GLOBAL EDITION

THOMAS' CALCULUS

Early Transcendentals

Fourteenth Edition in SI Units

Hass • Heil • Weir

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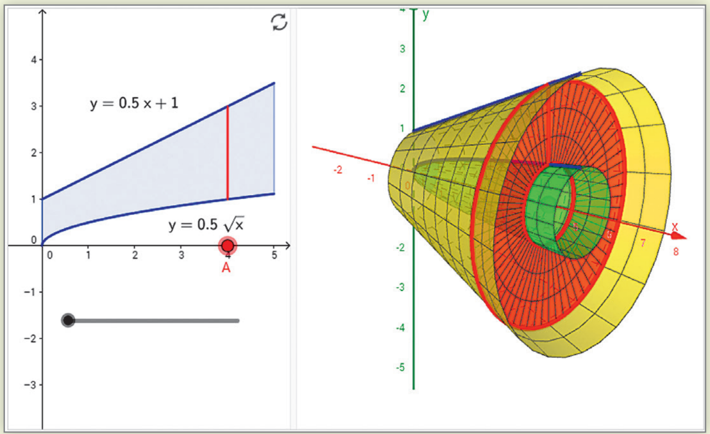

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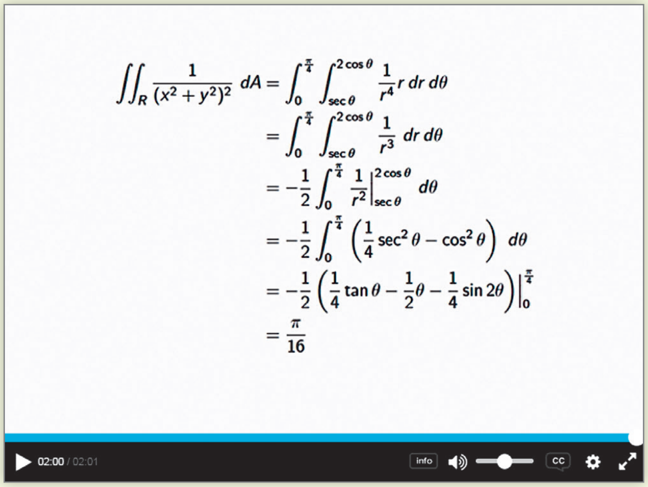

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THOMAS' CALCULUS
Early Transcendentals

FOURTEENTH EDITION IN SI UNITS

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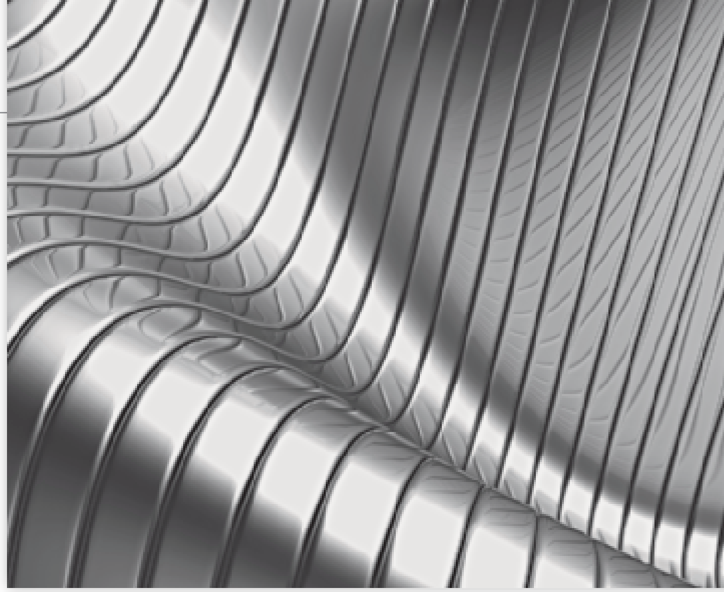


Preface

Thomas' Calculus: Early Transcendentals, Fourteenth Edition in SI Units, provides a modern introduction to calculus that focuses on developing conceptual understanding of the underlying mathematical ideas. This text supports a calculus sequence typically taken by students in STEM fields over several semesters. Intuitive and precise explanations, thoughtfully chosen examples, superior figures, and time-tested exercise sets are the foundation of this text. We continue to improve this text in keeping with shifts in both the preparation and the goals of today's students, and in the applications of calculus to a changing world.

Many of today's students have been exposed to calculus in high school. For some, this translates into a successful experience with calculus in college. For others, however, the result is an overconfidence in their computational abilities coupled with underlying gaps in algebra and trigonometry mastery, as well as poor conceptual understanding. In this text, we seek to meet the needs of the increasingly varied population in the calculus sequence. We have taken care to provide enough review material (in the text and appendices), detailed solutions, and a variety of examples and exercises, to support a complete understanding of calculus for students at varying levels. Within the text, we present the material in a way that supports the development of mathematical maturity, going beyond memorizing formulas and routine procedures, and we show students how to generalize key concepts once they are introduced. References are made throughout, tying new concepts to related ones that were studied earlier. After studying calculus from *Thomas*, students will have developed problem-solving and reasoning abilities that will serve them well in many important aspects of their lives. Mastering this beautiful and creative subject, with its many practical applications across so many fields, is its own reward. But the real gifts of studying calculus are acquiring the ability to think logically and precisely; understanding what is defined, what is assumed, and what is deduced; and learning how to generalize conceptually. We intend this book to encourage and support those goals.

New to This Edition

We welcome to this edition a new coauthor, Christopher Heil from the Georgia Institute of Technology. He has been involved in teaching calculus, linear algebra, analysis, and abstract algebra at Georgia Tech since 1993. He is an experienced author and served as a consultant on the previous edition of this text. His research is in harmonic analysis, including time-frequency analysis, wavelets, and operator theory.

This is a substantial revision. Every word, symbol, and figure was revisited to ensure clarity, consistency, and conciseness. Additionally, we made the following text-wide updates:

- Updated graphics to bring out clear visualization and mathematical correctness.
- Added examples (in response to user feedback) to overcome conceptual obstacles. See Example 3 in Section 16.1.
- Added new types of homework exercises throughout, including many with a geometric nature. The new exercises are not just more of the same, but rather give different perspectives on and approaches to each topic. We also analyzed aggregated student usage and performance data from MyLab Math for the previous edition of this text. The results of this analysis helped improve the quality and quantity of the exercises.
- Added short URLs to historical links that allow students to navigate directly to online information.
- Added new marginal notes throughout to guide the reader through the process of problem solution and to emphasize that each step in a mathematical argument is rigorously justified.

New to MyLab Math

Many improvements have been made to the overall functionality of MyLab Math since the previous edition. Beyond that, we have also increased and improved the content specific to this text.

- Instructors now have more exercises than ever to choose from in assigning homework.
- The MyLab Math exercise-scoring engine has been updated to allow for more robust coverage of certain topics, including differential equations.
- A full suite of Interactive Figures have been added to support teaching and learning. The figures are designed to be used in lecture, as well as by students independently. The figures are editable using the freely available GeoGebra software. The figures were created by Marc Renault (Shippensburg University), Kevin Hopkins (Southwest Baptist University), Steve Phelps (University of Cincinnati), and Tim Brzezinski (Berlin High School, CT).
- Enhanced Sample Assignments include just-in-time prerequisite review, help keep skills fresh with distributed practice of key concepts (based on research by Jeff Hieb, Keith Lyle, and Pat Ralston of University of Louisville), and provide opportunities to work exercises without learning aids (to help students develop confidence in their ability to solve problems independently).
- Additional Conceptual Questions augment text exercises to focus on deeper, theoretical understanding of the key concepts in calculus. These questions were written by faculty at Cornell University under an NSF grant. They are also assignable through Learning Catalytics.
- Setup & Solve exercises now appear in many sections. These exercises require students to show how they set up a problem as well as the solution, better mirroring what is required of students on tests.
- New instructional videos by Greg Wisloski and Dan Radelet (both of Indiana University of PA) augment the already robust collection within the course. These videos support the overall approach of the text—specifically, they go beyond routine procedures to show students how to generalize and connect key concepts.

Content Enhancements

Chapter 1

- Clarified explanation of definition of exponential function in 1.4.
- Replaced $\sin^{-1}$ notation for the inverse sine function with arcsin as default notation in 1.5, and similarly for other trig functions.
- Added new Exercises: **1.1:** 59–62, **1.2:** 21–22; **1.3:** 64–65, **1.5:** 61–64, 79cd; **PE:** 29–32.

Chapter 2

- Added definition of average speed in 2.1.
- Updated definition of limits to allow for arbitrary domains. The definition of limits is now consistent with the definition in multivariable domains later in the text and with more general mathematical usage.
- Reworded limit and continuity definitions to remove implication symbols and improve comprehension.
- Added new Example 7 in 2.4 to illustrate limits of ratios of trig functions.
- Rewrote 2.6 Example 11 to solve the equation by finding a zero, consistent with previous discussion.
- Added new Exercises: **2.1:** 15–18; **2.2:** 3h–k, 4f–I; **2.4:** 19–20, 45–46; **2.5:** 69–74; **2.6:** 31–32; **PE:** 57–58; **AAE:** 35–38.

Chapter 3

- Clarified relation of slope and rate of change.
- Added new Figure 3.9 using the square root function to illustrate vertical tangent lines.
- Added figure of $x \sin(1/x)$ in 3.2 to illustrate how oscillation can lead to non-existence of a derivative of a continuous function.
- Revised product rule to make order of factors consistent throughout text, including later dot product and cross product formulas.
- Added new Exercises: **3.2:** 36, 43–44; **3.3:** 65–66; **3.5:** 43–44, 61bc; **3.6:** 79–80, 111–113; **3.7:** 27–28; **3.8:** 97–100; **3.9:** 43–46; **3.10:** 47; **AAE:** 14–15, 26–27.

Chapter 4

- Added summary to 4.1.
- Added new Example 12 with new Figure 4.35 to give basic and advanced examples of concavity.
- Added new Exercises: **4.1:** 53–56, 67–70; **4.3:** 45–46, 67–68; **4.4:** 107–112; **4.6:** 37–42; **4.7:** 7–10; **4.8:** 115–118; **PE:** 1–16, 101–102; **AAE:** 19–20, 38–39. Moved Exercises 4.1: 53–68 to PE.

Chapter 5

- Improved discussion in 5.4 and added new Figure 5.18 to illustrate the Mean Value Theorem.
- Added new Exercises: **5.2:** 33–36; **5.4:** 71–72; **5.6:** 47–48; **PE:** 43–44, 75–76.

Chapter 6

- Clarified cylindrical shell method.
- Added introductory discussion of mass distribution along a line, with figure, in 6.6.
- Added new Exercises: **6.1:** 15; **6.2:** 49–50; **6.3:** 13–14; **6.5:** 1–2; **6.6:** 1–6, 21–22; **PE:** 17–18, 23–24, 37–38.

Chapter 7

- Clarified discussion of separable differential equations in 7.2.
- Added new Exercises: **7.1:** 61–62, 73; **PE:** 41–42.

Chapter 8

- Updated 8.2 Integration by Parts discussion to emphasize $u(x)v'(x)\,dx$ form rather than $u\,dv$. Rewrote Examples 1–3 accordingly.
- Removed discussion of tabular integration and associated exercises.
- Updated discussion in 8.5 on how to find constants in Partial Fraction method.
- Updated notation in 8.8 to align with standard usage in statistics.
- Added new Exercises: **8.1:** 41–44; **8.2:** 53–56, 72–73; **8.3:** 75–76; **8.4:** 49–52; **8.5:** 51–66, 73–74; **8.8:** 35–38, 77–78; **PE:** 69–88.

Chapter 9

- Clarified the different meaning of a sequence and a series.
- Added new Figure 9.9 to illustrate sum of a series as area of a histogram.
- Added to 9.3 a discussion on the importance of bounding errors in approximations.
- Added new Figure 9.13 illustrating how to use integrals to bound remainder terms of partial sums.
- Rewrote Theorem 10 in 9.4 to bring out similarity to the integral comparison test.
- Added new Figure 9.16 to illustrate the differing behaviors of the harmonic and alternating harmonic series.
- Renamed the nth term test the "nth term test for divergence" to emphasize that it says nothing about convergence.
- Added new Figure 9.19 to illustrate polynomials converging to $\ln(1 + x)$, which illustrates convergence on the half-open interval $(-1, 1]$.
- Used red dots and intervals to indicate intervals and points where divergence occurs and blue to indicate convergence throughout Chapter 9.
- Added new Figure 9.21 to show the six different possibilities for an interval of convergence.
- Added new Exercises: **9.1:** 27–30, 72–77; **9.2:** 19–22, 73–76, 105; **9.3:** 11–12, 39–42; **9.4:** 55–56; **9.5:** 45–46, 65–66; **9.6:** 57–82; **9.7:** 61–65; **9.8:** 23–24, 39–40; **9.9:** 11–12, 37–38; **PE:** 41–44, 97–102.

Chapter 10

- Added new Example 1 and Figure 10.2 in 10.1 to give a straightforward first example of a parametrized curve.
- Updated area formulas for polar coordinates to include conditions for positive r and non-overlapping θ.
- Added new Example 3 and Figure 10.37 in 10.4 to illustrate intersections of polar curves.
- Added new Exercises: **10.1:** 19–28; **10.2:** 49–50; **10.4:** 21–24.

Chapter 11

- Added new Figure 11.13(b) to show the effect of scaling a vector.
- Added new Example 7 and Figure 11.26 in 11.3 to illustrate projection of a vector.
- Added discussion on general quadric surfaces in 11.6, with new Example 4 and new Figure 11.48 illustrating the description of an ellipsoid not centered at the origin via completing the square.
- Added new Exercises: **11.1:** 31–34, 59–60, 73–76; **11.2:** 43–44; **11.3:** 17–18; **11.4:** 51–57; **11.5:** 49–52.

Chapter 12

- Added sidebars on how to pronounce Greek letters such as kappa, tau, etc.
- Added new Exercises: **12.1:** 1–4, 27–36; **12.2:** 15–16, 19–20; **12.4:** 27–28; **12.6:** 1–2.

Chapter 13

- Elaborated on discussion of open and closed regions in 13.1.
- Standardized notation for evaluating partial derivatives, gradients, and directional derivatives at a point, throughout the chapter.
- Renamed "branch diagrams" as "dependency diagrams" which clarifies that they capture dependence of variables.
- Added new Exercises: **13.2:** 51–54; **13.3:** 51–54, 59–60, 71–74, 103–104; **13.4:** 20–30, 43–46, 57–58; **13.5:** 41–44; **13.6:** 9–10, 61; **13.7:** 61–62.

Chapter 14

- Added new Figure 14.21b to illustrate setting up limits of a double integral.
- Added new 14.5 Example 1, modified Examples 2 and 3, and added new Figures 14.31, 14.32, and 14.33 to give basic examples of setting up limits of integration for a triple integral.
- Added new material on joint probability distributions as an application of multivariable integration.
- Added new Examples 5, 6 and 7 to Section 14.6.
- Added new Exercises: **14.1:** 15–16, 27–28; **14.6:** 39–44; **14.7:** 1–22.

Chapter 15

- Added new Figure 15.4 to illustrate a line integral of a function.
- Added new Figure 15.17 to illustrate a gradient field.
- Added new Figure 15.19 to illustrate a line integral of a vector field.
- Clarified notation for line integrals in 15.2.
- Added discussion of the sign of potential energy in 15.3.
- Rewrote solution of Example 3 in 15.4 to clarify connection to Green's Theorem.
- Updated discussion of surface orientation in 15.6 along with Figure 15.52.
- Added new Exercises: **15.2:** 37–38, 41–46; **15.4:** 1–6; **15.6:** 49–50; **15.7:** 1–6; **15.8:** 1–4.

Chapter 16

- Added new Example 3 with Figure 16.3 to illustrate how to construct a slope field.
- Added new Exercises: **16.1:** 11–14; **PE:** 17–22, 43–44.

Appendices: Rewrote Appendix 8 on complex numbers. Shortened Appendix 2 to focus on issues arising in use of mathematical software and potential pitfalls.

Continuing Features

Rigor The level of rigor is consistent with that of earlier editions. We continue to distinguish between formal and informal discussions and to point out their differences. Starting with a more intuitive, less formal approach helps students understand a new or difficult concept so they can then appreciate its full mathematical precision and outcomes. We pay attention to defining ideas carefully and to proving theorems appropriate for calculus students, while mentioning deeper or subtler issues they would study in a more advanced course. Our organization and distinctions between informal and formal discussions give the instructor a degree of flexibility in the amount and depth of coverage of the various topics. For example, while we do not prove the Intermediate Value Theorem or the Extreme Value Theorem for continuous functions on a closed finite interval, we do state these theorems precisely, illustrate their meanings in numerous examples, and use them to prove other important results. Furthermore, for those instructors who desire greater depth of coverage, in Appendix 7 we discuss the reliance of these theorems on the completeness of the real numbers.

Writing Exercises Writing exercises placed throughout the text ask students to explore and explain a variety of calculus concepts and applications. In addition, the end of each chapter contains a list of questions for students to review and summarize what they have learned. Many of these exercises make good writing assignments.

End-of-Chapter Reviews and Projects In addition to problems appearing after each section, each chapter culminates with review questions, practice exercises covering the entire chapter, and a series of Additional and Advanced Exercises with more challenging or synthesizing problems. Most chapters also include descriptions of several **Technology Application Projects** that can be worked by individual students or groups of students over a longer period of time. These projects require the use of *Mathematica* or *Maple*, along with pre-made files that are available for download within MyLab Math.

Writing and Applications This text continues to be easy to read, conversational, and mathematically rich. Each new topic is motivated by clear, easy-to-understand examples and is then reinforced by its application to real-world problems of immediate interest to students. A hallmark of this book has been the application of calculus to science and engineering. These applied problems have been updated, improved, and extended continually over the last several editions.

Technology In a course using the text, technology can be incorporated according to the taste of the instructor. Each section contains exercises requiring the use of technology; these are marked with a T if suitable for calculator or computer use, or they are labeled **Computer Explorations** if a computer algebra system (CAS, such as *Maple* or *Mathematica*) is required.

Additional Resources

MyLab Math® Online Course (access code required)

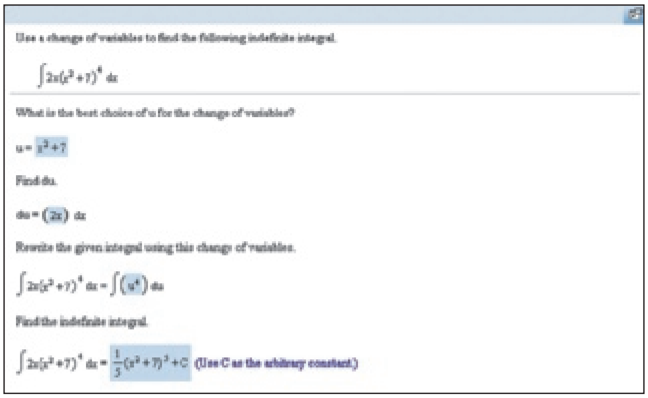
Built around Pearson's best-selling content, MyLab Math is an online homework, tutorial, and assessment program designed to work with this text to engage students and improve results. MyLab Math can be successfully implemented in any classroom environment—lab-based, hybrid, fully online, or traditional.

Used by more than 37 million students worldwide, MyLab Math delivers consistent, measurable gains in student learning outcomes, retention, and subsequent course success. Visit **www.mymathlab.com/results** to learn more.

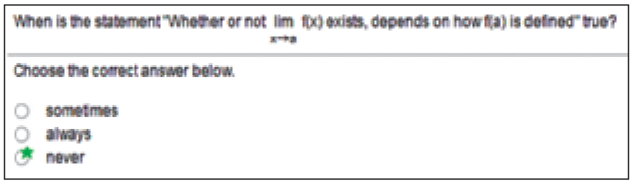
Motivation Students are motivated to succeed when they're engaged in the learning experience and understand the relevance and power of mathematics. MyLab Math's online homework offers students immediate feedback and tutorial assistance that motivates them to do more, which means they retain more knowledge and improve their test scores.

- **Exercises with immediate feedback**—assignable exercises for this text regenerate algorithmically to give students unlimited opportunity for practice and mastery. MyLab Math provides helpful feedback when students enter incorrect answers and includes optional learning aids such as Help Me Solve This, View an Example, videos, and an eText.

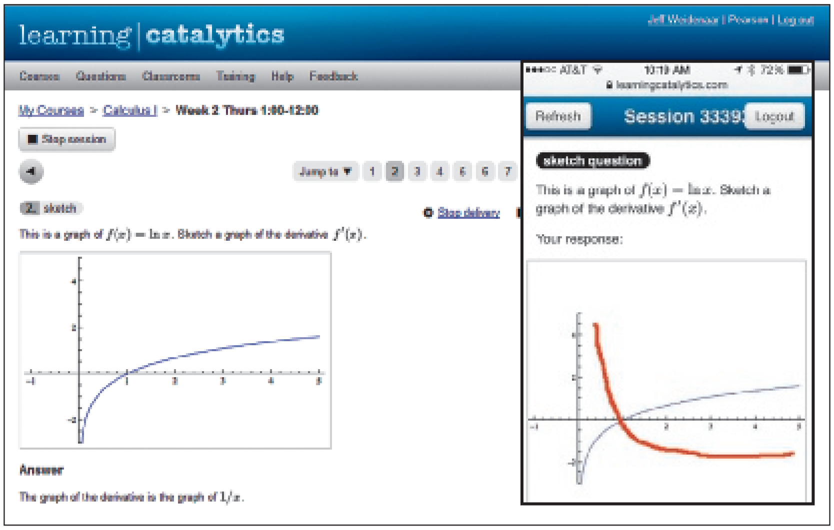
- **Setup and Solve Exercises** ask students to first describe how they will set up and approach the problem. This reinforces students' conceptual understanding of the process they are applying and promotes long-term retention of the skill.



- **Additional Conceptual Questions** focus on deeper, theoretical understanding of the key concepts in calculus. These questions were written by faculty at Cornell University under an NSF grant and are also assignable through Learning Catalytics.

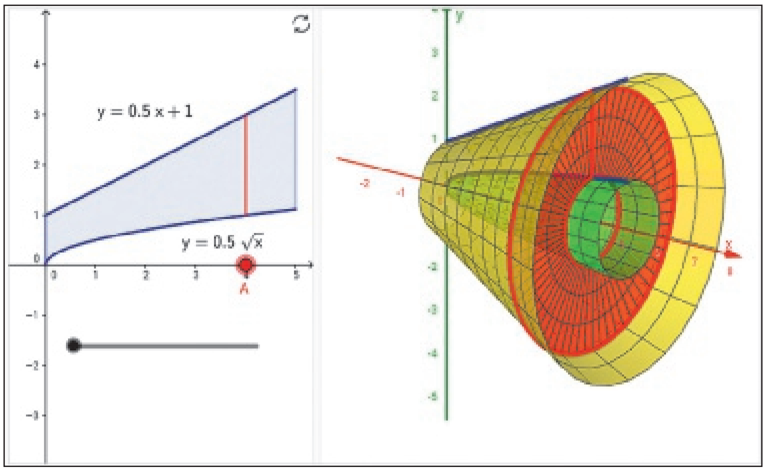


- **Learning Catalytics™** is a student response tool that uses students' smartphones, tablets, or laptops to engage them in more interactive tasks and thinking during lecture. Learning Catalytics fosters student engagement and peer-to-peer learning with real-time analytics. Learning Catalytics is available to all MyLab Math users.

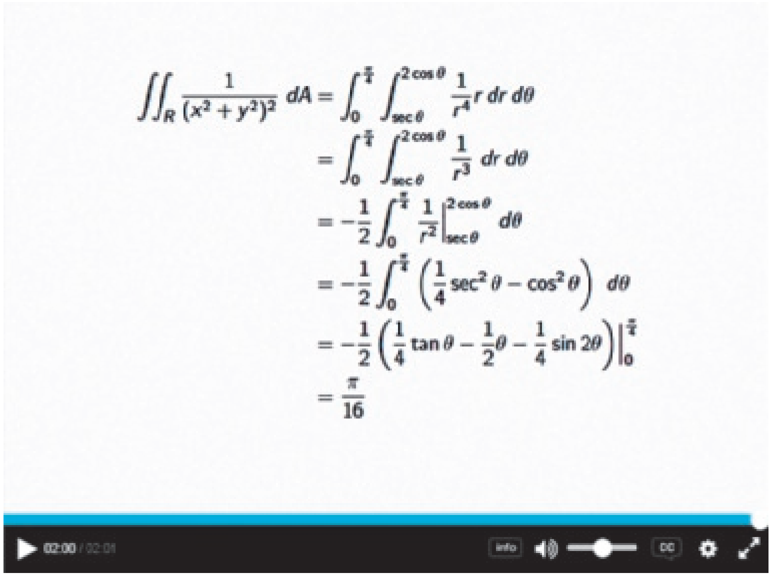


Learning and Teaching Tools

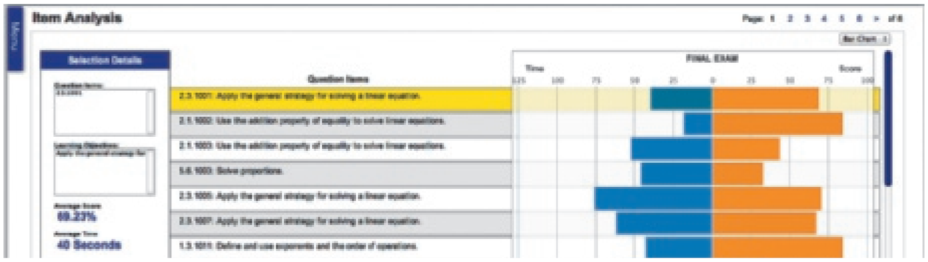
- **Interactive Figures** bring calculus concepts to life, helping you understand key ideas by working with their visual representations. They illustrate key concepts and allow manipulation for use as teaching and learning tools. We also include videos that use the Interactive Figures to explain key concepts.



- **Instructional videos**—Hundreds of videos are available as learning aids within exercises and for self-study. The tutorial videos cover key concepts from your text and are especially handy if you miss a lecture or just need another explanation. The Guide to Video-Based Assignments makes it easy to assign videos for homework by showing which MyLab Math exercises correspond to each video.



- **The complete eText** is available to students through their MyLab Math courses for the lifetime of the edition, giving students unlimited access to the eText within any course using that edition of the text.
- **Enhanced Sample Assignments** These assignments include just-in-time prerequisite review, help keep skills fresh with distributed practice of key concepts, and provide opportunities to work exercises without learning aids so students can check their understanding.
- **PowerPoint Presentations** that cover each section of the book are available for download.
- **Mathematica manual and projects, Maple manual and projects, TI Graphing Calculator manual**—These manuals cover *Maple* 17, *Mathematica 8,* and the TI-84 Plus and TI-89, respectively. Each provides detailed guidance for integrating the software package or graphing calculator throughout the course, including syntax and commands.
- **Accessibility** and achievement go hand in hand. MyLab Math is compatible with the JAWS screen reader, and it enables students to read and interact with multiple-choice and free-response problem types via keyboard controls and math notation input. MyLab Math also works with screen enlargers, including ZoomText, MAGic, and SuperNova. And, all MyLab Math videos have closed-captioning. More information is available at **http://mymathlab.com/accessibility**.
- **A comprehensive gradebook** with enhanced reporting functionality allows you to efficiently manage your course.
  - **The Reporting Dashboard** offers insight as you view, analyze, and report learning outcomes. Student performance data is presented at the class, section, and program levels in an accessible, visual manner so you'll have the information you need to keep your students on track.



- **Item Analysis** tracks class-wide understanding of particular exercises so you can refine your class lectures or adjust the course/department syllabus. Just-in-time teaching has never been easier!

MyLab Math comes from an experienced partner with educational expertise and an eye on the future. Whether you are just getting started with MyLab Math, or have a question along the way, we're here to help you learn about our technologies and how to incorporate them into your course. To learn more about how MyLab Math helps students succeed, visit **www.mymathlab.com** or contact your Pearson rep.

Instructor's Solutions Manual (downloadable)

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The Instructor's Solutions Manual contains complete worked-out solutions to all the exercises in *Thomas' Calculus: Early Transcendentals*. It can be downloaded from within MyLab Math or the Pearson Instructor Resource Center, **www.pearsonglobaleditions.com.**

Student's Solutions Manual

ISBN: 1-292-25319-3 | 978-1-292-25319-0

The Student's Solutions Manual contains worked-out solutions to all the odd-numbered exercises in *Thomas' Calculus: Early Transcendentals*. These manuals can be downloaded from within MyLab Math.

Companion Website

The companion Website, located at **www.pearsonglobaleditions.com**, includes opportunities for practice and review.

Just-In-Time Algebra and Trigonometry for Early Transcendentals Calculus, Fourth Edition

ISBN 0-321-67103-1 | 978-0-321-67103-5

Sharp algebra and trigonometry skills are critical to mastering calculus, and *Just-in-Time Algebra and Trigonometry for Early Transcendentals Calculus* by Guntram Mueller and Ronald I. Brent is designed to bolster these skills while students study calculus. As students make their way through calculus, this brief supplementary text is with them every step of the way, showing them the necessary algebra or trigonometry topics and pointing out potential problem spots. The easy-to-use table of contents has topics arranged in the order in which students will need them as they study calculus.

Technology Manuals and Projects (downloadable)

Maple Manual and Projects by Marie Vanisko, Carroll College
Mathematica Manual and Projects by Marie Vanisko, Carroll College
TI-Graphing Calculator Manual by Elaine McDonald-Newman, Sonoma State University

These manuals and projects cover Maple 17, Mathematica 9, and the TI-84 Plus and TI-89. Each manual provides detailed guidance for integrating a specific software package or graphing calculator throughout the course, including syntax and commands. The projects include instructions and ready-made application files for Maple and Mathematica. These materials are available to download within MyLab Math.

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PowerPoint® Lecture Slides

ISBN: 1-29-225318-5 | 978-1-29-225318-3

These classroom presentation slides were created for the *Thomas' Calculus* series. Key graphics from the book are included to help bring the concepts alive in the classroom. These files are available to qualified instructors through the Pearson Instructor Resource Center, **www.pearsonglobaleditions.com**, and within MyLab Math.

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Dedication

We regret that prior to the writing of this edition our coauthor Maurice Weir passed away. Maury was dedicated to achieving the highest possible standards in the presentation of mathematics. He insisted on clarity, rigor, and readability. Maury was a role model to his students, his colleagues, and his coauthors. He was very proud of his daughters, Maia Coyle and Renee Waina, and of his grandsons, Matthew Ryan and Andrew Dean Waina. He will be greatly missed.

1

Functions

OVERVIEW Functions are fundamental to the study of calculus. In this chapter we review what functions are and how they are visualized as graphs, how they are combined and transformed, and ways they can be classified.

1.1 Functions and Their Graphs

Functions are a tool for describing the real world in mathematical terms. A function can be represented by an equation, a graph, a numerical table, or a verbal description; we will use all four representations throughout this book. This section reviews these ideas.

Functions; Domain and Range

The temperature at which water boils depends on the elevation above sea level. The interest paid on a cash investment depends on the length of time the investment is held. The area of a circle depends on the radius of the circle. The distance an object travels depends on the elapsed time.

In each case, the value of one variable quantity, say y, depends on the value of another variable quantity, which we often call x. We say that "y is a function of x" and write this symbolically as

$$y = f(x) \qquad (\text{“}y \text{ equals } f \text{ of } x\text{”}).$$

The symbol f represents the function, the letter x is the **independent variable** representing the input value to f, and y is the **dependent variable** or output value of f at x.

> **DEFINITION** A **function** f from a set D to a set Y is a rule that assigns a *unique* value $f(x)$ in Y to each x in D.

The set D of all possible input values is called the **domain** of the function. The set of all output values of $f(x)$ as x varies throughout D is called the **range** of the function. The range might not include every element in the set Y. The domain and range of a function can be any sets of objects, but often in calculus they are sets of real numbers interpreted as points of a coordinate line. (In Chapters 12–15, we will encounter functions for which the elements of the sets are points in the plane, or in space.)

Often a function is given by a formula that describes how to calculate the output value from the input variable. For instance, the equation $A = \pi r^2$ is a rule that calculates the area A of a circle from its radius r. When we define a function $y = f(x)$ with a formula and the domain is not stated explicitly or restricted by context, the domain is assumed to be the largest set of real x-values for which the formula gives real y-values. This is called the **natural domain** of f. If we want to restrict the domain in some way, we must say so. The domain of $y = x^2$ is the entire set of real numbers. To restrict the domain of the function to, say, positive values of x, we would write "$y = x^2, x > 0$."

Changing the domain to which we apply a formula usually changes the range as well. The range of $y = x^2$ is $[0, \infty)$. The range of $y = x^2, x \geq 2$, is the set of all numbers obtained by squaring numbers greater than or equal to 2. In set notation (see Appendix 1), the range is $\{x^2 | x \geq 2\}$ or $\{y | y \geq 4\}$ or $[4, \infty)$.

When the range of a function is a set of real numbers, the function is said to be **real-valued**. The domains and ranges of most real-valued functions we consider are intervals or combinations of intervals. Sometimes the range of a function is not easy to find.

A function f is like a machine that produces an output value $f(x)$ in its range whenever we feed it an input value x from its domain (Figure 1.1). The function keys on a calculator give an example of a function as a machine. For instance, the $\sqrt{x}$ key on a calculator gives an output value (the square root) whenever you enter a nonnegative number x and press the $\sqrt{x}$ key.

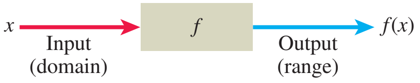


FIGURE 1.1 A diagram showing a function as a kind of machine.

A function can also be pictured as an **arrow diagram** (Figure 1.2). Each arrow associates to an element of the domain D a single element in the set Y. In Figure 1.2, the arrows indicate that $f(a)$ is associated with a, $f(x)$ is associated with x, and so on. Notice that a function can have the same *output value* for two different input elements in the domain (as occurs with $f(a)$ in Figure 1.2), but each input element x is assigned a *single* output value $f(x)$.

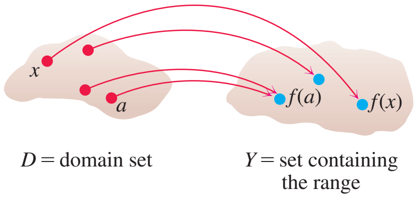


FIGURE 1.2 A function from a set D to a set Y assigns a unique element of Y to each element in D.

EXAMPLE 1 Verify the natural domains and associated ranges of some simple functions. The domains in each case are the values of x for which the formula makes sense.

Function	Domain (x)	Range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

Solution The formula $y = x^2$ gives a real y-value for any real number x, so the domain is $(-\infty, \infty)$. The range of $y = x^2$ is $[0, \infty)$ because the square of any real number is nonnegative and every nonnegative number y is the square of its own square root: $y = (\sqrt{y})^2$ for $y \geq 0$.

The formula $y = 1/x$ gives a real y-value for every x except $x = 0$. For consistency in the rules of arithmetic, *we cannot divide any number by zero*. The range of $y = 1/x$, the set of reciprocals of all nonzero real numbers, is the set of all nonzero real numbers, since $y = 1/(1/y)$. That is, for $y \neq 0$ the number $x = 1/y$ is the input that is assigned to the output value y.

The formula $y = \sqrt{x}$ gives a real y-value only if $x \geq 0$. The range of $y = \sqrt{x}$ is $[0, \infty)$ because every nonnegative number is some number's square root (namely, it is the square root of its own square).

In $y = \sqrt{4 - x}$, the quantity $4 - x$ cannot be negative. That is, $4 - x \geq 0$, or $x \leq 4$. The formula gives nonnegative real y-values for all $x \leq 4$. The range of $\sqrt{4 - x}$ is $[0, \infty)$, the set of all nonnegative numbers.

The formula $y = \sqrt{1 - x^2}$ gives a real y-value for every x in the closed interval from -1 to 1. Outside this domain, $1 - x^2$ is negative and its square root is not a real number. The values of $1 - x^2$ vary from 0 to 1 on the given domain, and the square roots of these values do the same. The range of $\sqrt{1 - x^2}$ is $[0, 1]$. ■

Graphs of Functions

If f is a function with domain D, its **graph** consists of the points in the Cartesian plane whose coordinates are the input-output pairs for f. In set notation, the graph is

$$\{(x, f(x)) \mid x \in D\}.$$

The graph of the function $f(x) = x + 2$ is the set of points with coordinates (x, y) for which $y = x + 2$. Its graph is the straight line sketched in Figure 1.3.

The graph of a function f is a useful picture of its behavior. If (x, y) is a point on the graph, then $y = f(x)$ is the height of the graph above (or below) the point x. The height may be positive or negative, depending on the sign of $f(x)$ (Figure 1.4).

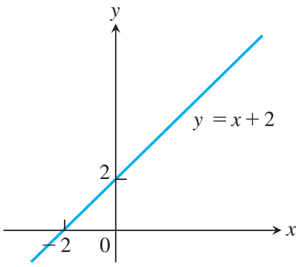


FIGURE 1.3 The graph of $f(x) = x + 2$ is the set of points (x, y) for which y has the value $x + 2$.

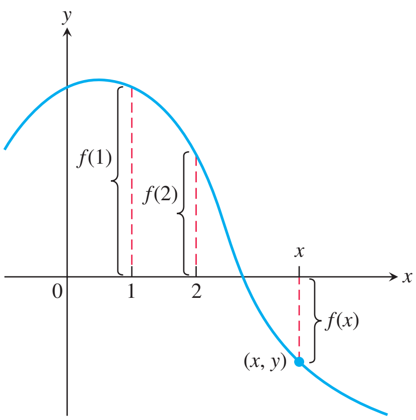


FIGURE 1.4 If (x, y) lies on the graph of f, then the value $y = f(x)$ is the height of the graph above the point x (or below x if $f(x)$ is negative).

x	$y = x^2$
-2	4
-1	1
0	0
1	1
$\frac{3}{2}$	$\frac{9}{4}$
2	4

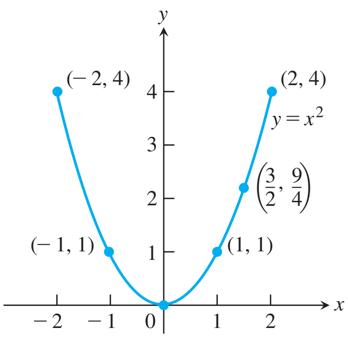
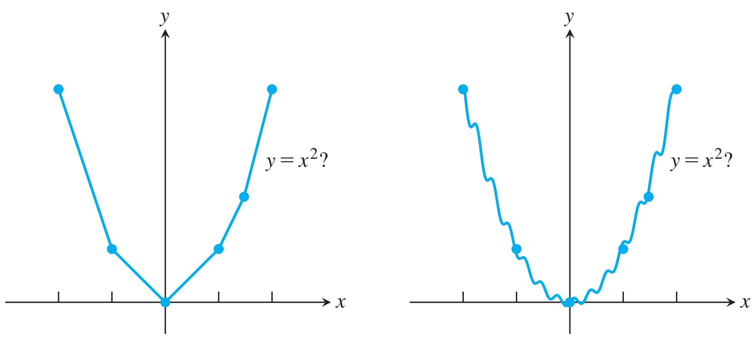


FIGURE 1.5 Graph of the function in Example 2.

EXAMPLE 2 Graph the function $y = x^2$ over the interval $[-2, 2]$.

Solution Make a table of xy-pairs that satisfy the equation $y = x^2$. Plot the points (x, y) whose coordinates appear in the table, and draw a *smooth* curve (labeled with its equation) through the plotted points (see Figure 1.5). ■

How do we know that the graph of $y = x^2$ doesn't look like one of these curves?



To find out, we could plot more points. But how would we then connect *them*? The basic question still remains: How do we know for sure what the graph looks like between the points we plot? Calculus answers this question, as we will see in Chapter 4. Meanwhile, we will have to settle for plotting points and connecting them as best we can.

Representing a Function Numerically

We have seen how a function may be represented algebraically by a formula and visually by a graph (Example 2). Another way to represent a function is **numerically**, through a table of values. Numerical representations are often used by engineers and experimental scientists. From an appropriate table of values, a graph of the function can be obtained using the method illustrated in Example 2, possibly with the aid of a computer. The graph consisting of only the points in the table is called a **scatterplot**.

EXAMPLE 3 Musical notes are pressure waves in the air. The data associated with Figure 1.6 give recorded pressure displacement versus time in seconds of a musical note produced by a tuning fork. The table provides a representation of the pressure function (in micropascals) over time. If we first make a scatterplot and then connect the data points (t, p) from the table, we obtain the graph shown in the figure.

Time	Pressure	Time	Pressure
0.00091	−0.080	0.00362	0.217
0.00108	0.200	0.00379	0.480
0.00125	0.480	0.00398	0.681
0.00144	0.693	0.00416	0.810
0.00162	0.816	0.00435	0.827
0.00180	0.844	0.00453	0.749
0.00198	0.771	0.00471	0.581
0.00216	0.603	0.00489	0.346
0.00234	0.368	0.00507	0.077
0.00253	0.099	0.00525	−0.164
0.00271	−0.141	0.00543	−0.320
0.00289	−0.309	0.00562	−0.354
0.00307	−0.348	0.00579	−0.248
0.00325	−0.248	0.00598	−0.035
0.00344	−0.041		

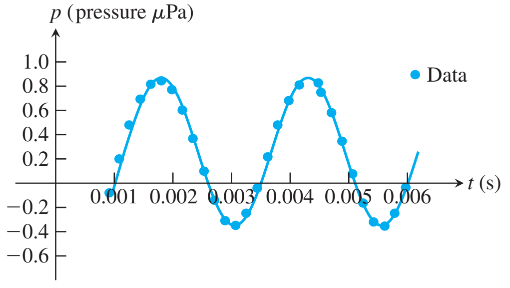


FIGURE 1.6 A smooth curve through the plotted points gives a graph of the pressure function represented by the accompanying tabled data (Example 3).

