Anthony Croft and Robert Davison

Foundation Maths

SEVENTH EDITION

Pearson

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Seventh edition

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PEARSON EDUCATION LIMITED
KAO Two
KAO Park
Harlow CM17 9SR
United Kingdom
Tel: +44 (0)1279 623623
Web: **www.pearson.com/uk**

First published 1995 (print)
Third edition published 2003 (print)
Fourth edition 2006 (print)
Fifth edition 2010 (print)
Sixth edition published 2016 (print and electronic)
Seventh edition published 2020 (print and electronic)

ISBN: 978-1-292-28968-7 (print)
978-1-292-28973-1 (PDF)
978-1-292-28969-4 (ePub)

British Library Cataloguing-in-Publication Data
A catalogue record for the print edition is available from the British Library

Library of Congress Cataloging-in-Publication Data
A catalog record for the print edition is available from the Library of Congress

10 9 8 7 6 5 4 3 2 1
24 23 22 21 20

Front cover image: © Shutterstock Premier / GrAI
Print edition typeset in 10/12.5 pt Times LT pro by SPi Global
Print edition printed and bound in Slovakia by Neografia

NOTE THAT ANY PAGE CROSS REFERENCES REFER TO THE PRINT EDITION

Brief contents

Supporting resources

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Preface

Today, a huge variety of disciplines require their students to have knowledge of certain mathematical tools in order to appreciate the quantitative aspects of their subjects. At the same time, higher education institutions have widened access so that there is much greater variety in the pre-university mathematical experiences of the student body. Some students are returning to education after many years in the workplace or at home bringing up families.

Foundation Maths has been written for those students in higher education who have not specialised in mathematics at A or AS level. It is intended for non-specialists who need some but not a great deal of mathematics as they embark upon their courses of higher education. It is likely to be especially useful to those students embarking upon a Foundation Degree with mathematical content. It takes students from around the lower levels of GCSE to a standard which will enable them to participate fully in a degree or diploma course. It is ideally suited for those studying marketing, business studies, management, science, engineering, social science, geography, computer science combined studies and design. It will be useful for those who lack confidence and need careful, steady guidance in mathematical methods. Even for those whose mathematical expertise is already established, the book will be a helpful revision and reference guide. The style of the book also makes it suitable for those who wish to engage in self-study or distance learning.

We have tried throughout to adopt an informal, user-friendly approach and have described mathematical processes in everyday language. Mathematical ideas are usually developed by example rather than by formal proof. This reflects our experience that students learn better from examples than from abstract development. Where appropriate, the examples contain a great deal of detail so that the student is not left wondering how one stage of a calculation leads to the next. In *Foundation Maths*, objectives are clearly stated at the beginning of each chapter, and key points and formulae are highlighted throughout the book. Self-assessment questions are provided at the end of most sections. These test understanding of important features in the section and answers are given at the back of the book. These are followed by exercises; it is essential that these are attempted as the only way to develop competence and understanding is through practice. Solutions to these exercises are given at the back of the book and should be consulted only after the exercises have been attempted. We have included in many of the chapters a number of *challenge exercises*. These exercises are intentionally demanding and require a considerable depth of understanding. Solutions to these exercises can be found at **go.pearson .com/uk/he/resources**. A further set of test and assignment exercises is given at the end of each chapter. These are provided so that the tutor can set regular assignments or tests throughout the course. Solutions to these are not provided. Feedback from students who have used earlier editions of this book indicates that they have found the style and pace of the book helpful in their study of mathematics at university.

In order to keep the size of the book reasonable we have endeavoured to include topics which we think are most important, cause the most problems for students, and have the widest applicability. We have started the book with materials on arithmetic including whole numbers, fractions and decimals. This is followed by several chapters which gradually introduce important and commonly used topics in algebra. There follows chapters on sets, number bases and logic, collectively known as discrete mathematics. The remaining chapters introduce functions, trigonometry, vectors, matrices, complex numbers, statistics, probability and calculus. These will be found useful in the courses previously listed.

The best strategy for those using the book would be to read through each section, carefully studying all of the worked examples and solutions. Many of these solutions develop important results needed later in the book. It is then a good idea to cover up the solution and try to work the example again independently. It is only by doing the calculation that the necessary techniques will be mastered. At the end of each section the self-assessment questions should be attempted. If these cannot be answered then the previous few pages should be worked through again in order to find the answers in the text, before checking with answers given at the back of the book. Finally, the exercises should be attempted and, again, answers should be checked regularly with those given at the back of the book.

Foundation Maths is enhanced by video clips (see **go.pearson.com/uk/he/resources**) in which we, the authors, work through some algebraic examples and exercises taken from the book, pointing out techniques and key points. The icon VIDEO next to an exercise signifies that there is a corresponding video clip.

New to this 7th edition is the inclusion of many examples which illustrate how readily-available software can be used to tackle the mathematical problems you will meet in *Foundation Maths*. These examples are marked with symbol .

Although many mathematical software packages and apps are available, the ones used here for the purposes of illustration are Excel and GeoGebra. Further details of this important aspect are given on p. xxiii.

In conclusion, remember that learning mathematics takes time and effort. Carrying out a large number of exercises allows the student to experience a greater variety of problems, thus building up expertise and confidence. Armed with these the student will be able to tackle more unfamiliar and demanding problems that arise in other aspects of their course.

We hope that you find *Foundation Maths* useful and wish you the very best of luck.

Anthony Croft, Robert Davison 2020

Publisher's acknowledgements

Text credit(s):

xxiv Advisory Committee on Mathematics Education: Mathematical Needs: Mathematics in the workplace and in Higher Education, June 2011. Advisory Committee on Mathematics Education.

Photo credit(s):

xxvi 9 13 87 88 109 295 297 298 341 344 353 354 358 377 464 476 502 GeoGebra: Screenshot of GeoGebra © GeoGebra 2019 **xxv 199 206 211 264 293 294 438 441 430 448 Microsoft Corporation:** Screenshot of Microsoft Excel © Microsoft 2019.

List of videos

The following table lists the videos which accompany selected exercises and examples in the book. You can view the videos at **go.pearson.com/uk/he/resources**

Name	**Reference**
Substitution of a value into a quadratic expression	Exercise 5.3 Q13
Simplification of expressions requiring use of the first law of indices	Exercise 6.1 Q8
Simplification of expressions requiring use of the second and third laws of indices	Exercise 6.1 Q10
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Removing the brackets from expressions 2	Example 7.24
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Simplifying products and quotients of algebraic fractions	Exercise 9.3 Q4
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Another example of partial fractions	Example 9.31
Transposition of a formula	Example 10.7
Solving simultaneous equations by elimination	Example 11.6
Solving a quadratic equation by factorisation	Example 11.10
Solving a quadratic equation using a formula	Example 11.16

Mathematical symbols

$+$	plus
$-$	minus
$\pm$	plus or minus
$\times$	multiply by
$\cdot$	multiply by
$\div$	divide by
$=$	is equal to
$\equiv$	is identically equal to
$\approx$	is approximately equal to
$\neq$	is not equal to
$>$	is greater than
$\geqslant$	is greater than or equal to
$<$	is less than
$\leqslant$	is less than or equal to
$\in$	is a member of set
ε	universal set
$\cap$	intersection
$\cup$	union
$\varnothing$	empty set
$\bar{A}$	complement of set A
$\subseteq$	subset
$\mathbb{R}$	all real numbers
$\mathbb{R}^+$	all numbers greater than 0
$\mathbb{R}^-$	all numbers less than 0
$\mathbb{Z}$	all integers
$\mathbb{N}$	all positive integers
$\mathbb{C}$	all complex numbers
$\mathbb{Q}$	rational numbers

$\prod$	irrational numbers
$\therefore$	therefore
∞	infinity
e	the base of natural logarithms (2.718 …)
ln	natural logarithm
log	logarithm of base 10
$\sum$	sum of terms
$\int$	integral
$\frac{\mathrm{d}y}{\mathrm{d}x}$	derivative of y with respect to x
π	‘pi’ $\approx$ 3.14159
$\neg$	negation (not)
$\wedge$	conjunction (and)
$\vee$	disjunction (or)
$\rightarrow$	implication

Using mathematical and statistical computer software and apps in *Foundation Maths*

Foundation Maths has been written for students taking further and higher education courses who have not specialised in mathematics on post-16 qualifications and who need to use mathematics or statistics in their courses. Our intentions are to provide a thorough, carefully-paced foundation in the mathematical methods needed for success, to develop understanding and to build confidence.

So what has computer software to do with *Foundation Maths*?

Computer software and apps available for use on tablets and smartphones which can be used to perform all of the mathematical and statistical calculations in *Foundation Maths* are now readily-available, either freely or at low-cost. You will probably come across a variety of such tools in your courses. They are able to go beyond arithmetical operations found on a calculator and can perform calculations using algebra. Two important topics that you will meet when studying calculus, namely differentiation and integration, give rise to problems which can be solved using software. Differentiation can be used to find the maximum and minimum values of a function, for example maximising profit or minimising cost in business analysis. Integration can be used, for example, in the solution of equations that describe the movement of a fluid or the vibration of an aerofoil. Software is used to apply these calculus techniques to such problems. Moreover, the software can produce visual representations of solutions which can supplement the information given in an algebraic answer and thereby provide more insight. For example, they can draw accurate graphs and enable the user to focus upon points of interest. Statistical apps can tackle the analysis of data sets and produce results in a wide variety of visually informative charts.

With all this software, why do I need to learn *Foundation Maths*?

At first sight it might appear that with access to these tools there is no longer a need to learn basic mathematical methods. However, and on the contrary, to be able to exploit their full capability and power, a firm understanding of the underlying mathematics is essential. In part, this is because to use such software requires the user to distinguish and understand mathematical and statistical terminology. For example, when using computer algebra software it is essential to understand the meaning of words such as *simplify*, *factorise*, *solve*, *expand*, …. Such words have precise mathematical meanings which inform important choices to be made by the user. Visualisation software is able to access user data in several ways (e.g. from a formula, from a set of data, from an external file) and display it using a variety of different graphs (for example, *polar*, *cartesian* and *logarithmic graphs*). So, it is important to understand what these words mean. Statistical software will allow you to interrogate large sets of data and to look for patterns in that data. You may be interested in whether two or more variables are associated, that is whether there is *correlation*, and if so, how strong is that association. Knowledge of the different ways in which this strength is measured, that is through *correlation coefficients*, is important if you are to make sensible choices and correct inferences when using the software. Whilst exceptionally powerful, software is not infallible! It is important that you can look critically at the output and make a judgement as to whether it is likely to be correct or not. Even when the output is correct, this output may be in a form that you do not recognise. Acquiring the mathematical fluency to compare and contrast different forms of output is a skill that you will develop by working through *Foundation Maths*.

