Add a visual requirement:* Visualization enhances understanding. Students can use two different manipulatives to justify their solution, or show how an equation fits a drawing of a story situation.
5. *Increase the number of entry points:* You can increase the entry points by asking students to write down everything they know about the problem, or listing any possible ideas for how they might solve the problem.
6. *Reason and convince:* Require students to create convincing mathematical arguments and to expect the same from their peers. Ask them to be skeptics and to ask clarifying questions of each other. (Modeling argument and questioning is important to be sure it focuses on supporting each other's thinking.)

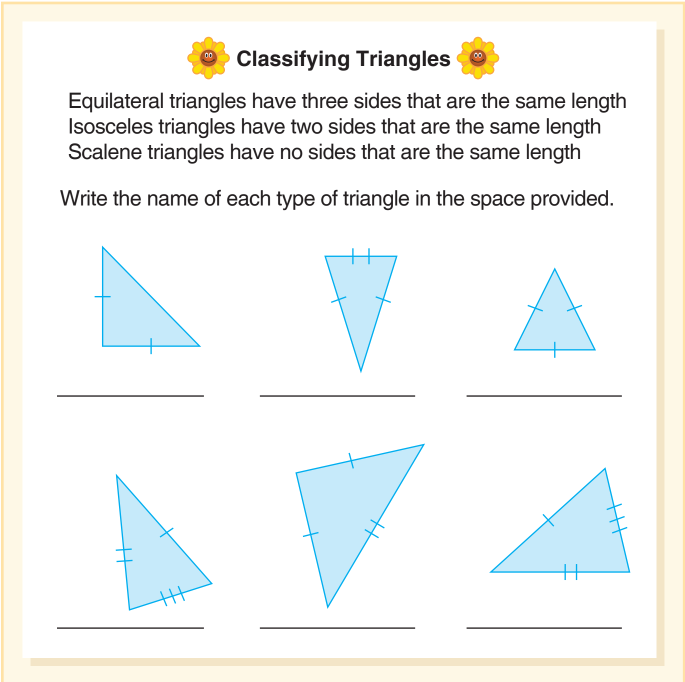
Just as students become adept at problem solving strategies, with time and commitment you will become adept at evaluating and adapting tasks to better support student learning. Imagine that you are teaching fourth grade and are seeking a worthwhile task for this goal:

4.G.A.2: Classify two-dimensional figures based on the presence or absence of parallel or perpendicular lines, or the presence or absence of angles of a specified size. Recognize right triangles as a category, and identify right triangles. (NGA Center & CCSSO, 2010, p. 32)

You type "classify triangles" into a web search. Hundreds of links and worksheet images appear, like the one in Figure 3.4.

Pause & Reflect

How does this task measure up on the Task Evaluation and Selection Guide? How might you adapt the task so that it rates higher on some of these measures of a worthwhile task? ●


Classifying Triangles

Equilateral triangles have three sides that are the same length
Isosceles triangles have two sides that are the same length
Scalene triangles have no sides that are the same length

Write the name of each type of triangle in the space provided.

FIGURE 3.4 Example of a categorizing triangles worksheet.

This worksheet appears to match the learning goals, but it does not include high-level cognitive demand, multiple entry points, or relevant contexts. Here are some ways you might adapt it:

1. Remove the tick marks telling which sides are the same, distribute just the grid of triangles, letter them, and ask students to write similarities and differences between pairs of triangles. This provides multiple entry points and a high level of cognitive demand.
2. Only use the list of terms at the top and ask students to identify examples of each in the room or in a picture book.
3. Cut out the triangles, ask students to create piles of what they consider the "same" triangles and to put names on their groups. In a later discussion, different possible ways to sort triangles can be discussed and appropriate terminology can be reinforced.

Extensions might be added to make this task a stronger task. These might include asking students, "Can you build two triangles of different sizes that are both isosceles?" "Can you create a triangle with three obtuse angles? Why or why not?" "If a triangle is classified as [right], then which classifications for sides are possible or impossible?" Each of these adaptations takes very little time, and greatly increases the tasks potential to develop a deep understanding of geometry concepts.