■

The Vertical Line Test for a Function

Not every curve in the coordinate plane can be the graph of a function. A function f can have only one value $f(x)$ for each x in its domain, so *no vertical* line can intersect the graph of a function more than once. If a is in the domain of the function f, then the vertical line $x = a$ will intersect the graph of f at the single point $(a, f(a))$.

A circle cannot be the graph of a function, since some vertical lines intersect the circle twice. The circle graphed in Figure 1.7a, however, contains the graphs of two functions of x, namely the upper semicircle defined by the function $f(x) = \sqrt{1 - x^2}$ and the lower semicircle defined by the function $g(x) = -\sqrt{1 - x^2}$ (Figures 1.7b and 1.7c).

Piecewise-Defined Functions

Sometimes a function is described in pieces by using different formulas on different parts of its domain. One example is the **absolute value function**

$$|x| = \begin{cases} x, & x \geq 0 \quad \text{First formula} \\ -x, & x < 0 \quad \text{Second formula} \end{cases}$$

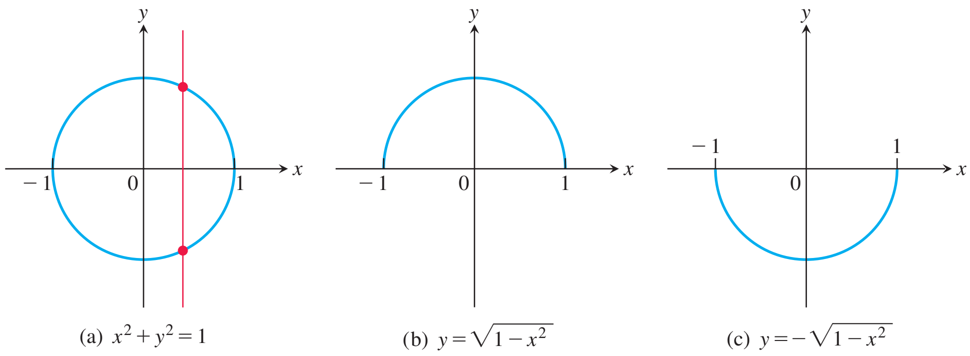


FIGURE 1.7 (a) The circle is not the graph of a function; it fails the vertical line test. (b) The upper semicircle is the graph of the function $f(x) = \sqrt{1 - x^2}$. (c) The lower semicircle is the graph of the function $g(x) = -\sqrt{1 - x^2}$.

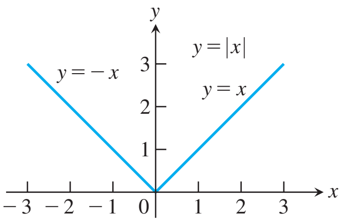


FIGURE 1.8 The absolute value function has domain $(-\infty, \infty)$ and range $[0, \infty)$.

whose graph is given in Figure 1.8. The right-hand side of the equation means that the function equals x if $x \geq 0$, and equals $-x$ if $x < 0$. Piecewise-defined functions often arise when real-world data are modeled. Here are some other examples.

EXAMPLE 4 The function

$$f(x) = \begin{cases} -x, & x < 0 \quad \text{First formula} \\ x^2, & 0 \leq x \leq 1 \quad \text{Second formula} \\ 1, & x > 1 \quad \text{Third formula} \end{cases}$$

is defined on the entire real line but has values given by different formulas, depending on the position of x. The values of f are given by $y = -x$ when $x < 0$, $y = x^2$ when $0 \leq x \leq 1$, and $y = 1$ when $x > 1$. The function, however, is *just one function* whose domain is the entire set of real numbers (Figure 1.9). ■

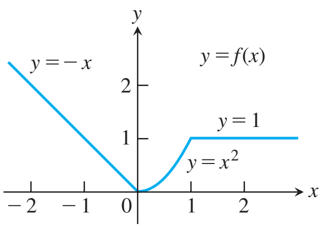


FIGURE 1.9 To graph the function $y = f(x)$ shown here, we apply different formulas to different parts of its domain (Example 4).

EXAMPLE 5 The function whose value at any number x is the *greatest integer less than or equal to* x is called the **greatest integer function** or the **integer floor function**. It is denoted $\lfloor x \rfloor$. Figure 1.10 shows the graph. Observe that

$$\lfloor 2.4 \rfloor = 2, \quad \lfloor 1.9 \rfloor = 1, \quad \lfloor 0 \rfloor = 0, \quad \lfloor -1.2 \rfloor = -2,$$
$$\lfloor 2 \rfloor = 2, \quad \lfloor 0.2 \rfloor = 0, \quad \lfloor -0.3 \rfloor = -1, \quad \lfloor -2 \rfloor = -2.$$ ■

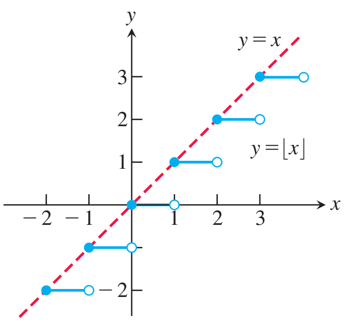


FIGURE 1.10 The graph of the greatest integer function $y = \lfloor x \rfloor$ lies on or below the line $y = x$, so it provides an integer floor for x (Example 5).

EXAMPLE 6 The function whose value at any number x is the *smallest integer greater than or equal to* x is called the **least integer function** or the **integer ceiling function**. It is denoted $\lceil x \rceil$. Figure 1.11 shows the graph. For positive values of x, this function might represent, for example, the cost of parking x hours in a parking lot that charges $1 for each hour or part of an hour. ■

Increasing and Decreasing Functions

If the graph of a function climbs or rises as you move from left to right, we say that the function is *increasing*. If the graph descends or falls as you move from left to right, the function is *decreasing*.

> **DEFINITIONS** Let f be a function defined on an interval I and let x_1 and x_2 be two distinct points in I.
>
> **1.** If $f(x_2) > f(x_1)$ whenever $x_1 < x_2$, then f is said to be **increasing** on I.
>
> **2.** If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then f is said to be **decreasing** on I.

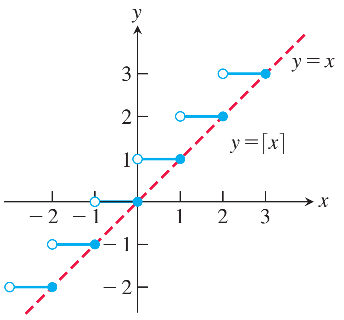


FIGURE 1.11 The graph of the least integer function $y = \lceil x \rceil$ lies on or above the line $y = x$, so it provides an integer ceiling for x (Example 6).

It is important to realize that the definitions of increasing and decreasing functions must be satisfied for *every* pair of points x_1 and x_2 in I with $x_1 < x_2$. Because we use the inequality $<$ to compare the function values, instead of $\leq$, it is sometimes said that f is *strictly* increasing or decreasing on I. The interval I may be finite (also called bounded) or infinite (unbounded).

EXAMPLE 7 The function graphed in Figure 1.9 is decreasing on $(-\infty, 0)$ and increasing on $(0, 1)$. The function is neither increasing nor decreasing on the interval $(1, \infty)$ because the function is constant on that interval, and hence the strict inequalities in the definition of increasing or decreasing are not satisfied on $(1, \infty)$. ■

Even Functions and Odd Functions: Symmetry

The graphs of *even* and *odd* functions have special symmetry properties.

DEFINITIONS A function $y = f(x)$ is an

even function of x if $f(-x) = f(x)$,
odd function of x if $f(-x) = -f(x)$,

for every x in the function's domain.

The names *even* and *odd* come from powers of x. If y is an even power of x, as in $y = x^2$ or $y = x^4$, it is an even function of x because $(-x)^2 = x^2$ and $(-x)^4 = x^4$. If y is an odd power of x, as in $y = x$ or $y = x^3$, it is an odd function of x because $(-x)^1 = -x$ and $(-x)^3 = -x^3$.

The graph of an even function is **symmetric about the y-axis**. Since $f(-x) = f(x)$, a point (x, y) lies on the graph if and only if the point $(-x, y)$ lies on the graph (Figure 1.12a). A reflection across the y-axis leaves the graph unchanged.

The graph of an odd function is **symmetric about the origin**. Since $f(-x) = -f(x)$, a point (x, y) lies on the graph if and only if the point $(-x, -y)$ lies on the graph (Figure 1.12b). Equivalently, a graph is symmetric about the origin if a rotation of 180° about the origin leaves the graph unchanged. Notice that the definitions imply that both x and $-x$ must be in the domain of f.

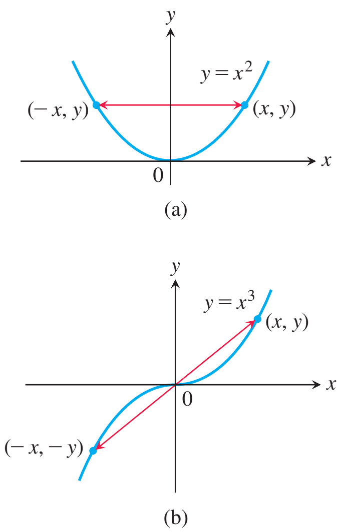


FIGURE 1.12 (a) The graph of $y = x^2$ (an even function) is symmetric about the y-axis. (b) The graph of $y = x^3$ (an odd function) is symmetric about the origin.

EXAMPLE 8 Here are several functions illustrating the definitions.

$f(x) = x^2$ — Even function: $(-x)^2 = x^2$ for all x; symmetry about y-axis. So $f(-3) = 9 = f(3)$. Changing the sign of x does not change the value of an even function.

$f(x) = x^2 + 1$ — Even function: $(-x)^2 + 1 = x^2 + 1$ for all x; symmetry about y-axis (Figure 1.13a).

$f(x) = x$ — Odd function: $(-x) = -x$ for all x; symmetry about the origin. So $f(-3) = -3$ while $f(3) = 3$. Changing the sign of x changes the sign of an odd function.

$f(x) = x + 1$ — Not odd: $f(-x) = -x + 1$, but $-f(x) = -x - 1$. The two are not equal.
Not even: $(-x) + 1 \neq x + 1$ for all $x \neq 0$ (Figure 1.13b). ■

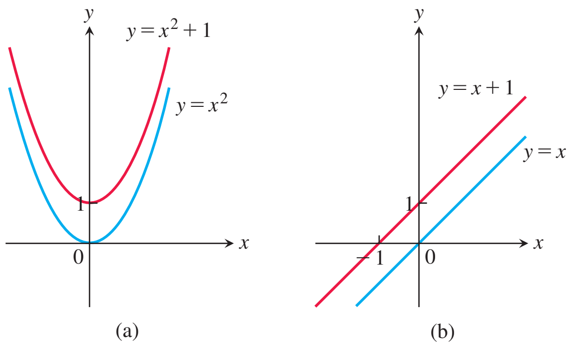


FIGURE 1.13 (a) When we add the constant term 1 to the function $y = x^2$, the resulting function $y = x^2 + 1$ is still even and its graph is still symmetric about the y-axis. (b) When we add the constant term 1 to the function $y = x$, the resulting function $y = x + 1$ is no longer odd, since the symmetry about the origin is lost. The function $y = x + 1$ is also not even (Example 8).

Common Functions

A variety of important types of functions are frequently encountered in calculus.

Linear Functions A function of the form $f(x) = mx + b$, where m and b are fixed constants, is called a **linear function**. Figure 1.14a shows an array of lines $f(x) = mx$. Each of these has $b = 0$, so these lines pass through the origin. The function $f(x) = x$ where $m = 1$ and $b = 0$ is called the **identity function**. Constant functions result when the slope is $m = 0$ (Figure 1.14b).

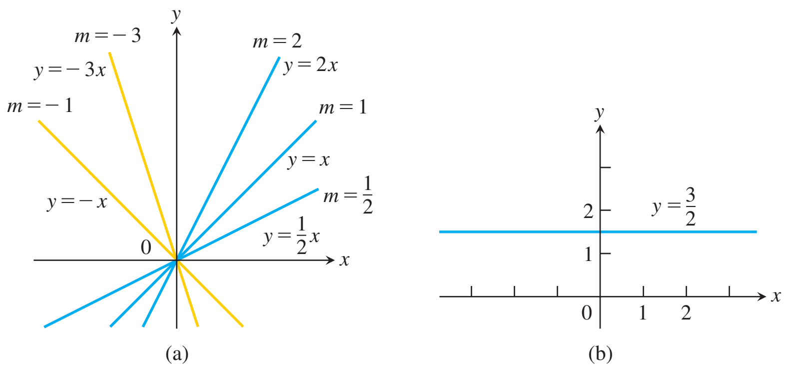


FIGURE 1.14 (a) Lines through the origin with slope m. (b) A constant function with slope $m = 0$.

> **DEFINITION** Two variables y and x are **proportional** (to one another) if one is always a constant multiple of the other—that is, if $y = kx$ for some nonzero constant k.

If the variable y is proportional to the reciprocal $1/x$, then sometimes it is said that y is **inversely proportional** to x (because $1/x$ is the multiplicative inverse of x).

Power Functions A function $f(x) = x^a$, where a is a constant, is called a **power function**. There are several important cases to consider.

(a) $f(x) = x^a$ with $a = n$, a positive integer.

The graphs of $f(x) = x^n$, for $n = 1, 2, 3, 4, 5$, are displayed in Figure 1.15. These functions are defined for all real values of x. Notice that as the power n gets larger, the curves tend to flatten toward the x-axis on the interval $(-1, 1)$ and to rise more steeply for $|x| > 1$. Each curve passes through the point $(1, 1)$ and through the origin. The graphs of functions with even powers are symmetric about the y-axis; those with odd powers are symmetric about the origin. The even-powered functions are decreasing on the interval $(-\infty, 0]$ and increasing on $[0, \infty)$; the odd-powered functions are increasing over the entire real line $(-\infty, \infty)$.

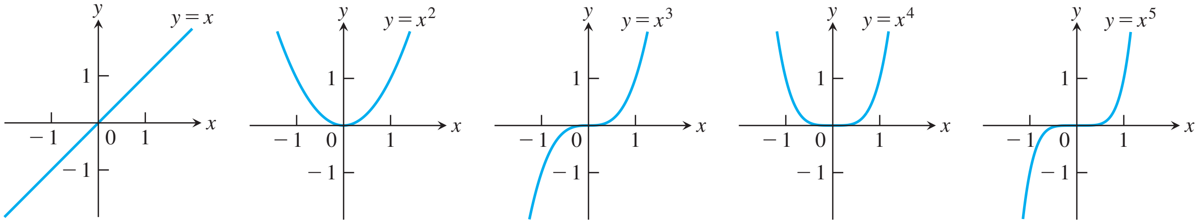


FIGURE 1.15 Graphs of $f(x) = x^n$, $n = 1, 2, 3, 4, 5$, defined for $-\infty < x < \infty$.

(b) $f(x) = x^a$ with $a = -1$ or $a = -2$.

The graphs of the functions $f(x) = x^{-1} = 1/x$ and $g(x) = x^{-2} = 1/x^2$ are shown in Figure 1.16. Both functions are defined for all $x \neq 0$ (you can never divide by zero). The graph of $y = 1/x$ is the hyperbola $xy = 1$, which approaches the coordinate axes far from the origin. The graph of $y = 1/x^2$ also approaches the coordinate axes. The graph of the function f is symmetric about the origin; f is decreasing on the intervals $(-\infty, 0)$ and $(0, \infty)$. The graph of the function g is symmetric about the y-axis; g is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.

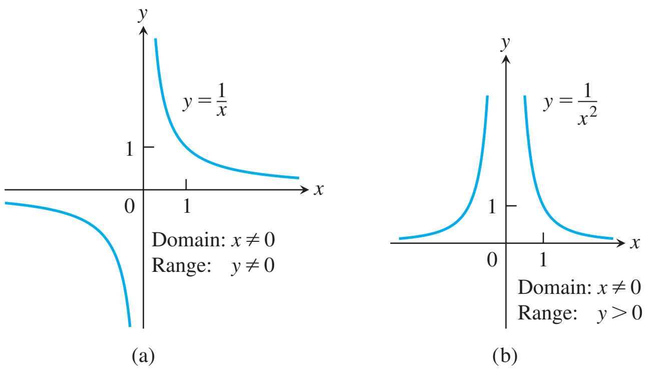


FIGURE 1.16 Graphs of the power functions $f(x) = x^a$. (a) $a = -1$, (b) $a = -2$.

(c) $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}$, and $\frac{2}{3}$.

The functions $f(x) = x^{1/2} = \sqrt{x}$ and $g(x) = x^{1/3} = \sqrt[3]{x}$ are the **square root** and **cube root** functions, respectively. The domain of the square root function is $[0, \infty)$, but the cube root function is defined for all real x. Their graphs are displayed in Figure 1.17, along with the graphs of $y = x^{3/2}$ and $y = x^{2/3}$. (Recall that $x^{3/2} = (x^{1/2})^3$ and $x^{2/3} = (x^{1/3})^2$.)

Polynomials A function p is a **polynomial** if

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where n is a nonnegative integer and the numbers $a_0, a_1, a_2, \ldots, a_n$ are real constants (called the **coefficients** of the polynomial). All polynomials have domain $(-\infty, \infty)$. If the

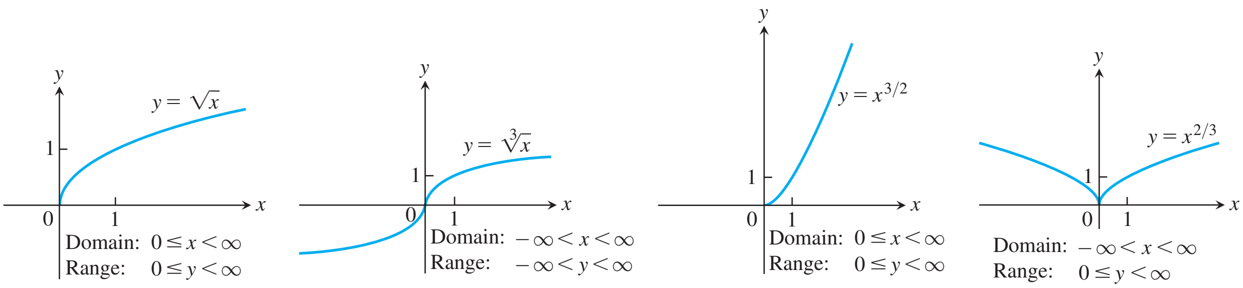


FIGURE 1.17 Graphs of the power functions $f(x) = x^a$ for $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}$, and $\frac{2}{3}$.

leading coefficient $a_n \neq 0$, then n is called the **degree** of the polynomial. Linear functions with $m \neq 0$ are polynomials of degree 1. Polynomials of degree 2, usually written as $p(x) = ax^2 + bx + c$, are called **quadratic functions**. Likewise, **cubic functions** are polynomials $p(x) = ax^3 + bx^2 + cx + d$ of degree 3. Figure 1.18 shows the graphs of three polynomials. Techniques to graph polynomials are studied in Chapter 4.

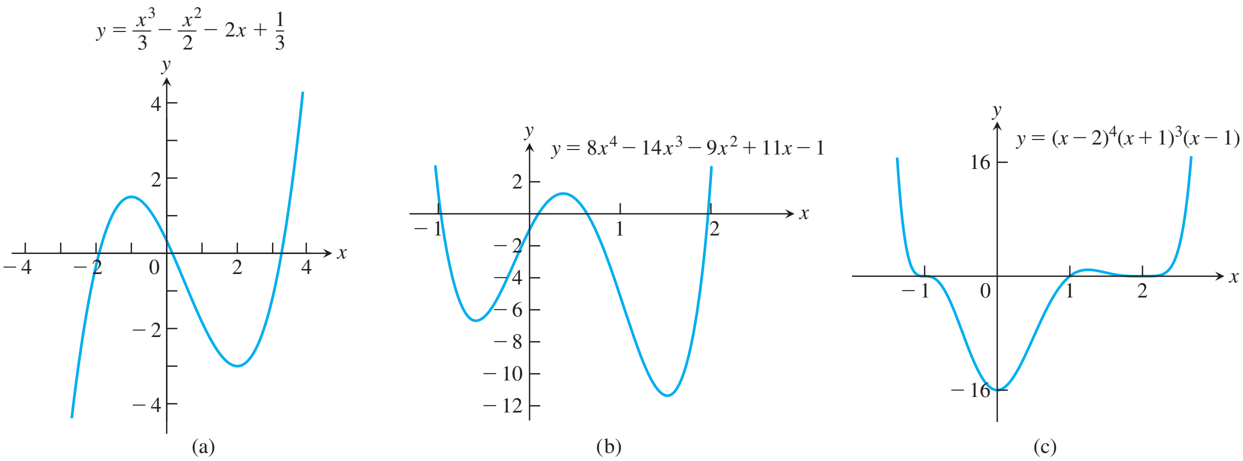


FIGURE 1.18 Graphs of three polynomial functions.

Rational Functions A **rational function** is a quotient or ratio $f(x) = p(x)/q(x)$, where p and q are polynomials. The domain of a rational function is the set of all real x for which $q(x) \neq 0$. The graphs of several rational functions are shown in Figure 1.19.

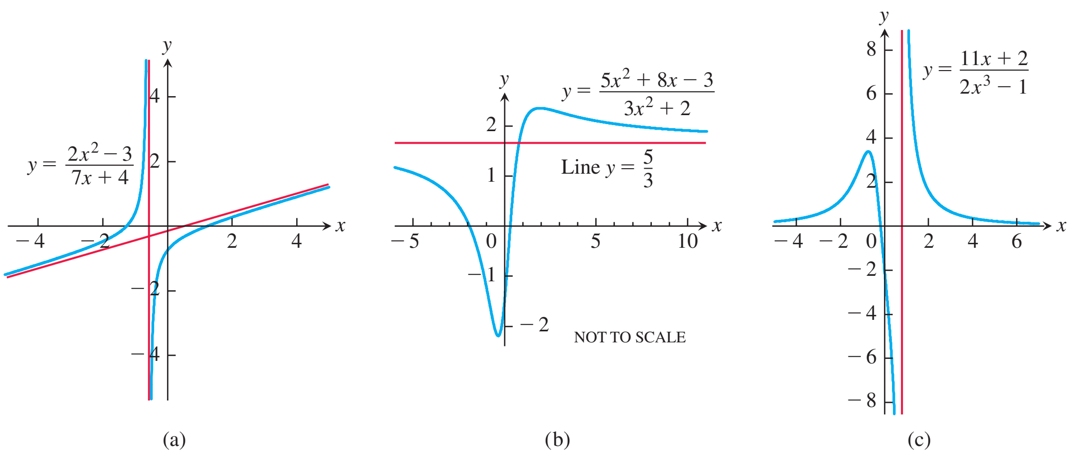


FIGURE 1.19 Graphs of three rational functions. The straight red lines approached by the graphs are called *asymptotes* and are not part of the graphs. We discuss asymptotes in Section 2.5.

Algebraic Functions Any function constructed from polynomials using algebraic operations (addition, subtraction, multiplication, division, and taking roots) lies within the class of **algebraic functions**. All rational functions are algebraic, but also included are more complicated functions (such as those satisfying an equation like $y^3 - 9xy + x^3 = 0$, studied in Section 3.7). Figure 1.20 displays the graphs of three algebraic functions.

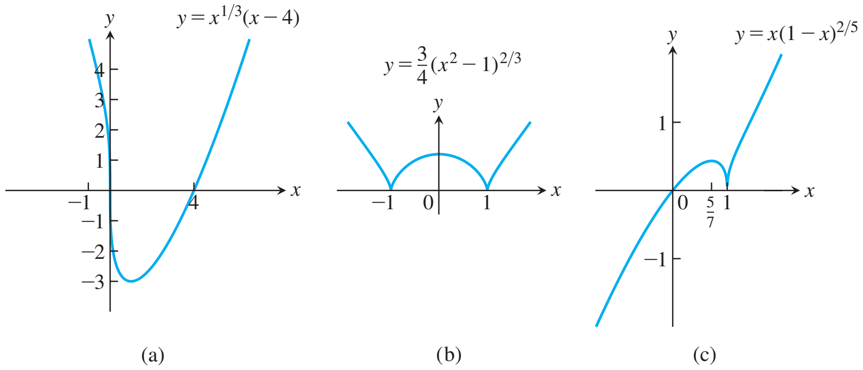


FIGURE 1.20 Graphs of three algebraic functions.

Trigonometric Functions The six basic trigonometric functions are reviewed in Section 1.3. The graphs of the sine and cosine functions are shown in Figure 1.21.

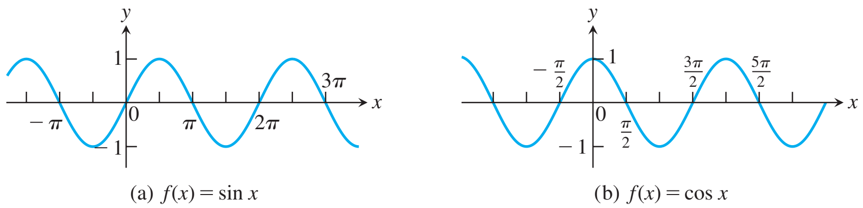


FIGURE 1.21 Graphs of the sine and cosine functions.

Exponential Functions A function of the form $f(x) = a^x$, where $a > 0$ and $a \neq 1$, is called an **exponential function** (with base a). All exponential functions have domain $(-\infty, \infty)$ and range $(0, \infty)$, so an exponential function never assumes the value 0. We discuss exponential functions in Section 1.4. The graphs of some exponential functions are shown in Figure 1.22.

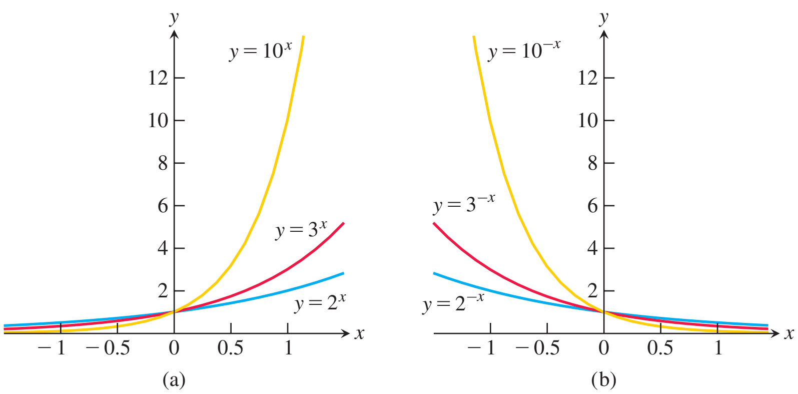


FIGURE 1.22 Graphs of exponential functions.

Logarithmic Functions These are the functions $f(x) = \log_a x$, where the base $a \neq 1$ is a positive constant. They are the *inverse functions* of the exponential functions, and we discuss these functions in Section 1.5. Figure 1.23 shows the graphs of four logarithmic functions with various bases. In each case the domain is $(0, \infty)$ and the range is $(-\infty, \infty)$.

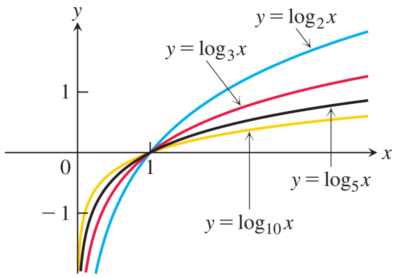


FIGURE 1.23 Graphs of four logarithmic functions.

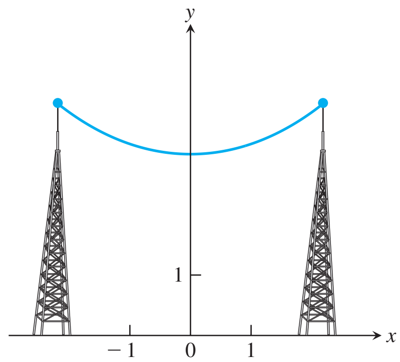


FIGURE 1.24 Graph of a catenary or hanging cable. (The Latin word *catena* means "chain.")

Transcendental Functions These are functions that are not algebraic. They include the trigonometric, inverse trigonometric, exponential, and logarithmic functions, and many other functions as well. The **catenary** is one example of a transcendental function. Its graph has the shape of a cable, like a telephone line or electric cable, strung from one support to another and hanging freely under its own weight (Figure 1.24). The function defining the graph is discussed in Section 7.3.

EXERCISES 1.1

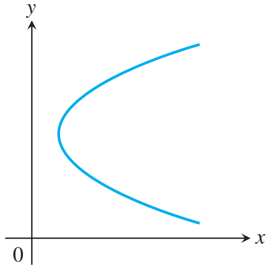
Functions

In Exercises 1–6, find the domain and range of each function.

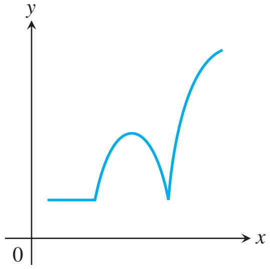
1. $f(x) = 1 + x^2$
2. $f(x) = 1 - \sqrt{x}$
3. $F(x) = \sqrt{5x + 10}$
4. $g(x) = \sqrt{x^2 - 3x}$
5. $f(t) = \dfrac{4}{3 - t}$
6. $G(t) = \dfrac{2}{t^2 - 16}$

In Exercises 7 and 8, which of the graphs are graphs of functions of x, and which are not? Give reasons for your answers.

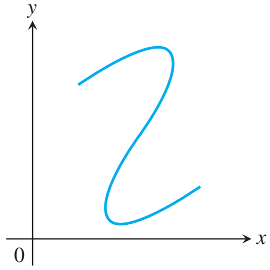
7. a.



b.



8. a.



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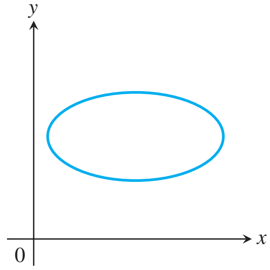


Finding Formulas for Functions

9. Express the area and perimeter of an equilateral triangle as a function of the triangle's side length x.
10. Express the side length of a square as a function of the length d of the square's diagonal. Then express the area as a function of the diagonal length.
11. Express the edge length of a cube as a function of the cube's diagonal length d. Then express the surface area and volume of the cube as a function of the diagonal length.

12. A point P in the first quadrant lies on the graph of the function $f(x) = \sqrt{x}$. Express the coordinates of P as functions of the slope of the line joining P to the origin.

13. Consider the point (x, y) lying on the graph of the line $2x + 4y = 5$. Let L be the distance from the point (x, y) to the origin $(0, 0)$. Write L as a function of x.

14. Consider the point (x, y) lying on the graph of $y = \sqrt{x - 3}$. Let L be the distance between the points (x, y) and $(4, 0)$. Write L as a function of y.

Functions and Graphs

Find the natural domain and graph the functions in Exercises 15–20.

15. $f(x) = 5 - 2x$ **16.** $f(x) = 1 - 2x - x^2$

17. $g(x) = \sqrt{|x|}$ **18.** $g(x) = \sqrt{-x}$

19. $F(t) = t/|t|$ **20.** $G(t) = 1/|t|$

21. Find the domain of $y = \dfrac{x + 3}{4 - \sqrt{x^2 - 9}}$.

22. Find the range of $y = 2 + \sqrt{9 + x^2}$.

23. Graph the following equations and explain why they are not graphs of functions of x.

a. $|y| = x$ **b.** $y^2 = x^2$

24. Graph the following equations and explain why they are not graphs of functions of x.

a. $|x| + |y| = 1$ **b.** $|x + y| = 1$

Piecewise-Defined Functions

Graph the functions in Exercises 25–28.

25. $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2 \end{cases}$

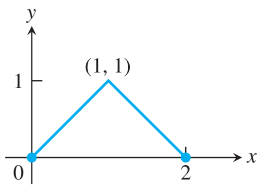
26. $g(x) = \begin{cases} 1 - x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2 \end{cases}$

27. $F(x) = \begin{cases} 4 - x^2, & x \le 1 \\ x^2 + 2x, & x > 1 \end{cases}$

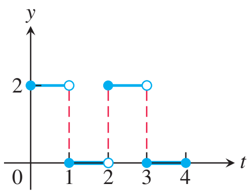
28. $G(x) = \begin{cases} 1/x, & x < 0 \\ x, & 0 \le x \end{cases}$

Find a formula for each function graphed in Exercises 29–32.

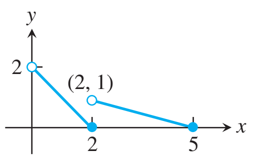
29. a.



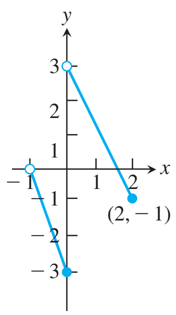
b.



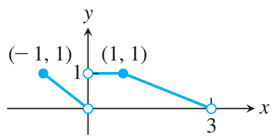
30. a.



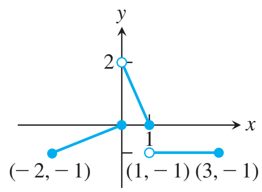
b.



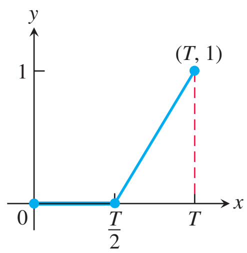
31. a.



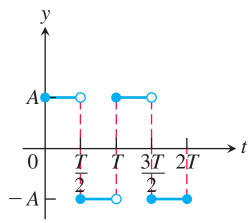
b.



32. a.



b.



The Greatest and Least Integer Functions

33. For what values of x is

a. $\lfloor x \rfloor = 0$? **b.** $\lceil x \rceil = 0$?

34. What real numbers x satisfy the equation $\lfloor x \rfloor = \lceil x \rceil$?

35. Does $\lceil -x \rceil = -\lfloor x \rfloor$ for all real x? Give reasons for your answer.

36. Graph the function

$$f(x) = \begin{cases} \lfloor x \rfloor, & x \ge 0 \\ \lceil x \rceil, & x < 0. \end{cases}$$

Why is $f(x)$ called the *integer part* of x?

Increasing and Decreasing Functions

Graph the functions in Exercises 37–46. What symmetries, if any, do the graphs have? Specify the intervals over which the function is increasing and the intervals where it is decreasing.

37. $y = -x^3$ **38.** $y = -\dfrac{1}{x^2}$

39. $y = -\dfrac{1}{x}$ **40.** $y = \dfrac{1}{|x|}$

41. $y = \sqrt{|x|}$ **42.** $y = \sqrt{-x}$

43. $y = x^3/8$ **44.** $y = -4\sqrt{x}$

45. $y = -x^{3/2}$ **46.** $y = (-x)^{2/3}$

Even and Odd Functions

In Exercises 47–58, say whether the function is even, odd, or neither. Give reasons for your answer.

47. $f(x) = 3$ **48.** $f(x) = x^{-5}$

49. $f(x) = x^2 + 1$ **50.** $f(x) = x^2 + x$

51. $g(x) = x^3 + x$ **52.** $g(x) = x^4 + 3x^2 - 1$

53. $g(x) = \dfrac{1}{x^2 - 1}$ **54.** $g(x) = \dfrac{x}{x^2 - 1}$

55. $h(t) = \dfrac{1}{t - 1}$ **56.** $h(t) = |t^3|$

57. $h(t) = 2t + 1$ **58.** $h(t) = 2|t| + 1$

59. $\sin 2x$ **60.** $\sin x^2$

61. $\cos 3x$ **62.** $1 + \cos x$

Theory and Examples

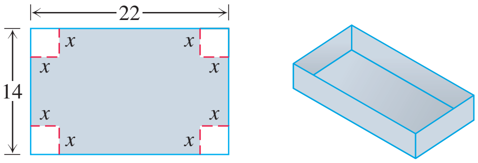
63. The variable s is proportional to t, and $s = 25$ when $t = 75$. Determine t when $s = 60$.

64. Kinetic energy The kinetic energy K of a mass is proportional to the square of its velocity v. If $K = 12{,}960$ joules when $v = 18\ \text{m/s}$, what is K when $v = 10\ \text{m/s}$?

65. The variables r and s are inversely proportional, and $r = 6$ when $s = 4$. Determine s when $r = 10$.

66. Boyle's Law Boyle's Law says that the volume V of a gas at constant temperature increases whenever the pressure P decreases, so that V and P are inversely proportional. If $P = 14.7\ \text{N/cm}^2$ when $V = 1000\ \text{cm}^3$, then what is V when $P = 23.4\ \text{N/cm}^2$?

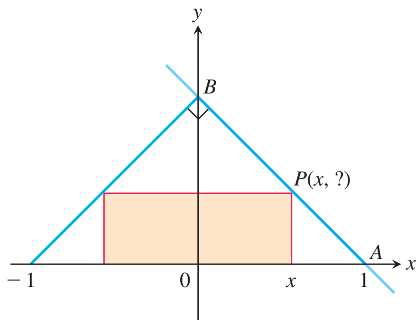
67. A box with an open top is to be constructed from a rectangular piece of cardboard with dimensions 14 cm. by 22 cm. by cutting out equal squares of side x at each corner and then folding up the sides as in the figure. Express the volume V of the box as a function of x.



68. The accompanying figure shows a rectangle inscribed in an isosceles right triangle whose hypotenuse is 2 units long.

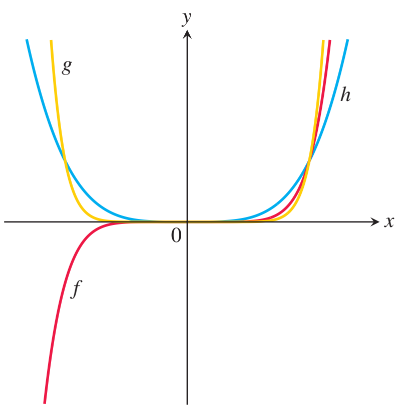
a. Express the y-coordinate of P in terms of x. (You might start by writing an equation for the line AB.)

b. Express the area of the rectangle in terms of x.

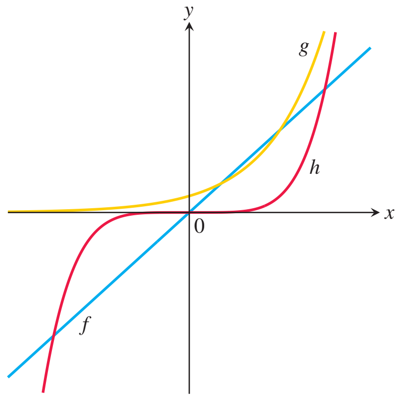


In Exercises 69 and 70, match each equation with its graph. Do not use a graphing device, and give reasons for your answer.

69. a. $y = x^4$ **b.** $y = x^7$ **c.** $y = x^{10}$



70. a. $y = 5x$ **b.** $y = 5^x$ **c.** $y = x^5$



T 71. a. Graph the functions $f(x) = x/2$ and $g(x) = 1 + (4/x)$ together to identify the values of x for which

$$\frac{x}{2} > 1 + \frac{4}{x}.$$

b. Confirm your findings in part (a) algebraically.

T 72. a. Graph the functions $f(x) = 3/(x - 1)$ and $g(x) = 2/(x + 1)$ together to identify the values of x for which

$$\frac{3}{x - 1} < \frac{2}{x + 1}.$$

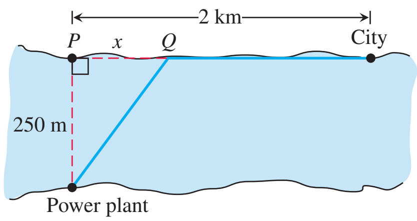
b. Confirm your findings in part (a) algebraically.

73. For a curve to be *symmetric about the x-axis*, the point (x, y) must lie on the curve if and only if the point $(x, -y)$ lies on the curve. Explain why a curve that is symmetric about the x-axis is not the graph of a function, unless the function is $y = 0$.

74. Three hundred books sell for \$40 each, resulting in a revenue of (300)(\$40) = \$12,000. For each \$5 increase in the price, 25 fewer books are sold. Write the revenue R as a function of the number x of \$5 increases.

75. A pen in the shape of an isosceles right triangle with legs of length x m and hypotenuse of length h m is to be built. If fencing costs \$5/m for the legs and \$10/m for the hypotenuse, write the total cost C of construction as a function of h.

76. Industrial costs A power plant sits next to a river where the river is 250 m wide. To lay a new cable from the plant to a location in the city 2 km downstream on the opposite side costs \$180 per meter across the river and \$100 per meter along the land.



NOT TO SCALE

a. Suppose that the cable goes from the plant to a point Q on the opposite side that is x m from the point P directly opposite the plant. Write a function $C(x)$ that gives the cost of laying the cable in terms of the distance x.

b. Generate a table of values to determine if the least expensive location for point Q is less than 300 m or greater than 300 m from point P.

1.2 Combining Functions; Shifting and Scaling Graphs

In this section we look at the main ways functions are combined or transformed to form new functions.

Sums, Differences, Products, and Quotients

Like numbers, functions can be added, subtracted, multiplied, and divided (except where the denominator is zero) to produce new functions. If f and g are functions, then for every x that belongs to the domains of both f and g (that is, for $x \in D(f) \cap D(g)$), we define functions $f + g$, $f - g$, and fg by the formulas

$$(f + g)(x) = f(x) + g(x)$$
$$(f - g)(x) = f(x) - g(x)$$
$$(fg)(x) = f(x)g(x).$$

Notice that the $+$ sign on the left-hand side of the first equation represents the operation of addition of *functions*, whereas the $+$ on the right-hand side of the equation means addition of the real numbers $f(x)$ and $g(x)$.