To quote from the ACME[1] Mathematical Needs report: *It is sometimes argued that the advent of computers has reduced the need for people to be able to do mathematics. Nothing could be further from the truth. Off-the-shelf and purpose-designed computer software packages are creating ever more data sets, statistics and graphs. Working with mathematical models, which people need to be able to understand, interpret, interrogate and use advantageously, is becoming commonplace. The use of quantitative data is now omnipresent and informs workplace practice*.

So how might I want to use software as I work through this book?

Firstly, you are able to verify the solutions you have already obtained 'by hand', and this gives you confidence that your methods are appropriate and your solutions are correct. Secondly, you can explore the effect of changing some of the values or 'parameters' in a mathematical expression. For example, what will happen to the graphs of $y = x^2 + 2x + c$ or $y = (x - c)^2$ when c is varied from negative to zero and then to positive values. Investigations like these are straightforward when you have access to software, and the results can be illuminating and aid understanding. Thirdly, using software means that you can attempt more complicated, and often more realistic problems that would be too lengthy or time-consuming to tackle by hand. Finally, you are able to produce mathematical and statistical output, for example graphs or charts, in a way that looks attractive and professional.

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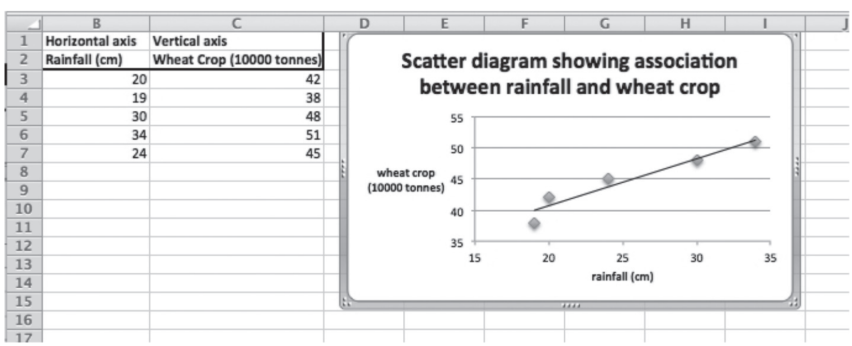
[1]ACME is the The Royal Society Advisory Committee on Mathematics Education, a distinguished body advising on mathematics education policy.

One of the purposes of this book is to help you to understand the mathematical foundations necessary to take advantage of this technology. We do not intend to teach you how to use the software – there are plenty of textbooks, user guides and on-line resources to help with this. However, we want to raise your awareness of what tools are available so that you become confident to explore these for yourself, or within your courses. Throughout the book we make reference to several pieces of software or apps outlined below. We do this solely for the purposes of illustration and are not making recommendations; there are numerous different tools available and we would encourage you to explore the field for yourself and to take advice from within your own institution.

The software and apps that we will refer to throughout *Foundation Maths*

We illustrate how *Microsoft Excel*[2] can be used for performing routine statistical calculations such as finding the mean and standard deviation of a set of data, for finding correlation coefficients and lines of best fit. Statistical charts and graphs such as the one shown in Figure i can be produced relatively easily from large sets of data.

Figure i
Using *Microsoft Excel* for producing a statistical chart



We illustrate how *GeoGebra*[3] can be used to perform calculations arising in *algebra* and *calculus*, and how it can be used to explore important mathematical objects such as *vectors* and *matrices*. Figure ii shows a screenshot depicting several algebraic operations that you will learn about in *Foundation Maths*: commands for simplification, factorisation, expansion of brackets, and solving equations. Further examples are given as you work through the book.

[2]Microsoft Excel is a spreadsheet developed by Microsoft Corporation

[3]GeoGebra is an interactive geometry, algebra, statistics and calculus application (**www.geogebra.org**)

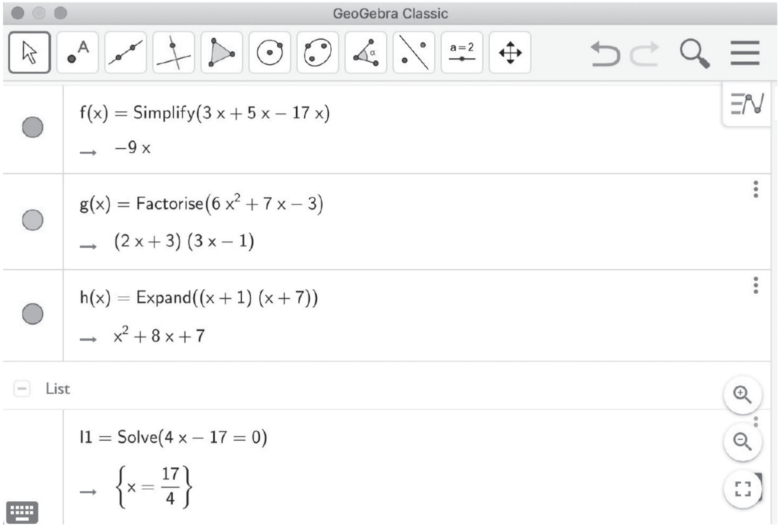


Figure ii
A selection of *GeoGebra* commands for algebraic manipulation

As we have noted earlier, there are many other commonly used software packages and apps. If you are studying, or intend to study, engineering, physics or mathematics you are likely to come across *Matlab*, *Mathematica* or *Maple*; these are extremely powerful technical computing systems that include additional toolboxes for tasks such as signal processing. If you are studying psychology or the social sciences it is likely that you will use statistical software such as *SPSS*, *minitab* or *R* in the analysis of large data sets. It will help your learning if you enquire about what packages are available for your use in the institution where you are studying and to explore how these can be put to use in the solution of exercises in *Foundation Maths*.

Examples illustrating use of software in *Foundation Maths*

The ability to use modern software in the solution of mathematical and statistical problems is an invaluable skill to develop. The first step in this development is to become aware of packages that are available, to appreciate how powerful they are, and how you can make use of them, particularly once you have acquired the necessary fundamental mathematical knowledge.

Throughout this edition of *Foundation Maths* we provide numerous illustrative examples of the wide-spread application and power of mathematical and statistical software. We would encourage you to try these and similar examples for yourself and to explore further.

Purpose	**Page**
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Arithmetic of whole numbers

1

Objectives: This chapter:

- explains the rules for adding, subtracting, multiplying and dividing positive and negative numbers
- explains what is meant by an integer
- explains what is meant by a prime number
- explains what is meant by a factor
- explains how to prime factorise an integer
- explains the terms 'highest common factor' and 'lowest common multiple'

1.1 Addition, subtraction, multiplication and division

Arithmetic is the study of numbers and their manipulation. A clear and firm understanding of the rules of arithmetic is essential for tackling everyday calculations. Arithmetic also serves as a springboard for tackling more abstract mathematics such as algebra and calculus.

The calculations in this chapter will involve mainly whole numbers, or **integers** as they are often called. The **positive integers** are the numbers

$$1, 2, 3, 4, 5 \ldots$$

and the **negative integers** are the numbers

$$\ldots -5, -4, -3, -2, -1$$

The dots (. . .) indicate that this sequence of numbers continues indefinitely. The number 0 is also an integer but is neither positive nor negative.

To find the **sum** of two or more numbers, the numbers are added together. To find the **difference** of two numbers, the second is subtracted from the first. The **product** of two numbers is found by multiplying the numbers together. Finally, the **quotient** of two numbers is found by dividing the first number by the second.

WORKED EXAMPLE

1.1

(a) Find the sum of 3, 6 and 4.

(b) Find the difference of 6 and 4.

(c) Find the product of 7 and 2.

(d) Find the quotient of 20 and 4.

Solution

(a) The sum of 3, 6 and 4 is

$$3 + 6 + 4 = 13$$

(b) The difference of 6 and 4 is

$$6 - 4 = 2$$

(c) The product of 7 and 2 is

$$7 \times 2 = 14$$

(d) The quotient of 20 and 4 is $\frac{20}{4}$, that is 5.

When writing products we sometimes replace the sign $\times$ by '$\cdot$' or even omit it completely. For example, $3 \times 6 \times 9$ could be written as $3 \cdot 6 \cdot 9$ or (3)(6)(9).

On occasions it is necessary to perform calculations involving negative numbers. To understand how these are added and subtracted consider Figure 1.1, which shows a **number line.**

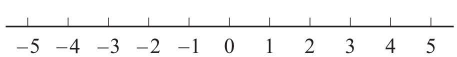

Figure 1.1
The number line

Any number can be represented by a point on the line. Positive numbers are on the right-hand side of the line and negative numbers are on the left. From any given point on the line, we can add a positive number by moving that number of places to the right. For example, to find the sum 5 + 3, start at the point 5 and move 3 places to the right, to arrive at 8. This is shown in Figure 1.2.

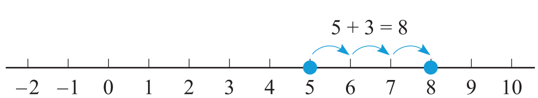

Figure 1.2
To add a positive number, move that number of places to the right

To subtract a positive number, we move that number of places to the left. For example, to find the difference $5 - 7$, start at the point 5 and move 7 places to the left to arrive at -2. Thus $5 - 7 = -2$. This is shown in Figure 1.3. The result of finding $-3 - 4$ is also shown to be -7.

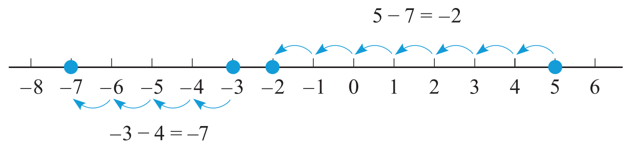

Figure 1.3
To subtract a positive number, move that number of places to the left

To add a negative number we move to the left. The result of finding $2 + (-3)$ is shown in Figure 1.4. Starting at 2, we move 3 places to the left, to arrive at -1.