Developing Procedural Fluency

In Chapter 2, we discussed the importance of developing mathematical proficiency through conceptual understanding and procedural proficiency. In Chapter 3 we have focused on problem solving and the features of a tasks that support problem solving. In this section, we focus on teaching concepts *and* procedures *through* problem solving, using five examples.

Example Tasks

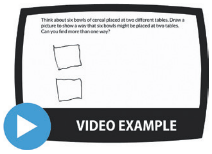
Developing procedural fluency involves developing conceptual understanding and making connections among mathematical ideas (Bay-Williams & Stokes Levine, 2017; NCTM, 2014.

As you read, consider how each task opens up opportunities to make such connections between concepts and procedures.

Topic 1: Partitioning **Grades: K–1**

Six bowls of cereal are placed at two different tables. Draw a picture to show a way that six bowls might be placed at two tables. Can you find more than one way? How many ways do you think there are?

MyLab Education **Application Exercise 3.1: Observing and Responding to Student Thinking** Click the link to access this exercise, then watch the video and answer the accompanying questions.



In kindergarten or grade 1, students may determine one or two ways to decompose 6 or may find all the ways (multiple entry points). Students can share how they thought about it and what patterns they noticed as they found new ways. For example, a student might note that as one table gets a bowl, the other table loses a bowl, so 1 bowl moves over. They may notice that there are then 7 possible ways. The task can be extended to other totals and other contexts, such as "How many ways can you put 10 toys into 2 baskets? Students begin to notice that numbers can be taken apart and put back together in different ways. They also begin to learn about addition, perhaps recording number sentences such as $1+5=6$, discussing the meaning of the symbols and how the numbers represent to the situation.

Topic 2: Adding Two-Digit Whole Numbers **Grades: 1–2**

What is the sum of 48 and 25? How did you figure it out?

Even though there is no story or situation to resolve, this task is problematic because students must figure out how they are going to approach the task. (They have not been taught the standard algorithm at this point.) Students might work on the problem using manipulatives, pictures, or mental strategies. This following list contains just some of the approaches created by students in one second-grade classroom:

4$\boxed{8}$ + 2$\boxed{5}$ (Boxed digits "help" them.)
40 + 20 = 60
8 + 2 = 10 $\boxed{3}$ (The 3 is left from the 5.)
60 + 10 = 70
70 + 3 = 73

---

40 + 20 = 60
60 + 8 = 68
68 + 5 = 73

---

48 + 20 = 68
68 + 2 ("from the 5") = 70
"Then I still have that 3 from the 5."
70 + 3 = 73

---

25 + 25 = 50 $\boxed{23}$
50 + 23 = 73
Teacher: Where does the 23 come from?
"It's sort of from the 48."
How did you split up the 48?
"20 and 20 and I split the 8 into 5 and 3."

---

48 − 3 = 45 $\boxed{3}$
45 + 25 = 70
70 + 3 = 73

As students share their ways, they are deepening their conceptual understanding of place value, they are learning place value strategies for solving addition tasks, and they are developing a foundation for learning the standard algorithm, which is also based on place value. Note: Operations for whole numbers, decimals, fractions, and integers can be explored in a similar manner.

Topic 3: Area of a Rectangle **Grades: 3–4**

Find the area of the cover of your math book by covering it with color tiles. Repeat for the areas of books of various sizes. What patterns do you notice in covering the book? Is this pattern or rule for covering any rectangle?

As students begin to cover surfaces, they may run out of tiles, or they may just get tired of placing tiles. They notice that each row has the same number of tiles, so they just need to know how many rows will cover the book and they can skip count or multiply to find the total number of tiles. This develops the concept of area, strengthens understanding of multiplication as repeated addition, and leads to the procedure for area of a rectangle. Most measurement formulas can be developed through problem solving, as you will read in Chapter 18.

Topic 4: Division of Fractions **Grades: 5–7**

Anthony is knitting scarves for gifts for his sisters. Each scarf is one yard long and he can knit $\frac{1}{4}$ of a scarf each day. How long will it take him to make 3 scarves?