At any point of $D(f) \cap D(g)$ at which $g(x) \neq 0$, we can also define the function f/g by the formula

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad (\text{where } g(x) \neq 0).$$

Functions can also be multiplied by constants: If c is a real number, then the function cf is defined for all x in the domain of f by

$$(cf)(x) = cf(x).$$

EXAMPLE 1 The functions defined by the formulas

$$f(x) = \sqrt{x} \qquad \text{and} \qquad g(x) = \sqrt{1 - x}$$

have domains $D(f) = [0, \infty)$ and $D(g) = (-\infty, 1]$. The points common to these domains are the points in

$$[0, \infty) \cap (-\infty, 1] = [0, 1].$$

The following table summarizes the formulas and domains for the various algebraic combinations of the two functions. We also write $f \cdot g$ for the product function fg.

Function	Formula	Domain
$f + g$	$(f + g)(x) = \sqrt{x} + \sqrt{1 - x}$	$[0, 1] = D(f) \cap D(g)$
$f - g$	$(f - g)(x) = \sqrt{x} - \sqrt{1 - x}$	$[0, 1]$
$g - f$	$(g - f)(x) = \sqrt{1 - x} - \sqrt{x}$	$[0, 1]$
$f \cdot g$	$(f \cdot g)(x) = f(x)g(x) = \sqrt{x(1 - x)}$	$[0, 1]$
f/g	$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \sqrt{\frac{x}{1 - x}}$	$[0, 1)$ $(x = 1$ excluded)
g/f	$\frac{g}{f}(x) = \frac{g(x)}{f(x)} = \sqrt{\frac{1 - x}{x}}$	$(0, 1]$ $(x = 0$ excluded)

■

The graph of the function $f + g$ is obtained from the graphs of f and g by adding the corresponding y-coordinates $f(x)$ and $g(x)$ at each point $x \in D(f) \cap D(g)$, as in Figure 1.25. The graphs of $f + g$ and $f \cdot g$ from Example 1 are shown in Figure 1.26.

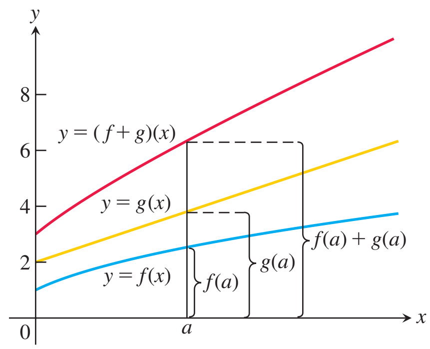


FIGURE 1.25 Graphical addition of two functions.

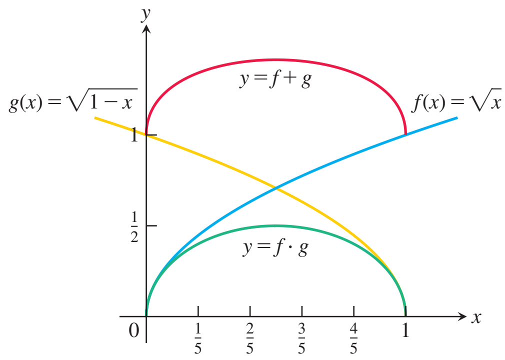


FIGURE 1.26 The domain of the function $f + g$ is the intersection of the domains of f and g, the interval $[0, 1]$ on the x-axis where these domains overlap. This interval is also the domain of the function $f \cdot g$ (Example 1).

Composite Functions

Composition is another method for combining functions. In this operation the output from one function becomes the input to a second function.

> **DEFINITION** If f and g are functions, the **composite** function $f \circ g$ ("f composed with g") is defined by
>
> $$(f \circ g)(x) = f(g(x)).$$
>
> The domain of $f \circ g$ consists of the numbers x in the domain of g for which $g(x)$ lies in the domain of f.

The definition implies that $f \circ g$ can be formed when the range of g lies in the domain of f. To find $(f \circ g)(x)$, *first* find $g(x)$ and *second* find $f(g(x))$. Figure 1.27 pictures $f \circ g$ as a machine diagram, and Figure 1.28 shows the composition as an arrow diagram.

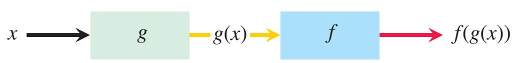


FIGURE 1.27 A composite function $f \circ g$ uses the output $g(x)$ of the first function g as the input for the second function f.

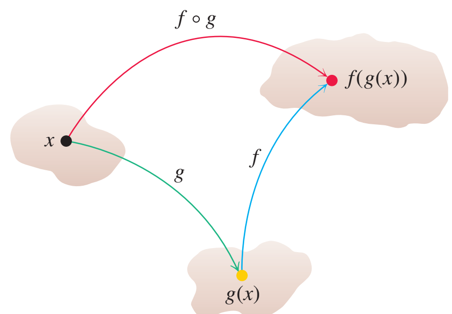


FIGURE 1.28 Arrow diagram for $f \circ g$. If x lies in the domain of g and $g(x)$ lies in the domain of f, then the functions f and g can be composed to form $(f \circ g)(x)$.

To evaluate the composite function $g \circ f$ (when defined), we find $f(x)$ first and then find $g(f(x))$. The domain of $g \circ f$ is the set of numbers x in the domain of f such that $f(x)$ lies in the domain of g.

The functions $f \circ g$ and $g \circ f$ are usually quite different.

EXAMPLE 2 If $f(x) = \sqrt{x}$ and $g(x) = x + 1$, find

(a) $(f \circ g)(x)$ **(b)** $(g \circ f)(x)$ **(c)** $(f \circ f)(x)$ **(d)** $(g \circ g)(x)$.

Solution

	Composition	Domain
(a)	$(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x + 1}$	$[-1, \infty)$
(b)	$(g \circ f)(x) = g(f(x)) = f(x) + 1 = \sqrt{x} + 1$	$[0, \infty)$
(c)	$(f \circ f)(x) = f(f(x)) = \sqrt{f(x)} = \sqrt{\sqrt{x}} = x^{1/4}$	$[0, \infty)$
(d)	$(g \circ g)(x) = g(g(x)) = g(x) + 1 = (x + 1) + 1 = x + 2$	$(-\infty, \infty)$

To see why the domain of $f \circ g$ is $[-1, \infty)$, notice that $g(x) = x + 1$ is defined for all real x but $g(x)$ belongs to the domain of f only if $x + 1 \geq 0$, that is to say, when $x \geq -1$. ■

Notice that if $f(x) = x^2$ and $g(x) = \sqrt{x}$, then $(f \circ g)(x) = \left(\sqrt{x}\right)^2 = x$. However, the domain of $f \circ g$ is $[0, \infty)$, not $(-\infty, \infty)$, since $\sqrt{x}$ requires $x \geq 0$.

Shifting a Graph of a Function

A common way to obtain a new function from an existing one is by adding a constant to each output of the existing function, or to its input variable. The graph of the new function is the graph of the original function shifted vertically or horizontally, as follows.

Shift Formulas

Vertical Shifts

$y = f(x) + k$ Shifts the graph of f *up* k units if $k > 0$
Shifts it *down* $|k|$ units if $k < 0$

Horizontal Shifts

$y = f(x + h)$ Shifts the graph of f *left* h units if $h > 0$
Shifts it *right* $|h|$ units if $h < 0$

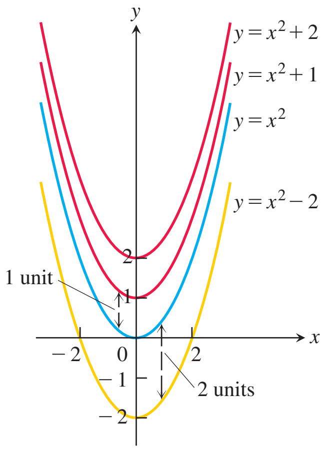


FIGURE 1.29 To shift the graph of $f(x) = x^2$ up (or down), we add positive (or negative) constants to the formula for f (Examples 3a and b).

EXAMPLE 3

(a) Adding 1 to the right-hand side of the formula $y = x^2$ to get $y = x^2 + 1$ shifts the graph up 1 unit (Figure 1.29).

(b) Adding -2 to the right-hand side of the formula $y = x^2$ to get $y = x^2 - 2$ shifts the graph down 2 units (Figure 1.29).

(c) Adding 3 to x in $y = x^2$ to get $y = (x + 3)^2$ shifts the graph 3 units to the left, while adding -2 shifts the graph 2 units to the right (Figure 1.30).

(d) Adding -2 to x in $y = |x|$, and then adding -1 to the result, gives $y = |x - 2| - 1$ and shifts the graph 2 units to the right and 1 unit down (Figure 1.31). ■

Scaling and Reflecting a Graph of a Function

To scale the graph of a function $y = f(x)$ is to stretch or compress it, vertically or horizontally. This is accomplished by multiplying the function f, or the independent variable x, by an appropriate constant c. Reflections across the coordinate axes are special cases where $c = -1$.

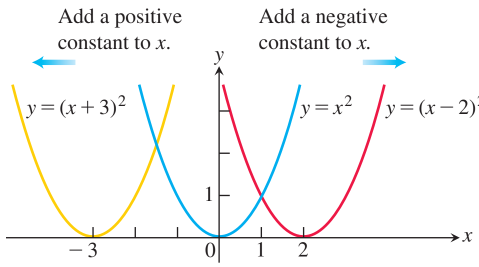


FIGURE 1.30 To shift the graph of $y = x^2$ to the left, we add a positive constant to x (Example 3c). To shift the graph to the right, we add a negative constant to x.

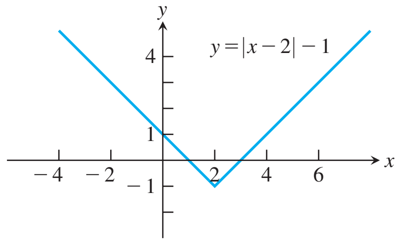


FIGURE 1.31 The graph of $y = |x|$ shifted 2 units to the right and 1 unit down (Example 3d).

Vertical and Horizontal Scaling and Reflecting Formulas

For $c > 1$, the graph is scaled:

$y = cf(x)$	Stretches the graph of f vertically by a factor of c.
$y = \frac{1}{c}f(x)$	Compresses the graph of f vertically by a factor of c.
$y = f(cx)$	Compresses the graph of f horizontally by a factor of c.
$y = f(x/c)$	Stretches the graph of f horizontally by a factor of c.

For $c = -1$, the graph is reflected:

$y = -f(x)$	Reflects the graph of f across the x-axis.
$y = f(-x)$	Reflects the graph of f across the y-axis.

EXAMPLE 4 Here we scale and reflect the graph of $y = \sqrt{x}$.

(a) **Vertical:** Multiplying the right-hand side of $y = \sqrt{x}$ by 3 to get $y = 3\sqrt{x}$ stretches the graph vertically by a factor of 3, whereas multiplying by $1/3$ compresses the graph vertically by a factor of 3 (Figure 1.32).

(b) **Horizontal:** The graph of $y = \sqrt{3x}$ is a horizontal compression of the graph of $y = \sqrt{x}$ by a factor of 3, and $y = \sqrt{x/3}$ is a horizontal stretching by a factor of 3 (Figure 1.33). Note that $y = \sqrt{3x} = \sqrt{3}\sqrt{x}$ so a horizontal compression *may* correspond to a vertical stretching by a different scaling factor. Likewise, a horizontal stretching may correspond to a vertical compression by a different scaling factor.

(c) **Reflection:** The graph of $y = -\sqrt{x}$ is a reflection of $y = \sqrt{x}$ across the x-axis, and $y = \sqrt{-x}$ is a reflection across the y-axis (Figure 1.34). ■

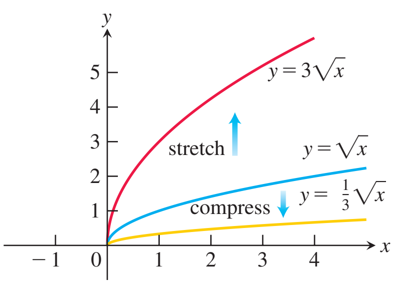


FIGURE 1.32 Vertically stretching and compressing the graph $y = \sqrt{x}$ by a factor of 3 (Example 4a).

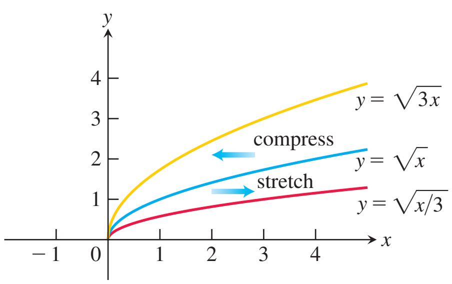


FIGURE 1.33 Horizontally stretching and compressing the graph $y = \sqrt{x}$ by a factor of 3 (Example 4b).

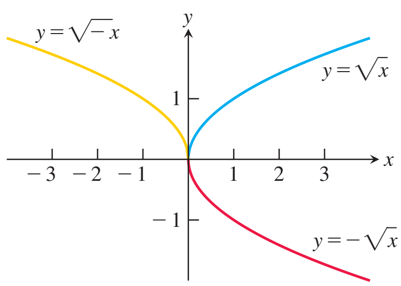


FIGURE 1.34 Reflections of the graph $y = \sqrt{x}$ across the coordinate axes (Example 4c).

EXAMPLE 5 Given the function $f(x) = x^4 - 4x^3 + 10$ (Figure 1.35a), find formulas to

(a) compress the graph horizontally by a factor of 2 followed by a reflection across the y-axis (Figure 1.35b).

(b) compress the graph vertically by a factor of 2 followed by a reflection across the x-axis (Figure 1.35c).

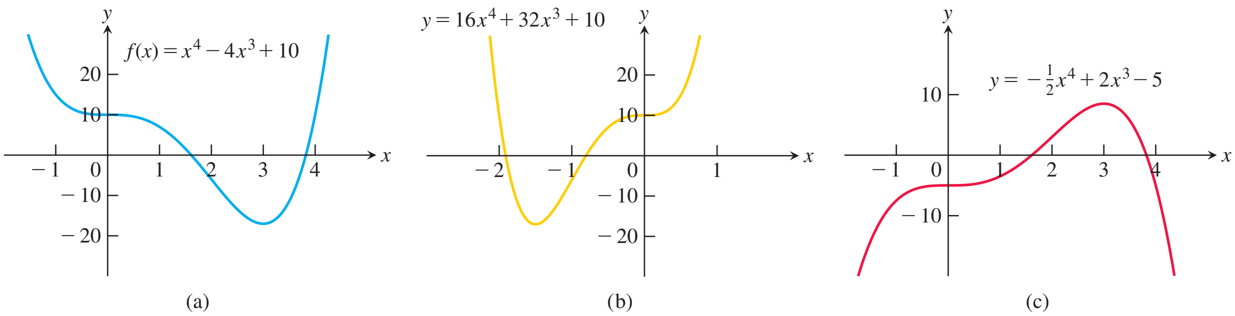


FIGURE 1.35 (a) The original graph of f. (b) The horizontal compression of $y = f(x)$ in part (a) by a factor of 2, followed by a reflection across the y-axis. (c) The vertical compression of $y = f(x)$ in part (a) by a factor of 2, followed by a reflection across the x-axis (Example 5).

Solution

(a) We multiply x by 2 to get the horizontal compression, and by -1 to give reflection across the y-axis. The formula is obtained by substituting $-2x$ for x in the right-hand side of the equation for f:

$$\begin{aligned} y &= f(-2x) = (-2x)^4 - 4(-2x)^3 + 10 \\ &= 16x^4 + 32x^3 + 10. \end{aligned}$$

(b) The formula is

$$y = -\frac{1}{2}f(x) = -\frac{1}{2}x^4 + 2x^3 - 5.$$ ■

EXERCISES 1.2

Algebraic Combinations

In Exercises 1 and 2, find the domains and ranges of f, g, $f + g$, and $f \cdot g$.

1. $f(x) = x, \quad g(x) = \sqrt{x - 1}$

2. $f(x) = \sqrt{x + 1}, \quad g(x) = \sqrt{x - 1}$

In Exercises 3 and 4, find the domains and ranges of f, g, f/g, and g/f.

3. $f(x) = 2, \quad g(x) = x^2 + 1$

4. $f(x) = 1, \quad g(x) = 1 + \sqrt{x}$

Compositions of Functions

5. If $f(x) = x + 5$ and $g(x) = x^2 - 3$, find the following.

a. $f(g(0))$ **b.** $g(f(0))$
c. $f(g(x))$ **d.** $g(f(x))$
e. $f(f(-5))$ **f.** $g(g(2))$
g. $f(f(x))$ **h.** $g(g(x))$

6. If $f(x) = x - 1$ and $g(x) = 1/(x + 1)$, find the following.

a. $f(g(1/2))$ **b.** $g(f(1/2))$
c. $f(g(x))$ **d.** $g(f(x))$
e. $f(f(2))$ **f.** $g(g(2))$
g. $f(f(x))$ **h.** $g(g(x))$

In Exercises 7–10, write a formula for $f \circ g \circ h$.

7. $f(x) = x + 1, \quad g(x) = 3x, \quad h(x) = 4 - x$

8. $f(x) = 3x + 4, \quad g(x) = 2x - 1, \quad h(x) = x^2$

9. $f(x) = \sqrt{x + 1}, \quad g(x) = \dfrac{1}{x + 4}, \quad h(x) = \dfrac{1}{x}$

10. $f(x) = \dfrac{x + 2}{3 - x}, \quad g(x) = \dfrac{x^2}{x^2 + 1}, \quad h(x) = \sqrt{2 - x}$

Let $f(x) = x - 3$, $g(x) = \sqrt{x}$, $h(x) = x^3$, and $j(x) = 2x$. Express each of the functions in Exercises 11 and 12 as a composition involving one or more of f, g, h, and j.

11. **a.** $y = \sqrt{x} - 3$ **b.** $y = 2\sqrt{x}$
c. $y = x^{1/4}$ **d.** $y = 4x$
e. $y = \sqrt{(x - 3)^3}$ **f.** $y = (2x - 6)^3$

12. **a.** $y = 2x - 3$ **b.** $y = x^{3/2}$
c. $y = x^9$ **d.** $y = x - 6$
e. $y = 2\sqrt{x - 3}$ **f.** $y = \sqrt{x^3 - 3}$

13. Copy and complete the following table.

	$g(x)$	$f(x)$	$(f \circ g)(x)$
a.	$x - 7$	$\sqrt{x}$	?
b.	$x + 2$	$3x$	?
c.	?	$\sqrt{x - 5}$	$\sqrt{x^2 - 5}$
d.	$\dfrac{x}{x - 1}$	$\dfrac{x}{x - 1}$	?
e.	?	$1 + \dfrac{1}{x}$	x
f.	$\dfrac{1}{x}$	?	x

14. Copy and complete the following table.

	$g(x)$	$f(x)$	$(f \circ g)(x)$
a.	$\dfrac{1}{x - 1}$	$\lvert x \rvert$	?
b.	?	$\dfrac{x - 1}{x}$	$\dfrac{x}{x + 1}$
c.	?	$\sqrt{x}$	$\lvert x \rvert$
d.	$\sqrt{x}$	?	$\lvert x \rvert$

15. Evaluate each expression using the given table of values:

x	−2	−1	0	1	2
$f(x)$	1	0	−2	1	2
$g(x)$	2	1	0	−1	0

a. $f(g(-1))$ **b.** $g(f(0))$ **c.** $f(f(-1))$
d. $g(g(2))$ **e.** $g(f(-2))$ **f.** $f(g(1))$

16. Evaluate each expression using the functions

$$f(x) = 2 - x, \quad g(x) = \begin{cases} -x, & -2 \le x < 0 \\ x - 1, & 0 \le x \le 2. \end{cases}$$

a. $f(g(0))$ **b.** $g(f(3))$ **c.** $g(g(-1))$
d. $f(f(2))$ **e.** $g(f(0))$ **f.** $f(g(1/2))$

In Exercises 17 and 18, **(a)** write formulas for $f \circ g$ and $g \circ f$ and find the **(b)** domain and **(c)** range of each.

17. $f(x) = \sqrt{x + 1}, \quad g(x) = \dfrac{1}{x}$

18. $f(x) = x^2, \quad g(x) = 1 - \sqrt{x}$

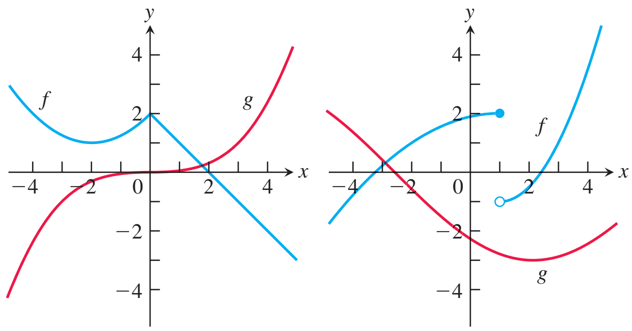
19. Let $f(x) = \dfrac{x}{x - 2}$. Find a function $y = g(x)$ so that $(f \circ g)(x) = x$.

20. Let $f(x) = 2x^3 - 4$. Find a function $y = g(x)$ so that $(f \circ g)(x) = x + 2$.

21. A balloon's volume V is given by $V = s^2 + 2s + 3$ cm^3, where s is the ambient temperature in °C. The ambient temperature s at time t minutes is given by $s = 2t - 3$ °C. Write the balloon's volume V as a function of time t.

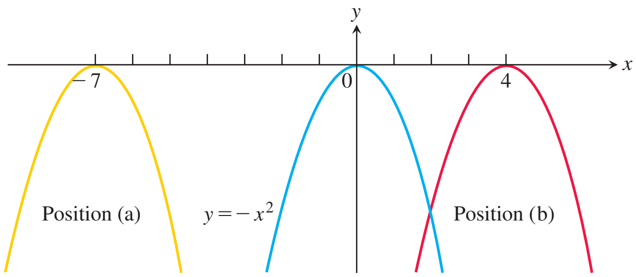
22. Use the graphs of f and g to sketch the graph of $y = f(g(x))$.

a. **b.**

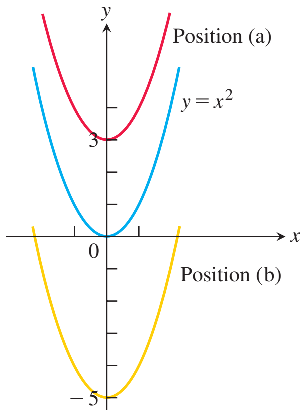


Shifting Graphs

23. The accompanying figure shows the graph of $y = -x^2$ shifted to two new positions. Write equations for the new graphs.



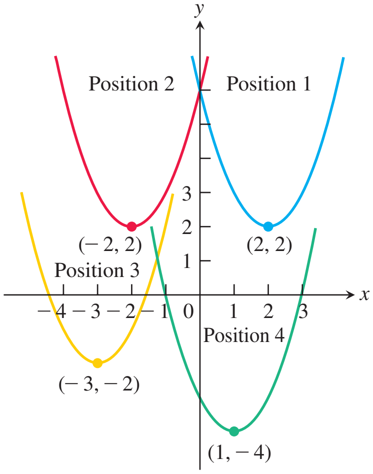
24. The accompanying figure shows the graph of $y = x^2$ shifted to two new positions. Write equations for the new graphs.



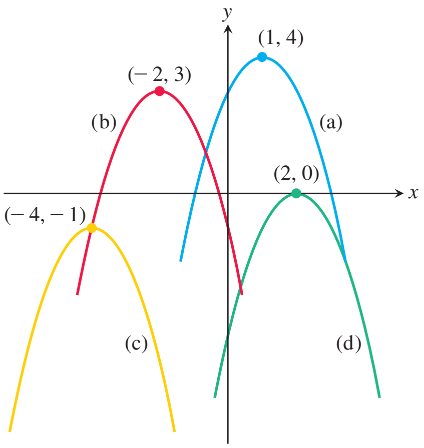
25. Match the equations listed in parts (a)–(d) to the graphs in the accompanying figure.

a. $y = (x - 1)^2 - 4$ **b.** $y = (x - 2)^2 + 2$

c. $y = (x + 2)^2 + 2$ **d.** $y = (x + 3)^2 - 2$



26. The accompanying figure shows the graph of $y = -x^2$ shifted to four new positions. Write an equation for each new graph.



Exercises 27–36 tell how many units and in what directions the graphs of the given equations are to be shifted. Give an equation for the shifted graph. Then sketch the original and shifted graphs together, labeling each graph with its equation.

27. $x^2 + y^2 = 49$ Down 3, left 2

28. $x^2 + y^2 = 25$ Up 3, left 4

29. $y = x^3$ Left 1, down 1

30. $y = x^{2/3}$ Right 1, down 1

31. $y = \sqrt{x}$ Left 0.81

32. $y = -\sqrt{x}$ Right 3

33. $y = 2x - 7$ Up 7

34. $y = \frac{1}{2}(x + 1) + 5$ Down 5, right 1

35. $y = 1/x$ Up 1, right 1

36. $y = 1/x^2$ Left 2, down 1

Graph the functions in Exercises 37–56.

37. $y = \sqrt{x + 4}$ **38.** $y = \sqrt{9 - x}$

39. $y = |x - 2|$ **40.** $y = |1 - x| - 1$

41. $y = 1 + \sqrt{x - 1}$ **42.** $y = 1 - \sqrt{x}$

43. $y = (x + 1)^{2/3}$ **44.** $y = (x - 8)^{2/3}$

45. $y = 1 - x^{2/3}$ **46.** $y + 4 = x^{2/3}$

47. $y = \sqrt[3]{x - 1} - 1$ **48.** $y = (x + 2)^{3/2} + 1$

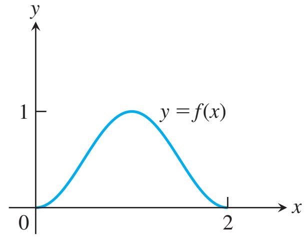
49. $y = \dfrac{1}{x - 2}$ **50.** $y = \dfrac{1}{x} - 2$

51. $y = \dfrac{1}{x} + 2$ **52.** $y = \dfrac{1}{x + 2}$

53. $y = \dfrac{1}{(x - 1)^2}$ **54.** $y = \dfrac{1}{x^2} - 1$

55. $y = \dfrac{1}{x^2} + 1$ **56.** $y = \dfrac{1}{(x + 1)^2}$

57. The accompanying figure shows the graph of a function $f(x)$ with domain $[0, 2]$ and range $[0, 1]$. Find the domains and ranges of the following functions, and sketch their graphs.



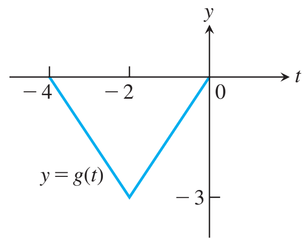
a. $f(x) + 2$ **b.** $f(x) - 1$

c. $2f(x)$ **d.** $-f(x)$

e. $f(x + 2)$ **f.** $f(x - 1)$

g. $f(-x)$ **h.** $-f(x + 1) + 1$

58. The accompanying figure shows the graph of a function $g(t)$ with domain $[-4, 0]$ and range $[-3, 0]$. Find the domains and ranges of the following functions, and sketch their graphs.



a. $g(-t)$ **b.** $-g(t)$

c. $g(t) + 3$ **d.** $1 - g(t)$

e. $g(-t + 2)$ **f.** $g(t - 2)$

g. $g(1 - t)$ **h.** $-g(t - 4)$

Vertical and Horizontal Scaling

Exercises 59–68 tell by what factor and direction the graphs of the given functions are to be stretched or compressed. Give an equation for the stretched or compressed graph.

59. $y = x^2 - 1$, stretched vertically by a factor of 3

60. $y = x^2 - 1$, compressed horizontally by a factor of 2

61. $y = 1 + \dfrac{1}{x^2}$, compressed vertically by a factor of 2

62. $y = 1 + \frac{1}{x^2}$, stretched horizontally by a factor of 3

63. $y = \sqrt{x + 1}$, compressed horizontally by a factor of 4

64. $y = \sqrt{x + 1}$, stretched vertically by a factor of 3

65. $y = \sqrt{4 - x^2}$, stretched horizontally by a factor of 2

66. $y = \sqrt{4 - x^2}$, compressed vertically by a factor of 3

67. $y = 1 - x^3$, compressed horizontally by a factor of 3

68. $y = 1 - x^3$, stretched horizontally by a factor of 2

Graphing

In Exercises 69–76, graph each function, not by plotting points, but by starting with the graph of one of the standard functions presented in Figures 1.14–1.17 and applying an appropriate transformation.

69. $y = -\sqrt{2x + 1}$

70. $y = \sqrt{1 - \frac{x}{2}}$

71. $y = (x - 1)^3 + 2$

72. $y = (1 - x)^3 + 2$

73. $y = \frac{1}{2x} - 1$

74. $y = \frac{2}{x^2} + 1$

75. $y = -\sqrt[3]{x}$

76. $y = (-2x)^{2/3}$

77. Graph the function $y = |x^2 - 1|$.

78. Graph the function $y = \sqrt{|x|}$.

Combining Functions

79. Assume that f is an even function, g is an odd function, and both f and g are defined on the entire real line $(-\infty, \infty)$. Which of the following (where defined) are even? odd?

a. fg **b.** f/g **c.** g/f
d. $f^2 = ff$ **e.** $g^2 = gg$ **f.** $f \circ g$
g. $g \circ f$ **h.** $f \circ f$ **i.** $g \circ g$

80. Can a function be both even and odd? Give reasons for your answer.

T 81. (*Continuation of* Example 1.) Graph the functions $f(x) = \sqrt{x}$ and $g(x) = \sqrt{1 - x}$ together with their (a) sum, (b) product, (c) two differences, (d) two quotients.

T 82. Let $f(x) = x - 7$ and $g(x) = x^2$. Graph f and g together with $f \circ g$ and $g \circ f$.

1.3 Trigonometric Functions

This section reviews radian measure and the basic trigonometric functions.

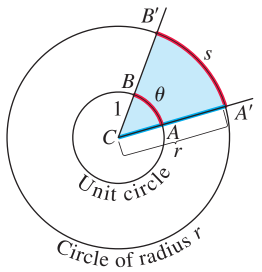


FIGURE 1.36 The radian measure of the central angle $A'CB'$ is the number $\theta = s/r$. For a unit circle of radius $r = 1$, θ is the length of arc AB that central angle ACB cuts from the unit circle.

Angles

Angles are measured in degrees or radians. The number of **radians** in the central angle $A'CB'$ within a circle of radius r is defined as the number of "radius units" contained in the arc s subtended by that central angle. If we denote this central angle by θ when measured in radians, this means that $\theta = s/r$ (Figure 1.36), or

$$s = r\theta \qquad (\theta \text{ in radians}). \tag{1}$$

If the circle is a unit circle having radius $r = 1$, then from Figure 1.36 and Equation (1), we see that the central angle θ measured in radians is just the length of the arc that the angle cuts from the unit circle. Since one complete revolution of the unit circle is 360° or 2π radians, we have

$$\pi \text{ radians} = 180° \tag{2}$$

and

$$1 \text{ radian} = \frac{180}{\pi} (\approx 57.3) \text{ degrees} \qquad \text{or} \qquad 1 \text{ degree} = \frac{\pi}{180} (\approx 0.017) \text{ radians}.$$

Table 1.1 shows the equivalence between degree and radian measures for some basic angles.

TABLE 1.1 Angles measured in degrees and radians

Degrees	−180	−135	−90	−45	0	30	45	60	90	120	135	150	180	270	360
θ (radians)	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

An angle in the xy-plane is said to be in **standard position** if its vertex lies at the origin and its initial ray lies along the positive x-axis (Figure 1.37). Angles measured counterclockwise from the positive x-axis are assigned positive measures; angles measured clockwise are assigned negative measures.

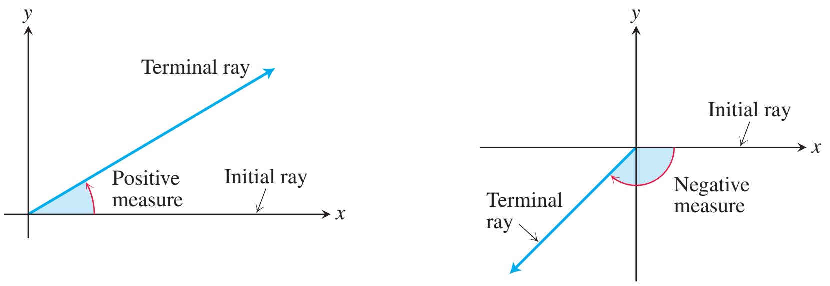


FIGURE 1.37 Angles in standard position in the xy-plane.

Angles describing counterclockwise rotations can go arbitrarily far beyond 2π radians or 360°. Similarly, angles describing clockwise rotations can have negative measures of all sizes (Figure 1.38).

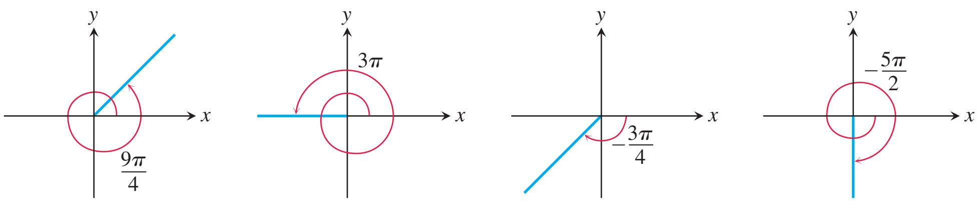
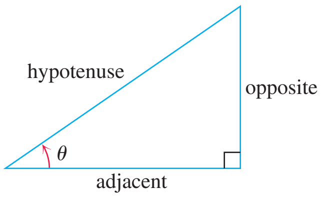


FIGURE 1.38 Nonzero radian measures can be positive or negative and can go beyond 2π.



$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$
$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$
$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

FIGURE 1.39 Trigonometric ratios of an acute angle.

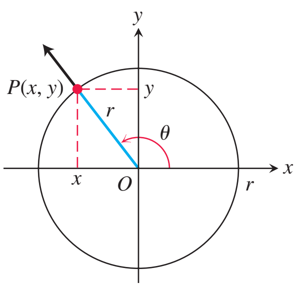


FIGURE 1.40 The trigonometric functions of a general angle θ are defined in terms of x, y, and r.

Angle Convention: Use Radians From now on, in this book it is assumed that all angles are measured in radians unless degrees or some other unit is stated explicitly. When we talk about the angle $\pi/3$, we mean $\pi/3$ radians (which is 60°), not $\pi/3$ degrees. Using radians simplifies many of the operations and computations in calculus.

The Six Basic Trigonometric Functions

The trigonometric functions of an acute angle are given in terms of the sides of a right triangle (Figure 1.39). We extend this definition to obtuse and negative angles by first placing the angle in standard position in a circle of radius r. We then define the trigonometric functions in terms of the coordinates of the point $P(x, y)$ where the angle's terminal ray intersects the circle (Figure 1.40).

$$\textbf{sine:} \quad \sin\theta = \frac{y}{r} \qquad \textbf{cosecant:} \quad \csc\theta = \frac{r}{y}$$
$$\textbf{cosine:} \quad \cos\theta = \frac{x}{r} \qquad \textbf{secant:} \quad \sec\theta = \frac{r}{x}$$
$$\textbf{tangent:} \quad \tan\theta = \frac{y}{x} \qquad \textbf{cotangent:} \quad \cot\theta = \frac{x}{y}$$

These extended definitions agree with the right-triangle definitions when the angle is acute.

Notice also that whenever the quotients are defined,

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$
$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

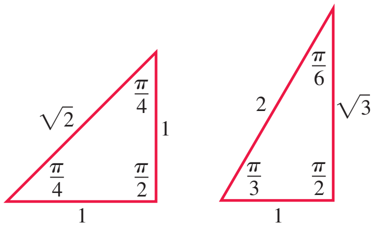


FIGURE 1.41 Radian angles and side lengths of two common triangles.

As you can see, tan θ and sec θ are not defined if $x = \cos\theta = 0$. This means they are not defined if θ is $\pm\pi/2, \pm 3\pi/2, \ldots$. Similarly, cot θ and csc θ are not defined for values of θ for which $y = 0$, namely $\theta = 0, \pm\pi, \pm 2\pi, \ldots$.

The exact values of these trigonometric ratios for some angles can be read from the triangles in Figure 1.41. For instance,

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \sin\frac{\pi}{6} = \frac{1}{2} \qquad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad \cos\frac{\pi}{3} = \frac{1}{2}$$

$$\tan\frac{\pi}{4} = 1 \qquad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \qquad \tan\frac{\pi}{3} = \sqrt{3}$$

The ASTC rule (Figure 1.42) is useful for remembering when the basic trigonometric functions are positive or negative. For instance, from the triangle in Figure 1.43, we see that

$$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}, \qquad \cos\frac{2\pi}{3} = -\frac{1}{2}, \qquad \tan\frac{2\pi}{3} = -\sqrt{3}.$$

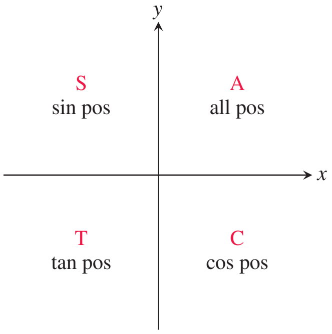


FIGURE 1.42 The ASTC rule, remembered by the statement "All Students Take Calculus," tells which trigonometric functions are positive in each quadrant.

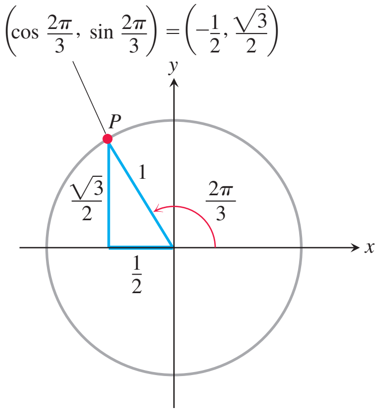


FIGURE 1.43 The triangle for calculating the sine and cosine of $2\pi/3$ radians. The side lengths come from the geometry of right triangles.

Using a similar method we obtain the values of sin θ, cos θ, and tan θ shown in Table 1.2.

TABLE 1.2 **Values of sin θ, cos θ, and tan θ for selected values of θ**

Degrees	**−180**	**−135**	**−90**	**−45**	**0**	**30**	**45**	**60**	**90**	**120**	**135**	**150**	**180**	**270**	**360**
θ (radians)	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
sin θ	0	$\frac{-\sqrt{2}}{2}$	-1	$\frac{-\sqrt{2}}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0
cos θ	-1	$\frac{-\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{3}}{2}$	-1	0	1
tan θ	0	1		-1	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$		$-\sqrt{3}$	-1	$\frac{-\sqrt{3}}{3}$	0		0

Periodicity and Graphs of the Trigonometric Functions

When an angle of measure θ and an angle of measure $\theta + 2\pi$ are in standard position, their terminal rays coincide. The two angles therefore have the same trigonometric function values: $\sin(\theta + 2\pi) = \sin\theta$, $\tan(\theta + 2\pi) = \tan\theta$, and so on. Similarly, $\cos(\theta - 2\pi) = \cos\theta$, $\sin(\theta - 2\pi) = \sin\theta$, and so on. We describe this repeating behavior by saying that the six basic trigonometric functions are *periodic*.

Periods of Trigonometric Functions

Period π: $\tan(x + \pi) = \tan x$
$\cot(x + \pi) = \cot x$

Period 2π: $\sin(x + 2\pi) = \sin x$
$\cos(x + 2\pi) = \cos x$
$\sec(x + 2\pi) = \sec x$
$\csc(x + 2\pi) = \csc x$

DEFINITION A function $f(x)$ is **periodic** if there is a positive number p such that $f(x + p) = f(x)$ for every value of x. The smallest such value of p is the **period** of f.

When we graph trigonometric functions in the coordinate plane, we usually denote the independent variable by x instead of θ. Figure 1.44 shows that the tangent and cotangent functions have period $p = \pi$, and the other four functions have period 2π. Also, the symmetries in these graphs reveal that the cosine and secant functions are even and the other four functions are odd (although this does not prove those results).

Even

$\cos(-x) = \cos x$
$\sec(-x) = \sec x$

Odd

$\sin(-x) = -\sin x$
$\tan(-x) = -\tan x$
$\csc(-x) = -\csc x$
$\cot(-x) = -\cot x$

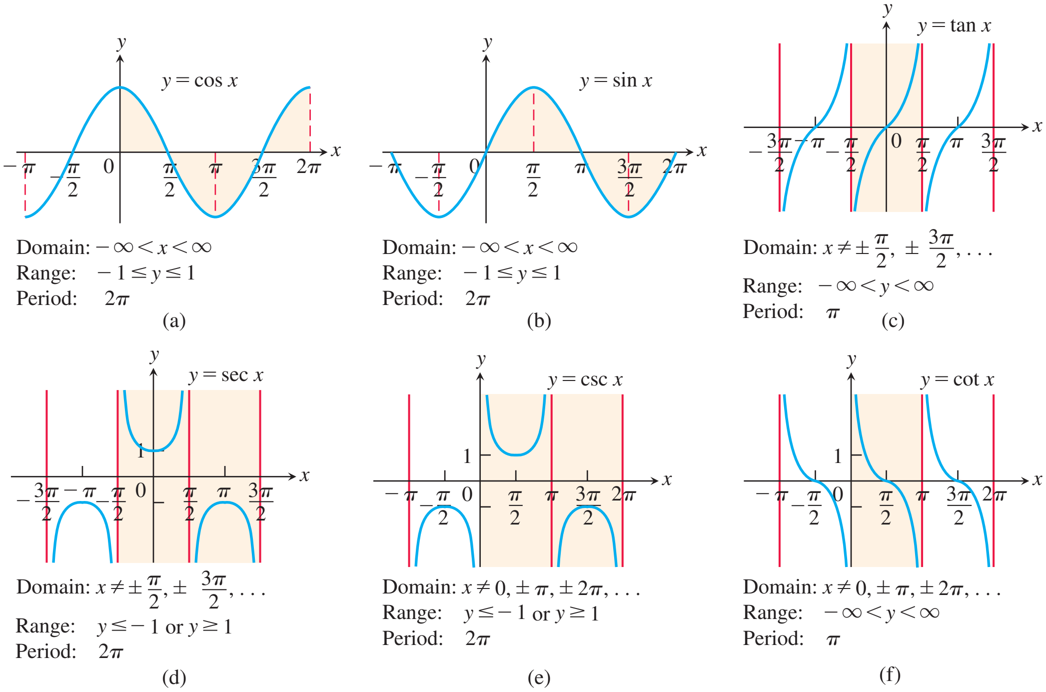


FIGURE 1.44 Graphs of the six basic trigonometric functions using radian measure. The shading for each trigonometric function indicates its periodicity.