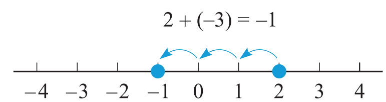

Figure 1.4
Adding a negative number involves moving to the left

We see that $2 + (-3) = -1$. Note that this is the same as the result of finding $2 - 3$, so that adding a negative number is equivalent to subtracting a positive number. For example

$$9+(-4) = 9-4 = 5,\ 3+(-7) = 3-7 = -4,\ -6+(-10) = -6-10 = -16$$

To subtract a negative number we move to the right. The result of finding $5 - (-3)$ is shown in Figure 1.5

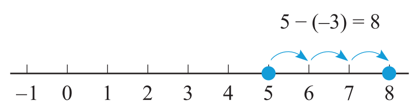

Figure 1.5
Subtracting a negative number involves moving to the right

We see that $5 - (-3) = 8$. This is the same as the result of finding $5+3$, so subtracting a negative number is equivalent to adding a positive number. For example

$$6-(-2) = 6+2 = 8,\ -5-(-3) = -5 + 3 = -2,\ -1-(-1) = -1+1 = 0$$

Key point

Adding a negative number is equivalent to subtracting a positive number.
Subtracting a negative number is equivalent to adding a positive number.

WORKED EXAMPLE

1.2 Evaluate (a) $8 + (-4)$, (b) $-15 + (-3)$, (c) $-15 - (-4)$.

Solution (a) $8 + (-4)$ is equivalent to $8 - 4$, that is 4.

(b) Because adding a negative number is equivalent to subtracting a positive number we find $-15 + (-3)$ is equivalent to $-15 - 3$, that is -18.

(c) $-15 - (-4)$ is equivalent to $-15 + 4$, that is -11.

When we need to multiply or divide negative numbers, care must be taken with the **sign** of the answer; that is, whether the result is positive or negative. The following rules apply for determining the sign of the answer when multiplying or dividing positive and negative numbers.

Key point

$$(\text{positive}) \times (\text{positive}) = \text{positive}$$
$$(\text{positive}) \times (\text{negative}) = \text{negative}$$
$$(\text{negative}) \times (\text{positive}) = \text{negative}$$
$$(\text{negative}) \times (\text{negative}) = \text{positive}$$

and

$$\frac{\text{positive}}{\text{positive}} = \text{positive}$$
$$\frac{\text{positive}}{\text{negative}} = \text{negative}$$
$$\frac{\text{negative}}{\text{positive}} = \text{negative}$$
$$\frac{\text{negative}}{\text{negative}} = \text{positive}$$

WORKED EXAMPLE

1.3 Evaluate

(a) $3 \times (-2)$ (b) $(-1) \times 7$ (c) $(-2) \times (-4)$ (d) $\dfrac{12}{(-4)}$ (e) $\dfrac{-8}{4}$ (f) $\dfrac{-6}{-2}$

Solution

(a) We have a positive number, 3, multiplied by a negative number, -2, and so the result will be negative:

$$3 \times (-2) = -6$$

(b) $(-1) \times 7 = -7$

(c) Here we have two negative numbers being multiplied and so the result will be positive:

$$(-2) \times (-4) = 8$$

(d) A positive number, 12, divided by a negative number, -4, gives a negative result:

$$\frac{12}{-4} = -3$$

(e) A negative number, -8, divided by a positive number, 4, gives a negative result:

$$\frac{-8}{4} = -2$$

(f) A negative number, -6, divided by a negative number, -2, gives a positive result:

$$\frac{-6}{-2} = 3$$

Self-assessment questions 1.1

1. Explain what is meant by an integer, a positive integer and a negative integer.
2. Explain the terms sum, difference, product and quotient.
3. State the sign of the result obtained after performing the following calculations:
   (a) $(-5) \times (-3)$ (b) $(-4) \times 2$ (c) $\frac{7}{-2}$ (d) $\frac{-8}{-4}$.

Exercise 1.1

1. Without using a calculator, evaluate each of the following:
   (a) $6 + (-3)$ (b) $6 - (-3)$
   (c) $16 + (-5)$ (d) $16 - (-5)$
   (e) $27 - (-3)$ (f) $27 - (-29)$
   (g) $-16 + 3$ (h) $-16 + (-3)$
   (i) $-16 - 3$ (j) $-16 - (-3)$
   (k) $-23 + 52$ (l) $-23 + (-52)$
   (m) $-23 - 52$ (n) $-23 - (-52)$
2. Without using a calculator, evaluate
   (a) $3 \times (-8)$ (b) $(-4) \times 8$ (c) $15 \times (-2)$
   (d) $(-2) \times (-8)$ (e) $14 \times (-3)$
3. Without using a calculator, evaluate
   (a) $\frac{15}{-3}$ (b) $\frac{21}{7}$ (c) $\frac{-21}{7}$ (d) $\frac{-21}{-7}$ (e) $\frac{21}{-7}$
   (f) $\frac{-12}{2}$ (g) $\frac{-12}{-2}$ (h) $\frac{12}{-2}$
4. Find the sum and product of (a) 3 and 6, (b) 10 and 7, (c) 2, 3 and 6.
5. Find the difference and quotient of (a) 18 and 9, (b) 20 and 5, (c) 100 and 20.

1.2 The BODMAS rule

When evaluating numerical expressions we need to know the order in which addition, subtraction, multiplication and division are carried out. As a simple example, consider evaluating $2 + 3 \times 4$. If the addition is carried out first we get $2 + 3 \times 4 = 5 \times 4 = 20$. If the multiplication is carried out first we get $2 + 3 \times 4 = 2 + 12 = 14$. Clearly the order of carrying out numerical operations is important. The BODMAS rule tells us the order in which we must carry out the operations of addition, subtraction, multiplication and division.

Key point

BODMAS stands for

Brackets ( )	First priority
Of $\times$	Second priority
Division $\div$	Second priority
Multiplication $\times$	Second priority
Addition $+$	Third priority
Subtraction $-$	Third priority

This is the order of carrying out arithmetical operations, with bracketed expressions having highest priority and subtraction and addition having the lowest priority. Note that 'Of', 'Division' and 'Multiplication' have equal priority, as do 'Addition' and 'Subtraction'. 'Of' is used to show multiplication when dealing with fractions: for example, find $\frac{1}{2}$ of 6 means $\frac{1}{2} \times 6$.

If an expression contains only multiplication and division, we evaluate by working from left to right. Similarly, if an expression contains only addition and subtraction, we also evaluate by working from left to right.

WORKED EXAMPLES

1.4 Evaluate

(a) $2 + 3 \times 4$ (b) $(2 + 3) \times 4$

Solution

(a) Using the BODMAS rule we see that multiplication is carried out first. So

$$2 + 3 \times 4 = 2 + 12 = 14$$

(b) Using the BODMAS rule we see that the bracketed expression takes priority over all else. Hence

$$(2 + 3) \times 4 = 5 \times 4 = 20$$

1.5 Evaluate

(a) $4 - 2 \div 2$ (b) $1 - 3 + 2 \times 2$

Solution (a) Division is carried out before subtraction, and so

$$4 - 2 \div 2 = 4 - \frac{2}{2} = 3$$

(b) Multiplication is carried out before subtraction or addition:

$$1 - 3 + 2 \times 2 = 1 - 3 + 4 = 2$$

1.6 Evaluate

(a) $(12 \div 4) \times 3$ (b) $12 \div (4 \times 3)$

Solution Recall that bracketed expressions are evaluated first.

(a) $(12 \div 4) \times 3 = \left(\frac{12}{4}\right) \times 3 = 3 \times 3 = 9$

(b) $12 \div (4 \times 3) = 12 \div 12 = 1$

Example 1.6 shows the importance of the position of brackets in an expression.

Self-assessment questions 1.2

1. State the BODMAS rule used to evaluate expressions.
2. The position of brackets in an expression is unimportant. True or false?

Exercise 1.2

1. Evaluate the following expressions:
   (a) $6 - 2 \times 2$ (b) $(6 - 2) \times 2$
   (c) $6 \div 2 - 2$ (d) $(6 \div 2) - 2$
   (e) $6 - 2 + 3 \times 2$ (f) $6 - (2 + 3) \times 2$
   (g) $(6 - 2) + 3 \times 2$ (h) $\frac{16}{-2}$ (i) $\frac{-24}{-3}$
   (j) $(-6) \times (-2)$ (k) $(-2)(-3)(-4)$

2. Place brackets in the following expressions to make them correct:
   (a) $6 \times 12 - 3 + 1 = 55$
   (b) $6 \times 12 - 3 + 1 = 68$
   (c) $6 \times 12 - 3 + 1 = 60$
   (d) $5 \times 4 - 3 + 2 = 7$
   (e) $5 \times 4 - 3 + 2 = 15$
   (f) $5 \times 4 - 3 + 2 = -5$

1.3 Prime numbers and factorisation

A **prime number** is a positive integer, larger than 1, which cannot be expressed as the product of two smaller positive integers. To put it another way, a prime number is one that can be divided exactly only by 1 and itself.

For example, $6 = 2 \times 3$, so 6 can be expressed as a product of smaller numbers and hence 6 is not a prime number. However, 7 is prime. Examples of prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23. Note that 2 is the only even prime.

Factorise means 'write as a product'. By writing 12 as 3×4 we have factorised 12. We say 3 is a **factor** of 12 and 4 is also a factor of 12. The way in which a number is factorised is not unique: for example, 12 may be expressed as 3×4 or 2×6. Note that 2 and 6 are also factors of 12.

When a number is written as a product of prime numbers we say the number has been **prime factorised.**

To prime factorise a number, consider the technique used in the following examples.

WORKED EXAMPLES

1.7 Prime factorise the following numbers:

(a) 12 (b) 42 (c) 40 (d) 70

Solution

(a) We begin with 2 and see whether this is a factor of 12. Clearly it is, so we write

$$12 = 2 \times 6$$

Now we consider 6. Again 2 is a factor so we write

$$12 = 2 \times 2 \times 3$$

All the factors are now prime, that is the prime factorisation of 12 is $2 \times 2 \times 3$.

(b) We begin with 2 and see whether this is a factor of 42. Clearly it is and so we can write

$$42 = 2 \times 21$$

Now we consider 21. Now 2 is not a factor of 21, so we examine the next prime, 3. Clearly 3 is a factor of 21 and so we can write

$$42 = 2 \times 3 \times 7$$

All the factors are now prime, and so the prime factorisation of 42 is $2 \times 3 \times 7$.

(c) Clearly 2 is a factor of 40,

$$40 = 2 \times 20$$

Clearly 2 is a factor of 20,

$$40 = 2 \times 2 \times 10$$

Again 2 is a factor of 10,

$$40 = 2 \times 2 \times 2 \times 5$$

All the factors are now prime. The prime factorisation of 40 is $2 \times 2 \times 2 \times 5$.

(d) Clearly 2 is a factor of 70,

$$70 = 2 \times 35$$

We consider 35: 2 is not a factor, 3 is not a factor, but 5 is:

$$70 = 2 \times 5 \times 7$$

All the factors are prime. The prime factorisation of 70 is $2 \times 5 \times 7$.