When students explore this task without the label "division of fractions," they can approach the problem in multiples ways. Leah, Kelly, Jaden, and MacKenna solved the task applying what they knew, including skip counting by fourths (Leah), measurement equivalencies (Kelly), ratios of yards to days (Jaden), and rates of days per yards (MacKenna). To extend their thinking about scarf-making, they were asked, "What would happen if Anthony decided to make $\frac{3}{4}$ of a scarf in one day?" A review of their second task shows that they used both their strategies and their answers from the first task in interesting ways. With more experiences with scarf-making with other rates, students can begin to connect to the concept of division and to generalize how solve such problems. Series of related tasks can help students connect concepts and procedures. The scarf length and the amount completed per day can vary to help students look for patterns across the problem set.

MyLab Education	**Leah's Solution**
MyLab Education	**Kelly's Solution**
MyLab Education	**Jaden's Solution**
MyLab Education	**MacKenna's Solution**

Topic 5: Ratios and Proportions **Grades: 6–8**

Jack and Jill were at the same spot at the bottom of a hill, hoping to fetch a pail of water. They both begin walking up the hill, Jack walking 5 yards every 25 seconds and Jill walking 3 yards every 10 seconds. Assuming constant walking rate, who will get to the pail of water first?

Students can engage in this task in a variety of ways. They can represent the problem visually with jumps on a number line or symbolically using a rate approach (determining the number of yards per second for each person). This task focuses on how students are able to compare ratios. By specializing and considering several examples, either in this context or another, students will begin to generalize a procedure for comparing ratios, which is the essence of proportional reasoning. See It's a Matter of Rates Expanded Lesson for a full lesson on this task.

MyLab Education **Expanded Lesson: It's a Matter of Rates**

Teaching mathematical concepts and procedures through problem solving helps students go beyond acquiring isolated ideas toward developing a connected and increasingly complex network of mathematical understanding (Cai, 2010; Carpenter, Franke, Jacobs, Fennema, & Empson, 1998; Fuson, Kalchman, & Bransford 2005; Lesh & Zawojewski, 2007; Schneider, Rittle-Johnson, & Star, 2011).

What about Drill and Practice?

The phrase "drill and practice" slips off the tongue so rapidly that the two words *drill* and *practice* appear to be synonyms—and, for the most part, they have been. In the interest of developing a new perspective on drill and practice, consider definitions that differentiate between these terms as different types of activities.

> *Practice* refers to varied tasks or experiences focused on a particular concept or procedure.
>
> *Drill* refers to repetitive exercises designed to replicate a procedure or algorithm.

Practice. Practice, as defined here, has numerous benefits, as this chapter has described. Students have an increased opportunity to develop conceptual ideas, alternative and flexible strategies for solving, and making connections between concepts and procedures. Importantly, using worthwhile tasks as practice sends a clear message that mathematics is about figuring things out.

Drill. Most textbooks include sets of exercises with every lesson. There is a seemingly endless amount of downloaded worksheets that focus on drill. What has all of this drill accomplished over the years? It has worked against developing mathematical practices, and created generations of people who don't remember the skills they learned, do not like mathematics, and do not pursue professions that involve mathematics.

What Does Appropriate Drill Look Like? In a review of research, Franke, Kazemi, and Battey (2007) report that drill improves procedural knowledge, but not conceptual understanding. But when the number of problems is reduced and time is then spent discussing problems, both procedural and conceptual knowledge are supported. The key is to keep drill short and to connect procedures to the related concepts. And, use the ideas listed above for adapting tasks to make drill more effective.

Technology makes possible differentiated drill. For example, *First in Math*® a subscription-based online game, offers students a self-paced approach to practicing basic math skills and complex problem-solving tasks. Programs such as this provide students with the opportunity to earn electronic incentives and move on to more difficult exercises based on their readiness.

Drill and Student Errors. As discussed earlier, the range of background experiences that students bring means that students will develop understanding in different ways and at different rates. For students who do not pick up new concepts quickly, there is a temptation to give in and "just drill 'em." In reality, when a student is making errors on a procedure, it is often a mis*concept*ion. Using a medical metaphor, students' computational errors are a symptom, not the problem. Therefore remediation should include activities that strengthen the student's conceptual knowledge and connecting that concept to the procedure.