Trigonometric Identities

The coordinates of any point $P(x, y)$ in the plane can be expressed in terms of the point's distance r from the origin and the angle θ that ray OP makes with the positive x-axis (Figure 1.40). Since $x/r = \cos\theta$ and $y/r = \sin\theta$, we have

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

When $r = 1$ we can apply the Pythagorean theorem to the reference right triangle in Figure 1.45 and obtain the equation

$$\cos^2\theta + \sin^2\theta = 1. \qquad (3)$$

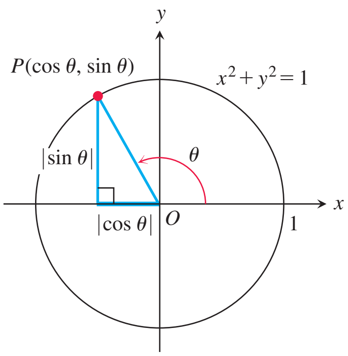


FIGURE 1.45 The reference triangle for a general angle θ.

This equation, true for all values of θ, is the most frequently used identity in trigonometry. Dividing this identity in turn by $\cos^2\theta$ and $\sin^2\theta$ gives

$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

The following formulas hold for all angles A and B (Exercise 58).

Addition Formulas

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$
$$\sin(A + B) = \sin A\cos B + \cos A\sin B \tag{4}$$

There are similar formulas for $\cos(A - B)$ and $\sin(A - B)$ (Exercises 35 and 36). All the trigonometric identities needed in this book derive from Equations (3) and (4). For example, substituting θ for both A and B in the addition formulas gives

Double-Angle Formulas

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\sin 2\theta = 2\sin\theta\cos\theta \tag{5}$$

Additional formulas come from combining the equations

$$\cos^2\theta + \sin^2\theta = 1, \qquad \cos^2\theta - \sin^2\theta = \cos 2\theta.$$

We add the two equations to get $2\cos^2\theta = 1 + \cos 2\theta$ and subtract the second from the first to get $2\sin^2\theta = 1 - \cos 2\theta$. This results in the following identities, which are useful in integral calculus.

Half-Angle Formulas

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2} \tag{6}$$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2} \tag{7}$$

The Law of Cosines

If a, b, and c are sides of a triangle ABC and if θ is the angle opposite c, then

$$c^2 = a^2 + b^2 - 2ab\cos\theta. \tag{8}$$

This equation is called the **law of cosines**.

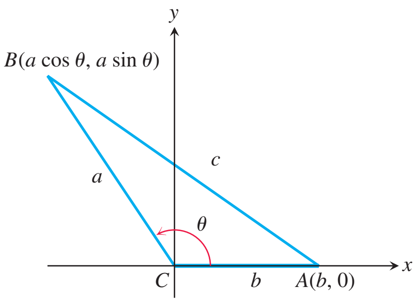


FIGURE 1.46 The square of the distance between A and B gives the law of cosines.

To see why the law holds, we position the triangle in the xy-plane with the origin at C and the positive x-axis along one side of the triangle, as in Figure 1.46. The coordinates of A are $(b, 0)$; the coordinates of B are $(a\cos\theta, a\sin\theta)$. The square of the distance between A and B is therefore

$$\begin{aligned} c^2 &= (a\cos\theta - b)^2 + (a\sin\theta)^2 \\ &= a^2\underbrace{(\cos^2\theta + \sin^2\theta)}_{1} + b^2 - 2ab\cos\theta \\ &= a^2 + b^2 - 2ab\cos\theta. \end{aligned}$$

The law of cosines generalizes the Pythagorean theorem. If $\theta = \pi/2$, then $\cos\theta = 0$ and $c^2 = a^2 + b^2$.

Two Special Inequalities

For any angle θ measured in radians, the sine and cosine functions satisfy

$$-|\theta| \le \sin\theta \le |\theta| \qquad \text{and} \qquad -|\theta| \le 1 - \cos\theta \le |\theta|.$$

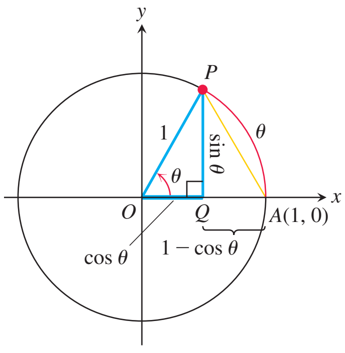


FIGURE 1.47 From the geometry of this figure, drawn for $\theta > 0$, we get the inequality $\sin^2\theta + (1 - \cos\theta)^2 \le \theta^2$.

To establish these inequalities, we picture θ as a nonzero angle in standard position (Figure 1.47). The circle in the figure is a unit circle, so $|\theta|$ equals the length of the circular arc AP. The length of line segment AP is therefore less than $|\theta|$.

Triangle APQ is a right triangle with sides of length

$$QP = |\sin\theta|, \qquad AQ = 1 - \cos\theta.$$

From the Pythagorean theorem and the fact that $AP < |\theta|$, we get

$$\sin^2\theta + (1 - \cos\theta)^2 = (AP)^2 \le \theta^2. \tag{9}$$

The terms on the left-hand side of Equation (9) are both positive, so each is smaller than their sum and hence is less than or equal to θ^2:

$$\sin^2\theta \le \theta^2 \qquad \text{and} \qquad (1 - \cos\theta)^2 \le \theta^2.$$

By taking square roots, this is equivalent to saying that

$$|\sin\theta| \le |\theta| \qquad \text{and} \qquad |1 - \cos\theta| \le |\theta|,$$

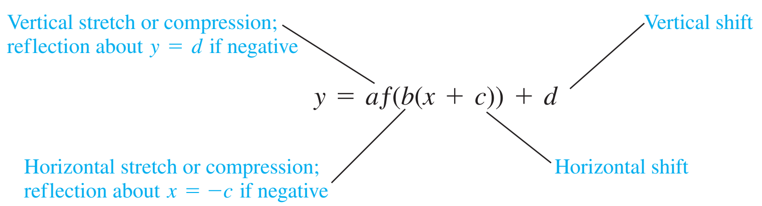
so

$$-|\theta| \le \sin\theta \le |\theta| \qquad \text{and} \qquad -|\theta| \le 1 - \cos\theta \le |\theta|.$$

These inequalities will be useful in the next chapter.

Transformations of Trigonometric Graphs

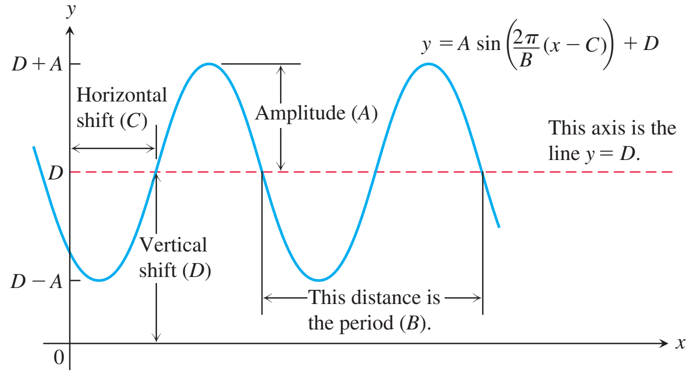
The rules for shifting, stretching, compressing, and reflecting the graph of a function summarized in the following diagram apply to the trigonometric functions we have discussed in this section.



The transformation rules applied to the sine function give the **general sine function** or **sinusoid** formula

$$f(x) = A\sin\left(\frac{2\pi}{B}(x - C)\right) + D,$$

where $|A|$ is the *amplitude*, $|B|$ is the *period*, C is the *horizontal shift*, and D is the *vertical shift*. A graphical interpretation of the various terms is given below.



EXERCISES 1.3

Radians and Degrees

1. On a circle of radius 10 m, how long is an arc that subtends a central angle of (**a**) $4\pi/5$ radians? (**b**) 110°?

2. A central angle in a circle of radius 8 is subtended by an arc of length 10π. Find the angle's radian and degree measures.

3. You want to make an 80° angle by marking an arc on the perimeter of a 12-cm-diameter disk and drawing lines from the ends of the arc to the disk's center. To the nearest millimeter, how long should the arc be?

4. If you roll a 1-m-diameter wheel forward 30 cm over level ground, through what angle will the wheel turn? Answer in radians (to the nearest tenth) and degrees (to the nearest degree).

Evaluating Trigonometric Functions

5. Copy and complete the following table of function values. If the function is undefined at a given angle, enter "UND." Do not use a calculator or tables.

θ	$-\pi$	$-2\pi/3$	0	$\pi/2$	$3\pi/4$
$\sin\theta$					
$\cos\theta$					
$\tan\theta$					
$\cot\theta$					
$\sec\theta$					
$\csc\theta$					

6. Copy and complete the following table of function values. If the function is undefined at a given angle, enter "UND." Do not use a calculator or tables.

θ	$-3\pi/2$	$-\pi/3$	$-\pi/6$	$\pi/4$	$5\pi/6$
$\sin\theta$					
$\cos\theta$					
$\tan\theta$					
$\cot\theta$					
$\sec\theta$					
$\csc\theta$					

In Exercises 7–12, one of $\sin x$, $\cos x$, and $\tan x$ is given. Find the other two if x lies in the specified interval.

7. $\sin x = \frac{3}{5}, \quad x \in \left[\frac{\pi}{2}, \pi\right]$

8. $\tan x = 2, \quad x \in \left[0, \frac{\pi}{2}\right]$

9. $\cos x = \frac{1}{3}, \quad x \in \left[-\frac{\pi}{2}, 0\right]$

10. $\cos x = -\frac{5}{13}, \quad x \in \left[\frac{\pi}{2}, \pi\right]$

11. $\tan x = \frac{1}{2}, \quad x \in \left[\pi, \frac{3\pi}{2}\right]$

12. $\sin x = -\frac{1}{2}, \quad x \in \left[\pi, \frac{3\pi}{2}\right]$

Graphing Trigonometric Functions

Graph the functions in Exercises 13–22. What is the period of each function?

13. $\sin 2x$

14. $\sin(x/2)$

15. $\cos \pi x$

16. $\cos \frac{\pi x}{2}$

17. $-\sin \frac{\pi x}{3}$

18. $-\cos 2\pi x$

19. $\cos\left(x - \frac{\pi}{2}\right)$

20. $\sin\left(x + \frac{\pi}{6}\right)$

21. $\sin\left(x - \frac{\pi}{4}\right) + 1$ **22.** $\cos\left(x + \frac{2\pi}{3}\right) - 2$

Graph the functions in Exercises 23–26 in the *ts*-plane (*t*-axis horizontal, *s*-axis vertical). What is the period of each function? What symmetries do the graphs have?

23. $s = \cot 2t$ **24.** $s = -\tan \pi t$

25. $s = \sec\left(\frac{\pi t}{2}\right)$ **26.** $s = \csc\left(\frac{t}{2}\right)$

T **27. a.** Graph $y = \cos x$ and $y = \sec x$ together for $-3\pi/2 \le x \le 3\pi/2$. Comment on the behavior of $\sec x$ in relation to the signs and values of $\cos x$.

b. Graph $y = \sin x$ and $y = \csc x$ together for $-\pi \le x \le 2\pi$. Comment on the behavior of $\csc x$ in relation to the signs and values of $\sin x$.

T **28.** Graph $y = \tan x$ and $y = \cot x$ together for $-7 \le x \le 7$. Comment on the behavior of $\cot x$ in relation to the signs and values of $\tan x$.

29. Graph $y = \sin x$ and $y = \lfloor \sin x \rfloor$ together. What are the domain and range of $\lfloor \sin x \rfloor$?

30. Graph $y = \sin x$ and $y = \lceil \sin x \rceil$ together. What are the domain and range of $\lceil \sin x \rceil$?

Using the Addition Formulas

Use the addition formulas to derive the identities in Exercises 31–36.

31. $\cos\left(x - \frac{\pi}{2}\right) = \sin x$ **32.** $\cos\left(x + \frac{\pi}{2}\right) = -\sin x$

33. $\sin\left(x + \frac{\pi}{2}\right) = \cos x$ **34.** $\sin\left(x - \frac{\pi}{2}\right) = -\cos x$

35. $\cos(A - B) = \cos A \cos B + \sin A \sin B$ (Exercise 57 provides a different derivation.)

36. $\sin(A - B) = \sin A \cos B - \cos A \sin B$

37. What happens if you take $B = A$ in the trigonometric identity $\cos(A - B) = \cos A \cos B + \sin A \sin B$? Does the result agree with something you already know?

38. What happens if you take $B = 2\pi$ in the addition formulas? Do the results agree with something you already know?

In Exercises 39–42, express the given quantity in terms of $\sin x$ and $\cos x$.

39. $\cos(\pi + x)$ **40.** $\sin(2\pi - x)$

41. $\sin\left(\frac{3\pi}{2} - x\right)$ **42.** $\cos\left(\frac{3\pi}{2} + x\right)$

43. Evaluate $\sin \frac{7\pi}{12}$ as $\sin\left(\frac{\pi}{4} + \frac{\pi}{3}\right)$.

44. Evaluate $\cos \frac{11\pi}{12}$ as $\cos\left(\frac{\pi}{4} + \frac{2\pi}{3}\right)$.

45. Evaluate $\cos \frac{\pi}{12}$. **46.** Evaluate $\sin \frac{5\pi}{12}$.

Using the Half-Angle Formulas

Find the function values in Exercises 47–50.

47. $\cos^2 \frac{\pi}{8}$ **48.** $\cos^2 \frac{5\pi}{12}$

49. $\sin^2 \frac{\pi}{12}$ **50.** $\sin^2 \frac{3\pi}{8}$

Solving Trigonometric Equations

For Exercises 51–54, solve for the angle θ, where $0 \le \theta \le 2\pi$.

51. $\sin^2 \theta = \frac{3}{4}$ **52.** $\sin^2 \theta = \cos^2 \theta$

53. $\sin 2\theta - \cos \theta = 0$ **54.** $\cos 2\theta + \cos \theta = 0$

Theory and Examples

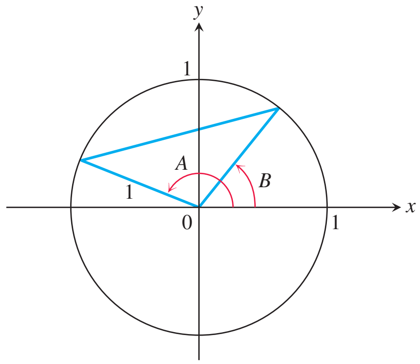
55. The tangent sum formula The standard formula for the tangent of the sum of two angles is

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

Derive the formula.

56. (*Continuation of Exercise 55.*) Derive a formula for $\tan(A - B)$.

57. Apply the law of cosines to the triangle in the accompanying figure to derive the formula for $\cos(A - B)$.



58. a. Apply the formula for $\cos(A - B)$ to the identity $\sin \theta = \cos\left(\frac{\pi}{2} - \theta\right)$ to obtain the addition formula for $\sin(A + B)$.

b. Derive the formula for $\cos(A + B)$ by substituting $-B$ for B in the formula for $\cos(A - B)$ from Exercise 35.

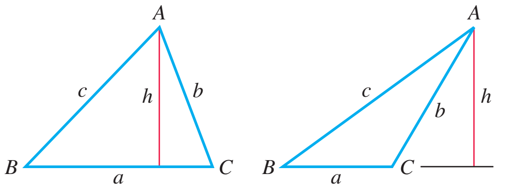
59. A triangle has sides $a = 2$ and $b = 3$ and angle $C = 60°$. Find the length of side c.

60. A triangle has sides $a = 2$ and $b = 3$ and angle $C = 40°$. Find the length of side c.

61. The law of sines *The law of sines* says that if a, b, and c are the sides opposite the angles A, B, and C in a triangle, then

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

Use the accompanying figures and the identity $\sin(\pi - \theta) = \sin \theta$, if required, to derive the law.

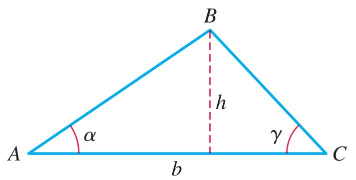


62. A triangle has sides $a = 2$ and $b = 3$ and angle $C = 60°$ (as in Exercise 59). Find the sine of angle B using the law of sines.

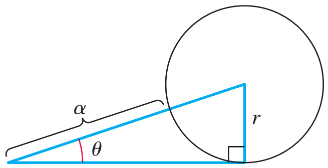
63. A triangle has side $c = 2$ and angles $A = \pi/4$ and $B = \pi/3$. Find the length a of the side opposite A.

64. Consider the length h of the perpendicular from point B to side b in the given triangle. Show that

$$h = \frac{b \tan \alpha \tan \gamma}{\tan \alpha + \tan \gamma}$$



65. Refer to the given figure. Write the radius r of the circle in terms of α and θ.



T 66. **The approximation $\sin x \approx x$** It is often useful to know that, when x is measured in radians, $\sin x \approx x$ for numerically small values of x. In Section 3.11, we will see why the approximation holds. The approximation error is less than 1 in 5000 if $|x| < 0.1$.

a. With your grapher in radian mode, graph $y = \sin x$ and $y = x$ together in a viewing window about the origin. What do you see happening as x nears the origin?

b. With your grapher in degree mode, graph $y = \sin x$ and $y = x$ together about the origin again. How is the picture different from the one obtained with radian mode?

General Sine Curves

For

$$f(x) = A \sin\left(\frac{2\pi}{B}(x - C)\right) + D,$$

identify A, B, C, and D for the sine functions in Exercises 67–70 and sketch their graphs.

67. $y = 2\sin(x + \pi) - 1$

68. $y = \frac{1}{2}\sin(\pi x - \pi) + \frac{1}{2}$

69. $y = -\frac{2}{\pi}\sin\left(\frac{\pi}{2}t\right) + \frac{1}{\pi}$

70. $y = \frac{L}{2\pi}\sin\frac{2\pi t}{L}, \quad L > 0$

COMPUTER EXPLORATIONS

In Exercises 71–74, you will explore graphically the general sine function

$$f(x) = A \sin\left(\frac{2\pi}{B}(x - C)\right) + D$$

as you change the values of the constants A, B, C, and D. Use a CAS or computer grapher to perform the steps in the exercises.

71. **The period B** Set the constants $A = 3$, $C = D = 0$.

a. Plot $f(x)$ for the values $B = 1, 3, 2\pi, 5\pi$ over the interval $-4\pi \le x \le 4\pi$. Describe what happens to the graph of the general sine function as the period increases.

b. What happens to the graph for negative values of B? Try it with $B = -3$ and $B = -2\pi$.

72. **The horizontal shift C** Set the constants $A = 3$, $B = 6$, $D = 0$.

a. Plot $f(x)$ for the values $C = 0$, 1, and 2 over the interval $-4\pi \le x \le 4\pi$. Describe what happens to the graph of the general sine function as C increases through positive values.

b. What happens to the graph for negative values of C?

c. What smallest positive value should be assigned to C so the graph exhibits no horizontal shift? Confirm your answer with a plot.

73. **The vertical shift D** Set the constants $A = 3$, $B = 6$, $C = 0$.

a. Plot $f(x)$ for the values $D = 0$, 1, and 3 over the interval $-4\pi \le x \le 4\pi$. Describe what happens to the graph of the general sine function as D increases through positive values.

b. What happens to the graph for negative values of D?

74. **The amplitude A** Set the constants $B = 6$, $C = D = 0$.

a. Describe what happens to the graph of the general sine function as A increases through positive values. Confirm your answer by plotting $f(x)$ for the values $A = 1$, 5, and 9.

b. What happens to the graph for negative values of A?

1.4 Exponential Functions

Exponential functions occur in a wide variety of applications, including interest rates, radioactive decay, population growth, the spread of a disease, consumption of natural resources, the earth's atmospheric pressure, temperature change of a heated object placed in a cooler environment, and the dating of fossils. In this section we introduce these functions informally, using an intuitive approach. We give a rigorous development of them in Chapter 7, based on the ideas of integral calculus.

Exponential Behavior

When a positive quantity P doubles, it increases by a factor of 2 and the quantity becomes $2P$. If it doubles again, it becomes $2(2P) = 2^2P$, and a third doubling gives $2(2^2P) = 2^3P$. Continuing to double in this fashion leads us to consider the function $f(x) = 2^x$. We call

Don't confuse the exponential 2^x with the power function x^2. In the exponential, the variable x is in the exponent, whereas the variable x is the base in the power function.

this an *exponential* function because the variable x appears in the exponent of 2^x. Functions such as $g(x) = 10^x$ and $h(x) = (1/2)^x$ are other examples of exponential functions. In general, if $a \neq 1$ is a positive constant, the function

$$f(x) = a^x, \quad a > 0$$

is the **exponential function with base *a***.

EXAMPLE 1 In 2014, \$100 is invested in a savings account, where it grows by accruing interest that is compounded annually (once a year) at an interest rate of 5.5%. Assuming no additional funds are deposited to the account and no money is withdrawn, give a formula for a function describing the amount A in the account after x years have elapsed.

Solution If $P = 100$, at the end of the first year the amount in the account is the original amount plus the interest accrued, or

$$P + \left(\frac{5.5}{100}\right)P = (1 + 0.055)P = (1.055)P.$$

At the end of the second year the account earns interest again and grows to

$$(1 + 0.055) \cdot (1.055P) = (1.055)^2 P = 100 \cdot (1.055)^2. \qquad P = 100$$

Continuing this process, after x years the value of the account is

$$A = 100 \cdot (1.055)^x.$$

This is a multiple of the exponential function $f(x) = (1.055)^x$ with base 1.055. Table 1.3 shows the amounts accrued over the first four years. Notice that the amount in the account each year is always 1.055 times its value in the previous year.

TABLE 1.3 Savings account growth

Year	Amount (dollars)	Yearly increase
2014	100	
2015	$100(1.055) = 105.50$	5.50
2016	$100(1.055)^2 = 111.30$	5.80
2017	$100(1.055)^3 = 117.42$	6.12
2018	$100(1.055)^4 = 123.88$	6.46

In general, the amount after x years is given by $P(1 + r)^x$, where P is the starting amount and r is the interest rate (expressed as a decimal). ■

For integer and rational exponents, the value of an exponential function $f(x) = a^x$ is obtained arithmetically by taking an appropriate number of products, quotients, or roots. If $x = n$ is a positive integer, the number a^n is given by multiplying a by itself n times:

$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}.$$

If $x = 0$, then we set $a^0 = 1$, and if $x = -n$ for some positive integer n, then

$$a^{-n} = \frac{1}{a^n} = \left(\frac{1}{a}\right)^n.$$

If $x = 1/n$ for some positive integer n, then

$$a^{1/n} = \sqrt[n]{a},$$

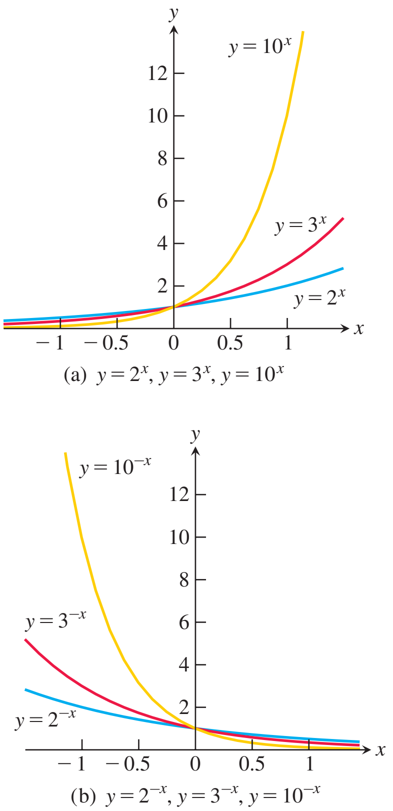


FIGURE 1.48 Graphs of exponential functions.

which is the positive number that when multiplied by itself n times gives a. If $x = p/q$ is any rational number, then

$$a^{p/q} = \sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p.$$

When x is *irrational*, the meaning of a^x is not immediately apparent. The value of a^x can be approximated by raising a to rational numbers that get closer and closer to the irrational number x. We will describe this informally now and will give a rigorous definition in Chapter 7.

The graphs of several exponential functions are shown in Figure 1.48. These graphs show the values of the exponential functions for real inputs x. We choose the value of a^x when x is irrational so that there are no "holes" or "jumps" in the graph of a^x (these words are not rigorous mathematical terms, but they informally convey the underlying idea). The value of a^x when x is irrational is chosen so that the function $f(x) = a^x$ is *continuous*, a notion that will be carefully developed in Chapter 2. This choice ensures that the graph is increasing when $a > 1$ and is decreasing when $0 < a < 1$ (see Figure 1.48).

We illustrate how to define the value of an exponential function at an irrational power using the exponential function $f(x) = 2^x$. How do we make sense of the expression $2^{\sqrt{3}}$? Any particular irrational number, say $x = \sqrt{3}$, has a decimal expansion

$$\sqrt{3} = 1.732050808\ldots.$$

We then consider the list of powers of 2 with more and more digits in the decimal expansion,

$$2^1, 2^{1.7}, 2^{1.73}, 2^{1.732}, 2^{1.7320}, 2^{1.73205}, \ldots. \qquad (1)$$

We know the meaning of each number in list (1) because the successive decimal approximations to $\sqrt{3}$ given by 1, 1.7, 1.73, 1.732, and so on are all *rational* numbers. As these decimal approximations get closer and closer to $\sqrt{3}$, it seems reasonable that the list of numbers in (1) gets closer and closer to some fixed number, which we specify to be $2^{\sqrt{3}}$.

Table 1.4 illustrates how taking better approximations to $\sqrt{3}$ gives better approximations to the number $2^{\sqrt{3}} \approx 3.321997086$. It is the *completeness property* of the real numbers (discussed in Appendix 7) which guarantees that this procedure gives a single number we define to be $2^{\sqrt{3}}$ (although it is beyond the scope of this text to give a proof). In a similar way, we can identify the number 2^x (or a^x, $a > 0$) for any irrational x. By identifying the number a^x for both rational and irrational x, we eliminate any "holes" or "gaps" in the graph of a^x.

TABLE 1.4 Values of $2^{\sqrt{3}}$ for rational r closer and closer to $\sqrt{3}$

r	2^r
1.0	2.000000000
1.7	3.249009585
1.73	3.317278183
1.732	3.321880096
1.7320	3.321880096
1.73205	3.321995226
1.732050	3.321995226
1.7320508	3.321997068
1.73205080	3.321997068
1.732050808	3.321997086

Rules for Exponents

If $a > 0$ and $b > 0$, the following rules hold for all real numbers x and y.

1. $a^x \cdot a^y = a^{x+y}$

2. $\dfrac{a^x}{a^y} = a^{x-y}$

3. $(a^x)^y = (a^y)^x = a^{xy}$

4. $a^x \cdot b^x = (ab)^x$

5. $\dfrac{a^x}{b^x} = \left(\dfrac{a}{b}\right)^x$

Exponential functions obey the rules of exponents listed below. It is easy to check these rules using algebra when the exponents are integers or rational numbers. We prove them for all real exponents in Chapter 7.

EXAMPLE 2 We use the rules for exponents to simplify some numerical expressions.

1. $3^{1.1} \cdot 3^{0.7} = 3^{1.1+0.7} = 3^{1.8}$ Rule 1

2. $\dfrac{\left(\sqrt{10}\right)^3}{\sqrt{10}} = \left(\sqrt{10}\right)^{3-1} = \left(\sqrt{10}\right)^2 = 10$ Rule 2

3. $\left(5^{\sqrt{2}}\right)^{\sqrt{2}} = 5^{\sqrt{2}\cdot\sqrt{2}} = 5^2 = 25$ Rule 3

4. $7^{\pi} \cdot 8^{\pi} = (56)^{\pi}$ Rule 4

5. $\left(\frac{4}{9}\right)^{1/2} = \frac{4^{1/2}}{9^{1/2}} = \frac{2}{3}$ Rule 5

The Natural Exponential Function e^x

The most important exponential function used for modeling natural, physical, and economic phenomena is the **natural exponential function**, whose base is the special number e. The number e is irrational, and its value to nine decimal places is 2.718281828. (In Section 3.8 we will see a way to calculate the value of e.) It might seem strange that we would use this number for a base rather than a simple number like 2 or 10. The advantage in using e as a base is that it greatly simplifies many of the calculations in calculus.

In Figure 1.48a you can see that the graphs of the exponential functions $y = a^x$ get steeper as the base a gets larger. This idea of steepness is conveyed by the slope of the tangent line to the graph at a point. Tangent lines to graphs of functions are defined precisely in the next chapter, but intuitively the tangent line to the graph at a point is the line that best approximates the graph at the point, like a tangent to a circle. Figure 1.49 shows the slope of the graph of $y = a^x$ as it crosses the y-axis for several values of a. Notice that the slope is exactly equal to 1 when a equals the number e. The slope is smaller than 1 if $a < e$, and larger than 1 if $a > e$. The graph of $y = e^x$ has slope 1 when it crosses the y-axis.

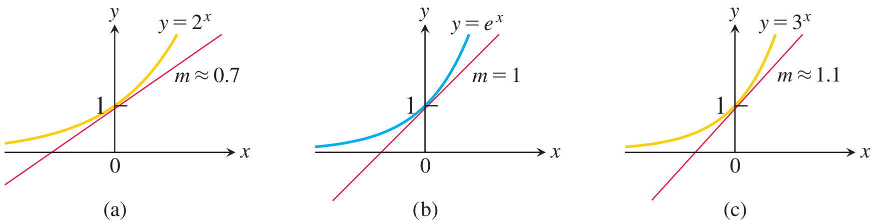


FIGURE 1.49 Among the exponential functions, the graph of $y = e^x$ has the property that the slope m of the tangent line to the graph is exactly 1 when it crosses the y-axis. The slope is smaller for a base less than e, such as 2^x, and larger for a base greater than e, such as 3^x.

Exponential Growth and Decay

The function $y = y_0 e^{kx}$, where k is a nonzero constant, is a model for **exponential growth** if $k > 0$ and a model for **exponential decay** if $k < 0$. Here y_0 is a constant that represents the value of the function when $x = 0$. An example of exponential growth occurs when computing interest **compounded continuously.** This is modeled by the formula $y = Pe^{rt}$, where P is the initial monetary investment, r is the interest rate as a decimal, and t is time in units consistent with r. An example of exponential decay is the model $y = Ae^{-1.2\times10^{-4}t}$, which represents how the radioactive isotope carbon-14 decays over time. Here A is the original amount of carbon-14 and t is the time in years. Carbon-14 decay is used to date

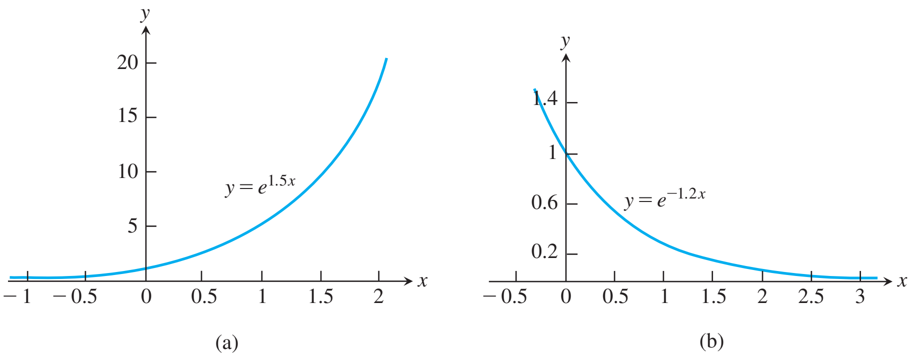


FIGURE 1.50 Graphs of (a) exponential growth, $k = 1.5 > 0$, and (b) exponential decay, $k = -1.2 < 0$.

the remains of dead organisms such as shells, seeds, and wooden artifacts. Figure 1.50 shows graphs of exponential growth and exponential decay.

EXAMPLE 3 Investment companies often use the model $y = Pe^{rt}$ in calculating the growth of an investment. Use this model to track the growth of \$100 invested in 2014 at an annual interest rate of 5.5%.

Solution Let $t = 0$ represent 2014, $t = 1$ represent 2015, and so on. Then the exponential growth model is $y(t) = Pe^{rt}$, where $P = 100$ (the initial investment), $r = 0.055$ (the annual interest rate expressed as a decimal), and t is time in years. To predict the amount in the account in 2018, after four years have elapsed, we take $t = 4$ and calculate

$$\begin{aligned} y(4) &= 100e^{0.055(4)} \\ &= 100e^{0.22} \\ &= 124.61. \qquad \text{Nearest cent using calculator} \end{aligned}$$

This compares with \$123.88 in the account when the interest is compounded annually, as was done in Example 1. ■

EXAMPLE 4 Laboratory experiments indicate that some atoms emit a part of their mass as radiation, with the remainder of the atom re-forming to make an atom of some new element. For example, radioactive carbon-14 decays into nitrogen; radium eventually decays into lead. If y_0 is the number of radioactive nuclei present at time zero, the number still present at any later time t will be

$$y = y_0 e^{-rt}, \qquad r > 0.$$

The number r is called the **decay rate** of the radioactive substance. (We will see how this formula is obtained in Section 7.2.) For carbon-14, the decay rate has been determined experimentally to be about $r = 1.2 \times 10^{-4}$ when t is measured in years. Predict the percent of carbon-14 present after 866 years have elapsed.

Solution If we start with an amount y_0 of carbon-14 nuclei, after 866 years we are left with the amount

$$\begin{aligned} y(866) &= y_0 e^{(-1.2\times 10^{-4})(866)} \\ &\approx (0.901)y_0. \qquad \text{Calculator evaluation} \end{aligned}$$

That is, after 866 years, we are left with about 90% of the original amount of carbon-14, so about 10% of the original nuclei have decayed. ■

You may wonder why we use the family of functions $y = e^{kx}$ for different values of the constant k instead of the general exponential functions $y = a^x$. In the next section, we show that the exponential function a^x is equal to e^{kx} for an appropriate value of k. So the formula $y = e^{kx}$ covers the entire range of possibilities, and it is generally easier to use.

EXERCISES 1.4

Sketching Exponential Curves

In Exercises 1–6, sketch the given curves together in the appropriate coordinate plane and label each curve with its equation.

1. $y = 2^x$, $y = 4^x$, $y = 3^{-x}$, $y = (1/5)^x$
2. $y = 3^x$, $y = 8^x$, $y = 2^{-x}$, $y = (1/4)^x$
3. $y = 2^{-t}$ and $y = -2^t$
4. $y = 3^{-t}$ and $y = -3^t$
5. $y = e^x$ and $y = 1/e^x$
6. $y = -e^x$ and $y = -e^{-x}$

In each of Exercises 7–10, sketch the shifted exponential curves.

7. $y = 2^x - 1$ and $y = 2^{-x} - 1$
8. $y = 3^x + 2$ and $y = 3^{-x} + 2$
9. $y = 1 - e^x$ and $y = 1 - e^{-x}$
10. $y = -1 - e^x$ and $y = -1 - e^{-x}$

Applying the Laws of Exponents

Use the laws of exponents to simplify the expressions in Exercises 11–20.

11. $16^2 \cdot 16^{-1.75}$

12. $9^{1/3} \cdot 9^{1/6}$

13. $\frac{4^{4.2}}{4^{3.7}}$

14. $\frac{3^{5/3}}{3^{2/3}}$

15. $(25^{1/8})^4$

16. $(13^{\sqrt{2}})^{\sqrt{2}/2}$

17. $2^{\sqrt{3}} \cdot 7^{\sqrt{3}}$

18. $(\sqrt{3})^{1/2} \cdot (\sqrt{12})^{1/2}$

19. $\left(\frac{2}{\sqrt{2}}\right)^4$

20. $\left(\frac{\sqrt{6}}{3}\right)^2$

Compositions Involving Exponential Functions

Find the domain and range for each of the functions in Exercises 21–24.

21. $f(x) = \frac{1}{2 + e^x}$

22. $g(t) = \cos(e^{-t})$

23. $g(t) = \sqrt{1 + 3^{-t}}$

24. $f(x) = \frac{3}{1 - e^{2x}}$

Applications

T In Exercises 25–28, use graphs to find approximate solutions.

25. $2^x = 5$

26. $e^x = 4$

27. $3^x - 0.5 = 0$

28. $3 - 2^{-x} = 0$

T In Exercises 29–36, use an exponential model and a graphing calculator to estimate the answer in each problem.

29. Population growth The population of Knoxville is 500,000 and is increasing at the rate of 3.75% each year. Approximately when will the population reach 1 million?

30. Population growth The population of Silver Run in the year 1890 was 6250. Assume the population increased at a rate of 2.75% per year.

a. Estimate the population in 1915 and 1940.

b. Approximately when did the population reach 50,000?

31. Radioactive decay The half-life of phosphorus-32 is about 14 days. There are 6.6 grams present initially.

a. Express the amount of phosphorus-32 remaining as a function of time t.

b. When will there be 1 gram remaining?

32. If Jean invests \$2300 in a retirement account with a 6% interest rate compounded annually, how long will it take until Jean's account has a balance of \$4150?

33. Doubling your money Determine how much time is required for an investment to double in value if interest is earned at the rate of 6.25% compounded annually.

34. Tripling your money Determine how much time is required for an investment to triple in value if interest is earned at the rate of 5.75% compounded continuously.

35. Cholera bacteria Suppose that a colony of bacteria starts with 1 bacterium and doubles in number every half hour. How many bacteria will the colony contain at the end of 24 hours?

36. Eliminating a disease Suppose that in any given year the number of cases of a disease is reduced by 20%. If there are 10,000 cases today, how many years will it take

a. to reduce the number of cases to 1000?

b. to eliminate the disease; that is, to reduce the number of cases to less than 1?

1.5 Inverse Functions and Logarithms

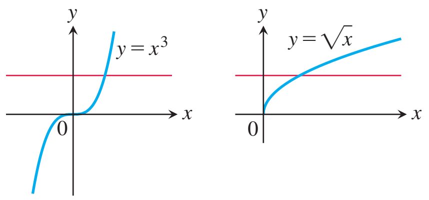
A function that undoes, or inverts, the effect of a function f is called the *inverse* of f. Many common functions, though not all, are paired with an inverse. In this section we present the natural logarithmic function $y = \ln x$ as the inverse of the exponential function $y = e^x$, and we also give examples of several inverse trigonometric functions.

One-to-One Functions

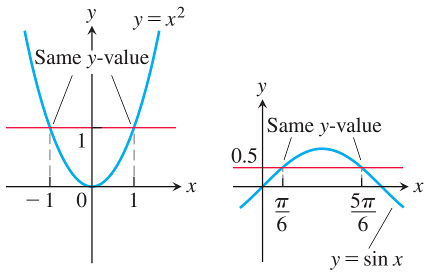
A function is a rule that assigns a value from its range to each element in its domain. Some functions assign the same range value to more than one element in the domain. The function $f(x) = x^2$ assigns the same value, 1, to both of the numbers -1 and $+1$. Similarly the sines of $\pi/3$ and $2\pi/3$ are both $\sqrt{3}/2$. Other functions assume each value in their range no more than once. The square roots and cubes of different numbers are always different. A function that has distinct values at distinct elements in its domain is called one-to-one.

DEFINITION A function $f(x)$ is **one-to-one** on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.

EXAMPLE 1 Some functions are one-to-one on their entire natural domain. Other functions are not one-to-one on their entire domain, but by restricting the function to a smaller domain we can create a function that is one-to-one. The original and restricted functions are not the same functions, because they have different domains. However, the two functions have the same values on the smaller domain.



(a) One-to-one: Graph meets each horizontal line at most once.



(b) Not one-to-one: Graph meets one or more horizontal lines more than once.

FIGURE 1.51 (a) $y = x^3$ and $y = \sqrt{x}$ are one-to-one on their domains $(-\infty, \infty)$ and $[0, \infty)$. (b) $y = x^2$ and $y = \sin x$ are not one-to-one on their domains $(-\infty, \infty)$.

(a) $f(x) = \sqrt{x}$ is one-to-one on any domain of nonnegative numbers because $\sqrt{x_1} \neq \sqrt{x_2}$ whenever $x_1 \neq x_2$.

(b) $g(x) = \sin x$ is *not* one-to-one on the interval $[0, \pi]$ because $\sin(\pi/6) = \sin(5\pi/6)$. In fact, for each element x_1 in the subinterval $[0, \pi/2)$ there is a corresponding element x_2 in the subinterval $(\pi/2, \pi]$ satisfying $\sin x_1 = \sin x_2$. The sine function *is* one-to-one on $[0, \pi/2]$, however, because it is an increasing function on $[0, \pi/2]$ and hence gives distinct outputs for distinct inputs in that interval. ■

The graph of a one-to-one function $y = f(x)$ can intersect a given horizontal line at most once. If the function intersects the line more than once, then it assumes the same y-value for at least two different x-values and is therefore not one-to-one (Figure 1.51).

The Horizontal Line Test for One-to-One Functions
A function $y = f(x)$ is one-to-one if and only if its graph intersects each horizontal line at most once.