1.8 Prime factorise 2299.

Solution We note that 2 is not a factor and so we try 3. Again 3 is not a factor and so we try 5. This process continues until we find the first prime factor. It is 11:

$$2299 = 11 \times 209$$

We now consider 209. The first prime factor is 11:

$$2299 = 11 \times 11 \times 19$$

All the factors are prime. The prime factorisation of 2299 is $11 \times 11 \times 19$.

Here we illustrate how software can be used to perform the prime factorisation of 2299 using GeoGebra Classic v6 with the CAS (Computer Algebra System) interface. Numerous other packages are available that perform the same calculation. Figure 1.6 illustrates the output following the command **`Factor( )`** which performs the prime factorisation. Compare this output to the solution of Worked Example 1.8, particularly the use of the power 2 in 11^2. Powers, or indices as they are also called, are explained in Chapter 5.

Figure 1.6
Syntax used to perform prime number calculations.

In addition, Figure 1.6 shows the commands **`IsPrime( )`** and **`NextPrime( )`** which will test whether a given number is prime and find the first prime number greater than a given number. You should consult the on-line help provided with your software to explore other prime number commands.

Self-assessment questions 1.3

1. Explain what is meant by a prime number.
2. List the first 10 prime numbers.
3. Explain why all even numbers other than 2 cannot be prime.

Exercise 1.3

1. State which of the following numbers are prime numbers:
   (a) 13 (b) 1000 (c) 2 (d) 29 (e) $\frac{1}{2}$
2. Prime factorise the following numbers:
   (a) 26 (b) 100 (c) 27 (d) 71 (e) 64 (f) 87 (g) 437 (h) 899
3. Prime factorise the two numbers 30 and 42. List any prime factors which are common to both numbers.

1.4 Highest common factor and lowest common multiple

Highest common factor

Suppose we prime factorise 12. This gives $12 = 2 \times 2 \times 3$. From this prime factorisation we can deduce all the factors of 12:

2 is a factor of 12
3 is a factor of 12
$2 \times 2 = 4$ is a factor of 12
$2 \times 3 = 6$ is a factor of 12

Hence 12 has factors 2, 3, 4 and 6, in addition to the obvious factors of 1 and 12.

Similarly we could prime factorise 18 to obtain $18 = 2 \times 3 \times 3$. From this we can list the factors of 18:

2 is a factor of 18
3 is a factor of 18
$2 \times 3 = 6$ is a factor of 18
$3 \times 3 = 9$ is a factor of 18

The factors of 18 are 1, 2, 3, 6, 9 and 18. Some factors are common to both 12 and 18. These are 2, 3 and 6. These are **common factors** of 12 and 18. The highest common factor of 12 and 18 is 6.

The highest common factor of 12 and 18 can be obtained directly from their prime factorisation. We simply note all the primes common to both factorisations:

$$12 = 2 \times 2 \times 3 \qquad 18 = 2 \times 3 \times 3$$

Common to both is 2×3. Thus the highest common factor is $2 \times 3 = 6$. Thus 6 is the highest number that divides exactly into both 12 and 18.

Key point

Given two or more numbers the **highest common factor** (h.c.f.) is the largest (highest) number that is a factor of all the given numbers.
The highest common factor is also referred to as the **greatest common divisor** (g.c.d).

WORKED EXAMPLES

1.9 Find the h.c.f. of 12 and 27.

Solution We prime factorise 12 and 27:

$$12 = 2 \times 2 \times 3 \qquad 27 = 3 \times 3 \times 3$$

Common to both is 3. Thus 3 is the h.c.f. of 12 and 27. This means that 3 is the highest number that divides both 12 and 27.

1.10 Find the h.c.f. of 28 and 210.

Solution The numbers are prime factorised:

$$28 = 2 \times 2 \times 7$$
$$210 = 2 \times 3 \times 5 \times 7$$

The factors that are common are identified: a 2 is common to both and a 7 is common to both. Hence both numbers are divisible by $2 \times 7 = 14$. Since this number contains all the common factors it is the highest common factor.

1.11 Find the h.c.f. of 90 and 108.

Solution The numbers are prime factorised:

$$90 = 2 \times 3 \times 3 \times 5$$
$$108 = 2 \times 2 \times 3 \times 3 \times 3$$

The common factors are 2, 3 and 3 and so the h.c.f. is $2 \times 3 \times 3$, that is 18. This is the highest number that divides both 90 and 108.

1.12 Find the h.c.f. of 12, 18 and 20.

Solution Prime factorisation yields

$$12 = 2 \times 2 \times 3 \qquad 18 = 2 \times 3 \times 3 \qquad 20 = 2 \times 2 \times 5$$

There is only one factor common to all three numbers: it is 2. Hence 2 is the h.c.f. of 12, 18 and 20.

Lowest common multiple

Suppose we are given two or more numbers and wish to find numbers into which all the given numbers will divide. For example, given 4 and 6 we see that they both divide exactly into 12, 24, 36, 48, 60 and so on. The smallest number into which they both divide is 12. We say 12 is the **lowest common multiple** of 4 and 6.

Key point

The lowest common multiple (l.c.m.) of a set of numbers is the smallest (lowest) number into which all the given numbers will divide exactly.

WORKED EXAMPLE

1.13 Find the l.c.m. of 6 and 10.

Solution We seek the smallest number into which both 6 and 10 will divide exactly. There are many numbers into which 6 and 10 will divide, for example 60, 120, 600, but we are seeking the smallest such number. By inspection, the smallest such number is 30. Thus the l.c.m. of 6 and 10 is 30.

A more systematic method of finding the l.c.m. involves the use of prime factorisation.

WORKED EXAMPLES

1.14 Find the l.c.m. of 15 and 20.

Solution As a first step, the numbers are prime factorised:

$$15 = 3 \times 5 \qquad 20 = 2 \times 2 \times 5$$

Since 15 must divide into the l.c.m., then the l.c.m. must contain the factors of 15, that is 3×5. Similarly, as 20 must divide into the l.c.m., then the l.c.m. must also contain the factors of 20, that is $2 \times 2 \times 5$. The l.c.m. is the smallest number that contains both of these sets of factors. Note that the l.c.m. will contain only 2s, 3s and 5s as its prime factors. We now need to determine how many of these particular factors are needed.

To determine the l.c.m. we ask ‘How many factors of 2 are required?’, ‘How many factors of 3 are required?’, ‘How many factors of 5 are required?’

The highest number of 2s occurs in the factorisation of 20. Hence the l.c.m. requires two factors of 2. Consider the number of 3s required. The highest number of 3s occurs in the factorisation of 15. Hence the l.c.m. requires one factor of 3. Consider the number of 5s required. The highest number of 5s is 1 and so the l.c.m. requires one factor of 5. Hence the l.c.m. is $2 \times 2 \times 3 \times 5 = 60$.

Hence 60 is the smallest number into which both 15 and 20 will divide exactly.

1.15 Find the l.c.m. of 20, 24 and 25.

Solution The numbers are prime factorised:

$$20 = 2 \times 2 \times 5 \qquad 24 = 2 \times 2 \times 2 \times 3 \qquad 25 = 5 \times 5$$

By considering the prime factorisations of 20, 24 and 25 we see that the only primes involved are 2, 3 and 5. Hence the l.c.m. will contain only 2s, 3s and 5s.

Consider the number of 2s required. The highest number of 2s required is three from factorising 24. The highest number of 3s required is one, again from factorising 24. The highest number of 5s required is two, found from factorising 25. Hence the l.c.m. is given by

$$\text{l.c.m.} = 2 \times 2 \times 2 \times 3 \times 5 \times 5 = 600$$

Hence 600 is the smallest number into which 20, 24 and 25 will all divide exactly.

GeoGebra can be used to find the highest common factor, `HCF( )`, and lowest common multiple, `LCM( )`, of a set or list of numbers, entered using braces, for example as `a={12,18,20}`. Use of this software for verification of Worked Examples 1.12 and 1.15 is shown in Figure 1.7.

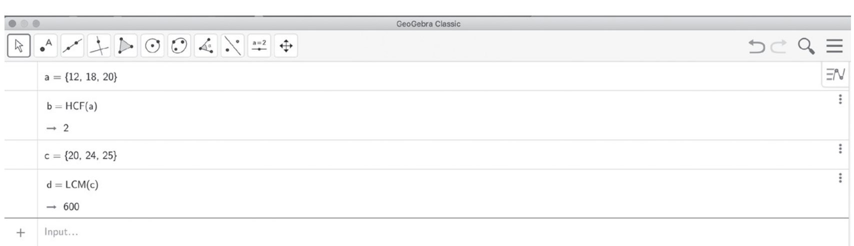

Figure 1.7
Using software to find the h.c.f and l.c.m.

Self-assessment questions 1.4

1. Explain what is meant by the h.c.f. of a set of numbers.
2. Explain what is meant by the l.c.m. of a set of numbers.

Exercise 1.4

1. Calculate the h.c.f. of the following sets of numbers:
   (a) 12, 15, 21 (b) 16, 24, 40 (c) 28, 70, 120, 160 (d) 35, 38, 42 (e) 96, 120, 144
2. Calculate the l.c.m. of the following sets of numbers:
   (a) 5, 6, 8 (b) 20, 30 (c) 7, 9, 12 (d) 100, 150, 235 (e) 96, 120, 144

Test and assignment exercises 1

1. Evaluate
   (a) $6 \div 2 + 1$ (b) $6 \div (2 + 1)$ (c) $12 + 4 \div 4$ (d) $(12 + 4) \div 4$
   (e) $3 \times 2 + 1$ (f) $3 \times (2 + 1)$ (g) $6 - 2 + 4 \div 2$ (h) $(6 - 2 + 4) \div 2$
   (i) $6 - (2 + 4 \div 2)$ (j) $6 - (2 + 4) \div 2$ (k) $2 \times 4 - 1$ (l) $2 \times (4 - 1)$
   (m) $2 \times 6 \div (3 - 1)$ (n) $2 \times (6 \div 3) - 1$ (o) $2 \times (6 \div 3 - 1)$
2. Prime factorise (a) 56, (b) 39, (c) 74.
3. Find the h.c.f. of
   (a) 8, 12, 14 (b) 18, 42, 66 (c) 20, 24, 30 (d) 16, 24, 32, 160
4. Find the l.c.m. of
   (a) 10, 15 (b) 11, 13 (c) 8, 14, 16 (d) 15, 24, 30

Fractions

2

Objectives: This chapter:

- explains what is meant by a fraction
- defines the terms 'improper fraction', 'proper fraction' and 'mixed fraction'
- explains how to write fractions in different but equivalent forms
- explains how to simplify fractions by cancelling common factors
- explains how to add, subtract, multiply and divide fractions

2.1 Introduction

The arithmetic of fractions is very important groundwork which must be mastered before topics in algebra such as formulae and equations can be understood. The same techniques that are used to manipulate fractions are used in these more advanced topics. You should use this chapter to ensure that you are confident at handling fractions before moving on to algebra. In all the examples and exercises it is important that you should carry out the calculations without the use of a calculator.