Orchestrating Classroom Discourse

While tasks that promote problem solving are a necessary component to an effective mathematics lesson, for the problem solving to occur requires skill at orchestrating classroom discourse. To help students become productive mathematical thinkers, teachers must be comfortable with uncertainty, ask key questions, be able to respond to students, probe student thinking, prompt students to reflect on their thinking, and know the difference between productive and nonproductive struggle (Heaton & Lewis, 2011; Kazemi & Hintz, 2014; Towers, 2010). Teachers, themselves, must display productive dispositions, showing students they are willing to explore, experiment and make conjectures, recognize multiple solution paths, make connections among strategies, and monitor and reflect on their work. The goal of productive discourse is to keep the cognitive demand high while students are learning and formalizing mathematical concepts (Breyfogle & Williams, 2008–2009; Kilic et al., 2010; Smith, Hughes, Engle, & Stein, 2009). Note that the purpose is not for students to tell their answers and get validation from the teacher. The aspects involved in orchestrating classroom discourse are so important, they directly involve three out of the eight teaching practices from *Principles to Actions* (NCTM, 2014): facilitate meaningful mathematical discourse; pose purposeful questions; and elicit and use evidence of student thinking.

Classroom Discussions

The value of student talk in mathematics lessons cannot be overemphasized. As students describe and evaluate solutions to tasks, share approaches, and make conjectures, learning will occur in ways that are otherwise unlikely to take place. Students—in particular English learners, other students with more limited language skills, and students with learning disabilities—need to use mathematical vocabulary and articulate mathematics concepts in order to learn both the language and the concepts of mathematics. Students begin to take ownership of ideas (strategic competence) and develop a sense of power in making sense of mathematics (productive disposition). As they listen to other students' ideas, they come to see the varied approaches in how mathematics can be solved and see mathematics as something that they can do.

Smith and Stein (2011) identified five teacher actions for orchestrating productive mathematics discussions: anticipating, monitoring, selecting, sequencing, and connecting. The first action, *anticipating* responses to the selected worthwhile task, takes place before the lesson even begins. As students are working, the teacher *monitors*, observing strategies students are using and asking questions, such as:

- How did you decide what to do? Did you use more than one strategy?
- What did you do that helped you make sense of the problem?
- Did you find any numbers or information you didn't need? How did you know that the information was not important?
- Did you try something that didn't work? How did you figure out it was not going to work?

These and similar questions are designed to help students reflect on their own strategies and help the teacher determine which strategies to *select* for a public discussion after the lesson. Having selected a range of strategies to be shared, the teacher strategically *sequences* the presentations so that particular mathematical ideas can be emphasized. Perhaps most importantly, the teacher designs questions and strategies that *connect* strategies and mathematical concepts. These tend to be questions that are specific to the task, but some general questions include:

- How did [Leslie] represent her solution? What mathematical terms, symbols, or tools did she use? How is this like/different from [Colin's] strategy?
- Was there something in the task that reminded you of another problem we've done?
- What might you do the same or differently the next time you encounter a similar problem?

Notice these questions focus on the problem solving process as well as the answer, and what worked as well as what didn't work.

Because of the benefits of talking about mathematics, it is critical to make sure that everyone participates in the classroom discussion. You may need to explicitly discuss with students why discussions are important and what it means to actively listen and respond to others' ideas. For example, children can demonstrate they are listening by making eye contact with the speaker and through nonverbal cues (e.g., nodding); letting the speaker finish before sharing questions or ideas; and responding appropriately and respectfully by asking questions or summarizing the speaker's ideas (Wagganer, 2015). Wait time is also critical to ensuring participation and thinking (Roake, 2013). Chapin, O'Conner, and Anderson (2013) describe specific *talk moves*, strategic ways to ask questions and invite participation in classroom discussions. These moves are briefly described in Table 3.3, including example question prompts.

The following exchange illustrates an example of discourse with a small group of students discussing how to solve $27 - 19 =$ ________. The teacher is asking two students (Tyler and Aleah) to think about their different answers.