Inverse Functions

Since each output of a one-to-one function comes from just one input, the effect of the function can be inverted to send each output back to the input from which it came.

DEFINITION Suppose that f is a one-to-one function on a domain D with range R. The **inverse function** f^{-1} is defined by

$$f^{-1}(b) = a \quad \text{if} \quad f(a) = b.$$

The domain of f^{-1} is R and the range of f^{-1} is D.

Caution
Do not confuse the inverse function f^{-1} with the reciprocal function $1/f$.

The symbol f^{-1} for the inverse of f is read "f inverse." The "-1" in f^{-1} is *not* an exponent; $f^{-1}(x)$ does not mean $1/f(x)$. Notice that the domains and ranges of f and f^{-1} are interchanged.

EXAMPLE 2 Suppose a one-to-one function $y = f(x)$ is given by a table of values

x	1	2	3	4	5	6	7	8
$f(x)$	3	4.5	7	10.5	15	20.5	27	34.5

A table for the values of $x = f^{-1}(y)$ can then be obtained by simply interchanging the values in each column of the table for f:

y	3	4.5	7	10.5	15	20.5	27	34.5
$f^{-1}(y)$	1	2	3	4	5	6	7	8

■

If we apply f to send an input x to the output $f(x)$ and follow by applying f^{-1} to $f(x)$, we get right back to x, just where we started. Similarly, if we take some number y in the range of f, apply f^{-1} to it, and then apply f to the resulting value $f^{-1}(y)$, we get back the value y from which we began. Composing a function and its inverse has the same effect as doing nothing.

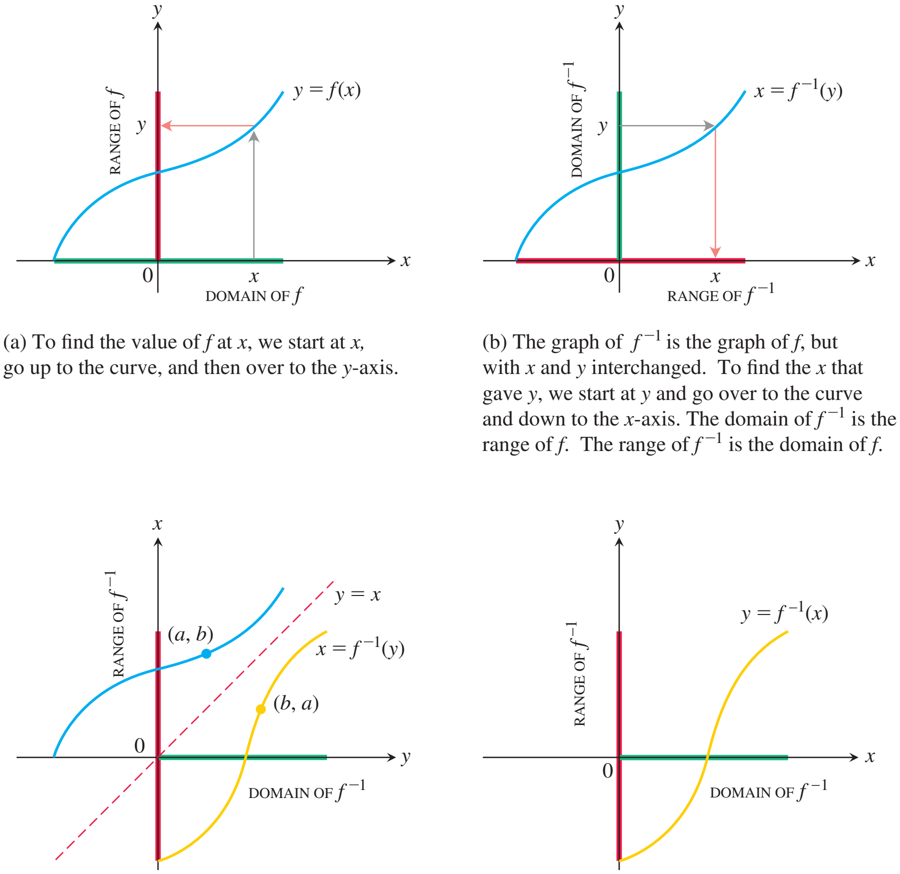
$$(f^{-1} \circ f)(x) = x, \qquad \text{for all } x \text{ in the domain of } f$$
$$(f \circ f^{-1})(y) = y, \qquad \text{for all } y \text{ in the domain of } f^{-1} \text{ (or range of } f)$$

Only a one-to-one function can have an inverse. The reason is that if $f(x_1) = y$ and $f(x_2) = y$ for two distinct inputs x_1 and x_2, then there is no way to assign a value to $f^{-1}(y)$ that satisfies both $f^{-1}(f(x_1)) = x_1$ and $f^{-1}(f(x_2)) = x_2$.

A function that is increasing on an interval satisfies the inequality $f(x_2) > f(x_1)$ when $x_2 > x_1$, so it is one-to-one and has an inverse. A function that is decreasing on an interval also has an inverse. Functions that are neither increasing nor decreasing may still be one-to-one and have an inverse, as with the function $f(x) = 1/x$ for $x \neq 0$ and $f(0) = 0$, defined on $(-\infty, \infty)$ and passing the horizontal line test.

Finding Inverses

The graphs of a function and its inverse are closely related. To read the value of a function from its graph, we start at a point x on the x-axis, go vertically to the graph, and then move horizontally to the y-axis to read the value of y. The inverse function can be read from the graph by reversing this process. Start with a point y on the y-axis, go horizontally to the graph of $y = f(x)$, and then move vertically to the x-axis to read the value of $x = f^{-1}(y)$ (Figure 1.52).



(a) To find the value of f at x, we start at x, go up to the curve, and then over to the y-axis.

(b) The graph of f^{-1} is the graph of f, but with x and y interchanged. To find the x that gave y, we start at y and go over to the curve and down to the x-axis. The domain of f^{-1} is the range of f. The range of f^{-1} is the domain of f.

(c) To draw the graph of f^{-1} in the more usual way, we reflect the system across the line $y = x$.

(d) Then we interchange the letters x and y. We now have a normal-looking graph of f^{-1} as a function of x.

FIGURE 1.52 The graph of $y = f^{-1}(x)$ is obtained by reflecting the graph of $y = f(x)$ about the line $y = x$.

We want to set up the graph of f^{-1} so that its input values lie along the x-axis, as is usually done for functions, rather than on the y-axis. To achieve this we interchange the x- and y-axes by reflecting across the 45° line $y = x$. After this reflection we have a new graph that represents f^{-1}. The value of $f^{-1}(x)$ can now be read from the graph in the usual way, by starting with a point x on the x-axis, going vertically to the graph, and then horizontally to the y-axis to get the value of $f^{-1}(x)$. Figure 1.52 indicates the relationship between the graphs of f and f^{-1}. The graphs are interchanged by reflection through the line $y = x$.

The process of passing from f to f^{-1} can be summarized as a two-step procedure.

1. Solve the equation $y = f(x)$ for x. This gives a formula $x = f^{-1}(y)$ where x is expressed as a function of y.
2. Interchange x and y, obtaining a formula $y = f^{-1}(x)$ where f^{-1} is expressed in the conventional format with x as the independent variable and y as the dependent variable.

EXAMPLE 3 Find the inverse of $y = \frac{1}{2}x + 1$, expressed as a function of x.

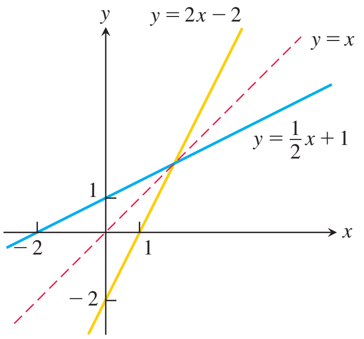


FIGURE 1.53 Graphing $f(x) = (1/2)x + 1$ and $f^{-1}(x) = 2x - 2$ together shows the graphs' symmetry with respect to the line $y = x$ (Example 3).

Solution

1. *Solve for x in terms of y*: $y = \frac{1}{2}x + 1$ — The graph satisfies the horizontal line test, so it is one-to-one (Fig. 1.58).

$$2y = x + 2$$
$$x = 2y - 2.$$

2. *Interchange x and y*: $y = 2x - 2.$ — Expresses the function in the usual form where y is the dependent variable.

The inverse of the function $f(x) = (1/2)x + 1$ is the function $f^{-1}(x) = 2x - 2$. (See Figure 1.53.) To check, we verify that both compositions give the identity function:

$$f^{-1}(f(x)) = 2\left(\frac{1}{2}x + 1\right) - 2 = x + 2 - 2 = x$$

$$f(f^{-1}(x)) = \frac{1}{2}(2x - 2) + 1 = x - 1 + 1 = x.$$ ■

EXAMPLE 4 Find the inverse of the function $y = x^2$, $x \geq 0$, expressed as a function of x.

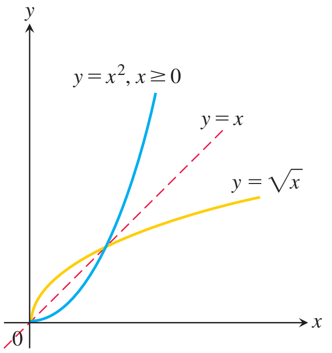


FIGURE 1.54 The functions $y = \sqrt{x}$ and $y = x^2$, $x \geq 0$, are inverses of one another (Example 4).

Solution For $x \geq 0$, the graph satisfies the horizontal line test, so the function is one-to-one and has an inverse. To find the inverse, we first solve for x in terms of y:

$$y = x^2$$
$$\sqrt{y} = \sqrt{x^2} = |x| = x \qquad |x| = x \text{ because } x \geq 0$$

We then interchange x and y, obtaining

$$y = \sqrt{x}.$$

The inverse of the function $y = x^2$, $x \geq 0$, is the function $y = \sqrt{x}$ (Figure 1.54). ■

Notice that the function $y = x^2$, $x \geq 0$, with domain *restricted* to the nonnegative real numbers, *is* one-to-one (Figure 1.54) and has an inverse. On the other hand, the function $y = x^2$, with no domain restrictions, *is not* one-to-one (Figure 1.51b) and therefore has no inverse.

Logarithmic Functions

If a is any positive real number other than 1, then the base a exponential function $f(x) = a^x$ is one-to-one. It therefore has an inverse. Its inverse is called the *logarithm function with base a*.

> **DEFINITION** The **logarithm function with base** a, written $y = \log_a x$, is the inverse of the base a exponential function $y = a^x$ $(a > 0, a \neq 1)$.

The domain of $\log_a x$ is $(0, \infty)$, the same as the range of a^x. The range of $\log_a x$ is $(-\infty, \infty)$, the same as the domain of a^x.

Figure 1.55a shows the graph of $y = \log_2 x$. The graph of $y = a^x$, $a > 1$, increases rapidly for $x > 0$, so its inverse, $y = \log_a x$, increases slowly for $x > 1$.

Because we have no technique yet for solving the equation $y = a^x$ for x in terms of y, we do not have an explicit formula for computing the logarithm at a given value of x. Nevertheless, we can obtain the graph of $y = \log_a x$ by reflecting the graph of the exponential $y = a^x$ across the line $y = x$. Figure 1.55a shows the graphs for $a = 2$ and $a = e$.

Logarithms with base 2 are often used when working with binary numbers, as is common in computer science. Logarithms with base e and base 10 are so important in applications that many calculators have special keys for them. They also have their own special notation and names:

$$\log_e x \text{ is written as } \ln x.$$
$$\log_{10} x \text{ is written as } \log x.$$

The function $y = \ln x$ is called the **natural logarithm function**, and $y = \log x$ is often called the **common logarithm function**. For the natural logarithm,

$$\ln x = y \iff e^y = x.$$

In particular, because $e^1 = e$, we obtain

$$\ln e = 1.$$

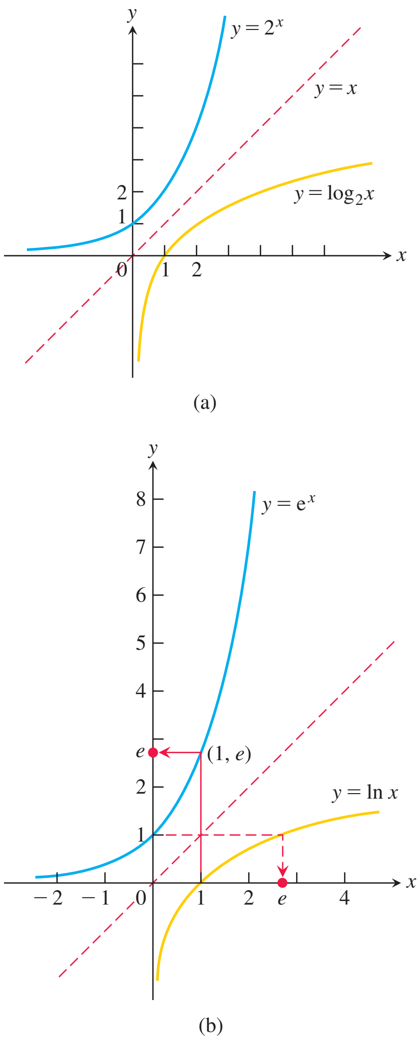


FIGURE 1.55 (a) The graph of 2^x and its inverse, $\log_2 x$. (b) The graph of e^x and its inverse, $\ln x$.

HISTORICAL BIOGRAPHY
John Napier
(1550–1617)
bit.ly/2IsZUN5

Properties of Logarithms

Logarithms, invented by John Napier, were the single most important improvement in arithmetic calculation before the modern electronic computer. The properties of logarithms reduce multiplication of positive numbers to addition of their logarithms, division of positive numbers to subtraction of their logarithms, and exponentiation of a number to multiplying its logarithm by the exponent.

We summarize these properties for the natural logarithm as a series of rules that we prove in Chapter 3. Although here we state the Power Rule for all real powers r, the case when r is an irrational number cannot be dealt with properly until Chapter 4. We establish the validity of the rules for logarithmic functions with any base a in Chapter 7.

THEOREM 1—Algebraic Properties of the Natural Logarithm
For any numbers $b > 0$ and $x > 0$, the natural logarithm satisfies the following rules:

1. *Product Rule*: $\ln bx = \ln b + \ln x$
2. *Quotient Rule*: $\ln \frac{b}{x} = \ln b - \ln x$
3. *Reciprocal Rule*: $\ln \frac{1}{x} = -\ln x$ Rule 2 with $b = 1$
4. *Power Rule*: $\ln x^r = r \ln x$

EXAMPLE 5 We use the properties in Theorem 1 to simplify three expressions.

(a) $\ln 4 + \ln \sin x = \ln (4 \sin x)$ — Product Rule

(b) $\ln \dfrac{x+1}{2x-3} = \ln (x+1) - \ln (2x-3)$ — Quotient Rule

(c) $\ln \dfrac{1}{8} = -\ln 8$ — Reciprocal Rule

$= -\ln 2^3 = -3 \ln 2$ — Power Rule ■

Because a^x and $\log_a x$ are inverses, composing them in either order gives the identity function.

> **Inverse Properties for a^x and $\log_a x$**
>
> **1.** Base a: $a^{\log_a x} = x, \quad \log_a a^x = x, \quad a > 0, a \neq 1, x > 0$
>
> **2.** Base e: $e^{\ln x} = x, \quad \ln e^x = x, \quad x > 0$

Substituting a^x for x in the equation $x = e^{\ln x}$ enables us to rewrite a^x as a power of e:

$$a^x = e^{\ln (a^x)} \qquad \text{Substitute } a^x \text{ for } x \text{ in } x = e^{\ln x}.$$
$$= e^{x \ln a} \qquad \text{Power Rule for logs}$$
$$= e^{(\ln a)x}. \qquad \text{Exponent rearranged}$$

Thus, the exponential function a^x is the same as e^{kx} with $k = \ln a$.

> Every exponential function is a power of the natural exponential function.
>
> $$a^x = e^{x \ln a}$$
>
> That is, a^x is the same as e^x raised to the power $\ln a$: $a^x = e^{kx}$ for $k = \ln a$.

For example,

$$2^x = e^{(\ln 2)x} = e^{x \ln 2}, \qquad \text{and} \qquad 5^{-3x} = e^{(\ln 5)(-3x)} = e^{-3x \ln 5}.$$

Returning once more to the properties of a^x and $\log_a x$, we have

$$\ln x = \ln (a^{\log_a x}) \qquad \text{Inverse Property for } a^x \text{ and } \log_a x$$
$$= (\log_a x)(\ln a). \qquad \text{Power Rule for logarithms, with } r = \log_a x$$

Rewriting this equation as $\log_a x = (\ln x)/(\ln a)$ shows that every logarithmic function is a constant multiple of the natural logarithm $\ln x$. This allows us to extend the algebraic properties for $\ln x$ to $\log_a x$. For instance, $\log_a bx = \log_a b + \log_a x$.

> **Change of Base Formula**
>
> Every logarithmic function is a constant multiple of the natural logarithm.
>
> $$\log_a x = \frac{\ln x}{\ln a} \qquad (a > 0, a \neq 1)$$

Applications

In Section 1.4 we looked at examples of exponential growth and decay problems. Here we use properties of logarithms to answer more questions concerning such problems.

EXAMPLE 6 If \$1000 is invested in an account that earns 5.25% interest compounded annually, how long will it take the account to reach \$2500?

Solution From Example 1, Section 1.4, with $P = 1000$ and $r = 0.0525$, the amount in the account at any time t in years is $1000(1.0525)^t$, so to find the time t when the account reaches \$2500 we need to solve the equation

$$1000(1.0525)^t = 2500.$$

Thus we have

$$\begin{aligned} (1.0525)^t &= 2.5 && \text{Divide by 1000.} \\ \ln(1.0525)^t &= \ln 2.5 && \text{Take logarithms of both sides.} \\ t \ln 1.0525 &= \ln 2.5 && \text{Power Rule} \\ t &= \frac{\ln 2.5}{\ln 1.0525} \approx 17.9 && \text{Values obtained by calculator} \end{aligned}$$

The amount in the account will reach \$2500 in 18 years, when the annual interest payment is deposited for that year. ■

EXAMPLE 7 The **half-life** of a radioactive element is the time expected to pass until half of the radioactive nuclei present in a sample decay. The half-life is a constant that does not depend on the number of radioactive nuclei initially present in the sample, but only on the radioactive substance.

To compute the half-life, let y_0 be the number of radioactive nuclei initially present in the sample. Then the number y present at any later time t will be $y = y_0 e^{-kt}$. We seek the value of t at which the number of radioactive nuclei present equals half the original number:

$$\begin{aligned} y_0 e^{-kt} &= \frac{1}{2} y_0 \\ e^{-kt} &= \frac{1}{2} \\ -kt &= \ln \frac{1}{2} = -\ln 2 && \text{Reciprocal Rule for logarithms} \\ t &= \frac{\ln 2}{k}. && (1) \end{aligned}$$

This value of t is the half-life of the element. It depends only on the value of k; the number y_0 does not have any effect.

The effective radioactive lifetime of polonium-210 is so short that we measure it in days rather than years. The number of radioactive atoms remaining after t days in a sample that starts with y_0 radioactive atoms is

$$y = y_0 e^{-5\times10^{-3} t}.$$

The element's half-life is

$$\begin{aligned} \text{Half-life} &= \frac{\ln 2}{k} && \text{Eq. (1)} \\ &= \frac{\ln 2}{5 \times 10^{-3}} && \text{The } k \text{ from polonium's decay equation} \\ &\approx 139 \text{ days.} \end{aligned}$$

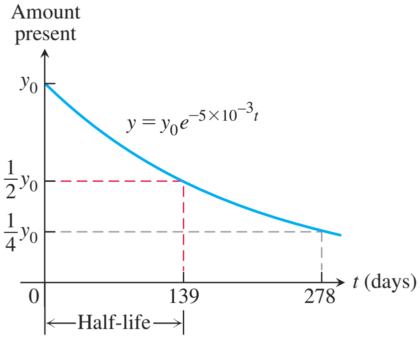


FIGURE 1.56 Amount of polonium-210 present at time t, where y_0 represents the number of radioactive atoms initially present (Example 7).

This means that after 139 days, $1/2$ of y_0 radioactive atoms remain; after another 139 days (278 days altogether) half of those remain, or $1/4$ of y_0 radioactive atoms remain, and so on (see Figure 1.56). ■

Inverse Trigonometric Functions

The six basic trigonometric functions are not one-to-one (since their values repeat periodically). However, we can restrict their domains to intervals on which they are one-to-one.

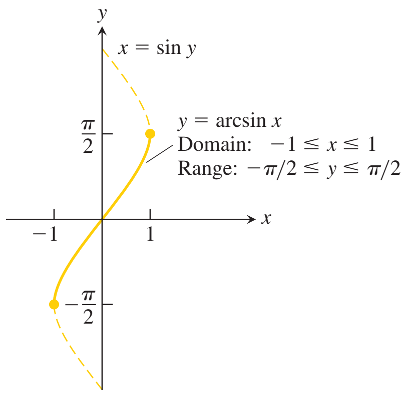
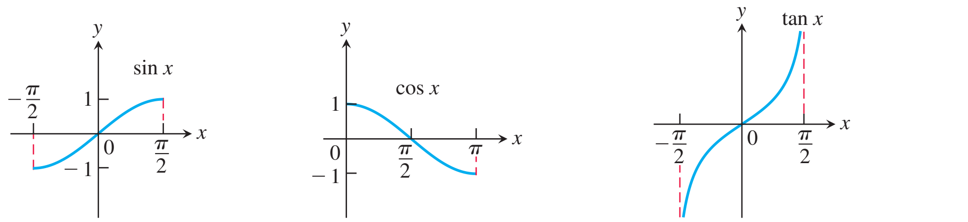


FIGURE 1.57 The graph of $y = \arcsin x$.

The sine function increases from -1 at $x = -\pi/2$ to $+1$ at $x = \pi/2$. By restricting its domain to the interval $[-\pi/2, \pi/2]$ we make it one-to-one, so that it has an inverse which is called $\arcsin x$ (Figure 1.57). Similar domain restrictions can be applied to all six trigonometric functions.

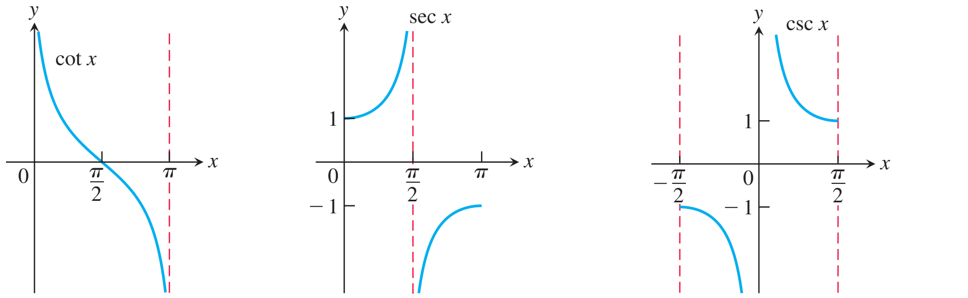
Domain restrictions that make the trigonometric functions one-to-one



$y = \sin x$
Domain: $[-\pi/2, \pi/2]$
Range: $[-1, 1]$

$y = \cos x$
Domain: $[0, \pi]$
Range: $[-1, 1]$

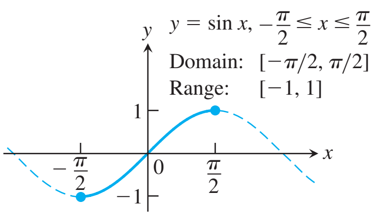
$y = \tan x$
Domain: $(-\pi/2, \pi/2)$
Range: $(-\infty, \infty)$



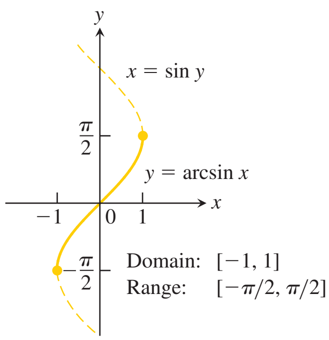
$y = \cot x$
Domain: $(0, \pi)$
Range: $(-\infty, \infty)$

$y = \sec x$
Domain: $[0, \pi/2) \cup (\pi/2, \pi]$
Range: $(-\infty, -1] \cup [1, \infty)$

$y = \csc x$
Domain: $[-\pi/2, 0) \cup (0, \pi/2]$
Range: $(-\infty, -1] \cup [1, \infty)$



(a)



(b)

FIGURE 1.58 The graphs of (a) $y = \sin x, -\pi/2 \le x \le \pi/2$, and (b) its inverse, $y = \arcsin x$. The graph of $\arcsin x$, obtained by reflection across the line $y = x$, is a portion of the curve $x = \sin y$.

Since these restricted functions are now one-to-one, they have inverses, which we denote by

$$y = \sin^{-1} x \quad \text{or} \quad y = \arcsin x, \qquad y = \cos^{-1} x \quad \text{or} \quad y = \arccos x$$
$$y = \tan^{-1} x \quad \text{or} \quad y = \arctan x, \qquad y = \cot^{-1} x \quad \text{or} \quad y = \operatorname{arccot} x$$
$$y = \sec^{-1} x \quad \text{or} \quad y = \operatorname{arcsec} x, \qquad y = \csc^{-1} x \quad \text{or} \quad y = \operatorname{arccsc} x$$

These equations are read "y equals the arcsine of x" or "y equals arcsin x" and so on.

Caution The -1 in the expressions for the inverse means "inverse." It does *not* mean reciprocal. For example, the *reciprocal* of $\sin x$ is $(\sin x)^{-1} = 1/\sin x = \csc x$.

The graphs of the six inverse trigonometric functions are obtained by reflecting the graphs of the restricted trigonometric functions through the line $y = x$. Figure 1.58b shows the graph of $y = \arcsin x$ and Figure 1.59 shows the graphs of all six functions. We now take a closer look at two of these functions.

The Arcsine and Arccosine Functions

We define the arcsine and arccosine as functions whose values are angles (measured in radians) that belong to restricted domains of the sine and cosine functions.

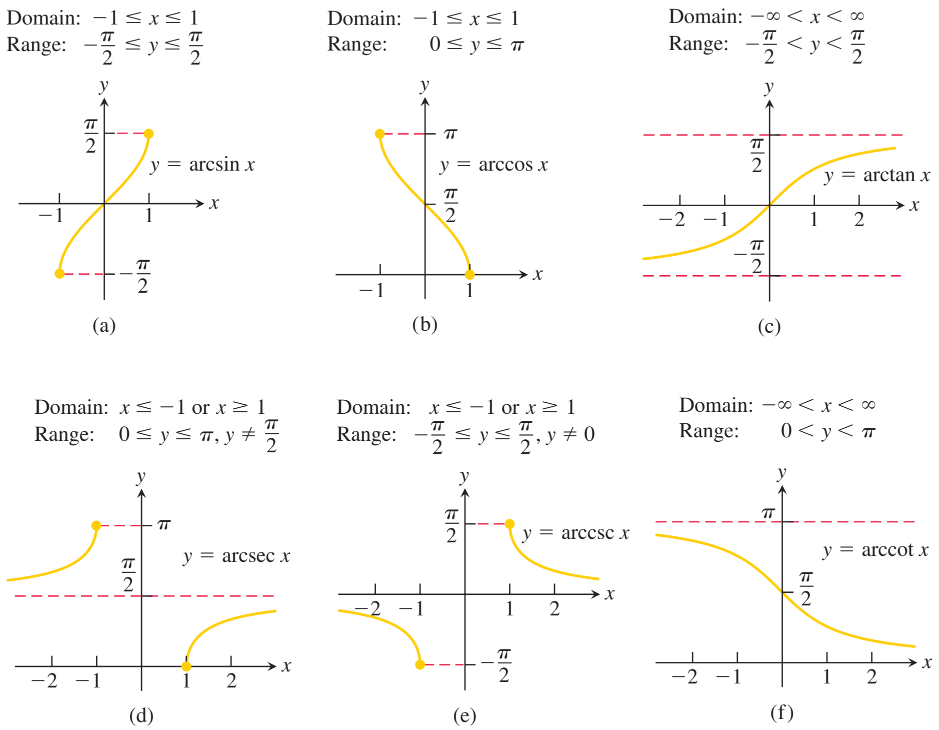


FIGURE 1.59 Graphs of the six basic inverse trigonometric functions.

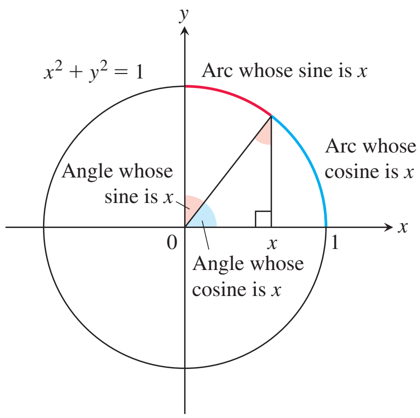
DEFINITION

$y = \arcsin x$ is the number in $[-\pi/2, \pi/2]$ for which $\sin y = x$.

$y = \arccos x$ is the number in $[0, \pi]$ for which $\cos y = x$.

The "Arc" in Arcsine and Arccosine

For a unit circle and radian angles, the arc length equation $s = r\theta$ becomes $s = \theta$, so central angles and the arcs they subtend have the same measure. If $x = \sin y$, then, in addition to being the angle whose sine is x, y is also the length of arc on the unit circle that subtends an angle whose sine is x. So we call y "the arc whose sine is x."



The graph of $y = \arcsin x$ (Figure 1.58b) is symmetric about the origin (it lies along the graph of $x = \sin y$). The arcsine is therefore an odd function:

$$\arcsin(-x) = -\arcsin x. \tag{2}$$

The graph of $y = \arccos x$ (Figure 1.60b) has no such symmetry.

EXAMPLE 8 Evaluate **(a)** $\arcsin\left(\frac{\sqrt{3}}{2}\right)$ and **(b)** $\arccos\left(-\frac{1}{2}\right)$.

Solution

(a) We see that

$$\arcsin\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$$

because $\sin(\pi/3) = \sqrt{3}/2$ and $\pi/3$ belongs to the range $[-\pi/2, \pi/2]$ of the arcsine function. See Figure 1.61a.

(b) We have

$$\arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$$

because $\cos(2\pi/3) = -1/2$ and $2\pi/3$ belongs to the range $[0, \pi]$ of the arccosine function. See Figure 1.61b. ■

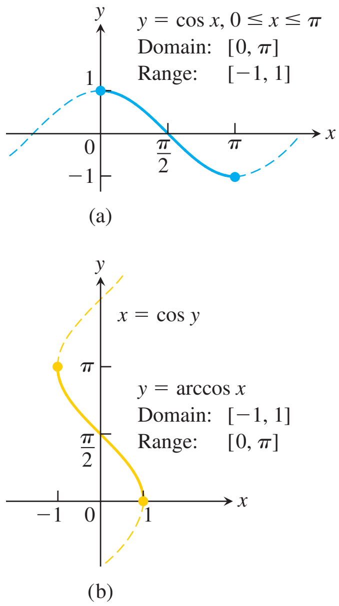


FIGURE 1.60 The graphs of (a) $y = \cos x$, $0 \le x \le \pi$, and (b) its inverse, $y = \arccos x$. The graph of $\arccos x$, obtained by reflection across the line $y = x$, is a portion of the curve $x = \cos y$.

Using the same procedure illustrated in Example 8, we can create the following table of common values for the arcsine and arccosine functions.

x	$\arcsin x$	$\arccos x$
$\sqrt{3}/2$	$\pi/3$	$\pi/6$
$\sqrt{2}/2$	$\pi/4$	$\pi/4$
$1/2$	$\pi/6$	$\pi/3$
$-1/2$	$-\pi/6$	$2\pi/3$
$-\sqrt{2}/2$	$-\pi/4$	$3\pi/4$
$-\sqrt{3}/2$	$-\pi/3$	$5\pi/6$

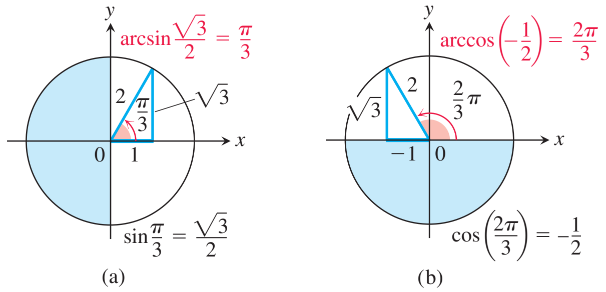


FIGURE 1.61 Values of the arcsine and arccosine functions (Example 8).

EXAMPLE 9 During a 240 km airplane flight from Zurich to Geneva after flying 180 km the navigator determines that the plane is 12 km off course, as shown in Figure 1.62. Find the angle a for a course parallel to the original correct course, the angle b, and the drift correction angle $c = a + b$.

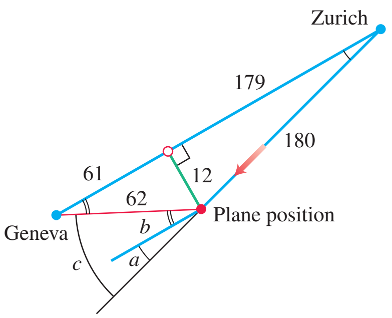


FIGURE 1.62 Diagram for drift correction (Example 9), with distances rounded to the nearest kilometer (drawing not to scale).

Solution From the Pythagorean theorem and given information, we compute an approximate hypothetical flight distance of 179 km, had the plane been flying along the original correct course (see Figure 1.62). Knowing the flight distance from Zurich to Geneva, we next calculate the remaining leg of the original course to be 61 km. Applying the Pythagorean theorem again then gives an approximate distance of 62 km from the position of the plane to Geneva. Finally, from Figure 1.62, we see that $180 \sin a = 12$ and $62 \sin b = 12$, so

$$a = \arcsin \frac{12}{180} \approx 0.067 \text{ radian} \approx 3.8°$$

$$b = \arcsin \frac{12}{62} \approx 0.195 \text{ radian} \approx 11.2°$$

$$c = a + b \approx 15°.$$

Identities Involving Arcsine and Arccosine

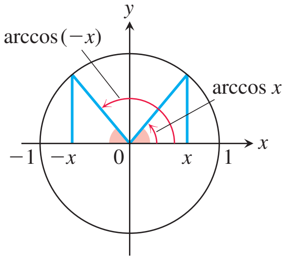


FIGURE 1.63 $\arccos x$ and $\arccos(-x)$ are supplementary angles (so their sum is π).

As we can see from Figure 1.63, the arccosine of x satisfies the identity

$$\arccos x + \arccos(-x) = \pi, \tag{3}$$

or

$$\arccos(-x) = \pi - \arccos x. \tag{4}$$

Also, we can see from the triangle in Figure 1.64 that for $x > 0$,

$$\arcsin x + \arccos x = \pi/2. \tag{5}$$

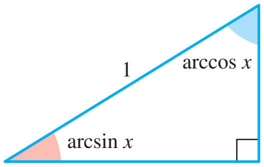


FIGURE 1.64 arcsin x and arccos x are complementary angles (so their sum is $\pi/2$).

Equation (5) holds for the other values of x in $[-1, 1]$ as well, but we cannot conclude this from the triangle in Figure 1.64. It is, however, a consequence of Equations (2) and (4) (Exercise 80).

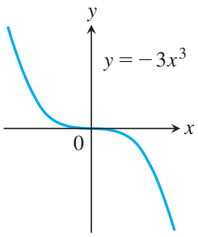
The arctangent, arccotangent, arcsecant, and arccosecant functions are defined in Section 3.9. There we develop additional properties of the inverse trigonometric functions using the identities discussed here.

EXERCISES 1.5

Identifying One-to-One Functions Graphically

Which of the functions graphed in Exercises 1–6 are one-to-one, and which are not?

1.



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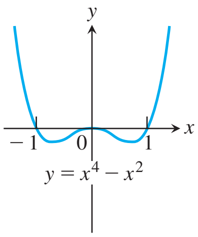

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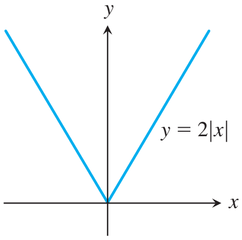

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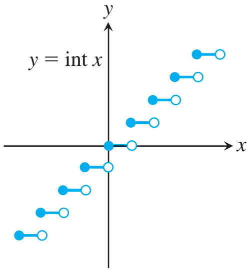

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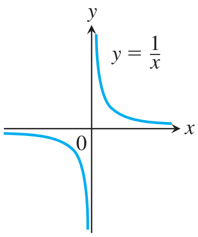

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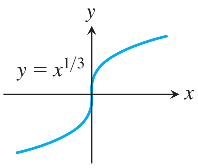


In Exercises 7–10, determine from its graph if the function is one-to-one.

7. $f(x) = \begin{cases} 3 - x, & x < 0 \\ 3, & x \geq 0 \end{cases}$

8. $f(x) = \begin{cases} 2x + 6, & x \leq -3 \\ x + 4, & x > -3 \end{cases}$

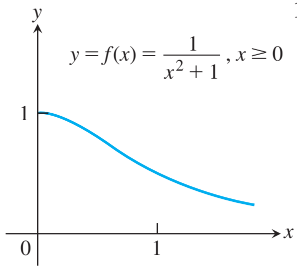
9. $f(x) = \begin{cases} 1 - \dfrac{x}{2}, & x \leq 0 \\ \dfrac{x}{x + 2}, & x > 0 \end{cases}$

10. $f(x) = \begin{cases} 2 - x^2, & x \leq 1 \\ x^2, & x > 1 \end{cases}$

Graphing Inverse Functions

Each of Exercises 11–16 shows the graph of a function $y = f(x)$. Copy the graph and draw in the line $y = x$. Then use symmetry with respect to the line $y = x$ to add the graph of f^{-1} to your sketch. (It is not necessary to find a formula for f^{-1}.) Identify the domain and range of f^{-1}.

11.



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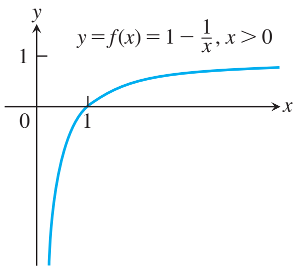

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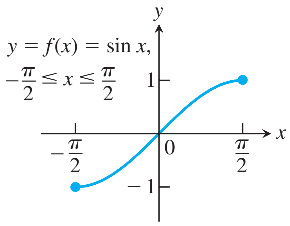

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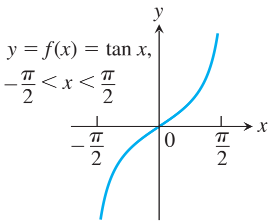
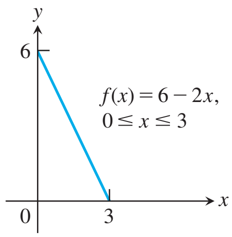


15.



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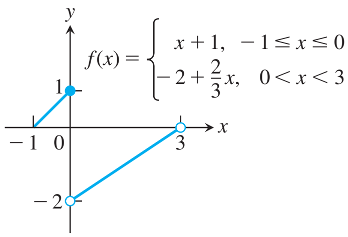


17. a. Graph the function $f(x) = \sqrt{1 - x^2}$, $0 \leq x \leq 1$. What symmetry does the graph have?

b. Show that f is its own inverse. (Remember that $\sqrt{x^2} = x$ if $x \geq 0$.)

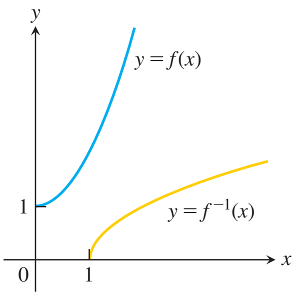
18. a. Graph the function $f(x) = 1/x$. What symmetry does the graph have?

b. Show that f is its own inverse.

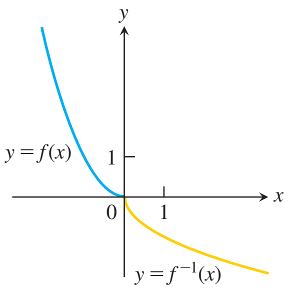
Formulas for Inverse Functions

Each of Exercises 19–24 gives a formula for a function $y = f(x)$ and shows the graphs of f and f^{-1}. Find a formula for f^{-1} in each case.

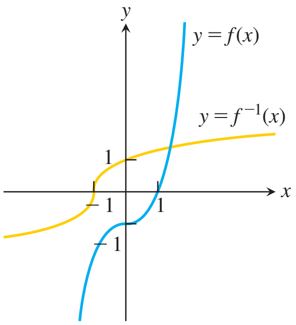
19. $f(x) = x^2 + 1, \quad x \geq 0$



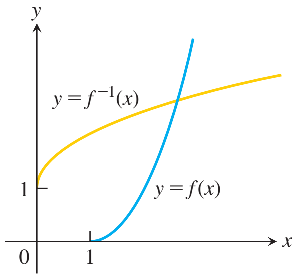
20. $f(x) = x^2, \quad x \leq 0$



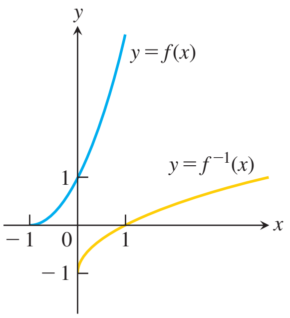
21. $f(x) = x^3 - 1$



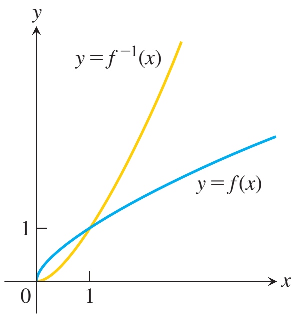
22. $f(x) = x^2 - 2x + 1, \quad x \geq 1$



23. $f(x) = (x + 1)^2, \quad x \geq -1$



24. $f(x) = x^{2/3}, \quad x \geq 0$



Each of Exercises 25–36 gives a formula for a function $y = f(x)$. In each case, find $f^{-1}(x)$ and identify the domain and range of f^{-1}. As a check, show that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

25. $f(x) = x^5$

26. $f(x) = x^4, \quad x \geq 0$

27. $f(x) = x^3 + 1$

28. $f(x) = (1/2)x - 7/2$

29. $f(x) = 1/x^2, \quad x > 0$

30. $f(x) = 1/x^3, \quad x \neq 0$

31. $f(x) = \dfrac{x + 3}{x - 2}$

32. $f(x) = \dfrac{\sqrt{x}}{\sqrt{x} - 3}$

33. $f(x) = x^2 - 2x, \quad x \leq 1$

(*Hint:* Complete the square.)