Fractions are numbers such as $\frac{1}{2}, \frac{3}{4}, \frac{11}{8}$ and so on. In general a fraction is a number of the form $\frac{p}{q}$, where the letters p and q represent whole numbers or integers. The integer q can never be zero because it is not possible to divide by zero.

In any fraction $\frac{p}{q}$ the number p is called the **numerator** and the number q is called the **denominator.**

Key point

$$\text{fraction} = \frac{\text{numerator}}{\text{denominator}} = \frac{p}{q}$$

Suppose that p and q are both positive numbers. If p is less than q, the fraction is said to be a **proper fraction.** So $\frac{1}{2}$ and $\frac{3}{4}$ are proper fractions since the numerator is less than the denominator. If p is greater than or equal to q, the fraction is said to be **improper.** So $\frac{11}{8}$, $\frac{7}{4}$ and $\frac{3}{3}$ are all improper fractions.

If either of p or q is negative, we simply ignore the negative sign when determining whether the fraction is proper or improper. So $-\frac{3}{5}$, $\frac{-7}{21}$ and $\frac{4}{-21}$ are proper fractions, but $\frac{3}{-3}$, $\frac{-8}{2}$ and $-\frac{11}{2}$ are improper.

Note that all proper fractions have a value less than 1.

The denominator of a fraction can take the value 1, as in $\frac{3}{1}$ and $\frac{7}{1}$. In these cases the result is a whole number, 3 and 7.

A fraction is **inverted** by interchanging its numerator and denominator. When $\frac{3}{2}$ is inverted this results in $\frac{2}{3}$. If 4 is inverted this results in $\frac{1}{4}$ since $4 = \frac{4}{1}$.

The **reciprocal** of a number is found by inverting it, so, for example, the reciprocal of $\frac{4}{5}$ is $\frac{5}{4}$.

Self-assessment questions 2.1

1. Explain the terms (a) fraction, (b) improper fraction, (c) proper fraction. In each case give an example of your own.
2. Explain the terms (a) numerator, (b) denominator.

Exercise 2.1

1. Classify each of the following as proper or improper:
   (a) $\frac{9}{17}$ (b) $\frac{-9}{17}$ (c) $\frac{8}{8}$ (d) $-\frac{7}{8}$ (e) $\frac{110}{77}$

2.2 Expressing a fraction in equivalent forms

Given a fraction, we may be able to express it in a different form. For example, you will know that $\frac{1}{2}$ is equivalent to $\frac{2}{4}$. Note that multiplying both numerator and denominator by the same number leaves the value of the fraction unchanged. So, for example,

$$\frac{1}{2} = \frac{1 \times 2}{2 \times 2} = \frac{2}{4}$$

We say that $\frac{1}{2}$ and $\frac{2}{4}$ are **equivalent fractions.** Although they might look different, they have the same value.

Similarly, given the fraction $\frac{8}{12}$ we can divide both numerator and denominator by 4 to obtain

$$\frac{8}{12} = \frac{8/4}{12/4} = \frac{2}{3}$$

so $\frac{8}{12}$ and $\frac{2}{3}$ have the same value and are equivalent fractions.

Key point

Multiplying or dividing both numerator and denominator of a fraction by the same number produces a fraction having the same value, called an equivalent fraction.

A fraction is in its **simplest form** when there are no factors common to both numerator and denominator. For example, $\frac{5}{12}$ is in its simplest form, but $\frac{3}{6}$ is not since 3 is a factor common to both numerator and denominator. Its simplest form is the equivalent fraction $\frac{1}{2}$.

To express a fraction in its simplest form we look for factors that are common to both the numerator and denominator. This is done by prime factorising both of these. Dividing both the numerator and denominator by any common factors removes them but leaves an equivalent fraction. This is equivalent to cancelling any common factors. For example, to simplify $\frac{4}{6}$ we prime factorise to produce

$$\frac{4}{6} = \frac{2 \times 2}{2 \times 3}$$

Dividing both numerator and denominator by 2 leaves $\frac{2}{3}$. This is equivalent to cancelling the common factor of 2.

WORKED EXAMPLES

2.1 Express $\frac{24}{36}$ in its simplest form.

Solution We seek factors common to both numerator and denominator. To do this we prime factorise 24 and 36:

Prime factorisation has been described in §1.3.

$$24 = 2 \times 2 \times 2 \times 3 \qquad 36 = 2 \times 2 \times 3 \times 3$$

The factors $2 \times 2 \times 3$ are common to both 24 and 36 and so these may be cancelled. Note that only common factors may be cancelled when simplifying a fraction. Hence

Finding the highest common factor (h.c.f.) of two numbers is detailed in §1.4.

$$\frac{24}{36} = \frac{\cancel{2} \times \cancel{2} \times 2 \times \cancel{3}}{\cancel{2} \times \cancel{2} \times \cancel{3} \times 3} = \frac{2}{3}$$

In its simplest form $\frac{24}{36}$ is $\frac{2}{3}$. In effect we have divided 24 and 36 by 12, which is their h.c.f.

2.2 Express $\frac{49}{21}$ in its simplest form.

Solution Prime factorising 49 and 21 gives

$$49 = 7 \times 7 \qquad 21 = 3 \times 7$$

Their h.c.f. is 7. Dividing 49 and 21 by 7 gives

$$\frac{49}{21} = \frac{7}{3}$$

Hence the simplest form of $\frac{49}{21}$ is $\frac{7}{3}$.

Before we can start to add and subtract fractions it is necessary to be able to convert fractions into a variety of equivalent forms. Work through the following examples.

WORKED EXAMPLES

2.3 Express $\frac{3}{4}$ as an equivalent fraction having a denominator of 20.

Solution To achieve a denominator of 20, the existing denominator must be multiplied by 5. To produce an equivalent fraction both numerator and denominator must be multiplied by 5, so

$$\frac{3}{4} = \frac{3 \times 5}{4 \times 5} = \frac{15}{20}$$

2.4 Express 7 as an equivalent fraction with a denominator of 3.

Solution Note that 7 is the same as the fraction $\frac{7}{1}$. To achieve a denominator of 3, the existing denominator must be multiplied by 3. To produce an equivalent fraction both numerator and denominator must be multiplied by 3, so

$$7 = \frac{7}{1} = \frac{7 \times 3}{1 \times 3} = \frac{21}{3}$$

Self-assessment questions 2.2

1. All integers can be thought of as fractions. True or false?
2. Explain the use of h.c.f. in the simplification of fractions.
3. Give an example of three fractions that are equivalent.

Exercise 2.2

1. Express the following fractions in their simplest form:
(a) $\frac{18}{27}$ (b) $\frac{12}{20}$ (c) $\frac{15}{45}$ (d) $\frac{25}{80}$ (e) $\frac{15}{60}$
(f) $\frac{90}{200}$ (g) $\frac{15}{20}$ (h) $\frac{2}{18}$ (i) $\frac{16}{24}$ (j) $\frac{30}{65}$
(k) $\frac{12}{21}$ (l) $\frac{100}{45}$ (m) $\frac{6}{9}$ (n) $\frac{12}{16}$ (o) $\frac{13}{42}$
(p) $\frac{13}{39}$ (q) $\frac{11}{33}$ (r) $\frac{14}{30}$ (s) $-\frac{12}{16}$ (t) $\frac{11}{-33}$
(u) $\frac{-14}{-30}$
2. Express $\frac{3}{4}$ as an equivalent fraction having a denominator of 28.
3. Express 4 as an equivalent fraction with a denominator of 5.
4. Express $\frac{5}{12}$ as an equivalent fraction having a denominator of 36.
5. Express 2 as an equivalent fraction with a denominator of 4.
6. Express 6 as an equivalent fraction with a denominator of 3.
7. Express each of the fractions $\frac{2}{3}$, $\frac{5}{4}$ and $\frac{5}{6}$ as an equivalent fraction with a denominator of 12.
8. Express each of the fractions $\frac{4}{9}$, $\frac{1}{2}$ and $\frac{5}{6}$ as an equivalent fraction with a denominator of 18.
9. Express each of the following numbers as an equivalent fraction with a denominator of 12:
(a) $\frac{1}{2}$ (b) $\frac{3}{4}$ (c) $\frac{5}{2}$ (d) 5 (e) 4 (f) 12

2.3 Addition and subtraction of fractions

To add and subtract fractions we first rewrite each fraction so that they all have the same denominator. This is known as the **common denominator.** The denominator is chosen to be the lowest common multiple of the original denominators. Then the numerators only are added or subtracted as appropriate, and the result is divided by the common denominator.

WORKED EXAMPLES

2.5 Find $\frac{2}{3} + \frac{5}{4}$.

Solution The denominators are 3 and 4. The l.c.m. of 3 and 4 is 12. We need to express both fractions with a denominator of 12.

Finding the lowest common multiple (l.c.m.) is detailed in §1.4.

To express $\frac{2}{3}$ with a denominator of 12 we multiply both numerator and denominator by 4. Hence $\frac{2}{3}$ is the same as $\frac{8}{12}$. To express $\frac{5}{4}$ with a denominator of 12 we multiply both numerator and denominator by 3. Hence $\frac{5}{4}$ is the same as $\frac{15}{12}$. So

$$\frac{2}{3} + \frac{5}{4} = \frac{8}{12} + \frac{15}{12} = \frac{8 + 15}{12} = \frac{23}{12}$$

2.6 Find $\frac{4}{9} - \frac{1}{2} + \frac{5}{6}$.

Solution The denominators are 9, 2 and 6. Their l.c.m. is 18. Each fraction is expressed with 18 as the denominator:

$$\frac{4}{9} = \frac{8}{18} \quad \frac{1}{2} = \frac{9}{18} \quad \frac{5}{6} = \frac{15}{18}$$

Then

$$\frac{4}{9} - \frac{1}{2} + \frac{5}{6} = \frac{8}{18} - \frac{9}{18} + \frac{15}{18} = \frac{8 - 9 + 15}{18} = \frac{14}{18}$$

The fraction $\frac{14}{18}$ can be simplified to $\frac{7}{9}$. Hence

$$\frac{4}{9} - \frac{1}{2} + \frac{5}{6} = \frac{7}{9}$$

2.7 Find $\frac{1}{4} - \frac{5}{9}$.

Solution The l.c.m. of 4 and 9 is 36. Each fraction is expressed with a denominator of 36. Thus

$$\frac{1}{4} = \frac{9}{36} \quad \text{and} \quad \frac{5}{9} = \frac{20}{36}$$

Then

$$\frac{1}{4} - \frac{5}{9} = \frac{9}{36} - \frac{20}{36}$$

$$= \frac{9 - 20}{36}$$

$$= \frac{-11}{36}$$

$$= -\frac{11}{36}$$

Consider the number $2\frac{3}{4}$. This is referred to as a **mixed fraction** because it contains a whole number part, 2, and a fractional part, $\frac{3}{4}$. We can convert this mixed fraction into an improper fraction as follows. Recognise that 2 is equivalent to $\frac{8}{4}$, and so $2\frac{3}{4}$ is $\frac{8}{4} + \frac{3}{4} = \frac{11}{4}$.