Tyler: Well, I added one to nineteen to get twenty. So then I did twenty-seven take away twenty and got seven. But I added one, so I needed to take one away from the seven, and I got six.

Teacher: What do you think of that, Aleah?

Aleah: That is not what I got.

Teacher: Yes, I know that, but what do you think of Tyler's explanation?

Aleah: Well, it can't be right, because I just counted up. I added one to nineteen to get twenty and then added seven more to get twenty-seven. So, I counted eight altogether. Six can't be right.

Teacher: Tyler, what do you think of Aleah's explanation?

Tyler: That makes sense, too. I should have counted.

Teacher: So, do you think both answers are right?

Tyler: No.

Aleah: No. If it was twenty-seven minus twenty, the answer would be seven, because you count up seven. So, if it is nineteen, it has to be eight.

Tyler: Oh, wait. I see something. I did get the seven . . . . ? See, I got the twenty-seven take away twenty is seven. But then . . . ? I see . . . ? it's twenty-seven take away nineteen. I took away twenty! I took away too many so I have to add one to the seven. I get eight, just like Aleah! (Kline, 2008 , p. 148)

TABLE 3.3 TALK MOVES FOR SUPPORTING CLASSROOM DISCUSSIONS

Talk Moves	What It Means	Example Prompts
UNDERSTANDING IDEAS		
1. Wait time	Ironically, one "talk move" is to not talk. Quiet time should not feel uncomfortable, but should feel like thinking time. High-level thinking requires processing time. Give students time to think before responding.	"This question is important. Let's take some time to think about it." "In one minute I am going to ask you the answer to this question: . . . "
2. Partner talk	This moves gives students a chance to verbalize and refine their ideas before sharing with the whole class. It also gives students a chance to hear someone else's ideas to both inform their own thinking and to support the thinking of their partner.	"Is the answer going to be greater than or less than 1? Turn and talk to your partner for 30 seconds." "Rosa proposes that we can use the hundreds chart. Talk to your elbow partner about how the hundreds chart can help us solve this problem."
3. Revoicing	This move involves restating a students statement as a question in order to clarify what a student said, apply appropriate language to a student's idea, and involve more students in hearing a student's idea.	"So, you were saying . . . " "You used the hundreds chart and counted on?"
4. Say more	Sometimes students (especially young ones or ones new to talking about mathematics) give 2- to 3-word responses. This move involves eliciting more information to get to their thinking.	"You said you added these two numbers. How did you add them?" "Can you say more about why you used ___________ strategy?"
5. Who can repeat?	Asking students to restate someone else's ideas in their own words will ensure that ideas are stated in a variety of ways and encourage students to listen to each other.	"Who can repeat what Ricardo just said?" "Who can explain Emma's strategy but in your own words?"
DEEPENING STUDENT REASONING AND UNDERSTANDING		
6. "Why . . . "& "When . . . "	This move us designed to deepen a student's understanding of a strategy or idea. The teacher presses students to tell *why* a strategy works and *when* it works.	"Why do you think that is true?" "When will that strategy work?"
7. What do you think?	Rather than restate, as in revoicing, this move asks the student to critique the idea proposed by another student.	"What do you think about Amalia's approach?" "Do you agree or disagree with Johanna? Why?"
8. Tell me more . . .	This is a request for students to add on to someone's idea, along with inviting students to give examples and make connections. It is intended to get more participation from students and deepen student understanding.	"What might we add to Jerod's explanation?" "Can you give an example?" "Do you see a connection between Julio and Briana's strategies?"

Source: Adapted from Chapin, S., O'Conner, C., & Anderson, N. (2013). *Classroom Discussions: Using Math Talk to Help Students Learn* (3rd ed.). Sausalito, CA: Math Solutions.

Pause & Reflect

What "talk moves" do you notice in the previous vignette? See if you can identify two.