34. $f(x) = (2x^3 + 1)^{1/5}$

35. $f(x) = \dfrac{x + b}{x - 2}, \quad b > -2$ and constant

36. $f(x) = x^2 - 2bx, \quad b > 0$ and constant, $x \leq b$

Inverses of Lines

37. a. Find the inverse of the function $f(x) = mx$, where m is a constant different from zero.

b. What can you conclude about the inverse of a function $y = f(x)$ whose graph is a line through the origin with a nonzero slope m?

38. Show that the graph of the inverse of $f(x) = mx + b$, where m and b are constants and $m \neq 0$, is a line with slope $1/m$ and y-intercept $-b/m$.

39. a. Find the inverse of $f(x) = x + 1$. Graph f and its inverse together. Add the line $y = x$ to your sketch, drawing it with dashes or dots for contrast.

b. Find the inverse of $f(x) = x + b$ (b constant). How is the graph of f^{-1} related to the graph of f?

c. What can you conclude about the inverses of functions whose graphs are lines parallel to the line $y = x$?

40. a. Find the inverse of $f(x) = -x + 1$. Graph the line $y = -x + 1$ together with the line $y = x$. At what angle do the lines intersect?

b. Find the inverse of $f(x) = -x + b$ (b constant). What angle does the line $y = -x + b$ make with the line $y = x$?

c. What can you conclude about the inverses of functions whose graphs are lines perpendicular to the line $y = x$?

Logarithms and Exponentials

41. Express the following logarithms in terms of ln 2 and ln 3.

a. $\ln 0.75$ **b.** $\ln (4/9)$

c. $\ln (1/2)$ **d.** $\ln \sqrt[3]{9}$

e. $\ln 3\sqrt{2}$ **f.** $\ln \sqrt{13.5}$

42. Express the following logarithms in terms of ln 5 and ln 7.

a. $\ln (1/125)$ **b.** $\ln 9.8$

c. $\ln 7\sqrt{7}$ **d.** $\ln 1225$

e. $\ln 0.056$ **f.** $(\ln 35 + \ln (1/7))/(\ln 25)$

Use the properties of logarithms to write the expressions in Exercises 43 and 44 as a single term.

43. a. $\ln \sin \theta - \ln \left(\dfrac{\sin \theta}{5}\right)$ **b.** $\ln (3x^2 - 9x) + \ln \left(\dfrac{1}{3x}\right)$

c. $\dfrac{1}{2} \ln (4t^4) - \ln b$

44. a. $\ln \sec \theta + \ln \cos \theta$ **b.** $\ln (8x + 4) - 2 \ln c$

c. $3 \ln \sqrt[3]{t^2 - 1} - \ln (t + 1)$

Find simpler expressions for the quantities in Exercises 45–48.

45. a. $e^{\ln 7.2}$ **b.** $e^{-\ln x^2}$ **c.** $e^{\ln x - \ln y}$

46. a. $e^{\ln (x^2 + y^2)}$ **b.** $e^{-\ln 0.3}$ **c.** $e^{\ln \pi x - \ln 2}$

47. a. $2 \ln \sqrt{e}$ **b.** $\ln (\ln e^e)$ **c.** $\ln (e^{-x^2 - y^2})$

48. a. $\ln (e^{\sec \theta})$ **b.** $\ln (e^{(e^x)})$ **c.** $\ln (e^{2 \ln x})$

In Exercises 49–54, solve for y in terms of t or x, as appropriate.

49. $\ln y = 2t + 4$

50. $\ln y = -t + 5$

51. $\ln (y - b) = 5t$

52. $\ln (c - 2y) = t$

53. $\ln (y - 1) - \ln 2 = x + \ln x$

54. $\ln (y^2 - 1) - \ln (y + 1) = \ln (\sin x)$

In Exercises 55 and 56, solve for k.

55. **a.** $e^{2k} = 4$ **b.** $100e^{10k} = 200$ **c.** $e^{k/1000} = a$

56. **a.** $e^{5k} = \frac{1}{4}$ **b.** $80e^{k} = 1$ **c.** $e^{(\ln 0.8)k} = 0.8$

In Exercises 57–64, solve for t.

57. **a.** $e^{-0.3t} = 27$ **b.** $e^{kt} = \frac{1}{2}$ **c.** $e^{(\ln 0.2)t} = 0.4$

58. **a.** $e^{-0.01t} = 1000$ **b.** $e^{kt} = \frac{1}{10}$ **c.** $e^{(\ln 2)t} = \frac{1}{2}$

59. $e^{\sqrt{t}} = x^2$

60. $e^{(x^2)}e^{(2x+1)} = e^t$

61. $e^{2t} - 3e^t = 0$

62. $e^{-2t} + 6 = 5e^{-t}$

63. $\ln\left(\frac{t}{t-1}\right) = 2$

64. $\ln(t-2) = \ln 8 - \ln t$

Simplify the expressions in Exercises 65–68.

65. **a.** $5^{\log_5 7}$ **b.** $8^{\log_8 \sqrt{2}}$ **c.** $1.3^{\log_{1.3} 75}$
d. $\log_4 16$ **e.** $\log_3 \sqrt{3}$ **f.** $\log_4\left(\frac{1}{4}\right)$

66. **a.** $2^{\log_2 3}$ **b.** $10^{\log_{10}(1/2)}$ **c.** $\pi^{\log_\pi 7}$
d. $\log_{11} 121$ **e.** $\log_{121} 11$ **f.** $\log_3\left(\frac{1}{9}\right)$

67. **a.** $2^{\log_4 x}$ **b.** $9^{\log_3 x}$ **c.** $\log_2(e^{(\ln 2)(\sin x)})$

68. **a.** $25^{\log_5(3x^2)}$ **b.** $\log_e(e^x)$ **c.** $\log_4(2^{e^x \sin x})$

Express the ratios in Exercises 69 and 70 as ratios of natural logarithms and simplify.

69. **a.** $\frac{\log_2 x}{\log_3 x}$ **b.** $\frac{\log_2 x}{\log_8 x}$ **c.** $\frac{\log_x a}{\log_{x^2} a}$

70. **a.** $\frac{\log_9 x}{\log_3 x}$ **b.** $\frac{\log_{\sqrt{10}} x}{\log_{\sqrt{2}} x}$ **c.** $\frac{\log_a b}{\log_b a}$

Arcsine and Arccosine

In Exercises 71–74, find the exact value of each expression.

71. **a.** $\sin^{-1}\left(\frac{-1}{2}\right)$ **b.** $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ **c.** $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)$

72. **a.** $\cos^{-1}\left(\frac{1}{2}\right)$ **b.** $\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$ **c.** $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

73. **a.** $\arccos(-1)$ **b.** $\arccos(0)$

74. **a.** $\arcsin(-1)$ **b.** $\arcsin\left(-\frac{1}{\sqrt{2}}\right)$

Theory and Examples

75. If $f(x)$ is one-to-one, can anything be said about $g(x) = -f(x)$? Is it also one-to-one? Give reasons for your answer.

76. If $f(x)$ is one-to-one and $f(x)$ is never zero, can anything be said about $h(x) = 1/f(x)$? Is it also one-to-one? Give reasons for your answer.

77. Suppose that the range of g lies in the domain of f so that the composition $f \circ g$ is defined. If f and g are one-to-one, can anything be said about $f \circ g$? Give reasons for your answer.

78. If a composition $f \circ g$ is one-to-one, must g be one-to-one? Give reasons for your answer.

79. Find a formula for the inverse function f^{-1} and verify that $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$.

a. $f(x) = \frac{100}{1 + 2^{-x}}$ **b.** $f(x) = \frac{50}{1 + 1.1^{-x}}$

c. $f(x) = \frac{e^x - 1}{e^x + 1}$ **d.** $f(x) = \frac{\ln x}{2 - \ln x}$

80. **The identity $\sin^{-1}x + \cos^{-1}x = \pi/2$** Figure 1.64 establishes the identity for $0 < x < 1$. To establish it for the rest of $[-1, 1]$, verify by direct calculation that it holds for $x = 1$, 0, and -1. Then, for values of x in $(-1, 0)$, let $x = -a$, $a > 0$, and apply Eqs. (3) and (5) to the sum $\sin^{-1}(-a) + \cos^{-1}(-a)$.

81. Start with the graph of $y = \ln x$. Find an equation of the graph that results from

a. shifting down 3 units.

b. shifting right 1 unit.

c. shifting left 1, up 3 units.

d. shifting down 4, right 2 units.

e. reflecting about the y-axis.

f. reflecting about the line $y = x$.

82. Start with the graph of $y = \ln x$. Find an equation of the graph that results from

a. vertical stretching by a factor of 2.

b. horizontal stretching by a factor of 3.

c. vertical compression by a factor of 4.

d. horizontal compression by a factor of 2.

83. The equation $x^2 = 2^x$ has three solutions: $x = 2$, $x = 4$, and one other. Estimate the third solution as accurately as you can by graphing.

84. Could $x^{\ln 2}$ possibly be the same as $2^{\ln x}$ for $x > 0$? Graph the two functions and explain what you see.

85. **Radioactive decay** The half-life of a certain radioactive substance is 12 hours. There are 8 grams present initially.

a. Express the amount of substance remaining as a function of time t.

b. When will there be 1 gram remaining?

86. **Doubling your money** Determine how much time is required for a \$500 investment to double in value if interest is earned at the rate of 4.75% compounded annually.

87. **Population growth** The population of Glenbrook is 375,000 and is increasing at the rate of 2.25% per year. Predict when the population will be 1 million.

88. **Radon-222** The decay equation for radon-222 gas is known to be $y = y_0 e^{-0.18t}$, with t in days. About how long will it take the radon in a sealed sample of air to fall to 90% of its original value?

CHAPTER 1 Questions to Guide Your Review

1. What is a function? What is its domain? Its range? What is an arrow diagram for a function? Give examples.
2. What is the graph of a real-valued function of a real variable? What is the vertical line test?
3. What is a piecewise-defined function? Give examples.
4. What are the important types of functions frequently encountered in calculus? Give an example of each type.
5. What is meant by an increasing function? A decreasing function? Give an example of each.
6. What is an even function? An odd function? What symmetry properties do the graphs of such functions have? What advantage can we take of this? Give an example of a function that is neither even nor odd.
7. If f and g are real-valued functions, how are the domains of $f + g$, $f - g$, fg, and f/g related to the domains of f and g? Give examples.
8. When is it possible to compose one function with another? Give examples of compositions and their values at various points. Does the order in which functions are composed ever matter?
9. How do you change the equation $y = f(x)$ to shift its graph vertically up or down by $|k|$ units? Horizontally to the left or right? Give examples.
10. How do you change the equation $y = f(x)$ to compress or stretch the graph by a factor $c > 1$? Reflect the graph across a coordinate axis? Give examples.
11. What is radian measure? How do you convert from radians to degrees? Degrees to radians?
12. Graph the six basic trigonometric functions. What symmetries do the graphs have?
13. What is a periodic function? Give examples. What are the periods of the six basic trigonometric functions?
14. Starting with the identity $\sin^2\theta + \cos^2\theta = 1$ and the formulas for $\cos(A + B)$ and $\sin(A + B)$, show how a variety of other trigonometric identities may be derived.
15. How does the formula for the general sine function $f(x) = A\sin((2\pi/B)(x - C)) + D$ relate to the shifting, stretching, compressing, and reflection of its graph? Give examples. Graph the general sine curve and identify the constants A, B, C, and D.
16. Name three issues that arise when functions are graphed using a calculator or computer with graphing software. Give examples.
17. What is an exponential function? Give examples. What laws of exponents does it obey? How does it differ from a simple power function like $f(x) = x^n$? What kind of real-world phenomena are modeled by exponential functions?
18. What is the number e, and how is it defined? What are the domain and range of $f(x) = e^x$? What does its graph look like? How do the values of e^x relate to x^2, x^3, and so on?
19. What functions have inverses? How do you know if two functions f and g are inverses of one another? Give examples of functions that are (are not) inverses of one another.
20. How are the domains, ranges, and graphs of functions and their inverses related? Give an example.
21. What procedure can you sometimes use to express the inverse of a function of x as a function of x?
22. What is a logarithmic function? What properties does it satisfy? What is the natural logarithm function? What are the domain and range of $y = \ln x$? What does its graph look like?
23. How is the graph of $\log_a x$ related to the graph of $\ln x$? What truth is in the statement that there is really only one exponential function and one logarithmic function?
24. How are the inverse trigonometric functions defined? How can you sometimes use right triangles to find values of these functions? Give examples.

CHAPTER 1 Practice Exercises

Functions and Graphs

1. Express the area and circumference of a circle as functions of the circle's radius. Then express the area as a function of the circumference.
2. Express the radius of a sphere as a function of the sphere's surface area. Then express the surface area as a function of the volume.
3. A point P in the first quadrant lies on the parabola $y = x^2$. Express the coordinates of P as functions of the angle of inclination of the line joining P to the origin.
4. A hot-air balloon rising straight up from a level field is tracked by a range finder located 500 m from the point of liftoff. Express the balloon's height as a function of the angle the line from the range finder to the balloon makes with the ground.

In Exercises 5–8, determine whether the graph of the function is symmetric about the y-axis, the origin, or neither.

5. $y = x^{1/5}$
6. $y = x^{2/5}$
7. $y = x^2 - 2x - 1$
8. $y = e^{-x^2}$

In Exercises 9–16, determine whether the function is even, odd, or neither.

9. $y = x^2 + 1$
10. $y = x^5 - x^3 - x$
11. $y = 1 - \cos x$
12. $y = \sec x \tan x$
13. $y = \dfrac{x^4 + 1}{x^3 - 2x}$
14. $y = x - \sin x$
15. $y = x + \cos x$
16. $y = x\cos x$

17. Suppose that f and g are both odd functions defined on the entire real line. Which of the following (where defined) are even? odd?

a. fg **b.** f^3 **c.** $f(\sin x)$ **d.** $g(\sec x)$ **e.** $|g|$

18. If $f(a - x) = f(a + x)$, show that $g(x) = f(x + a)$ is an even function.

In Exercises 19–32, find the **(a)** domain and **(b)** range.

19. $y = |x| - 2$

20. $y = -2 + \sqrt{1 - x}$

21. $y = \sqrt{16 - x^2}$

22. $y = 3^{2-x} + 1$

23. $y = 2e^{-x} - 3$

24. $y = \tan(2x - \pi)$

25. $y = 2\sin(3x + \pi) - 1$

26. $y = x^{2/5}$

27. $y = \ln(x - 3) + 1$

28. $y = -1 + \sqrt[3]{2 - x}$

29. $y = 5 - \sqrt{x^2 - 2x - 3}$

30. $y = 2 + \dfrac{3x^2}{x^2 + 4}$

31. $y = 4\sin\left(\dfrac{1}{x}\right)$

32. $y = 3\cos x + 4\sin x$ (*Hint:* A trig identity is required.)

33. State whether each function is increasing, decreasing, or neither.

a. Volume of a sphere as a function of its radius

b. Greatest integer function

c. Height above Earth's sea level as a function of atmospheric pressure (assumed nonzero)

d. Kinetic energy as a function of a particle's velocity

34. Find the largest interval on which the given function is increasing.

a. $f(x) = |x - 2| + 1$ **b.** $f(x) = (x + 1)^4$

c. $g(x) = (3x - 1)^{1/3}$ **d.** $R(x) = \sqrt{2x - 1}$

Piecewise-Defined Functions

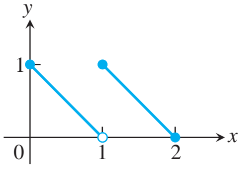
In Exercises 35 and 36, find the **(a)** domain and **(b)** range.

35. $y = \begin{cases} \sqrt{-x}, & -4 \le x \le 0 \\ \sqrt{x}, & 0 < x \le 4 \end{cases}$

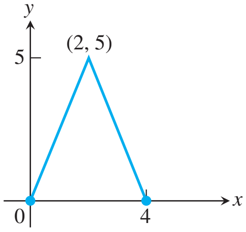
36. $y = \begin{cases} -x - 2, & -2 \le x \le -1 \\ x, & -1 < x \le 1 \\ -x + 2, & 1 < x \le 2 \end{cases}$

In Exercises 37 and 38, write a piecewise formula for the function.

37.



38.



Composition of Functions

In Exercises 39 and 40, find

a. $(f \circ g)(-1)$. **b.** $(g \circ f)(2)$.

c. $(f \circ f)(x)$. **d.** $(g \circ g)(x)$.

39. $f(x) = \dfrac{1}{x}$, $g(x) = \dfrac{1}{\sqrt{x + 2}}$

40. $f(x) = 2 - x$, $g(x) = \sqrt[3]{x + 1}$

In Exercises 41 and 42, **(a)** write formulas for $f \circ g$ and $g \circ f$ and find the **(b)** domain and **(c)** range of each.

41. $f(x) = 2 - x^2$, $g(x) = \sqrt{x + 2}$

42. $f(x) = \sqrt{x}$, $g(x) = \sqrt{1 - x}$

For Exercises 43 and 44, sketch the graphs of f and $f \circ f$.

43. $f(x) = \begin{cases} -x - 2, & -4 \le x \le -1 \\ -1, & -1 < x \le 1 \\ x - 2, & 1 < x \le 2 \end{cases}$

44. $f(x) = \begin{cases} x + 1, & -2 \le x < 0 \\ x - 1, & 0 \le x \le 2 \end{cases}$

Composition with absolute values In Exercises 45–52, graph f_1 and f_2 together. Then describe how applying the absolute value function in f_2 affects the graph of f_1.

	$f_1(x)$	$f_2(x)$
45.	x	$\lvert x\rvert$
46.	x^2	$\lvert x\rvert^2$
47.	x^3	$\lvert x^3\rvert$
48.	$x^2 + x$	$\lvert x^2 + x\rvert$
49.	$4 - x^2$	$\lvert 4 - x^2\rvert$
50.	$\dfrac{1}{x}$	$\dfrac{1}{\lvert x\rvert}$
51.	$\sqrt{x}$	$\sqrt{\lvert x\rvert}$
52.	$\sin x$	$\sin\lvert x\rvert$

Shifting and Scaling Graphs

53. Suppose the graph of g is given. Write equations for the graphs that are obtained from the graph of g by shifting, scaling, or reflecting, as indicated.

a. Up $\frac{1}{2}$ unit, right 3

b. Down 2 units, left $\frac{2}{3}$

c. Reflect about the y-axis

d. Reflect about the x-axis

e. Stretch vertically by a factor of 5

f. Compress horizontally by a factor of 5

54. Describe how each graph is obtained from the graph of $y = f(x)$.

a. $y = f(x - 5)$ **b.** $y = f(4x)$

c. $y = f(-3x)$ **d.** $y = f(2x + 1)$

e. $y = f\left(\dfrac{x}{3}\right) - 4$ **f.** $y = -3f(x) + \dfrac{1}{4}$

In Exercises 55–58, graph each function, not by plotting points, but by starting with the graph of one of the standard functions presented in Figures 1.15–1.17, and applying an appropriate transformation.

55. $y = -\sqrt{1 + \dfrac{x}{2}}$

56. $y = 1 - \dfrac{x}{3}$

57. $y = \dfrac{1}{2x^2} + 1$

58. $y = (-5x)^{1/3}$

Trigonometry

In Exercises59–62, sketch the graph of the given function. What is the period of the function?

59. $y = \cos 2x$

60. $y = \sin \frac{x}{2}$

61. $y = \sin \pi x$

62. $y = \cos \frac{\pi x}{2}$

63. Sketch the graph $y = 2\cos\left(x - \frac{\pi}{3}\right)$.

64. Sketch the graph $y = 1 + \sin\left(x + \frac{\pi}{4}\right)$.

In Exercises 65–68, *ABC* is a right triangle with the right angle at *C*. The sides opposite angles *A*, *B*, and *C* are *a*, *b*, and *c*, respectively.

65. a. Find *a* and *b* if $c = 2, B = \pi/3$.

b. Find *a* and *c* if $b = 2, B = \pi/3$.

66. a. Express *a* in terms of *A* and *c*.

b. Express *a* in terms of *A* and *b*.

67. a. Express *a* in terms of *B* and *b*.

b. Express *c* in terms of *A* and *a*.

68. a. Express sin *A* in terms of *a* and *c*.

b. Express sin *A* in terms of *b* and *c*.

69. Height of a pole Two wires stretch from the top *T* of a vertical pole to points *B* and *C* on the ground, where *C* is 10 m closer to the base of the pole than is *B*. If wire *BT* makes an angle of 35° with the horizontal and wire *CT* makes an angle of 50° with the horizontal, how high is the pole?

70. Height of a weather balloon Observers at positions *A* and *B* 2 km apart simultaneously measure the angle of elevation of a weather balloon to be 40° and 70°, respectively. If the balloon is directly above a point on the line segment between *A* and *B*, find the height of the balloon.

T **71. a.** Graph the function $f(x) = \sin x + \cos(x/2)$.

b. What appears to be the period of this function?

c. Confirm your finding in part (b) algebraically.

T **72. a.** Graph $f(x) = \sin(1/x)$.

b. What are the domain and range of *f*?

c. Is *f* periodic? Give reasons for your answer.

Transcendental Functions

In Exercises 73–76, find the domain of each function.

73. a. $f(x) = 1 + e^{-\sin x}$ **b.** $g(x) = e^x + \ln\sqrt{x}$

74. a. $f(x) = e^{1/x^2}$ **b.** $g(x) = \ln|4 - x^2|$

75. a. $h(x) = \sin^{-1}\left(\frac{x}{3}\right)$ **b.** $f(x) = \cos^{-1}(\sqrt{x} - 1)$

76. a. $h(x) = \ln(\cos^{-1} x)$ **b.** $f(x) = \sqrt{\pi - \sin^{-1} x}$

77. If $f(x) = \ln x$ and $g(x) = 4 - x^2$, find the functions $f \circ g$, $g \circ f$, $f \circ f$, $g \circ g$, and their domains.

78. Determine whether *f* is even, odd, or neither.

a. $f(x) = e^{-x^2}$ **b.** $f(x) = 1 + \sin^{-1}(-x)$

c. $f(x) = |e^x|$ **d.** $f(x) = e^{\ln|x|+1}$

T **79.** Graph ln *x*, ln 2*x*, ln 4*x*, ln 8*x*, and ln 16*x* (as many as you can) together for $0 < x \le 10$. What is going on? Explain.

T **80.** Graph $y = \ln(x^2 + c)$ for $c = -4, -2, 0, 3$, and 5. How does the graph change when *c* changes?

T **81.** Graph $y = \ln|\sin x|$ in the window $0 \le x \le 22, -2 \le y \le 0$. Explain what you see. How could you change the formula to turn the arches upside down?

T **82.** Graph the three functions $y = x^a$, $y = a^x$, and $y = \log_a x$ together on the same screen for $a = 2$, 10, and 20. For large values of *x*, which of these functions has the largest values and which has the smallest values?

Theory and Examples

In Exercises 83 and 84, find the domain and range of each composite function. Then graph the compositions on separate screens. Do the graphs make sense in each case? Give reasons for your answers and comment on any differences you see.

83. a. $y = \sin^{-1}(\sin x)$ **b.** $y = \sin(\sin^{-1} x)$

84. a. $y = \cos^{-1}(\cos x)$ **b.** $y = \cos(\cos^{-1} x)$

85. Use a graph to decide whether *f* is one-to-one.

a. $f(x) = x^3 - \frac{x}{2}$ **b.** $f(x) = x^3 + \frac{x}{2}$

T **86.** Use a graph to find to 3 decimal places the values of *x* for which $e^x > 10{,}000{,}000$.

87. a. Show that $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$ are inverses of one another.

T **b.** Graph *f* and *g* over an *x*-interval large enough to show the graphs intersecting at (1, 1) and (−1, −1). Be sure the picture shows the required symmetry in the line $y = x$.

88. a. Show that $h(x) = x^3/4$ and $k(x) = (4x)^{1/3}$ are inverses of one another.

T **b.** Graph *h* and *k* over an *x*-interval large enough to show the graphs intersecting at (2, 2) and (−2, −2). Be sure the picture shows the required symmetry in the line $y = x$.

CHAPTER 1 Additional and Advanced Exercises

Functions and Graphs

1. Are there two functions *f* and *g* such that $f \circ g = g \circ f$? Give reasons for your answer.

2. Are there two functions *f* and *g* with the following property? The graphs of *f* and *g* are not straight lines but the graph of $f \circ g$ is a straight line. Give reasons for your answer.

3. If $f(x)$ is odd, can anything be said of $g(x) = f(x) - 2$? What if *f* is even instead? Give reasons for your answer.

4. If $g(x)$ is an odd function defined for all values of *x*, can anything be said about $g(0)$? Give reasons for your answer.

5. Graph the equation $|x| + |y| = 1 + x$.

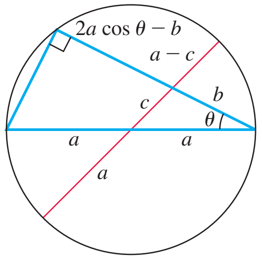
6. Graph the equation $y + |y| = x + |x|$.

Derivations and Proofs

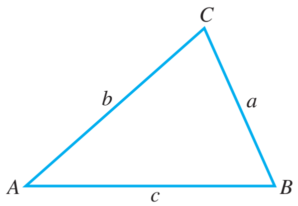
7. Prove the following identities.

a. $\dfrac{1 - \cos x}{\sin x} = \dfrac{\sin x}{1 + \cos x}$ **b.** $\dfrac{1 - \cos x}{1 + \cos x} = \tan^2 \dfrac{x}{2}$

8. Explain the following "proof without words" of the law of cosines. (*Source:* Kung, Sidney H., "Proof Without Words: The Law of Cosines," *Mathematics Magazine*, Vol. 63, no. 5, Dec. 1990, p. 342.)



9. Show that the area of triangle ABC is given by $(1/2)ab \sin C = (1/2)bc \sin A = (1/2)ca \sin B$.



10. Show that the area of triangle ABC is given by $\sqrt{s(s - a)(s - b)(s - c)}$ where $s = (a + b + c)/2$ is the semiperimeter of the triangle.

11. Show that if f is both even and odd, then $f(x) = 0$ for every x in the domain of f.

12. **a. Even-odd decompositions** Let f be a function whose domain is symmetric about the origin, that is, $-x$ belongs to the domain whenever x does. Show that f is the sum of an even function and an odd function:

$$f(x) = E(x) + O(x),$$

where E is an even function and O is an odd function. (*Hint:* Let $E(x) = (f(x) + f(-x))/2$. Show that $E(-x) = E(x)$, so that E is even. Then show that $O(x) = f(x) - E(x)$ is odd.)

b. Uniqueness Show that there is only one way to write f as the sum of an even and an odd function. (*Hint:* One way is given in part (a). If also $f(x) = E_1(x) + O_1(x)$ where E_1 is even and O_1 is odd, show that $E - E_1 = O_1 - O$. Then use Exercise 11 to show that $E = E_1$ and $O = O_1$.)

Effects of Parameters on Graphs

T 13. What happens to the graph of $y = ax^2 + bx + c$ as

a. a changes while b and c remain fixed?

b. b changes (a and c fixed, $a \neq 0$)?

c. c changes (a and b fixed, $a \neq 0$)?

T 14. What happens to the graph of $y = a(x + b)^3 + c$ as

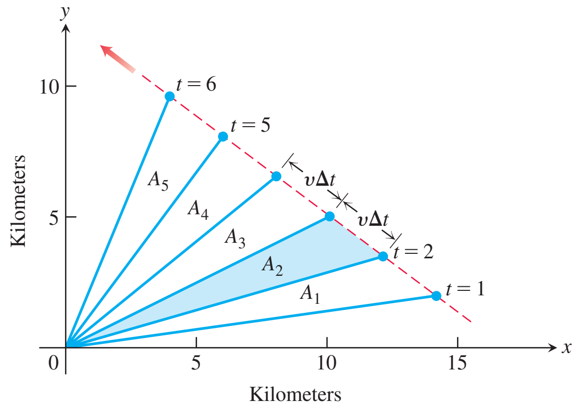
a. a changes while b and c remain fixed?

b. b changes (a and c fixed, $a \neq 0$)?

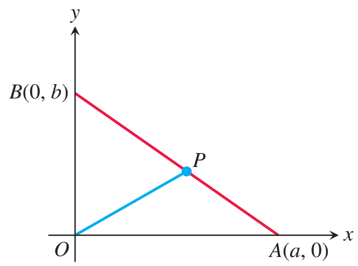
c. c changes (a and b fixed, $a \neq 0$)?

Geometry

15. An object's center of mass moves at a constant velocity v along a straight line past the origin. The accompanying figure shows the coordinate system and the line of motion. The dots show positions that are 1 second apart. Why are the areas $A_1, A_2, \ldots, A_5$ in the figure all equal? As in Kepler's equal area law (see Section 12.6), the line that joins the object's center of mass to the origin sweeps out equal areas in equal times.



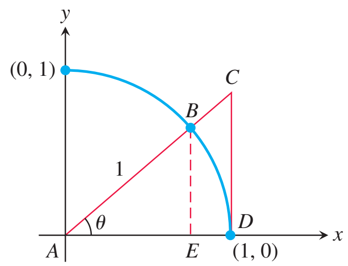
16. **a.** Find the slope of the line from the origin to the midpoint P of side AB in the triangle in the accompanying figure ($a, b > 0$).



b. When is OP perpendicular to AB?

17. Consider the quarter-circle of radius 1 and right triangles ABE and ACD given in the accompanying figure. Use standard area formulas to conclude that

$$\frac{1}{2} \sin \theta \cos \theta < \frac{\theta}{2} < \frac{1}{2} \frac{\sin \theta}{\cos \theta}.$$



18. Let $f(x) = ax + b$ and $g(x) = cx + d$. What condition must be satisfied by the constants a, b, c, d in order that $(f \circ g)(x) = (g \circ f)(x)$ for every value of x?

Theory and Examples

19. Domain and range Suppose that $a \neq 0, b \neq 1$, and $b > 0$. Determine the domain and range of the function.

a. $y = a(b^{c-x}) + d$ **b.** $y = a \log_b(x - c) + d$

20. Inverse functions Let

$$f(x) = \frac{ax + b}{cx + d}, \qquad c \neq 0, \qquad ad - bc \neq 0.$$

a. Give a convincing argument that f is one-to-one.

b. Find a formula for the inverse of f.

21. Depreciation Smith Hauling purchased an 18-wheel truck for \$100,000. The truck depreciates at the constant rate of \$10,000 per year for 10 years.

a. Write an expression that gives the value y after x years.

b. When is the value of the truck \$55,000?

22. Drug absorption A drug is administered intravenously for pain. The function

$$f(t) = 90 - 52 \ln(1 + t), \qquad 0 \leq t \leq 4$$

gives the number of units of the drug remaining in the body after t hours.

a. What was the initial number of units of the drug administered?

b. How much is present after 2 hours?

c. Draw the graph of f.

23. Finding investment time If Juanita invests \$1500 in a retirement account that earns 8% compounded annually, how long will it take this single payment to grow to \$5000?

24. The rule of 70 If you use the approximation $\ln 2 \approx 0.70$ (in place of $0.69314\ldots$), you can derive a rule of thumb that says, "To estimate how many years it will take an amount of money to double when invested at r percent compounded continuously, divide r into 70." For instance, an amount of money invested at 5% will double in about $70/5 = 14$ years. If you want it to double in 10 years instead, you have to invest it at $70/10 = 7\%$. Show how the rule of 70 is derived. (A similar "rule of 72" uses 72 instead of 70, because 72 has more integer factors.)

25. For what $x > 0$ does $x^{(x^x)} = (x^x)^x$? Give reasons for your answer.

T **26. a.** If $(\ln x)/x = (\ln 2)/2$, must $x = 2$?

b. If $(\ln x)/x = -2 \ln 2$, must $x = 1/2$?

Give reasons for your answers.

27. The quotient $(\log_4 x)/(\log_2 x)$ has a constant value. What value? Give reasons for your answer.

T **28. $\log_x(2)$ vs. $\log_2(x)$** How does $f(x) = \log_x(2)$ compare with $g(x) = \log_2(x)$? Here is one way to find out.

a. Use the equation $\log_a b = (\ln b)/(\ln a)$ to express $f(x)$ and $g(x)$ in terms of natural logarithms.

b. Graph f and g together. Comment on the behavior of f in relation to the signs and values of g.

CHAPTER 1 Technology Application Projects

Mathematica/Maple Projects

Projects can be found within MyLab Math.

- ***An Overview of Mathematica***
  An overview of *Mathematica* sufficient to complete the *Mathematica* modules appearing on the Web site.
- ***Modeling Change: Springs, Driving Safety, Radioactivity, Trees, Fish, and Mammals***
  Construct and interpret mathematical models, analyze and improve them, and make predictions using them.

2

Limits and Continuity

OVERVIEW In this chapter we develop the concept of a limit, first intuitively and then formally. We use limits to describe the way a function varies. Some functions vary *continuously*; small changes in x produce only small changes in $f(x)$. Other functions can have values that jump, vary erratically, or tend to increase or decrease without bound. The notion of limit gives a precise way to distinguish among these behaviors.

2.1 Rates of Change and Tangent Lines to Curves

Average and Instantaneous Speed

HISTORICAL BIOGRAPHY
Galileo Galilei
(1564–1642)
bit.ly/2OpdNBs

In the late sixteenth century, Galileo discovered that a solid object dropped from rest (initially not moving) near the surface of the earth and allowed to fall freely will fall a distance proportional to the square of the time it has been falling. This type of motion is called **free fall**. It assumes negligible air resistance to slow the object down, and that gravity is the only force acting on the falling object. If y denotes the distance fallen in meters after t seconds, then Galileo's law is

$$y = 4.9t^2 \text{ m},$$

where 4.9 is the (approximate) constant of proportionality.

More generally, suppose that a moving object has traveled distance $f(t)$ at time t. The object's **average speed** during an interval of time $[t_1, t_2]$ is found by dividing the distance traveled $f(t_2) - f(t_1)$ by the time elapsed $t_2 - t_1$. The unit of measure is length per unit time: kilometers per hour, or whatever is appropriate to the problem at hand.

Average Speed

When $f(t)$ measures the distance traveled at time t,

$$\text{Average speed over } [t_1, t_2] = \frac{\text{distance traveled}}{\text{elapsed time}} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

EXAMPLE 1 A rock breaks loose from the top of a tall cliff. What is its average speed

(a) during the first 2 seconds of fall?

(b) during the 1-second interval between second 1 and second 2?

Δ is the capital Greek letter Delta

Solution The average speed of the rock during a given time interval is the change in distance, Δy, divided by the length of the time interval, Δt. (Increments like Δy and Δt are reviewed in Appendix 4, and pronounced "delta *y*" and "delta *t*.") Measuring distance in meters and time in seconds, we have the following calculations:

(a) For the first 2 seconds: $\dfrac{\Delta y}{\Delta t} = \dfrac{4.9(2)^2 - 4.9(0)^2}{2 - 0} = 9.8\,\dfrac{\text{m}}{\text{s}}$

(b) From second 1 to second 2: $\dfrac{\Delta y}{\Delta t} = \dfrac{4.9(2)^2 - 4.9(1)^2}{2 - 1} = 14.7\,\dfrac{\text{m}}{\text{s}}$ ■

We want a way to determine the speed of a falling object at a single instant t_0, instead of using its average speed over an interval of time. To do this, we examine what happens when we calculate the average speed over shorter and shorter time intervals starting at t_0. The next example illustrates this process. Our discussion is informal here but will be made precise in Chapter 3.

EXAMPLE 2 Find the speed of the falling rock in Example 1 at $t = 1$ and $t = 2$ s.

Solution We can calculate the average speed of the rock over a time interval $[t_0, t_0 + h]$, having length $\Delta t = h$, as

$$\frac{\Delta y}{\Delta t} = \frac{4.9(t_0 + h)^2 - 4.9t_0^{\,2}}{h}. \tag{1}$$

We cannot use this formula to calculate the "instantaneous" speed at the exact moment t_0 by simply substituting $h = 0$, because we cannot divide by zero. But we *can* use it to calculate average speeds over increasingly short time intervals starting at $t_0 = 1$ and $t_0 = 2$. When we do so, by taking smaller and smaller values of h, we see a pattern (Table 2.1).

TABLE 2.1 Average speeds over short time intervals $[t_0, t_0 + h]$

Average speed: $\dfrac{\Delta y}{\Delta t} = \dfrac{4.9(t_0 + h)^2 - 4.9t_0^{\,2}}{h}$

Length of time interval h	Average speed over interval of length h starting at $t_0 = 1$	Average speed over interval of length h starting at $t_0 = 2$
1	14.7	24.5
0.1	10.29	20.09
0.01	9.849	19.649
0.001	9.8049	19.6049
0.0001	9.80049	19.60049

The average speed on intervals starting at $t_0 = 1$ seems to approach a limiting value of 9.8 as the length of the interval decreases. This suggests that the rock is falling at a speed of 9.8 m/s at $t_0 = 1$ s. Let's confirm this algebraically.

If we set $t_0 = 1$ and then expand the numerator in Equation (1) and simplify, we find that

$$\frac{\Delta y}{\Delta t} = \frac{4.9(1 + h)^2 - 4.9(1)^2}{h} = \frac{4.9(1 + 2h + h^2) - 4.9}{h}$$
$$= \frac{9.8h + 4.9h^2}{h} = 9.8 + 4.9h.$$

For values of h different from 0, the expressions on the right and left are equivalent and the average speed is $9.8 + 4.9h$ m/s. We can now see why the average speed has the limiting value $9.8 + 4.9(0) = 9.8$ m/s as h approaches 0.

Similarly, setting $t_0 = 2$ in Equation (1), the procedure yields

$$\frac{\Delta y}{\Delta t} = 19.6 + 4.9h$$

for values of h different from 0. As h gets closer and closer to 0, the average speed has the limiting value 19.6 m/s when $t_0 = 2$ s, as suggested by Table 2.1. ■

The average speed of a falling object is an example of a more general idea, an average rate of change.

Average Rates of Change and Secant Lines

Given any function $y = f(x)$, we calculate the average rate of change of y with respect to x over the interval $[x_1, x_2]$ by dividing the change in the value of y, $\Delta y = f(x_2) - f(x_1)$, by the length $\Delta x = x_2 - x_1 = h$ of the interval over which the change occurs. (We use the symbol h for Δx to simplify the notation here and later on.)

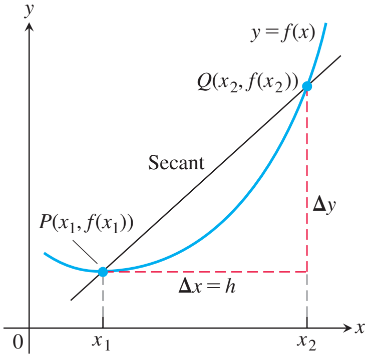


FIGURE 2.1 A secant to the graph $y = f(x)$. Its slope is $\Delta y/\Delta x$, the average rate of change of f over the interval $[x_1, x_2]$.

DEFINITION The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \qquad h \neq 0.$$

Geometrically, the rate of change of f over $[x_1, x_2]$ is the slope of the line through the points $P(x_1, f(x_1))$ and $Q(x_2, f(x_2))$ (Figure 2.1). In geometry, a line joining two points of a curve is called a **secant line**. Thus, the average rate of change of f from x_1 to x_2 is identical with the slope of secant line PQ. As the point Q approaches the point P along the curve, the length h of the interval over which the change occurs approaches zero. We will see that this procedure leads to the definition of the slope of a curve at a point.

Defining the Slope of a Curve

We know what is meant by the slope of a straight line, which tells us the rate at which it rises or falls—its rate of change as a linear function. But what is meant by the *slope of a curve* at a point P on the curve? If there is a *tangent line* to the curve at P—a line that grazes the curve like the tangent line to a circle—it would be reasonable to identify the slope of the tangent line as the slope of the curve at P. We will see that, among all the lines that pass through the point P, the tangent line is the one that gives the best approximation to the curve at P. We need a precise way to specify the tangent line at a point on a curve.

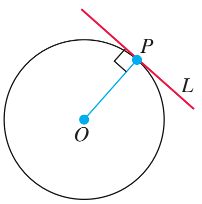


FIGURE 2.2 L is tangent to the circle at P if it passes through P perpendicular to radius OP.

Specifying a tangent line to a circle is straightforward. A line L is tangent to a circle at a point P if L passes through P and is perpendicular to the radius at P (Figure 2.2). But what does it mean to say that a line L is tangent to a more general curve at a point P?

HISTORICAL BIOGRAPHY
Pierre de Fermat
(1601–1665)
bit.ly/2NRJEeC

To define tangency for general curves, we use an approach that analyzes the behavior of the secant lines that pass through P and nearby points Q as Q moves toward P along the curve (Figure 2.3). We start with what we *can* calculate, namely the slope of the secant line PQ. We then compute the limiting value of the secant line's slope as Q approaches P along the curve. (We clarify the limit idea in the next section.) If the limit exists, we take it to be the slope of the curve at P and *define* the tangent line to the curve at P to be the line through P with this slope.