The reverse of this process is to convert an improper fraction into a mixed fraction. Consider the improper fraction $\frac{11}{4}$. We seek the largest multiple of the denominator that is less than the numerator. The numerator is then split into

$$\text{Numerator} = \text{Multiple of denominator} + \text{Remainder}$$

In this example, the numerator is 11, the denominator is 4. So the largest multiple of 4 that is less than 11 is $2 \times 4 = 8$. So we write

$$11 = 8 + \text{Remainder} = 8 + 3$$

Then

$$\frac{11}{4} = \frac{8 + 3}{4} = \frac{8}{4} + \frac{3}{4} = 2 + \frac{3}{4} = 2\,\frac{3}{4}$$

WORKED EXAMPLE

2.8

(a) Express $4\frac{2}{5}$ as an improper fraction.

(b) Find $4\frac{2}{5} + \frac{1}{3}$.

(c) Convert to mixed fractions i. $\frac{38}{5}$ ii. $\frac{118}{7}$

Solution

(a) $4\frac{2}{5}$ is a mixed fraction. Note that $4\frac{2}{5}$ is equal to $4 + \frac{2}{5}$. We can write 4 as the equivalent fraction $\frac{20}{5}$. Therefore

$$4\,\frac{2}{5} = \frac{20}{5} + \frac{2}{5}$$

$$= \frac{22}{5}$$

(b)
$$4\frac{2}{5}+\frac{1}{3}=\frac{22}{5}+\frac{1}{3}$$
$$=\frac{66}{15}+\frac{5}{15}$$
$$=\frac{71}{15}$$

(c) i. The largest multiple of 5 that is less than 38 is $7 \times 5 = 35$. So we write $38 = 35 + 3$ and then

$$\frac{38}{5}=\frac{35+3}{5}=\frac{35}{5}+\frac{3}{5}=7+\frac{3}{5}=7\frac{3}{5}$$

ii. The largest multiple of 7 that is less than 118 is $7 \times 16 = 112$. Hence we write $118 = 112 + 6$. Then

$$\frac{118}{7}=\frac{112}{7}+\frac{6}{7}=16+\frac{6}{7}=16\frac{6}{7}$$

Self-assessment question 2.3

1. Explain the use of l.c.m. when adding and subtracting fractions.

Exercise 2.3

1. Find

(a) $\frac{1}{4}+\frac{2}{3}$ (b) $\frac{3}{5}+\frac{5}{3}$ (c) $\frac{12}{14}-\frac{2}{7}$

(d) $\frac{3}{7}-\frac{1}{2}+\frac{2}{21}$ (e) $1\frac{1}{2}+\frac{4}{9}$

(f) $2\frac{1}{4}-1\frac{1}{3}+\frac{1}{2}$ (g) $\frac{10}{15}-1\frac{2}{5}+\frac{8}{3}$

(h) $\frac{9}{10}-\frac{7}{16}+\frac{1}{2}-\frac{2}{5}$

2. Find

(a) $\frac{7}{8}+\frac{1}{3}$ (b) $\frac{1}{2}-\frac{3}{4}$ (c) $\frac{3}{5}+\frac{2}{3}+\frac{1}{2}$

(d) $\frac{3}{8}+\frac{1}{3}+\frac{1}{4}$ (e) $\frac{2}{3}-\frac{4}{7}$ (f) $\frac{1}{11}-\frac{1}{2}$

(g) $\frac{3}{11}-\frac{5}{8}$

3. Express as improper fractions:

(a) $2\frac{1}{2}$ (b) $3\frac{2}{3}$ (c) $10\frac{1}{4}$ (d) $5\frac{2}{7}$

(e) $6\frac{2}{9}$ (f) $11\frac{1}{3}$ (g) $15\frac{1}{2}$ (h) $13\frac{3}{4}$

(i) $12\frac{1}{11}$ (j) $13\frac{2}{3}$ (k) $56\frac{1}{2}$

4. Without using a calculator express these improper fractions as mixed fractions:

(a) $\frac{10}{3}$ (b) $\frac{7}{2}$ (c) $\frac{15}{4}$ (d) $\frac{25}{6}$

2.4 Multiplication of fractions

The product of two or more fractions is found by multiplying their numerators to form a new numerator, and then multiplying their denominators to form a new denominator.

WORKED EXAMPLES

2.9 Find $\frac{4}{9} \times \frac{3}{8}$.

Solution The numerators are multiplied: $4 \times 3 = 12$. The denominators are multiplied: $9 \times 8 = 72$. Hence

$$\frac{4}{9} \times \frac{3}{8} = \frac{12}{72}$$

This may now be expressed in its simplest form:

$$\frac{12}{72} = \frac{1}{6}$$

Hence

$$\frac{4}{9} \times \frac{3}{8} = \frac{1}{6}$$

An alternative, but equivalent, method is to cancel any factors common to both numerator and denominator at the outset:

$$\frac{4}{9} \times \frac{3}{8} = \frac{4 \times 3}{9 \times 8}$$

A factor of 4 is common to the 4 and the 8. Hence

$$\frac{4 \times 3}{9 \times 8} = \frac{1 \times 3}{9 \times 2}$$

A factor of 3 is common to the 3 and the 9. Hence

$$\frac{1 \times 3}{9 \times 2} = \frac{1 \times 1}{3 \times 2} = \frac{1}{6}$$

2.10 Find $\frac{12}{25} \times \frac{2}{7} \times \frac{10}{9}$.

Solution We cancel factors common to both numerator and denominator. A factor of 5 is common to 10 and 25. Cancelling this gives

$$\frac{12}{25} \times \frac{2}{7} \times \frac{10}{9} = \frac{12}{5} \times \frac{2}{7} \times \frac{2}{9}$$

A factor of 3 is common to 12 and 9. Cancelling this gives

$$\frac{12}{5} \times \frac{2}{7} \times \frac{2}{9} = \frac{4}{5} \times \frac{2}{7} \times \frac{2}{3}$$

There are no more common factors. Hence

$$\frac{12}{25} \times \frac{2}{7} \times \frac{10}{9} = \frac{4}{5} \times \frac{2}{7} \times \frac{2}{3} = \frac{16}{105}$$

2.11 Find $\frac{3}{4}$ of $\frac{5}{9}$.

Recall that 'of' means multiply.

Solution $\frac{3}{4}$ of $\frac{5}{9}$ is the same as $\frac{3}{4} \times \frac{5}{9}$. Cancelling a factor of 3 from numerator and denominator gives $\frac{1}{4} \times \frac{5}{3}$, that is $\frac{5}{12}$. Hence $\frac{3}{4}$ of $\frac{5}{9}$ is $\frac{5}{12}$.

2.12 Find $\frac{5}{6}$ of 70.

Solution We can write 70 as $\frac{70}{1}$. So

$$\frac{5}{6} \text{ of } 70 = \frac{5}{6} \times \frac{70}{1} = \frac{5}{3} \times \frac{35}{1} = \frac{175}{3} = 58\,\frac{1}{3}$$

2.13 Find $2\frac{7}{8} \times \frac{2}{3}$.

Solution In this example the first fraction is a mixed fraction. We convert it to an improper fraction before performing the multiplication. Note that $2\frac{7}{8} = \frac{23}{8}$. Then

$$\begin{aligned} \frac{23}{8} \times \frac{2}{3} &= \frac{23}{4} \times \frac{1}{3} \\ &= \frac{23}{12} \\ &= 1\,\frac{11}{12} \end{aligned}$$

Self-assessment question 2.4

1. Describe how to multiply fractions together.

Exercise 2.4

1. Evaluate

(a) $\frac{2}{3} \times \frac{6}{7}$ (b) $\frac{8}{15} \times \frac{25}{32}$ (c) $\frac{1}{4} \times \frac{8}{9}$

(d) $\frac{16}{17} \times \frac{34}{48}$ (e) $2 \times \frac{3}{5} \times \frac{5}{12}$

(f) $2\,\frac{1}{3} \times 1\,\frac{1}{4}$ (g) $1\,\frac{3}{4} \times 2\,\frac{1}{2}$

(h) $\frac{3}{4} \times 1\,\frac{1}{2} \times 3\,\frac{1}{2}$

2. Evaluate

(a) $\frac{2}{3}$ of $\frac{3}{4}$ (b) $\frac{4}{7}$ of $\frac{21}{30}$

(c) $\frac{9}{10}$ of 80 (d) $\frac{6}{7}$ of 42

3. Is $\frac{3}{4}$ of $\frac{12}{15}$ the same as $\frac{12}{15}$ of $\frac{3}{4}$?

4. Find

(a) $-\frac{1}{3} \times \frac{5}{7}$ (b) $\frac{3}{4} \times -\frac{1}{2}$

(c) $\left(-\frac{5}{8}\right) \times \frac{8}{11}$ (d) $\left(-\frac{2}{3}\right) \times \left(-\frac{15}{7}\right)$

5. Find

(a) $5\frac{1}{2} \times \frac{1}{2}$ (b) $3\frac{3}{4} \times \frac{1}{3}$ (c) $\frac{2}{3} \times 5\frac{1}{9}$

(d) $\frac{3}{4} \times 11\frac{1}{2}$

6. Find

(a) $\frac{3}{5}$ of $11\frac{1}{4}$ (b) $\frac{2}{3}$ of $15\frac{1}{2}$

(c) $\frac{1}{4}$ of $-8\frac{1}{3}$

2.5 Division by a fraction

To divide one fraction by another fraction, we invert the second fraction and then multiply. When we invert a fraction we interchange the numerator and denominator.