Considerable research into how mathematical communities develop and operate provides us with additional insight for developing effective classroom discourse (e.g., Kazemi & Hintz, 2014; Rasmussen, Yackel, & King, 2003; Stephan & Whitenack, 2003; Wood, Williams, & McNeal, 2006; Yackel & Cobb, 1996). Suggestions from this collection of research include the following recommendations:

- Encourage student—student dialogue rather than student—teacher conversations that exclude the rest of the class. When students have differing solutions, have students work these ideas out as a class. "George, I noticed that you got a different answer than Tomeka. What do you think about her explanation?"
- Encourage students to ask questions. "Pete, did you understand how they did that? Do you want to ask Antonio a question?"
- Ask follow-up questions whether the answer is right or wrong. Your role is to understand student thinking, not to lead students to the correct answer. So follow up with probes to learn more about their answers. Sometimes you will find that what you assumed they were thinking is not accurate. And if you only follow up on wrong answers, students quickly figure this out and get nervous when you ask them to explain their thinking.

- Call on students in such a way that, over time, all students are able to participate. Use time when students are working in small groups to identify interesting solutions that you will highlight during the sharing time. Be intentional about the order in which the solutions are shared; for example, select two that you would like to compare presented back-to-back. All students should be prepared to share their strategies.
- Demonstrate to students that it is okay to be confused and that asking clarifying questions is appropriate. This confusion, or disequilibrium, just means they are engaged in doing real mathematics and is an indication they are learning.
- Move students to more conceptually based explanations when appropriate. For example, if a student says that he knows 4.17 is more than 4.1638, you can ask him (or another student) to explain why this is so. Say, "I see *what* you did but I think some of us are confused about *why* you did it that way."
- Be sure *all* students are involved in the discussion. ELs, in particular, need more than vocabulary support; they need support with mathematical discussions (Moschkovich, 1998). For example, you can use sentence starters or examples to help students know what kind of responses you are hoping to hear and to reduce the language demands. Sentence starters can also be helpful for students with disabilities because it adds structure. You can have students practice their explanations with a peer. You can invite students to use illustrations and actual objects to support their explanations. These strategies benefit not just the ELs and other students in the class who struggle with language, but all students.

Orchestrating productive discourse is difficult, especially if you are used to teaching through a teacher-directed approach. Remember the purpose is to unveil student thinking and to help students understand each others' thinking. Warning signs that you are taking over children's thinking include interrupting a child's strategy or explanation, manipulating the tools instead of allowing the child to do so, and asking a string of closed questions (Jacobs, Martin, Ambrose, & Philipp, 2014). Taking over children's thinking sends the message that you do not believe they are capable and can inhibit the discourse you are trying to encourage.

Questioning Considerations

Questions are important in learning about student thinking, challenging conclusions, and extending the inquiry to help generalize patterns. Questioning is very complex and something that effective teachers continue to improve on throughout their careers. Here are some major considerations in questioning that influence student learning.

1. *The "level" of the question.* Questions are leveled in various models. For example, Bloom's Taxonomy (revised) includes six levels, with remembering considered to be low level, and each of the others one more cognitively demanding than the previous ones (Anderson & Krathwohl, 2001). Smith and Stein's (1998) *Levels of Cognitive Demand* include two low-level demand categories and two high-level demand categories. The key is to ask high level questions (see Table 3.2). This is critical if students are to think at high levels about mathematics.
2. *The type of understanding that is targeted.* Both procedural and conceptual knowledge are important, and questions must target both and connect the two. If questions are limited to questions, such as "How did you solve this?" or "What are the steps?" then students will be thinking about procedures, but not about related concepts. Questions focused on conceptual knowledge and making connections include, "Will this rule always work? (Why?)" "When will this strategy work?" "How does the equation you wrote connect to the picture?" and "Why use common denominators to add fractions?"
3. *The pattern of questioning.* Some patterns of questioning are not as effective as others in encourage student reasoning and problem solving (Herbel-Eisenmann & Breyfogle, 2005). One common pattern goes like this: teacher asks a question, student answers the question, teacher confirms or challenges answer. This "initiation-response-feedback" or "IRF" pattern does not lead to classroom discussions that encourage all students to think. Another pattern is "funneling," when a teacher continues to probe students in order to get them to a particular answer. This is different than a "focusing" pattern, which uses probing questions to help students understand the mathematics. The talk moves described previously are intended for a focusing pattern of questioning.