The next example illustrates the geometric idea for finding the tangent line to a curve.

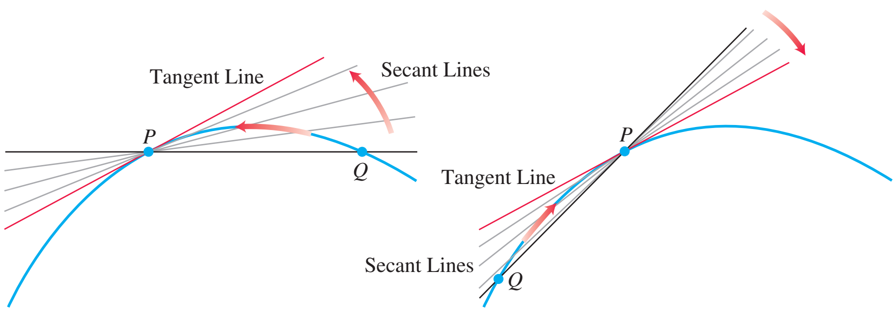


FIGURE 2.3 The tangent line to the curve at P is the line through P whose slope is the limit of the secant line slopes as $Q \to P$ from either side.

EXAMPLE 3 Find the slope of the tangent line to the parabola $y = x^2$ at the point $(2, 4)$ by analyzing the slopes of secant lines through $(2, 4)$. Write an equation for the tangent line to the parabola at this point.

Solution We begin with a secant line through $P(2, 4)$ and a nearby point $Q(2 + h, (2 + h)^2)$. We then write an expression for the slope of the secant line PQ and investigate what happens to the slope as Q approaches P along the curve:

$$\text{Secant line slope} = \frac{\Delta y}{\Delta x} = \frac{(2 + h)^2 - 2^2}{h} = \frac{h^2 + 4h + 4 - 4}{h}$$
$$= \frac{h^2 + 4h}{h} = h + 4.$$

If $h > 0$, then Q lies above and to the right of P, as in Figure 2.4. If $h < 0$, then Q lies to the left of P (not shown). In either case, as Q approaches P along the curve, h approaches zero and the secant line slope $h + 4$ approaches 4. We take 4 to be the parabola's slope at P.

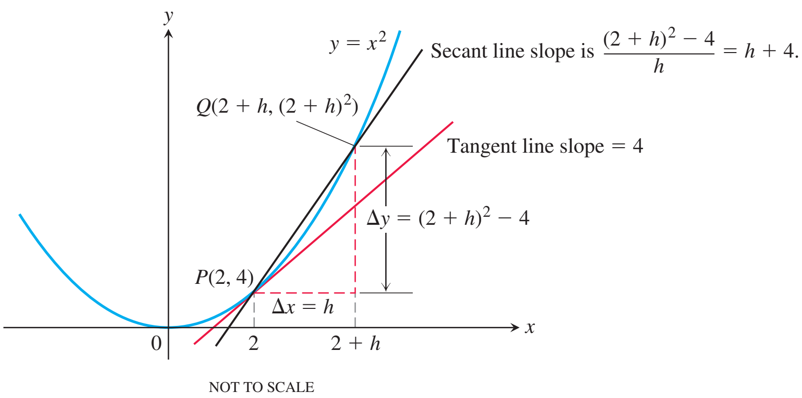


FIGURE 2.4 Finding the slope of the parabola $y = x^2$ at the point $P(2, 4)$ as the limit of secant line slopes (Example 3).

The tangent line to the parabola at P is the line through P with slope 4:

$$y = 4 + 4(x - 2) \qquad \text{Point-slope equation}$$
$$y = 4x - 4.$$

Rates of Change and Tangent Lines

The rates at which the rock in Example 2 was falling at the instants $t = 1$ and $t = 2$ are called *instantaneous rates of change*. Instantaneous rates of change and slopes of tangent lines are closely connected, as we see in the following examples.

EXAMPLE 4 Figure 2.5 shows how a population p of fruit flies (*Drosophila*) grew in a 50-day experiment. The number of flies was counted at regular intervals, the counted values plotted with respect to the number of elapsed days t, and the points joined by a smooth curve (colored blue in Figure 2.5). Find the average growth rate from day 23 to day 45.

Solution There were 150 flies on day 23 and 340 flies on day 45. Thus the number of flies increased by $340 - 150 = 190$ in $45 - 23 = 22$ days. The average rate of change of the population from day 23 to day 45 was

$$\text{Average rate of change: } \frac{\Delta p}{\Delta t} = \frac{340 - 150}{45 - 23} = \frac{190}{22} \approx 8.6 \text{ flies/day}.$$

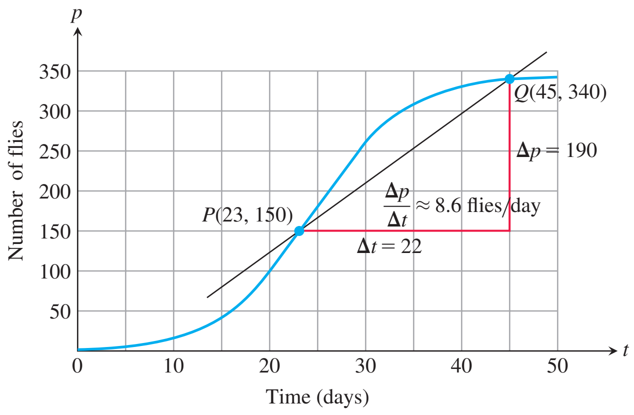


FIGURE 2.5 Growth of a fruit fly population in a controlled experiment. The average rate of change over 22 days is the slope $\Delta p/\Delta t$ of the secant line (Example 4).

This average is the slope of the secant line through the points P and Q on the graph in Figure 2.5. ■

The average rate of change from day 23 to day 45 calculated in Example 4 does not tell us how fast the population was changing on day 23 itself. For that we need to examine time intervals closer to the day in question.

EXAMPLE 5 How fast was the number of flies in the population of Example 4 growing on day 23?

Solution To answer this question, we examine the average rates of change over shorter and shorter time intervals starting at day 23. In geometric terms, we find these rates by calculating the slopes of secant lines from P to Q, for a sequence of points Q approaching P along the curve (Figure 2.6).

Q	Slope of $PQ = \Delta p/\Delta t$ (flies / day)
(45, 340)	$\frac{340 - 150}{45 - 23} \approx 8.6$
(40, 330)	$\frac{330 - 150}{40 - 23} \approx 10.6$
(35, 310)	$\frac{310 - 150}{35 - 23} \approx 13.3$
(30, 265)	$\frac{265 - 150}{30 - 23} \approx 16.4$

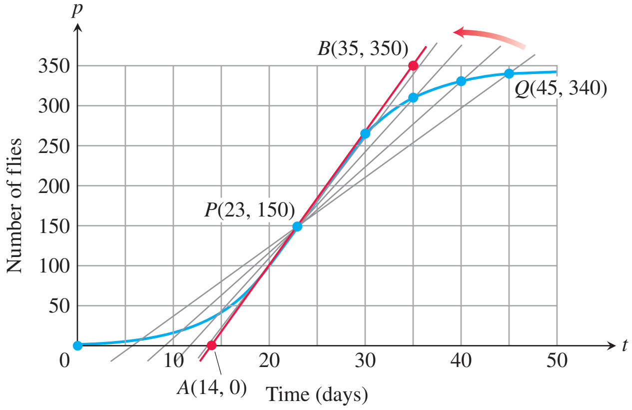


FIGURE 2.6 The positions and slopes of four secant lines through the point P on the fruit fly graph (Example 5).

The values in the table show that the secant line slopes rise from 8.6 to 16.4 as the t-coordinate of Q decreases from 45 to 30, and we would expect the slopes to rise slightly higher as t continued decreasing toward 23. Geometrically, the secant lines rotate counterclockwise about P and seem to approach the red tangent line in the figure. Since the line appears to pass through the points (14, 0) and (35, 350), its slope is approximately

$$\frac{350 - 0}{35 - 14} = 16.7 \text{ flies/day}.$$

On day 23 the population was increasing at a rate of about 16.7 flies / day. ■

The instantaneous rate of change is the value the average rate of change approaches as the length h of the interval over which the change occurs approaches zero. The average rate of change corresponds to the slope of a secant line; the instantaneous rate corresponds to the slope of the tangent line at a fixed value. So instantaneous rates and slopes of tangent lines are closely connected. We give a precise definition for these terms in the next chapter, but to do so we first need to develop the concept of a *limit*.

EXERCISES 2.1

Average Rates of Change

In Exercises 1–6, find the average rate of change of the function over the given interval or intervals.

1. $f(x) = x^3 + 1$
 a. $[2, 3]$ **b.** $[-1, 1]$

2. $g(x) = x^2 - 2x$
 a. $[1, 3]$ **b.** $[-2, 4]$

3. $h(t) = \cot t$
 a. $[\pi/4, 3\pi/4]$ **b.** $[\pi/6, \pi/2]$

4. $g(t) = 2 + \cos t$
 a. $[0, \pi]$ **b.** $[-\pi, \pi]$

5. $R(\theta) = \sqrt{4\theta + 1};\quad [0, 2]$

6. $P(\theta) = \theta^3 - 4\theta^2 + 5\theta;\quad [1, 2]$

Slope of a Curve at a Point

In Exercises 7–18, use the method in Example 3 to find **(a)** the slope of the curve at the given point P, and **(b)** an equation of the tangent line at P.

7. $y = x^2 - 5, \quad P(2, -1)$

8. $y = 7 - x^2, \quad P(2, 3)$

9. $y = x^2 - 2x - 3, \quad P(2, -3)$

10. $y = x^2 - 4x, \quad P(1, -3)$

11. $y = x^3, \quad P(2, 8)$

12. $y = 2 - x^3, \quad P(1, 1)$

13. $y = x^3 - 12x, \quad P(1, -11)$

14. $y = x^3 - 3x^2 + 4, \quad P(2, 0)$

15. $y = \frac{1}{x}, \quad P(-2, -1/2)$

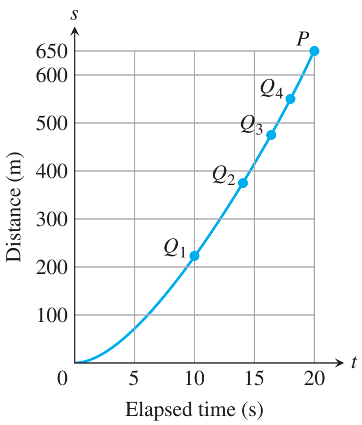
16. $y = \dfrac{x}{2 - x}, \quad P(4, -2)$

17. $y = \sqrt{x}, \quad P(4, 2)$

18. $y = \sqrt{7 - x}, \quad P(-2, 3)$

Instantaneous Rates of Change

19. Speed of a car The accompanying figure shows the time-to-distance graph for a sports car accelerating from a standstill.



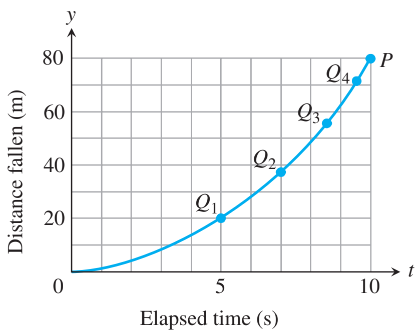
a. Estimate the slopes of secant lines PQ_1, PQ_2, PQ_3, and PQ_4, arranging them in order in a table like the one in Figure 2.6. What are the appropriate units for these slopes?

b. Then estimate the car's speed at time $t = 20$ s.

20. The accompanying figure shows the plot of distance fallen versus time for an object that fell from the lunar landing module a distance 80 m to the surface of the moon.

a. Estimate the slopes of the secant lines PQ_1, PQ_2, PQ_3, and PQ_4, arranging them in a table like the one in Figure 2.6.

b. About how fast was the object going when it hit the surface?



T **21.** The profits of a small company for each of the first five years of its operation are given in the following table:

Year	Profit in $1000s
2010	6
2011	27
2012	62
2013	111
2014	174

a. Plot points representing the profit as a function of year, and join them by as smooth a curve as you can.

b. What is the average rate of increase of the profits between 2012 and 2014?

c. Use your graph to estimate the rate at which the profits were changing in 2012.

T **22.** Make a table of values for the function $F(x) = (x + 2)/(x - 2)$ at the points $x = 1.2$, $x = 11/10$, $x = 101/100$, $x = 1001/1000$, $x = 10001/10000$, and $x = 1$.

a. Find the average rate of change of $F(x)$ over the intervals $[1, x]$ for each $x \neq 1$ in your table.

b. Extending the table if necessary, try to determine the rate of change of $F(x)$ at $x = 1$.

T **23.** Let $g(x) = \sqrt{x}$ for $x \geq 0$.

a. Find the average rate of change of $g(x)$ with respect to x over the intervals $[1, 2]$, $[1, 1.5]$ and $[1, 1 + h]$.

b. Make a table of values of the average rate of change of g with respect to x over the interval $[1, 1 + h]$ for some values of h approaching zero, say $h = 0.1, 0.01, 0.001, 0.0001, 0.00001$, and 0.000001.

c. What does your table indicate is the rate of change of $g(x)$ with respect to x at $x = 1$?

d. Calculate the limit as h approaches zero of the average rate of change of $g(x)$ with respect to x over the interval $[1, 1 + h]$.

T **24.** Let $f(t) = 1/t$ for $t \neq 0$.

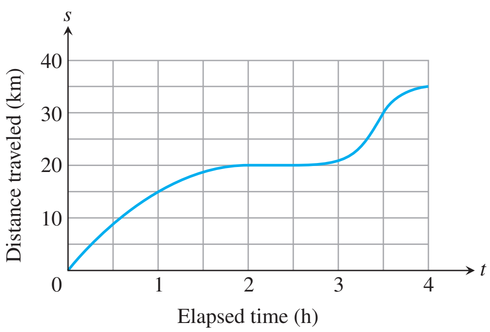
a. Find the average rate of change of f with respect to t over the intervals (i) from $t = 2$ to $t = 3$, and (ii) from $t = 2$ to $t = T$.

b. Make a table of values of the average rate of change of f with respect to t over the interval $[2, T]$, for some values of T approaching 2, say $T = 2.1, 2.01, 2.001, 2.0001, 2.00001$, and 2.000001.

c. What does your table indicate is the rate of change of f with respect to t at $t = 2$?

d. Calculate the limit as T approaches 2 of the average rate of change of f with respect to t over the interval from 2 to T. You will have to do some algebra before you can substitute $T = 2$.

25. The accompanying graph shows the total distance s traveled by a bicyclist after t hours.

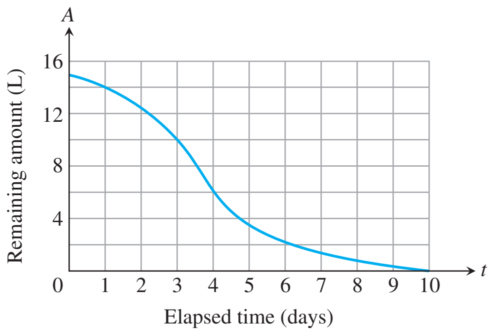


a. Estimate the bicyclist's average speed over the time intervals $[0, 1]$, $[1, 2.5]$, and $[2.5, 3.5]$.

b. Estimate the bicyclist's instantaneous speed at the times $t = \frac{1}{2}$, $t = 2$, and $t = 3$.

c. Estimate the bicyclist's maximum speed and the specific time at which it occurs.

26. The accompanying graph shows the total amount of gasoline A in the gas tank of a motorcycle after being driven for t days.



a. Estimate the average rate of gasoline consumption over the time intervals $[0, 3]$, $[0, 5]$, and $[7, 10]$.

b. Estimate the instantaneous rate of gasoline consumption at the times $t = 1$, $t = 4$, and $t = 8$.

c. Estimate the maximum rate of gasoline consumption and the specific time at which it occurs.

2.2 Limit of a Function and Limit Laws

In Section 2.1 we saw how limits arise when finding the instantaneous rate of change of a function or the tangent line to a curve. We begin this section by presenting an informal definition of the limit of a function. We then describe laws that capture the behavior of limits. These laws enable us to quickly compute limits for a variety of functions, including polynomials and rational functions. We present the precise definition of a limit in the next section.

HISTORICAL ESSAY
Limits
bit.ly/2P04ZyV

Limits of Function Values

Frequently when studying a function $y = f(x)$, we find ourselves interested in the function's behavior *near* a particular point c, but not *at* c itself. An important example occurs when the process of trying to evaluate a function at c leads to division by zero, which is undefined. We encountered this when seeking the instantaneous rate of change in y by considering the quotient function $\Delta y/h$ for h closer and closer to zero. In the next example we explore numerically how a function behaves near a particular point at which we cannot directly evaluate the function.

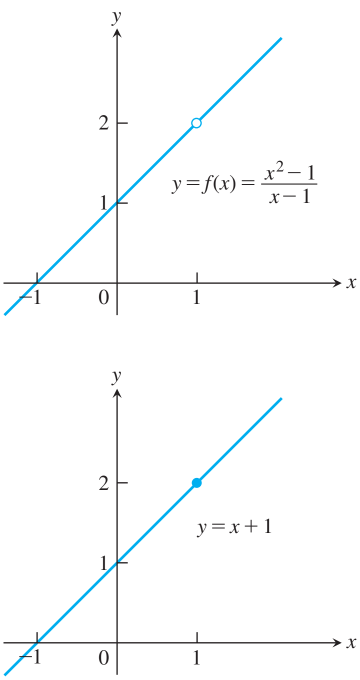


FIGURE 2.7 The graph of f is identical with the line $y = x + 1$ except at $x = 1$, where f is not defined (Example 1).

EXAMPLE 1 How does the function

$$f(x) = \frac{x^2 - 1}{x - 1}$$

behave near $x = 1$?

Solution The given formula defines f for all real numbers x except $x = 1$ (since we cannot divide by zero). For any $x \neq 1$, we can simplify the formula by factoring the numerator and canceling common factors:

$$f(x) = \frac{(x - 1)(x + 1)}{x - 1} = x + 1 \qquad \text{for} \qquad x \neq 1.$$

The graph of f is the line $y = x + 1$ with the point $(1, 2)$ *removed*. This removed point is shown as a "hole" in Figure 2.7. Even though $f(1)$ is not defined, it is clear that we can make the value of $f(x)$ as close as we want to 2 by choosing x close enough to 1 (Table 2.2). ■

An Informal Description of the Limit of a Function

We now give an informal definition of the limit of a function f at an interior point of the domain of f. Suppose that $f(x)$ is defined on an open interval about c, *except possibly at c*

TABLE 2.2 As x gets closer to 1, $f(x)$ gets closer to 2.

x	$f(x) = \dfrac{x^2-1}{x-1}$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001
0.999999	1.999999
1.000001	2.000001

itself. If $f(x)$ is arbitrarily close to the number L (as close to L as we like) for all x sufficiently close to c, other than c itself, then we say that f approaches the **limit** L as x approaches c, and write

$$\lim_{x \to c} f(x) = L,$$

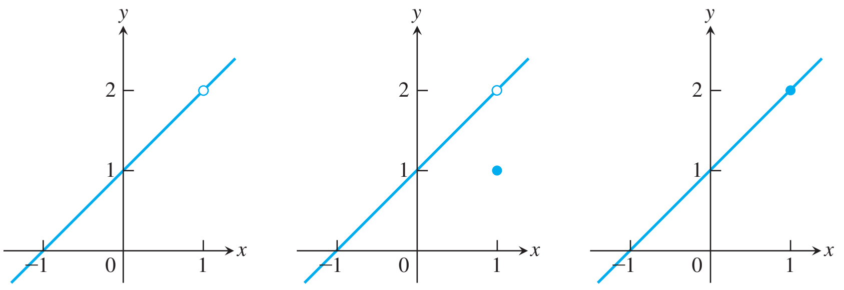
which is read "the limit of $f(x)$ as x approaches c is L." In Example 1 we would say that $f(x)$ approaches the *limit* 2 as x approaches 1, and write

$$\lim_{x \to 1} f(x) = 2, \quad \text{or} \quad \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2.$$

Essentially, the definition says that the values of $f(x)$ are close to the number L whenever x is close to c. The value of the function at c itself is not considered.

Our definition here is informal, because phrases like *arbitrarily close* and *sufficiently close* are imprecise; their meaning depends on the context. (To a machinist manufacturing a piston, *close* may mean *within a few hundredths of a millimeter*. To an astronomer studying distant galaxies, *close* may mean *within a few thousand light-years*.) Nevertheless, the definition is clear enough to enable us to recognize and evaluate limits of many specific functions. We will need the precise definition given in Section 2.3, when we set out to prove theorems about limits or study complicated functions. Here are several more examples exploring the idea of limits.

EXAMPLE 2 The limit of a function does not depend on how the function is defined at the point being approached. Consider the three functions in Figure 2.8. The function f has limit 2 as $x \to 1$ even though f is not defined at $x = 1$. The function g has limit 2 as $x \to 1$ even though $2 \neq g(1)$. The function h is the only one of the three functions in Figure 2.8 whose limit as $x \to 1$ equals its value at $x = 1$. For h, we have $\lim_{x \to 1} h(x) = h(1)$. This equality of limit and function value has an important meaning. As illustrated by the three examples in Figure 2.8, equality of limit and function value captures the notion of "continuity." We study this in detail in Section 2.6. ■



(a) $f(x) = \dfrac{x^2-1}{x-1}$ (b) $g(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$ (c) $h(x) = x + 1$

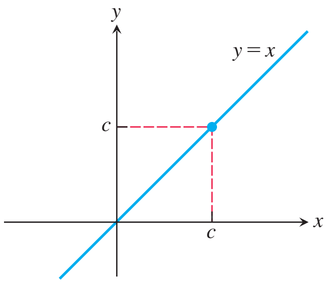
FIGURE 2.8 The limits of $f(x)$, $g(x)$, and $h(x)$ all equal 2 as x approaches 1. However, only $h(x)$ has the same function value as its limit at $x = 1$ (Example 2).

The process of finding a limit can be broken up into a series of steps involving limits of basic functions, which are combined using a sequence of simple operations that we will develop. We start with two basic functions.

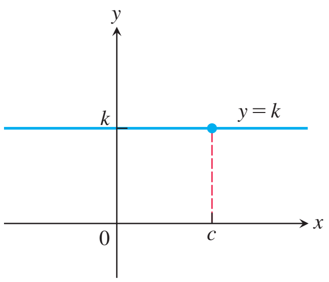
EXAMPLE 3 We find the limits of the identity function and of a constant function as x approaches $x = c$.

(a) If f is the **identity function** $f(x) = x$, then for any value of c (Figure 2.9a),

$$\lim_{x \to c} f(x) = \lim_{x \to c} x = c.$$



(a) Identity function



(b) Constant function

FIGURE 2.9 The functions in Example 3 have limits at all points c.

(b) If f is the **constant function** $f(x) = k$ (function with the constant value k), then for any value of c (Figure 2.9b),

$$\lim_{x \to c} f(x) = \lim_{x \to c} k = k.$$

For instances of each of these rules we have

$$\lim_{x \to 3} x = 3$$

Limit of identity function at $x = 3$

and

$$\lim_{x \to -7} (4) = \lim_{x \to 2} (4) = 4.$$

Limit of constant function $f(x) = 4$ at $x = -7$ or at $x = 2$

We prove these rules in Example 3 in Section 2.3. ■

A function may not have a limit at a particular point. Some ways that limits can fail to exist are illustrated in Figure 2.10 and described in the next example.

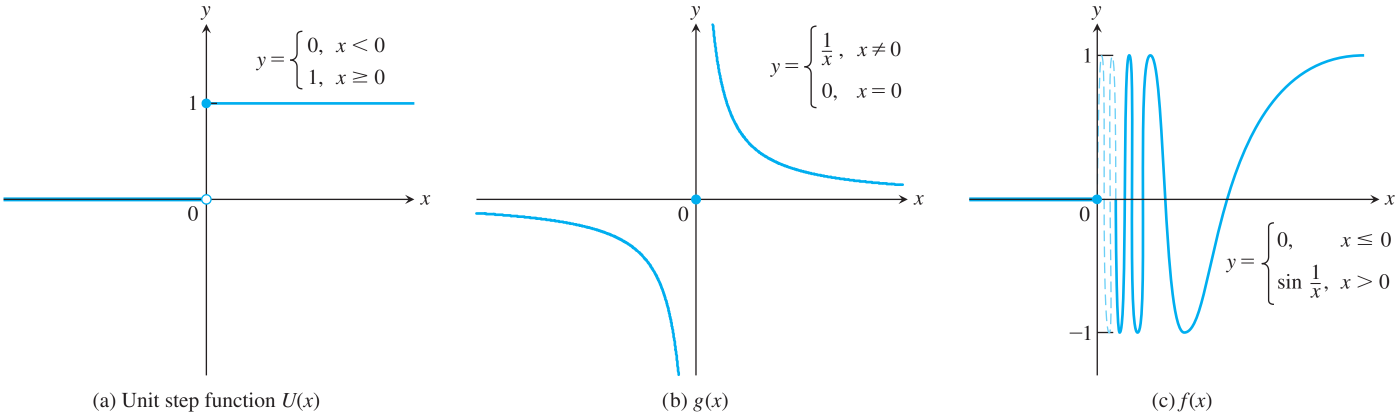


FIGURE 2.10 None of these functions has a limit as x approaches 0 (Example 4).

EXAMPLE 4 Discuss the behavior of the following functions, explaining why they have no limit as $x \to 0$.

(a) $U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$

(b) $g(x) = \begin{cases} \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

(c) $f(x) = \begin{cases} 0, & x \leq 0 \\ \sin \dfrac{1}{x}, & x > 0 \end{cases}$

Solution

(a) The function *jumps*: The **unit step function** $U(x)$ has no limit as $x \to 0$ because its values jump at $x = 0$. For negative values of x arbitrarily close to zero, $U(x) = 0$. For positive values of x arbitrarily close to zero, $U(x) = 1$. There is no *single* value L approached by $U(x)$ as $x \to 0$ (Figure 2.10a).

(b) The function *grows too "large" to have a limit*: $g(x)$ has no limit as $x \to 0$ because the values of g grow arbitrarily large in absolute value as $x \to 0$ and therefore do not stay close to *any* fixed real number (Figure 2.10b). We say the function is *not bounded*.

(c) The function *oscillates too much to have a limit*: $f(x)$ has no limit as $x \to 0$ because the function's values oscillate between $+1$ and -1 in every open interval containing 0. The values do not stay close to any single number as $x \to 0$ (Figure 2.10c). ■

The Limit Laws

A few basic rules allow us to break down complicated functions into simple ones when calculating limits. By using these laws, we can greatly simplify many limit computations.

THEOREM 1—Limit Laws
If L, M, c, and k are real numbers and

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x\to c}(f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x\to c}(f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x\to c}(k \cdot f(x)) = k \cdot L$
4. *Product Rule:* $\lim_{x\to c}(f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:* $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x\to c} [f(x)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

(If n is even, we assume that $f(x) \geq 0$ for x in an interval containing c.)

The Sum Rule says that the limit of a sum is the sum of the limits. Similarly, the next rules say that the limit of a difference is the difference of the limits; the limit of a constant times a function is the constant times the limit of the function; the limit of a product is the product of the limits; the limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0); the limit of a positive integer power (or root) of a function is the integer power (or root) of the limit (provided that the root of the limit is a real number).

There are simple intuitive arguments for why the properties in Theorem 1 are true (although these do not constitute proofs). If x is sufficiently close to c, then $f(x)$ is close to L and $g(x)$ is close to M, from our informal definition of a limit. It is then reasonable that $f(x) + g(x)$ is close to $L + M$; $f(x) - g(x)$ is close to $L - M$; $kf(x)$ is close to kL; $f(x)g(x)$ is close to LM; and $f(x)/g(x)$ is close to L/M if M is not zero. We prove the Sum Rule in Section 2.3, based on a rigorous definition of the limit. Rules 2–5 are proved in Appendix 5. Rule 6 is obtained by applying Rule 4 repeatedly. Rule 7 is proved in more advanced texts. The Sum, Difference, and Product Rules can be extended to any number of functions, not just two.

EXAMPLE 5 Use the observations $\lim_{x\to c} k = k$ and $\lim_{x\to c} x = c$ (Example 3) and the limit laws in Theorem 1 to find the following limits.

(a) $\lim_{x\to c}(x^3 + 4x^2 - 3)$

(b) $\lim_{x\to c} \dfrac{x^4 + x^2 - 1}{x^2 + 5}$

(c) $\lim_{x\to -2} \sqrt{4x^2 - 3}$

Solution

(a) $\lim_{x \to c}(x^3 + 4x^2 - 3) = \lim_{x \to c} x^3 + \lim_{x \to c} 4x^2 - \lim_{x \to c} 3$ — Sum and Difference Rules

$= c^3 + 4c^2 - 3$ — Power and Multiple Rules

(b) $\lim_{x \to c} \frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{\lim_{x \to c}(x^4 + x^2 - 1)}{\lim_{x \to c}(x^2 + 5)}$ — Quotient Rule

$= \frac{\lim_{x \to c} x^4 + \lim_{x \to c} x^2 - \lim_{x \to c} 1}{\lim_{x \to c} x^2 + \lim_{x \to c} 5}$ — Sum and Difference Rules

$= \frac{c^4 + c^2 - 1}{c^2 + 5}$ — Power or Product Rule

(c) $\lim_{x \to -2} \sqrt{4x^2 - 3} = \sqrt{\lim_{x \to -2}(4x^2 - 3)}$ — Root Rule with $n = 2$

$= \sqrt{\lim_{x \to -2} 4x^2 - \lim_{x \to -2} 3}$ — Difference Rule

$= \sqrt{4(-2)^2 - 3}$ — Product and Multiple Rules and limit of a constant function

$= \sqrt{16 - 3}$

$= \sqrt{13}$ ■

Evaluating Limits of Polynomials and Rational Functions

Theorem 1 simplifies the task of calculating limits of polynomials and rational functions. To evaluate the limit of a polynomial function as x approaches c, just substitute c for x in the formula for the function. To evaluate the limit of a rational function as x approaches a point *c at which the denominator is not zero*, substitute c for x in the formula for the function. (See Examples 5a and 5b.) We state these results formally as theorems.

THEOREM 2—Limits of Polynomials
If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0.$$

THEOREM 3—Limits of Rational Functions
If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

EXAMPLE 6 The following calculation illustrates Theorems 2 and 3:

$$\lim_{x \to -1} \frac{x^3 + 4x^2 - 3}{x^2 + 5} = \frac{(-1)^3 + 4(-1)^2 - 3}{(-1)^2 + 5} = \frac{0}{6} = 0$$

Since the denominator of this rational expression does not equal 0 when we substitute -1 for x, we can just compute the value of the expression at $x = -1$ to evaluate the limit. ■

Eliminating Common Factors from Zero Denominators

Theorem 3 applies only if the denominator of the rational function is not zero at the limit point c. If the denominator is zero, canceling common factors in the numerator and

Identifying Common Factors

If $Q(x)$ is a polynomial and $Q(c) = 0$, then $(x - c)$ is a factor of $Q(x)$. Thus, if the numerator and denominator of a rational function of x are both zero at $x = c$, they have $(x - c)$ as a common factor.

denominator may reduce the fraction to one whose denominator is no longer zero at c. If this happens, we can find the limit by substitution in the simplified fraction.

EXAMPLE 7 Evaluate

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x}.$$

Solution We cannot substitute $x = 1$ because it makes the denominator zero. We test the numerator to see if it, too, is zero at $x = 1$. It is, so it has a factor of $(x - 1)$ in common with the denominator. Canceling this common factor gives a simpler fraction with the same values as the original for $x \neq 1$:

$$\frac{x^2 + x - 2}{x^2 - x} = \frac{(x - 1)(x + 2)}{x(x - 1)} = \frac{x + 2}{x}, \qquad \text{if } x \neq 1.$$

Using the simpler fraction, we find the limit of these values as $x \to 1$ by evaluating the function at $x = 1$, as in Theorem 3:

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x \to 1} \frac{x + 2}{x} = \frac{1 + 2}{1} = 3.$$

See Figure 2.11. ■

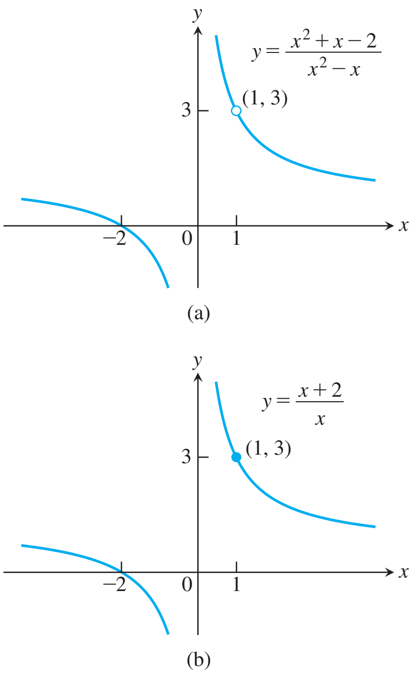


FIGURE 2.11 The graph of $f(x) = (x^2 + x - 2)/(x^2 - x)$ in part (a) is the same as the graph of $g(x) = (x + 2)/x$ in part (b) except at $x = 1$, where f is undefined. The functions have the same limit as $x \to 1$ (Example 7).

Using Calculators and Computers to Estimate Limits

We can try using a calculator or computer to guess a limit numerically. However, calculators and computers can sometimes give false values and misleading evidence about limits. Usually the problem is associated with rounding errors, as we now illustrate.

EXAMPLE 8 Estimate the value of $\lim_{x \to 0} \dfrac{\sqrt{x^2 + 100} - 10}{x^2}$.

Solution Table 2.3 lists values of the function obtained on a calculator for several points approaching $x = 0$. As x approaches 0 through the points ± 1, ± 0.5, ± 0.10, and ± 0.01, the function seems to approach the number 0.05.

As we take even smaller values of x, ± 0.0005, ± 0.0001, ± 0.00001, and ± 0.000001, the function appears to approach the number 0.

Is the answer 0.05 or 0, or some other value? We resolve this question in the next example. ■

TABLE 2.3 **Computed values of** $f(x) = \dfrac{\sqrt{x^2 + 100} - 10}{x^2}$ **near** $x = 0$

x	$f(x)$	
± 1	0.049876	approaches 0.05?
± 0.5	0.049969	
± 0.1	0.049999	
± 0.01	0.050000	
± 0.0005	0.050000	approaches 0?
± 0.0001	0.000000	
± 0.00001	0.000000	
± 0.000001	0.000000	

Using a computer or calculator may give ambiguous results, as in Example 8. A computer cannot always keep track of enough digits to avoid rounding errors in computing the values of $f(x)$ when x is very small. We cannot substitute $x = 0$ in the problem, and the numerator and denominator have no obvious common factors (as they did in Example 7). Sometimes, however, we can create a common factor algebraically.

EXAMPLE 9 Evaluate

$$\lim_{x \to 0} \frac{\sqrt{x^2 + 100} - 10}{x^2}.$$

Solution This is the limit we considered in Example 8. We can create a common factor by multiplying both numerator and denominator by the conjugate radical expression $\sqrt{x^2 + 100} + 10$ (obtained by changing the sign after the square root). The preliminary algebra rationalizes the numerator:

$$\begin{aligned}\frac{\sqrt{x^2 + 100} - 10}{x^2} &= \frac{\sqrt{x^2 + 100} - 10}{x^2} \cdot \frac{\sqrt{x^2 + 100} + 10}{\sqrt{x^2 + 100} + 10} && \text{Multiply and divide by the conjugate.}\\ &= \frac{x^2 + 100 - 100}{x^2(\sqrt{x^2 + 100} + 10)} && \text{Simplify}\\ &= \frac{x^2}{x^2(\sqrt{x^2 + 100} + 10)} && \text{Common factor } x^2\\ &= \frac{1}{\sqrt{x^2 + 100} + 10}. && \text{Cancel } x^2 \text{ for } x \neq 0.\end{aligned}$$

Therefore,

$$\begin{aligned}\lim_{x \to 0} \frac{\sqrt{x^2 + 100} - 10}{x^2} &= \lim_{x \to 0} \frac{1}{\sqrt{x^2 + 100} + 10}\\ &= \frac{1}{\sqrt{0^2 + 100} + 10} && \text{Limit Quotient Rule: Denominator not 0 at } x = 0 \text{ so can substitute.}\\ &= \frac{1}{20} = 0.05.\end{aligned}$$

This calculation provides the correct answer, resolving the ambiguous computer results in Example 8. ■

We cannot always manipulate the terms in an expression to find the limit of a quotient where the denominator becomes zero. In some cases the limit might then be found with geometric arguments (see the proof of Theorem 7 in Section 2.4), or through methods of calculus (developed in Section 4.5). The next theorem shows how to evaluate difficult limits by comparing them with functions having known limits.

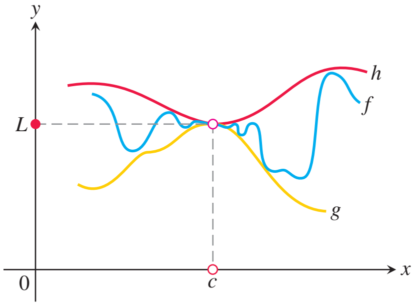


FIGURE 2.12 The graph of f is sandwiched between the graphs of g and h.

The Sandwich Theorem

The following theorem enables us to calculate a variety of limits. It is called the Sandwich Theorem because it refers to a function f whose values are sandwiched between the values of two other functions g and h that have the same limit L at a point c. Being trapped between the values of two functions that approach L, the values of f must also approach L (Figure 2.12). A proof is given in Appendix 5.

THEOREM 4—The Sandwich Theorem
Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L.$$

Then $\lim_{x\to c} f(x) = L$.

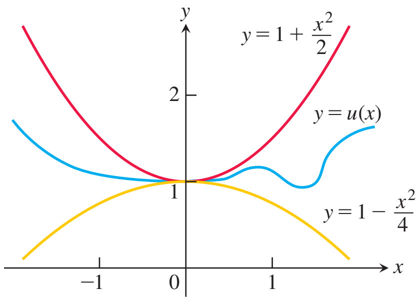


FIGURE 2.13 Any function $u(x)$ whose graph lies in the region between $y = 1 + (x^2/2)$ and $y = 1 - (x^2/4)$ has limit 1 as $x \to 0$ (Example 10).

The Sandwich Theorem is also called the Squeeze Theorem or the Pinching Theorem.

EXAMPLE 10 Given a function u that satisfies

$$1 - \frac{x^2}{4} \le u(x) \le 1 + \frac{x^2}{2} \quad \text{for all } x \ne 0,$$

find $\lim_{x\to 0} u(x)$, no matter how complicated u is.

Solution Since

$$\lim_{x\to 0}(1 - (x^2/4)) = 1 \quad \text{and} \quad \lim_{x\to 0}(1 + (x^2/2)) = 1,$$

the Sandwich Theorem implies that $\lim_{x\to 0} u(x) = 1$ (Figure 2.13). ■

EXAMPLE 11 The Sandwich Theorem helps us establish several important limit rules:

(a) $\lim_{\theta\to 0} \sin\theta = 0$

(b) $\lim_{\theta\to 0} \cos\theta = 1$

(c) For any function f, $\lim_{x\to c} |f(x)| = 0$ implies $\lim_{x\to c} f(x) = 0$.

Solution

(a) In Section 1.3 we established that $-|\theta| \le \sin\theta \le |\theta|$ for all θ (see Figure 2.14a). Since $\lim_{\theta\to 0}(-|\theta|) = \lim_{\theta\to 0}|\theta| = 0$, we have

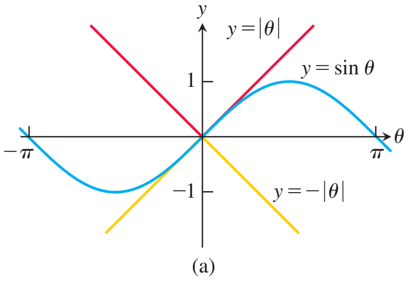
$$\lim_{\theta\to 0} \sin\theta = 0.$$

(b) From Section 1.3, $0 \le 1 - \cos\theta \le |\theta|$ for all θ (see Figure 2.14b), and we have $\lim_{\theta\to 0}(1 - \cos\theta) = 0$ so

$$\lim_{\theta\to 0} 1 - (1 - \cos\theta) = 1 - \lim_{\theta\to 0}(1 - \cos\theta) = 1 - 0,$$

$$\lim_{\theta\to 0} \cos\theta = 1. \qquad \text{Simplify}$$

(c) Since $-|f(x)| \le f(x) \le |f(x)|$ and $-|f(x)|$ and $|f(x)|$ have limit 0 as $x \to c$, it follows that $\lim_{x\to c} f(x) = 0$. ■



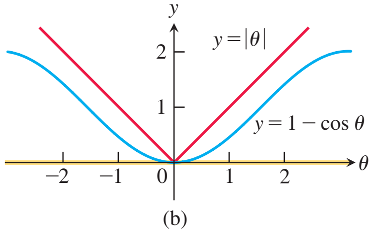


FIGURE 2.14 The Sandwich Theorem confirms the limits in Example 11.

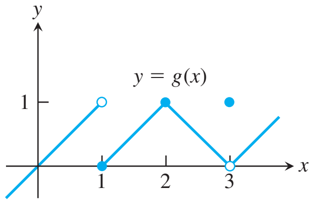
Example 11 shows that the sine and cosine functions are equal to their limits at $\theta = 0$. We have not yet established that for any c, $\lim_{\theta\to c} \sin\theta = \sin c$, and $\lim_{\theta\to c} \cos\theta = \cos c$. These limit formulas do hold, as will be shown in Section 2.6.