WORKED EXAMPLES

2.14 Find $\frac{6}{25} \div \frac{2}{5}$.

Solution We invert $\frac{2}{5}$ to obtain $\frac{5}{2}$. Multiplication is then performed. So

$$\frac{6}{25} \div \frac{2}{5} = \frac{6}{25} \times \frac{5}{2} = \frac{3}{25} \times \frac{5}{1} = \frac{3}{5} \times \frac{1}{1} = \frac{3}{5}$$

2.15 Evaluate (a) $1\frac{1}{3} \div \frac{8}{3}$, (b) $\frac{20}{21} \div \frac{5}{7}$.

Solution (a) First we express $1\frac{1}{3}$ as an improper fraction:

$$1\frac{1}{3} = 1 + \frac{1}{3} = \frac{3}{3} + \frac{1}{3} = \frac{4}{3}$$

So we calculate

$$\frac{4}{3} \div \frac{8}{3} = \frac{4}{3} \times \frac{3}{8} = \frac{4}{8} = \frac{1}{2}$$

Hence

$$1\frac{1}{3} \div \frac{8}{3} = \frac{1}{2}$$

(b) $$\frac{20}{21} \div \frac{5}{7} = \frac{20}{21} \times \frac{7}{5} = \frac{4}{21} \times \frac{7}{1} = \frac{4}{3}$$

Self-assessment question 2.5

1. Explain the process of division by a fraction.

Exercise 2.5

1. Evaluate

(a) $\frac{3}{4} \div \frac{1}{8}$ (b) $\frac{8}{9} \div \frac{4}{3}$ (c) $\frac{-2}{7} \div \frac{4}{21}$ (d) $\frac{9}{4} \div 1\frac{1}{2}$

(e) $\frac{5}{6} \div \frac{5}{12}$ (f) $\frac{99}{100} \div 1\frac{4}{5}$ (g) $3\frac{1}{4} \div 1\frac{1}{8}$ (h) $\left(2\frac{1}{4} \div \frac{3}{4}\right) \times 2$

(i) $2\frac{1}{4} \div \left(\frac{3}{4} \times 2\right)$ (j) $6\frac{1}{4} \div 2\frac{1}{2} + 5$ (k) $6\frac{1}{4} \div \left(2\frac{1}{2} + 5\right)$

Test and assignment exercises 2

1. Evaluate

(a) $\frac{3}{4} + \frac{1}{6}$ (b) $\frac{2}{3} + \frac{3}{5} - \frac{1}{6}$ (c) $\frac{5}{7} - \frac{2}{3}$ (d) $2\frac{1}{3} - \frac{9}{10}$

(e) $5\frac{1}{4} + 3\frac{1}{6}$ (f) $\frac{9}{8} - \frac{7}{6} + 1$ (g) $\frac{5}{6} - \frac{5}{3} + \frac{5}{4}$ (h) $\frac{4}{5} + \frac{1}{3} - \frac{3}{4}$

2. Evaluate

(a) $\frac{4}{7} \times \frac{21}{32}$ (b) $\frac{5}{6} \times \frac{8}{15}$ (c) $\frac{3}{11} \times \frac{20}{21}$ (d) $\frac{9}{14} \times \frac{8}{18}$

(e) $\frac{5}{4} \div \frac{10}{13}$ (f) $\frac{7}{16} \div \frac{21}{32}$ (g) $\frac{-24}{25} \div \frac{51}{50}$ (h) $\frac{45}{81} \div \frac{25}{27}$

3. Evaluate the following expressions using the BODMAS rule:

(a) $\frac{1}{2} + \frac{1}{3} \times 2$ (b) $\frac{3}{4} \times \frac{2}{3} + \frac{1}{4}$ (c) $\frac{5}{6} \div \frac{2}{3} + \frac{3}{4}$ (d) $\left(\frac{2}{3} + \frac{1}{4}\right) \div 4 + \frac{3}{5}$

(e) $\left(\frac{4}{3} - \frac{2}{5} \times \frac{1}{3}\right) \times \frac{1}{4} + \frac{1}{2}$ (f) $\frac{3}{4}$ of $\left(1 + \frac{2}{3}\right)$ (g) $\frac{2}{3}$ of $\frac{1}{2} + 1$ (h) $\frac{1}{5} \times \frac{2}{3} + \frac{2}{5} \div \frac{4}{5}$

4. Express in their simplest form:

(a) $\frac{21}{84}$ (b) $\frac{6}{80}$ (c) $\frac{34}{85}$ (d) $\frac{22}{143}$ (e) $\frac{69}{253}$

3 Decimal numbers

Objectives: This chapter:

- revises the decimal number system
- shows how to write a number to a given number of significant figures
- shows how to write a number to a given number of decimal places

3.1 Decimal numbers

Consider the whole number 478. We can regard it as the sum

$$400 + 70 + 8$$

In this way we see that, in the number 478, the 8 represents eight ones, or 8 units, the 7 represents seven tens, or 70, and the number 4 represents four hundreds or 400. Thus we have the system of hundreds, tens and units familiar from early years in school. All whole numbers can be thought of in this way.

When we wish to deal with proper fractions and mixed fractions, we extend the hundreds, tens and units system as follows. A **decimal point,** '.', marks the end of the whole number part, and the numbers that follow it, to the right, form the fractional part.

A number immediately to the right of the decimal point, that is in the **first decimal place,** represents tenths, so

$$0.1 = \frac{1}{10}$$

$$0.2 = \frac{2}{10} \qquad \text{or } \frac{1}{5}$$

$$0.3 = \frac{3}{10} \qquad \text{and so on}$$

Note that when there are no whole numbers involved it is usual to write a zero in front of the decimal point, thus, .2 would be written 0.2.

WORKED EXAMPLE

3.1 Express the following decimal numbers as proper fractions in their simplest form

(a) 0.4 (b) 0.5 (c) 0.6

Solution The first number after the decimal point represents tenths.

(a) $0.4 = \frac{4}{10}$, which simplifies to $\frac{2}{5}$

(b) $0.5 = \frac{5}{10}$ or simply $\frac{1}{2}$

(c) $0.6 = \frac{6}{10} = \frac{3}{5}$

Frequently we will deal with numbers having a whole number part and a fractional part. Thus

$$
\begin{aligned}
5.2 &= 5 \text{ units} + 2 \text{ tenths} \\
&= 5 + \frac{2}{10} \\
&= 5 + \frac{1}{5} \\
&= 5\,\frac{1}{5}
\end{aligned}
$$

Similarly,

$$175.8 = 175\,\frac{8}{10} = 175\,\frac{4}{5}$$

Numbers in the second position after the decimal point, or the **second decimal place,** represent hundredths, so

$$0.01 = \frac{1}{100}$$

$$0.02 = \frac{2}{100} \quad \text{or} \quad \frac{1}{50}$$

$$0.03 = \frac{3}{100} \quad \text{and so on}$$

Consider 0.25. We can think of this as

$$
\begin{aligned}
0.25 &= 0.2 + 0.05 \\
&= \frac{2}{10} + \frac{5}{100} \\
&= \frac{25}{100}
\end{aligned}
$$

We see that 0.25 is equivalent to $\frac{25}{100}$, which in its simplest form is $\frac{1}{4}$.

In fact we can regard any numbers occupying the first two decimal places as hundredths, so that

$$0.25 = \frac{25}{100} \quad \text{or simply } \frac{1}{4}$$

$$0.50 = \frac{50}{100} \quad \text{or } \frac{1}{2}$$

$$0.75 = \frac{75}{100} = \frac{3}{4}$$

WORKED EXAMPLES

3.2 Express the following decimal numbers as proper fractions in their simplest form:

(a) 0.35 (b) 0.56 (c) 0.68

Solution The first two decimal places represent hundredths:

(a) $0.35 = \frac{35}{100} = \frac{7}{20}$

(b) $0.56 = \frac{56}{100} = \frac{14}{25}$

(c) $0.68 = \frac{68}{100} = \frac{17}{25}$

3.3 Express 37.25 as a mixed fraction in its simplest form.

Solution

$$\begin{aligned} 37.25 &= 37 + 0.25 \\ &= 37 + \frac{25}{100} \\ &= 37 + \frac{1}{4} \\ &= 37\frac{1}{4} \end{aligned}$$

Numbers in the third position after the decimal point, or **third decimal place,** represent thousandths, so

$$0.001 = \frac{1}{1000}$$

$$0.002 = \frac{2}{1000} \quad \text{or } \frac{1}{500}$$

$$0.003 = \frac{3}{1000} \quad \text{and so on}$$

In fact we can regard any numbers occupying the first three positions after the decimal point as thousandths, so that

$$0.356 = \frac{356}{1000} \quad \text{or} \quad \frac{89}{250}$$

$$0.015 = \frac{15}{1000} \quad \text{or} \quad \frac{3}{200}$$

$$0.075 = \frac{75}{1000} = \frac{3}{40}$$

WORKED EXAMPLE

3.4 Write each of the following as a decimal number:

(a) $\frac{3}{10} + \frac{7}{100}$ (b) $\frac{8}{10} + \frac{3}{1000}$

Solution

(a) $\frac{3}{10} + \frac{7}{100} = 0.3 + 0.07 = 0.37$
(b) $\frac{8}{10} + \frac{3}{1000} = 0.8 + 0.003 = 0.803$

You will normally use a calculator to add, subtract, multiply and divide decimal numbers. Generally the more decimal places used, the more precisely we can state a number. This idea is developed in the next section.

Self-assessment questions 3.1

1. State which is the largest and which is the smallest of the following numbers:
   23.001, 23.0, 23.00001, 23.0008, 23.01

2. Which is the largest of the following numbers?
   0.1, 0.02, 0.003, 0.0004, 0.00005

Exercise 3.1

1. Express the following decimal numbers as proper fractions in their simplest form:
   (a) 0.7 (b) 0.8 (c) 0.9

2. Express the following decimal numbers as proper fractions in their simplest form:
   (a) 0.55 (b) 0.158 (c) 0.98 (d) 0.099

3. Express each of the following as a mixed fraction in its simplest form:
   (a) 4.6 (b) 5.2 (c) 8.05 (d) 11.59
   (e) 121.09

4. Write each of the following as a decimal number:
   (a) $\frac{6}{10} + \frac{9}{100} + \frac{7}{1000}$ (b) $\frac{8}{100} + \frac{3}{1000}$
   (c) $\frac{17}{1000} + \frac{5}{10}$

3.2 Significant figures and decimal places

The precision to which we state a number often depends upon the context in which the number is being used. The volume of a petrol tank is usually given to the nearest litre. It is of no practical use to give such a volume to the nearest cubic centimetre.

When writing a number we often give the precision by stating the **number of significant figures** or the **number of decimal places** used. These terms are now explained.

Significant figures

Suppose we are asked to write down the number nearest to 857 using at most two non-zero digits, or numbers. We would write 860. This number is nearer to 857 than any other number with two non-zero digits. We say that 857 to 2 **significant figures** is 860. The words 'significant figures' are usually abbreviated to s.f. Because 860 is larger than 857 we say that the 857 has been **rounded up** to 860.

To write a number to three significant figures we can use no more than three non-zero digits. For example, the number closest to 1784 which has no more than three non-zero digits is 1780. We say that 1784 to 3 significant figures is 1780. In this case, because 1780 is less than 1784 we say that 1784 has been **rounded down** to 1780.

WORKED EXAMPLES

3.5 Write down the number nearest to 86 using only one non-zero digit. Has 86 been rounded up or down?

Solution The number 86 written to one significant figure is 90. This number is nearer to 86 than any other number having only one non-zero digit. Thus 86 has been rounded up to 90.

3.6 Write down the number nearest to 999 which uses only one non-zero digit.

Solution The number 999 to one significant figure is 1000. This number is nearer to 999 than any other number having only one non-zero digit.

We now explain the process of writing to a given number of significant figures.

When asked to write a number to, say, three significant figures, 3 s.f., the first step is to look at the first four digits. If asked to write a number to two significant figures we look at the first three digits and so on. We always look at one more digit than the number of significant figures required.

For example, to write 6543.19 to 2 s.f. we would consider the number 6540.00; the digits 3, 1 and 9 are effectively ignored. The next step is to round up or down. If the final digit is a 5 or more then we round up by increasing the previous digit by 1. If the final digit is 4 or less we round down by leaving the previous digit unchanged. Hence when considering 6543.19 to 2 s.f., the 4 in the third place means that we round down to 6500.

To write 23865 to 3 s.f. we would consider the number 23860. The next step is to increase the 8 to a 9. Thus 23865 is rounded up to 23900.

Zeros at the beginning of a number are ignored. To write 0.004693 to 2 s.f. we would first consider the number 0.00469. Note that the zeros at the beginning of the number have not been counted. We then round the 6 to a 7, producing 0.0047.

The following examples illustrate the process.

WORKED EXAMPLES

3.7 Write 36.482 to 3 s.f.

Solution We consider the first four digits, that is 36.48. The final digit is 8 and so we round up 36.48 to 36.5. To 3 s.f. 36.482 is 36.5.

3.8 Write 1.0049 to 4 s.f.

Solution To write to 4 s.f. we consider the first five digits, that is 1.0049. The final digit is a 9 and so 1.0049 is rounded up to 1.005.

3.9 Write 695.3 to 2 s.f.

Solution We consider 695. The final digit is a 5 and so we round up. We cannot round up the 9 to a 10 and so the 69 is rounded up to 70. Hence to 2 s.f. the number is 700.

3.10 Write 0.0473 to 1 s.f.

Solution We do not count the initial zeros and consider 0.047. The final digit tells us to round up. Hence to 1 s.f. we have 0.05.

3.11 A number is given to 2 s.f. as 67.

(a) What is the maximum value the number could have?

(b) What is the minimum value the number could have?

Solution

(a) To 2 s.f. 67.5 is 68. Any number just below 67.5, for example 67.49 or 67.499, to 2 s.f. is 67. Hence the maximum value of the number is 67.4999. . . .

(b) To 2 s.f. 66.4999. . . is 66. However, 66.5 to 2 s.f. is 67. The minimum value of the number is thus 66.5.

Decimal places

When asked to write a number to 3 decimal places (3 d.p.) we consider the first 4 decimal places, that is numbers after the decimal point. If asked to write to 2 d.p. we consider the first 3 decimal places and so on. If the final digit is 5 or more we round up, otherwise we round down.

WORKED EXAMPLES

3.12 Write 63.4261 to 2 d.p.

Solution We consider the number to 3 d.p., that is 63.426. The final digit is 6 and so we round up 63.426 to 63.43. Hence 63.4261 to 2 d.p. is 63.43.

3.13 Write 1.97 to 1 d.p.

Solution In order to write to 1 d.p. we consider the number to 2 d.p., that is we consider 1.97. The final digit is a 7 and so we round up. The 9 cannot be rounded up and so we look at 1.9. This can be rounded up to 2.0. Hence 1.97 to 1 d.p. is 2.0. Note that it is crucial to write 2.0 and not simply 2, as this shows that the number is written to 1 d.p.

3.14 Write −6.0439 to 2 d.p.

Solution We consider −6.043. As the final digit is a 3 the number is rounded down to −6.04.

Self-assessment questions 3.2

1. Explain the meaning of ‘significant figures’.
2. Explain the process of writing a number to so many decimal places.

Exercise 3.2

1. Write to 3 s.f.
   (a) 6962 (b) 70.406 (c) 0.0123
   (d) 0.010991 (e) 45.607 (f) 2345
2. Write 65.999 to
   (a) 4 s.f. (b) 3 s.f. (c) 2 s.f.
   (d) 1 s.f. (e) 2 d.p. (f) 1 d.p.
3. Write 9.99 to
   (a) 1 s.f. (b) 1 d.p.
4. Write 65.4555 to
   (a) 3 d.p. (b) 2 d.p. (c) 1 d.p.
   (d) 5 s.f. (e) 4 s.f. (f) 3 s.f. (g) 2 s.f.
   (h) 1 s.f.

Test and assignment exercises 3

1. Express the following numbers as proper fractions in their simplest form:
   (a) 0.74 (b) 0.96 (c) 0.05 (d) 0.25

2. Express each of the following as a mixed fraction in its simplest form:
   (a) 2.5 (b) 3.25 (c) 3.125 (d) 6.875

3. Write each of the following as a decimal number:
   (a) $\frac{3}{10} + \frac{1}{100} + \frac{7}{1000}$ (b) $\frac{5}{1000} + \frac{9}{100}$ (c) $\frac{4}{1000} + \frac{9}{10}$

4. Write 0.09846 to (a) 1 d.p, (b) 2 s.f., (c) 1 s.f.

5. Write 9.513 to (a) 3 s.f., (b) 2 s.f., (c) 1 s.f.

6. Write 19.96 to (a) 1 d.p., (b) 2 s.f., (c) 1 s.f.

4 Percentage and ratio

Objectives: This chapter:

- explains the terms 'percentage' and 'ratio'
- shows how to perform calculations using percentages and ratios
- explains how to calculate the percentage change in a quantity

4.1 Percentage

In everyday life we come across percentages regularly. During sales periods shops offer discounts – for example, we might hear expressions like 'everything reduced by 50%'. Students often receive examination marks in the form of percentages – for example, to achieve a pass grade in a university examination, a student may be required to score at least 40%. Banks and building societies charge interest on loans, and the interest rate quoted is usually given as a percentage, for example 4.75%. Percentages also provide a way of comparing two or more quantities. For example, suppose we want to know which is the better mark: 40 out of 70, or 125 out of 200? By expressing these marks as percentages we will be able to answer this question.

Consequently an understanding of what a percentage is, and an ability to perform calculations involving percentages, are not only useful in mathematical applications, but also essential life skills.

Fundamentally, a **percentage** is a fraction whose denominator is 100. In fact you can think of the phrase 'per cent' meaning 'out of 100'. We use the symbol % to represent a percentage, as earlier. The following three fractions all have a denominator of 100, and are expressed as percentages as shown:

$$\frac{17}{100} \quad \text{may be expressed as} \quad 17\%$$

$$\frac{50}{100} \text{ may be expressed as } 50\%$$

$$\frac{3}{100} \text{ may be expressed as } 3\%$$

WORKED EXAMPLE

4.1 Express $\frac{19}{100}$, $\frac{35}{100}$ and $\frac{17.5}{100}$ as percentages.

Solution All of these fractions have a denominator of 100. So it is straightforward to write down their percentage form:

$$\frac{19}{100} = 19\% \quad \frac{35}{100} = 35\% \quad \frac{17.5}{100} = 17.5\%$$

Sometimes it is necessary to convert a fraction whose denominator is not 100, for example $\frac{2}{5}$, into a percentage. This could be done by expressing the fraction as an equivalent fraction with denominator 100, as was explained in Section 2.2 on page 16. However, with calculators readily available, the calculation can be done as follows.

We can use the calculator to divide the numerator of the fraction by the denominator. The answer is then multiplied by 100. The resulting number is the required percentage. So, to convert $\frac{2}{5}$ we perform the following key strokes:

$$2 \div 5 \times 100 = 40$$

and so $\frac{2}{5} = 40\%$. You should check this now using your own calculator,

Key point

To convert a fraction to a percentage, divide the numerator by the denominator, multiply by 100 and then label the result as a percentage.

WORKED EXAMPLES

4.2 Convert $\frac{5}{8}$ into a percentage.

Solution Using the method described above we find

$$5 \div 8 \times 100 = 62.5$$

Labelling the answer as a percentage, we see that $\frac{5}{8}$ is equivalent to 62.5%.

4.3 Bill scores $\frac{13}{17}$ in a test. In a different test, Mary scores $\frac{14}{19}$. Express the scores as percentages, and thereby make a comparison of the two marks.

Solution Use your calculator to perform the division and then multiply the result by 100.

Bill's score: $13 \div 17 \times 100 = 76.5$ (1 d.p.)

Mary's score: $14 \div 19 \times 100 = 73.7$ (1 d.p.)

So we see that Bill scores 76.5% and Mary scores 73.7%. Notice that in these percentage forms it is easy to compare the two marks. We see that Bill has achieved the higher score. Making easy comparisons like this is one of the reasons why percentages are used so frequently.

We have seen that percentages are fractions with a denominator of 100, so that, for example, $\frac{19}{100} = 19\%$. Sometimes a fraction may be given not as a numerator divided by a denominator, but in its decimal form. For example, the decimal form of $\frac{19}{100}$ is 0.19. To convert a decimal fraction into a percentage we simply multiply by 100. So

$$0.19 = 0.19 \times 100\% = 19\%$$

Key point

To convert a decimal fraction to a percentage, multiply by 100 and then label the result as a percentage.

We may also want to reverse the process. Frequently in business calculations involving formulae for interest it is necessary to express a percentage in its decimal form. To convert a percentage to its equivalent decimal form we divide the percentage by 100.

WORKED EXAMPLE

4.4 Express 50% as a decimal.

Solution We divide the percentage by 100:

$$50 \div 100 = 0.5$$

So 50% is equivalent to 0.5. To see why this is the case, remember that 'per cent' literally means 'out of 100' so 50% means 50 out of 100, or $\frac{50}{100}$, or in its simplest form 0.5.

Key point

To convert a percentage to its equivalent decimal fraction form, divide by 100.