EXERCISES 2.2

Limits from Graphs

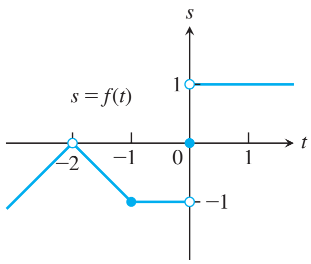
1. For the function $g(x)$ graphed here, find the following limits or explain why they do not exist.

   a. $\lim_{x \to 1} g(x)$ b. $\lim_{x \to 2} g(x)$ c. $\lim_{x \to 3} g(x)$ d. $\lim_{x \to 2.5} g(x)$



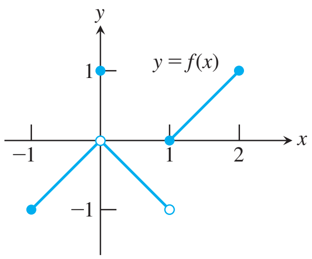
2. For the function $f(t)$ graphed here, find the following limits or explain why they do not exist.

   a. $\lim_{t \to -2} f(t)$ b. $\lim_{t \to -1} f(t)$ c. $\lim_{t \to 0} f(t)$ d. $\lim_{t \to -0.5} f(t)$



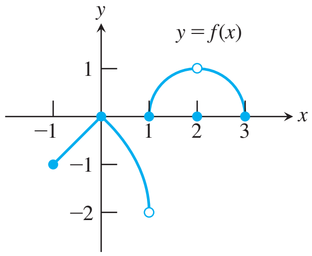
3. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?

   a. $\lim_{x \to 0} f(x)$ exists.
   b. $\lim_{x \to 0} f(x) = 0$
   c. $\lim_{x \to 0} f(x) = 1$
   d. $\lim_{x \to 1} f(x) = 1$
   e. $\lim_{x \to 1} f(x) = 0$
   f. $\lim_{x \to c} f(x)$ exists at every point c in $(-1, 1)$.
   g. $\lim_{x \to 1} f(x)$ does not exist.
   h. $f(0) = 0$
   i. $f(0) = 1$
   j. $f(1) = 0$
   k. $f(1) = -1$



4. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?

   a. $\lim_{x \to 2} f(x)$ does not exist.
   b. $\lim_{x \to 2} f(x) = 2$
   c. $\lim_{x \to 1} f(x)$ does not exist.
   d. $\lim_{x \to c} f(x)$ exists at every point c in $(-1, 1)$.
   e. $\lim_{x \to c} f(x)$ exists at every point c in $(1, 3)$.
   f. $f(1) = 0$
   g. $f(1) = -2$
   h. $f(2) = 0$
   i. $f(2) = 1$



Existence of Limits

In Exercises 5 and 6, explain why the limits do not exist.

5. $\lim_{x \to 0} \dfrac{x}{|x|}$

6. $\lim_{x \to 1} \dfrac{1}{x - 1}$

7. Suppose that a function $f(x)$ is defined for all real values of x except $x = c$. Can anything be said about the existence of $\lim_{x \to c} f(x)$? Give reasons for your answer.

8. Suppose that a function $f(x)$ is defined for all x in $[-1, 1]$. Can anything be said about the existence of $\lim_{x \to 0} f(x)$? Give reasons for your answer.

9. If $\lim_{x \to 1} f(x) = 5$, must f be defined at $x = 1$? If it is, must $f(1) = 5$? Can we conclude *anything* about the values of f at $x = 1$? Explain.

10. If $f(1) = 5$, must $\lim_{x \to 1} f(x)$ exist? If it does, then must $\lim_{x \to 1} f(x) = 5$? Can we conclude *anything* about $\lim_{x \to 1} f(x)$? Explain.

Calculating Limits

Find the limits in Exercises 11–22.

11. $\lim_{x \to -3} (x^2 - 13)$

12. $\lim_{x \to 2} (-x^2 + 5x - 2)$

13. $\lim_{t \to 6} 8(t - 5)(t - 7)$

14. $\lim_{x \to -2} (x^3 - 2x^2 + 4x + 8)$

15. $\lim_{x \to 2} \dfrac{2x + 5}{11 - x^3}$

16. $\lim_{s \to 2/3} (8 - 3s)(2s - 1)$

17. $\lim_{x \to -1/2} 4x(3x + 4)^2$

18. $\lim_{y \to 2} \dfrac{y + 2}{y^2 + 5y + 6}$

19. $\lim_{y \to -3} (5 - y)^{4/3}$

20. $\lim_{z \to 4} \sqrt{z^2 - 10}$

21. $\lim_{h \to 0} \dfrac{3}{\sqrt{3h + 1} + 1}$

22. $\lim_{h \to 0} \dfrac{\sqrt{5h + 4} - 2}{h}$

Limits of quotients Find the limits in Exercises 23–42.

23. $\lim_{x \to 5} \dfrac{x - 5}{x^2 - 25}$

24. $\lim_{x \to -3} \dfrac{x + 3}{x^2 + 4x + 3}$

25. $\lim_{x \to -5} \dfrac{x^2 + 3x - 10}{x + 5}$

26. $\lim_{x \to 2} \dfrac{x^2 - 7x + 10}{x - 2}$

27. $\lim_{t \to 1} \dfrac{t^2 + t - 2}{t^2 - 1}$

28. $\lim_{t \to -1} \dfrac{t^2 + 3t + 2}{t^2 - t - 2}$

29. $\lim_{x \to -2} \dfrac{-2x - 4}{x^3 + 2x^2}$

30. $\lim_{y \to 0} \dfrac{5y^3 + 8y^2}{3y^4 - 16y^2}$

31. $\lim_{x\to1}\dfrac{x^{-1}-1}{x-1}$

32. $\lim_{x\to0}\dfrac{\frac{1}{x-1}+\frac{1}{x+1}}{x}$

33. $\lim_{u\to1}\dfrac{u^4-1}{u^3-1}$

34. $\lim_{v\to2}\dfrac{v^3-8}{v^4-16}$

35. $\lim_{x\to9}\dfrac{\sqrt{x}-3}{x-9}$

36. $\lim_{x\to4}\dfrac{4x-x^2}{2-\sqrt{x}}$

37. $\lim_{x\to1}\dfrac{x-1}{\sqrt{x+3}-2}$

38. $\lim_{x\to-1}\dfrac{\sqrt{x^2+8}-3}{x+1}$

39. $\lim_{x\to2}\dfrac{\sqrt{x^2+12}-4}{x-2}$

40. $\lim_{x\to-2}\dfrac{x+2}{\sqrt{x^2+5}-3}$

41. $\lim_{x\to-3}\dfrac{2-\sqrt{x^2-5}}{x+3}$

42. $\lim_{x\to4}\dfrac{4-x}{5-\sqrt{x^2+9}}$

Limits with trigonometric functions Find the limits in Exercises 43–50.

43. $\lim_{x\to0}(2\sin x-1)$

44. $\lim_{x\to0}\sin^2 x$

45. $\lim_{x\to0}\sec x$

46. $\lim_{x\to0}\tan x$

47. $\lim_{x\to0}\dfrac{1+x+\sin x}{3\cos x}$

48. $\lim_{x\to0}(x^2-1)(2-\cos x)$

49. $\lim_{x\to-\pi}\sqrt{x+4}\cos(x+\pi)$

50. $\lim_{x\to0}\sqrt{7+\sec^2 x}$

Using Limit Rules

51. Suppose $\lim_{x\to0}f(x)=1$ and $\lim_{x\to0}g(x)=-5$. Name the rules in Theorem 1 that are used to accomplish steps (a), (b), and (c) of the following calculation.

$$\lim_{x\to0}\frac{2f(x)-g(x)}{(f(x)+7)^{2/3}}=\frac{\lim_{x\to0}(2f(x)-g(x))}{\lim_{x\to0}(f(x)+7)^{2/3}}\qquad\text{(a)}$$

$$=\frac{\lim_{x\to0}2f(x)-\lim_{x\to0}g(x)}{\left(\lim_{x\to0}(f(x)+7)\right)^{2/3}}\qquad\text{(b)}$$

$$=\frac{2\lim_{x\to0}f(x)-\lim_{x\to0}g(x)}{\left(\lim_{x\to0}f(x)+\lim_{x\to0}7\right)^{2/3}}\qquad\text{(c)}$$

$$=\frac{(2)(1)-(-5)}{(1+7)^{2/3}}=\frac{7}{4}$$

52. Let $\lim_{x\to1}h(x)=5$, $\lim_{x\to1}p(x)=1$, and $\lim_{x\to1}r(x)=2$. Name the rules in Theorem 1 that are used to accomplish steps (a), (b), and (c) of the following calculation.

$$\lim_{x\to1}\frac{\sqrt{5h(x)}}{p(x)(4-r(x))}=\frac{\lim_{x\to1}\sqrt{5h(x)}}{\lim_{x\to1}(p(x)(4-r(x)))}\qquad\text{(a)}$$

$$=\frac{\sqrt{\lim_{x\to1}5h(x)}}{\left(\lim_{x\to1}p(x)\right)\left(\lim_{x\to1}(4-r(x))\right)}\qquad\text{(b)}$$

$$=\frac{\sqrt{5\lim_{x\to1}h(x)}}{\left(\lim_{x\to1}p(x)\right)\left(\lim_{x\to1}4-\lim_{x\to1}r(x)\right)}\qquad\text{(c)}$$

$$=\frac{\sqrt{(5)(5)}}{(1)(4-2)}=\frac{5}{2}$$

53. Suppose $\lim_{x\to c}f(x)=5$ and $\lim_{x\to c}g(x)=-2$. Find

a. $\lim_{x\to c}f(x)g(x)$

b. $\lim_{x\to c}2f(x)g(x)$

c. $\lim_{x\to c}(f(x)+3g(x))$

d. $\lim_{x\to c}\dfrac{f(x)}{f(x)-g(x)}$

54. Suppose $\lim_{x\to4}f(x)=0$ and $\lim_{x\to4}g(x)=-3$. Find

a. $\lim_{x\to4}(g(x)+3)$

b. $\lim_{x\to4}xf(x)$

c. $\lim_{x\to4}(g(x))^2$

d. $\lim_{x\to4}\dfrac{g(x)}{f(x)-1}$

55. Suppose $\lim_{x\to b}f(x)=7$ and $\lim_{x\to b}g(x)=-3$. Find

a. $\lim_{x\to b}(f(x)+g(x))$

b. $\lim_{x\to b}f(x)\cdot g(x)$

c. $\lim_{x\to b}4g(x)$

d. $\lim_{x\to b}f(x)/g(x)$

56. Suppose that $\lim_{x\to-2}p(x)=4$, $\lim_{x\to-2}r(x)=0$, and $\lim_{x\to-2}s(x)=-3$. Find

a. $\lim_{x\to-2}(p(x)+r(x)+s(x))$

b. $\lim_{x\to-2}p(x)\cdot r(x)\cdot s(x)$

c. $\lim_{x\to-2}(-4p(x)+5r(x))/s(x)$

Limits of Average Rates of Change

Because of their connection with secant lines, tangents, and instantaneous rates, limits of the form

$$\lim_{h\to0}\frac{f(x+h)-f(x)}{h}$$

occur frequently in calculus. In Exercises 57–62, evaluate this limit for the given value of x and function f.

57. $f(x)=x^2,\quad x=1$

58. $f(x)=x^2,\quad x=-2$

59. $f(x)=3x-4,\quad x=2$

60. $f(x)=1/x,\quad x=-2$

61. $f(x)=\sqrt{x},\quad x=7$

62. $f(x)=\sqrt{3x+1},\quad x=0$

Using the Sandwich Theorem

63. If $\sqrt{5-2x^2}\le f(x)\le\sqrt{5-x^2}$ for $-1\le x\le1$, find $\lim_{x\to0}f(x)$.

64. If $2-x^2\le g(x)\le 2\cos x$ for all x, find $\lim_{x\to0}g(x)$.

65. a. It can be shown that the inequalities

$$1-\frac{x^2}{6}<\frac{x\sin x}{2-2\cos x}<1$$

hold for all values of x close to zero. What, if anything, does this tell you about

$$\lim_{x\to0}\frac{x\sin x}{2-2\cos x}?$$

Give reasons for your answer.

T **b.** Graph $y=1-(x^2/6)$, $y=(x\sin x)/(2-2\cos x)$, and $y=1$ together for $-2\le x\le2$. Comment on the behavior of the graphs as $x\to0$.

66. **a.** Suppose that the inequalities

$$\frac{1}{2} - \frac{x^2}{24} < \frac{1 - \cos x}{x^2} < \frac{1}{2}$$

hold for values of x close to zero. (They do, as you will see in Section 9.9.) What, if anything, does this tell you about

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2}?$$

Give reasons for your answer.

T **b.** Graph the equations $y = (1/2) - (x^2/24)$, $y = (1 - \cos x)/x^2$, and $y = 1/2$ together for $-2 \le x \le 2$. Comment on the behavior of the graphs as $x \to 0$.

Estimating Limits

T You will find a graphing calculator useful for Exercises 67–76.

67. Let $f(x) = (x^2 - 9)/(x + 3)$.

a. Make a table of the values of f at the points $x = -3.1, -3.01, -3.001$, and so on as far as your calculator can go. Then estimate $\lim_{x \to -3} f(x)$. What estimate do you arrive at if you evaluate f at $x = -2.9, -2.99, -2.999, \ldots$ instead?

b. Support your conclusions in part (a) by graphing f near $c = -3$ and using Zoom and Trace to estimate y-values on the graph as $x \to -3$.

c. Find $\lim_{x \to -3} f(x)$ algebraically, as in Example 7.

68. Let $g(x) = (x^2 - 2)/(x - \sqrt{2})$.

a. Make a table of the values of g at the points $x = 1.4, 1.41, 1.414$, and so on through successive decimal approximations of $\sqrt{2}$. Estimate $\lim_{x \to \sqrt{2}} g(x)$.

b. Support your conclusion in part (a) by graphing g near $c = \sqrt{2}$ and using Zoom and Trace to estimate y-values on the graph as $x \to \sqrt{2}$.

c. Find $\lim_{x \to \sqrt{2}} g(x)$ algebraically.

69. Let $G(x) = (x + 6)/(x^2 + 4x - 12)$.

a. Make a table of the values of G at $x = -5.9, -5.99, -5.999$, and so on. Then estimate $\lim_{x \to -6} G(x)$. What estimate do you arrive at if you evaluate G at $x = -6.1, -6.01, -6.001, \ldots$ instead?

b. Support your conclusions in part (a) by graphing G and using Zoom and Trace to estimate y-values on the graph as $x \to -6$.

c. Find $\lim_{x \to -6} G(x)$ algebraically.

70. Let $h(x) = (x^2 - 2x - 3)/(x^2 - 4x + 3)$.

a. Make a table of the values of h at $x = 2.9, 2.99, 2.999$, and so on. Then estimate $\lim_{x \to 3} h(x)$. What estimate do you arrive at if you evaluate h at $x = 3.1, 3.01, 3.001, \ldots$ instead?

b. Support your conclusions in part (a) by graphing h near $c = 3$ and using Zoom and Trace to estimate y-values on the graph as $x \to 3$.

c. Find $\lim_{x \to 3} h(x)$ algebraically.

71. Let $f(x) = (x^2 - 1)/(|x| - 1)$.

a. Make tables of the values of f at values of x that approach $c = -1$ from above and below. Then estimate $\lim_{x \to -1} f(x)$.

b. Support your conclusion in part (a) by graphing f near $c = -1$ and using Zoom and Trace to estimate y-values on the graph as $x \to -1$.

c. Find $\lim_{x \to -1} f(x)$ algebraically.

72. Let $F(x) = (x^2 + 3x + 2)/(2 - |x|)$.

a. Make tables of values of F at values of x that approach $c = -2$ from above and below. Then estimate $\lim_{x \to -2} F(x)$.

b. Support your conclusion in part (a) by graphing F near $c = -2$ and using Zoom and Trace to estimate y-values on the graph as $x \to -2$.

c. Find $\lim_{x \to -2} F(x)$ algebraically.

73. Let $g(\theta) = (\sin \theta)/\theta$.

a. Make a table of the values of g at values of θ that approach $\theta_0 = 0$ from above and below. Then estimate $\lim_{\theta \to 0} g(\theta)$.

b. Support your conclusion in part (a) by graphing g near $\theta_0 = 0$.

74. Let $G(t) = (1 - \cos t)/t^2$.

a. Make tables of values of G at values of t that approach $t_0 = 0$ from above and below. Then estimate $\lim_{t \to 0} G(t)$.

b. Support your conclusion in part (a) by graphing G near $t_0 = 0$.

75. Let $f(x) = x^{1/(1-x)}$.

a. Make tables of values of f at values of x that approach $c = 1$ from above and below. Does f appear to have a limit as $x \to 1$? If so, what is it? If not, why not?

b. Support your conclusions in part (a) by graphing f near $c = 1$.

76. Let $f(x) = (3^x - 1)/x$.

a. Make tables of values of f at values of x that approach $c = 0$ from above and below. Does f appear to have a limit as $x \to 0$? If so, what is it? If not, why not?

b. Support your conclusions in part (a) by graphing f near $c = 0$.

Theory and Examples

77. If $x^4 \le f(x) \le x^2$ for x in $[-1, 1]$ and $x^2 \le f(x) \le x^4$ for $x < -1$ and $x > 1$, at what points c do you automatically know $\lim_{x \to c} f(x)$? What can you say about the value of the limit at these points?

78. Suppose that $g(x) \le f(x) \le h(x)$ for all $x \ne 2$ and suppose that

$$\lim_{x \to 2} g(x) = \lim_{x \to 2} h(x) = -5.$$

Can we conclude anything about the values of f, g, and h at $x = 2$? Could $f(2) = 0$? Could $\lim_{x \to 2} f(x) = 0$? Give reasons for your answers.

79. If $\lim_{x \to 4} \frac{f(x) - 5}{x - 2} = 1$, find $\lim_{x \to 4} f(x)$.

80. If $\lim_{x \to -2} \frac{f(x)}{x^2} = 1$, find

a. $\lim_{x \to -2} f(x)$ **b.** $\lim_{x \to -2} \frac{f(x)}{x}$

81. **a.** If $\lim_{x \to 2} \frac{f(x) - 5}{x - 2} = 3$, find $\lim_{x \to 2} f(x)$.

b. If $\lim_{x \to 2} \frac{f(x) - 5}{x - 2} = 4$, find $\lim_{x \to 2} f(x)$.

82. If $\lim_{x \to 0} \frac{f(x)}{x^2} = 1$, find

a. $\lim_{x \to 0} f(x)$ **b.** $\lim_{x \to 0} \frac{f(x)}{x}$

T **83.** **a.** Graph $g(x) = x \sin(1/x)$ to estimate $\lim_{x \to 0} g(x)$, zooming in on the origin as necessary.

b. Confirm your estimate in part (a) with a proof.

T **84.** **a.** Graph $h(x) = x^2 \cos(1/x^3)$ to estimate $\lim_{x \to 0} h(x)$, zooming in on the origin as necessary.

b. Confirm your estimate in part (a) with a proof.

COMPUTER EXPLORATIONS

Graphical Estimates of Limits

In Exercises 85–90, use a CAS to perform the following steps:

a. Plot the function near the point c being approached.

b. From your plot guess the value of the limit.

85. $\lim_{x \to 2} \frac{x^4 - 16}{x - 2}$

86. $\lim_{x \to -1} \frac{x^3 - x^2 - 5x - 3}{(x + 1)^2}$

87. $\lim_{x \to 0} \frac{\sqrt[3]{1 + x} - 1}{x}$

88. $\lim_{x \to 3} \frac{x^2 - 9}{\sqrt{x^2 + 7} - 4}$

89. $\lim_{x \to 0} \frac{1 - \cos x}{x \sin x}$

90. $\lim_{x \to 0} \frac{2x^2}{3 - 3\cos x}$

2.3 The Precise Definition of a Limit

We now turn our attention to the precise definition of a limit. The early history of calculus saw controversy about the validity of the basic concepts underlying the theory. Apparent contradictions were argued over by both mathematicians and philosophers. These controversies were resolved by the precise definition, which allows us to replace vague phrases like "gets arbitrarily close to" in the informal definition with specific conditions that can be applied to any particular example. With a rigorous definition, we can avoid misunderstandings, prove the limit properties given in the preceding section, and establish many important limits.

To show that the limit of $f(x)$ as $x \to c$ equals the number L, we need to show that the gap between $f(x)$ and L can be made "as small as we choose" if x is kept "close enough" to c. Let us see what this requires if we specify the size of the gap between $f(x)$ and L.

EXAMPLE 1 Consider the function $y = 2x - 1$ near $x = 4$. Intuitively it seems clear that y is close to 7 when x is close to 4, so $\lim_{x \to 4}(2x - 1) = 7$. However, how close to $x = 4$ does x have to be so that $y = 2x - 1$ differs from 7 by, say, less than 2 units?

Solution We are asked: For what values of x is $|y - 7| < 2$? To find the answer we first express $|y - 7|$ in terms of x:

$$|y - 7| = |(2x - 1) - 7| = |2x - 8|.$$

The question then becomes: what values of x satisfy the inequality $|2x - 8| < 2$? To find out, we solve the inequality:

$$|2x - 8| < 2$$

$$-2 < 2x - 8 < 2 \quad \text{Removing absolute value gives two inequalities.}$$

$$6 < 2x < 10 \quad \text{Add 8 to each term.}$$

$$3 < x < 5 \quad \text{Solve for } x.$$

$$-1 < x - 4 < 1. \quad \text{Solve for } x - 4.$$

Keeping x within 1 unit of $x = 4$ will keep y within 2 units of $y = 7$ (Figure 2.15). ■

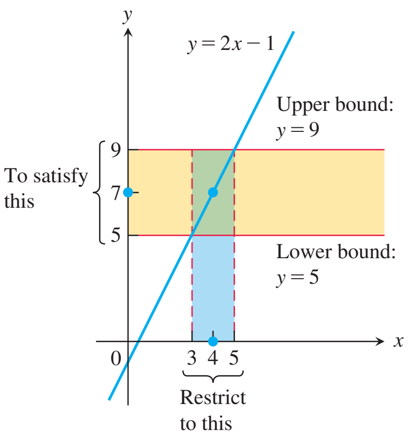


FIGURE 2.15 Keeping x within 1 unit of $x = 4$ will keep y within 2 units of $y = 7$ (Example 1).

In the previous example we determined how close x must be to a particular value c to ensure that the outputs $f(x)$ of some function lie within a prescribed interval about a limit value L. To show that the limit of $f(x)$ as $x \to c$ actually equals L, we must be able to show that the gap between $f(x)$ and L can be made less than *any prescribed error*, no matter how

δ is the Greek letter delta
ε is the Greek letter epsilon

small, by holding x close enough to c. To describe arbitrary prescribed errors, we introduce two constants, δ (delta) and ε (epsilon). These Greek letters are traditionally used to represent small changes in a variable or a function.

Definition of Limit

Suppose we are watching the values of a function $f(x)$ as x approaches c (without taking on the value c itself). Certainly we want to be able to say that $f(x)$ stays within one-tenth of a unit from L as soon as x stays within some distance δ of c (Figure 2.16). But that in itself is not enough, because as x continues on its course toward c, what is to prevent $f(x)$ from jumping around within the interval from $L - (1/10)$ to $L + (1/10)$ without tending toward L? We can be told that the error can be no more than $1/100$ or $1/1000$ or $1/100{,}000$. Each time, we find a new δ-interval about c so that keeping x within that interval satisfies the new error tolerance. And each time the possibility exists that $f(x)$ might jump away from L at some later stage.

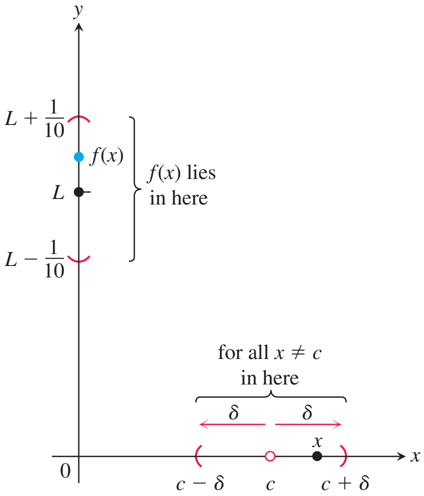


FIGURE 2.16 How should we define $\delta > 0$ so that keeping x within the interval $(c - \delta, c + \delta)$ will keep $f(x)$ within the interval $\left(L - \frac{1}{10}, L + \frac{1}{10}\right)$?

The figures on the next page illustrate the problem. You can think of this as a quarrel between a skeptic and a scholar. The skeptic presents ε-challenges to show there is room for doubt that the limit exists. The scholar counters every challenge with a δ-interval around c which ensures that the function takes values within ε of L.

How do we stop this seemingly endless series of challenges and responses? We can do so by proving that for *every* error tolerance ε that the challenger can produce, we can present a matching distance δ that keeps x "close enough" to c to keep $f(x)$ within that ε-tolerance of L (Figure 2.17). This leads us to the precise definition of a limit.

DEFINITION Let $f(x)$ be defined on an open interval about c, except possibly at c itself. We say that the **limit of $f(x)$ as x approaches c is the number L**, and write

$$\lim_{x \to c} f(x) = L,$$

if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

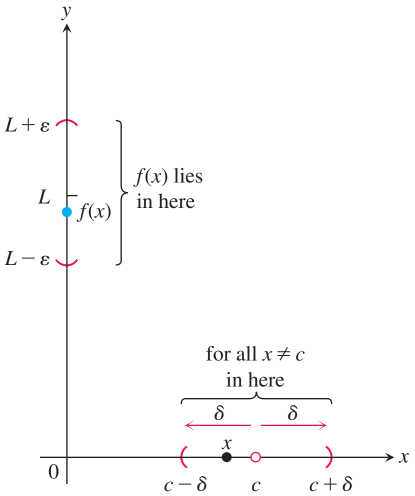


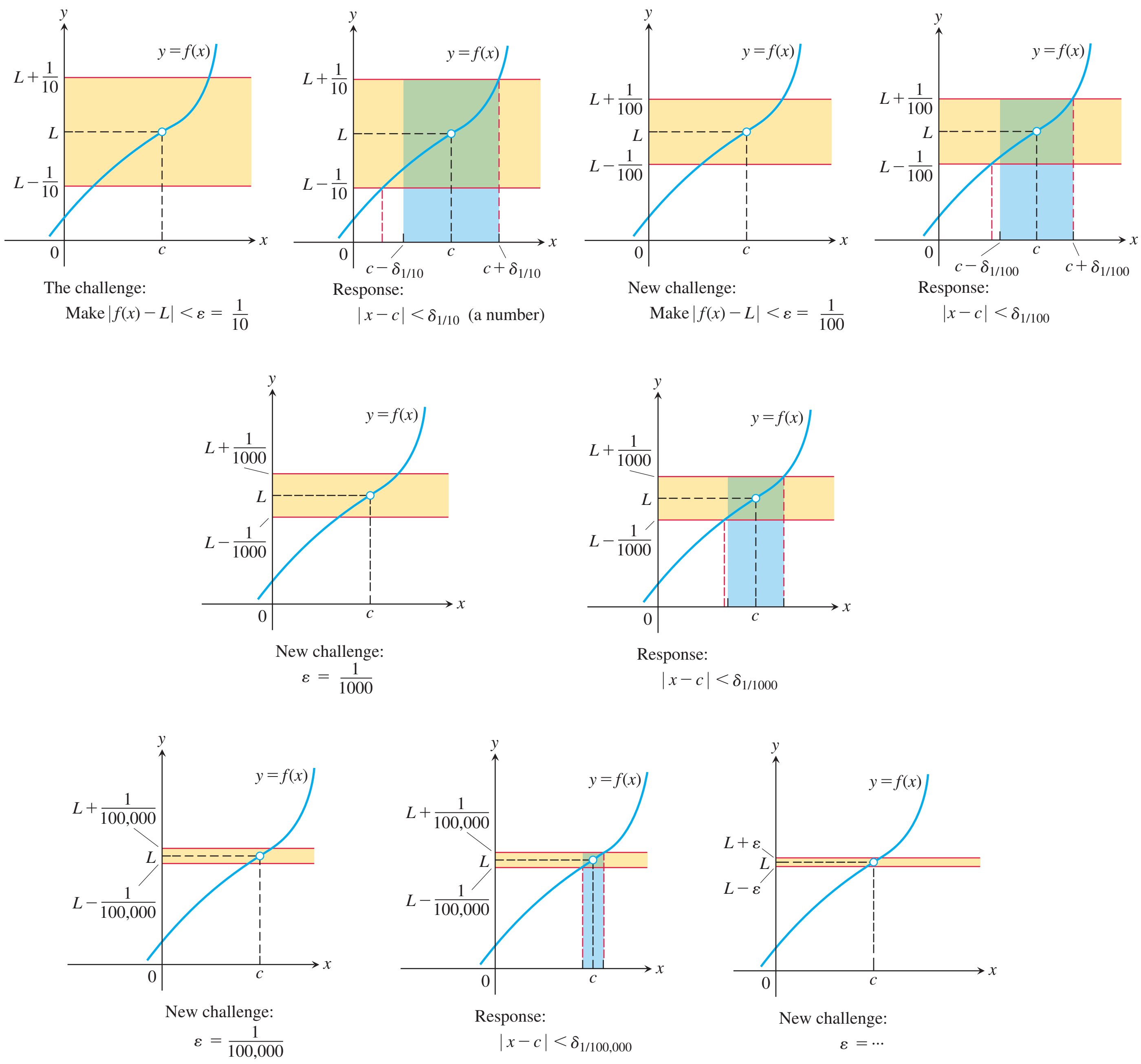
FIGURE 2.17 The relation of δ and ε in the definition of limit.

To visualize the definition, imagine machining a cylindrical shaft to a close tolerance. The diameter of the shaft is determined by turning a dial to a setting measured by a variable x. We try for diameter L, but since nothing is perfect we must be satisfied with a diameter $f(x)$ somewhere between $L - \varepsilon$ and $L + \varepsilon$. The number δ is our control tolerance for the dial; it tells us how close our dial setting must be to the setting $x = c$ in order to guarantee that the diameter $f(x)$ of the shaft will be accurate to within ε of L. As the tolerance for error becomes stricter, we may have to adjust δ. The value of δ, how tight our control setting must be, depends on the value of ε, the error tolerance.

The definition of limit extends to functions on more general domains. It is only required that each open interval around c contains points in the domain of the function other than c. See Additional and Advanced Exercises 39–43 for examples of limits for functions with complicated domains. In the next section we will see how the definition of limit applies at points lying on the boundary of an interval.

Examples: Testing the Definition

The formal definition of limit does not tell how to find the limit of a function, but it does enable us to verify that a conjectured limit value is correct. The following examples show how the definition can be used to verify limit statements for specific functions. However, the real purpose of the definition is not to do calculations like this, but rather to prove general theorems so that the calculation of specific limits can be simplified, such as the theorems stated in the previous section.



EXAMPLE 2 Show that

$$\lim_{x \to 1} (5x - 3) = 2.$$

Solution Set $c = 1$, $f(x) = 5x - 3$, and $L = 2$ in the definition of limit. For any given $\varepsilon > 0$, we have to find a suitable $\delta > 0$ so that if $x \neq 1$ and x is within distance δ of $c = 1$, that is, whenever

$$0 < |x - 1| < \delta,$$

it is true that $f(x)$ is within distance ε of $L = 2$, so

$$|f(x) - 2| < \varepsilon.$$

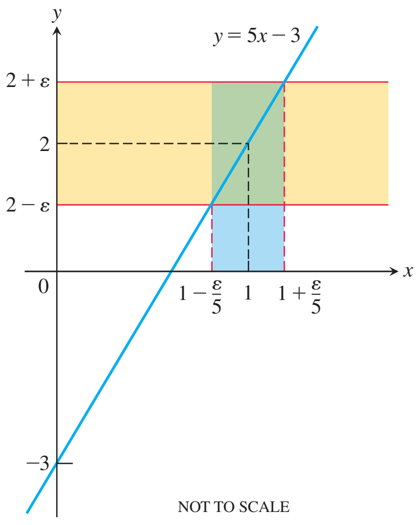


FIGURE 2.18 If $f(x) = 5x - 3$, then $0 < |x - 1| < \varepsilon/5$ guarantees that $|f(x) - 2| < \varepsilon$ (Example 2).

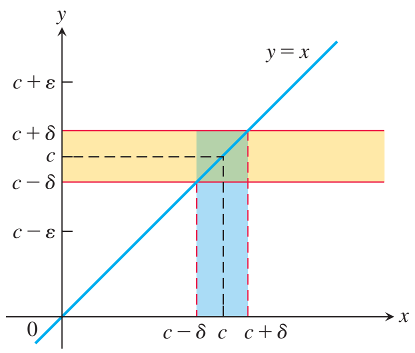


FIGURE 2.19 For the function $f(x) = x$, we find that $0 < |x - c| < \delta$ will guarantee $|f(x) - c| < \varepsilon$ whenever $\delta \le \varepsilon$ (Example 3a).

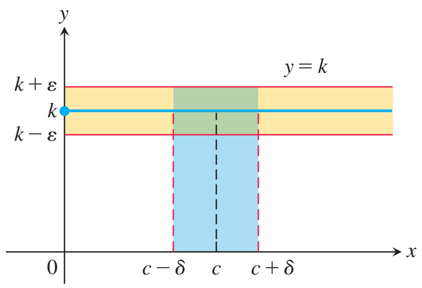


FIGURE 2.20 For the function $f(x) = k$, we find that $|f(x) - k| < \varepsilon$ for any positive δ (Example 3b).

We find δ by working backward from the ε-inequality:

$$\begin{aligned} |(5x-3)-2| = |5x-5| &< \varepsilon \\ 5|x-1| &< \varepsilon \\ |x-1| &< \varepsilon/5. \end{aligned}$$

Thus, we can take $\delta = \varepsilon/5$ (Figure 2.18). If $0 < |x - 1| < \delta = \varepsilon/5$, then

$$|(5x-3)-2| = |5x-5| = 5\,|x-1| < 5(\varepsilon/5) = \varepsilon,$$

which proves that $\lim_{x\to1}(5x-3) = 2$.

The value of $\delta = \varepsilon/5$ is not the only value that will make $0 < |x - 1| < \delta$ imply $|5x - 5| < \varepsilon$. Any smaller positive δ will do as well. The definition does not ask for the "best" positive δ, just one that will work. ■

EXAMPLE 3 Prove the following results presented graphically in Section 2.2.

(a) $\lim\limits_{x\to c} x = c$

(b) $\lim\limits_{x\to c} k = k$ (k constant)

Solution

(a) Let $\varepsilon > 0$ be given. We must find $\delta > 0$ such that

$$|x - c| < \varepsilon \qquad \text{whenever} \qquad 0 < |x - c| < \delta.$$

The implication will hold if δ equals ε or any smaller positive number (Figure 2.19). This proves that $\lim_{x\to c} x = c$.

(b) Let $\varepsilon > 0$ be given. We must find $\delta > 0$ such that

$$|k - k| < \varepsilon \qquad \text{whenever} \qquad 0 < |x - c| < \delta.$$

Since $k - k = 0$, we can use any positive number for δ and the implication will hold (Figure 2.20). This proves that $\lim_{x\to c} k = k$. ■

Finding Deltas Algebraically for Given Epsilons

In Examples 2 and 3, the interval of values about c for which $|f(x) - L|$ was less than ε was symmetric about c and we could take δ to be half the length of that interval. When the interval around c on which we have $|f(x) - L| < \varepsilon$ is not symmetric about c, we can take δ to be the distance from c to the interval's *nearer* endpoint.

EXAMPLE 4 For the limit $\lim_{x\to5}\sqrt{x-1} = 2$, find a $\delta > 0$ that works for $\varepsilon = 1$. That is, find a $\delta > 0$ such that

$$\left|\sqrt{x-1} - 2\right| < 1 \qquad \text{whenever} \qquad 0 < |x - 5| < \delta.$$

Solution We organize the search into two steps.

1. *Solve the inequality* $\left|\sqrt{x-1} - 2\right| < 1$ *to find an interval containing* $x = 5$ *on which the inequality holds for all* $x \ne 5$.

$$\begin{aligned} \left|\sqrt{x-1} - 2\right| &< 1 \\ -1 < \sqrt{x-1} - 2 &< 1 \\ 1 < \sqrt{x-1} &< 3 \\ 1 < x - 1 &< 9 \\ 2 < x &< 10 \end{aligned}$$

FIGURE 2.21 An open interval of radius 3 about $x = 5$ will lie inside the open interval (2, 10).

The inequality holds for all x in the open interval (2, 10), so it holds for all $x \neq 5$ in this interval as well.

2. *Find a value of* $\delta > 0$ *to place the centered interval* $5 - \delta < x < 5 + \delta$ (centered at $x = 5$) *inside the interval* (2, 10). The distance from 5 to the nearer endpoint of (2, 10) is 3 (Figure 2.21). If we take $\delta = 3$ or any smaller positive number, then the inequality $0 < |x - 5| < \delta$ will automatically place x between 2 and 10 and imply that $|\sqrt{x-1} - 2| < 1$ (Figure 2.22):

$$|\sqrt{x-1} - 2| < 1 \qquad \text{whenever} \qquad 0 < |x - 5| < 3.$$

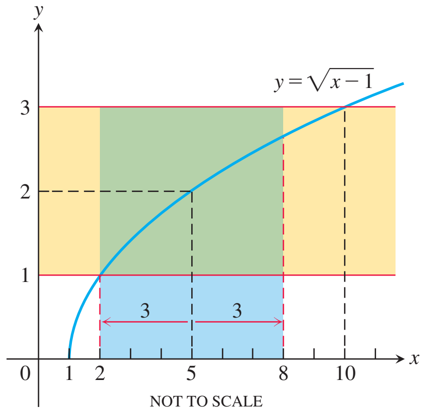


FIGURE 2.22 The function and intervals in Example 4.

How to Find Algebraically a δ for a Given f, L, c, and $\varepsilon > 0$

The process of finding a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \qquad \text{whenever} \qquad 0 < |x - c| < \delta$$

can be accomplished in two steps.

1. *Solve the inequality* $|f(x) - L| < \varepsilon$ to find an open interval (a, b) containing c on which the inequality holds for all $x \neq c$. Note that we do not require the inequality to hold at $x = c$. It may hold there or it may not, but the value of f at $x = c$ does not influence the existence of a limit.
2. *Find a value of* $\delta > 0$ that places the open interval $(c - \delta, c + \delta)$ centered at c inside the interval (a, b). The inequality $|f(x) - L| < \varepsilon$ will hold for all $x \neq c$ in this δ-interval.

EXAMPLE 5 Prove that $\lim_{x \to 2} f(x) = 4$ if

$$f(x) = \begin{cases} x^2, & x \neq 2 \\ 1, & x = 2. \end{cases}$$

Solution Our task is to show that given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - 4| < \varepsilon \qquad \text{whenever} \qquad 0 < |x - 2| < \delta.$$

1. *Solve the inequality* $|f(x) - 4| < \varepsilon$ *to find an open interval containing* $x = 2$ *on which the inequality holds for all* $x \neq 2$.

   For $x \neq c = 2$, we have $f(x) = x^2$, and the inequality to solve is $|x^2 - 4| < \varepsilon$:

$$\begin{aligned} |x^2 - 4| &< \varepsilon \\ -\varepsilon < x^2 - 4 &< \varepsilon \\ 4 - \varepsilon < x^2 &< 4 + \varepsilon \\ \sqrt{4-\varepsilon} < |x| &< \sqrt{4+\varepsilon} \\ \sqrt{4-\varepsilon} < x &< \sqrt{4+\varepsilon}. \end{aligned}$$

Assumes $\varepsilon < 4$; see below.

An open interval about $x = 2$ that solves the inequality.

   The inequality $|f(x) - 4| < \varepsilon$ holds for all $x \neq 2$ in the open interval $\left(\sqrt{4-\varepsilon}, \sqrt{4+\varepsilon}\right)$ (Figure 2.23).

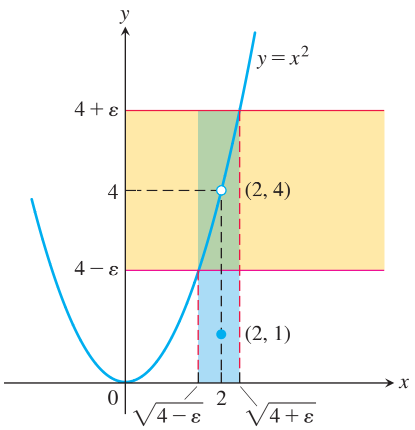


FIGURE 2.23 An interval containing $x = 2$ so that the function in Example 5 satisfies $|f(x) - 4| < \varepsilon$.

2. *Find a value of* $\delta > 0$ *that places the centered interval* $(2 - \delta, 2 + \delta)$ *inside the interval* $\left(\sqrt{4-\varepsilon}, \sqrt{4+\varepsilon}\right)$.

   Take δ to be the distance from $x = 2$ to the nearer endpoint of $\left(\sqrt{4-\varepsilon}, \sqrt{4+\varepsilon}\right)$. In other words, take $\delta = \min\left\{2 - \sqrt{4-\varepsilon}, \sqrt{4+\varepsilon} - 2\right\}$, the *minimum* (the smaller) of the two numbers $2 - \sqrt{4 - \varepsilon}$ and $\sqrt{4 + \varepsilon} - 2$. If δ has this or any smaller positive value, the inequality $0 < |x - 2| < \delta$ will automatically place x between $\sqrt{4 - \varepsilon}$ and $\sqrt{4 + \varepsilon}$ to make $|f(x) - 4| < \varepsilon$. For all x,

$$|f(x) - 4| < \varepsilon \qquad \text{whenever} \qquad 0 < |x - 2| < \delta.$$

This completes the proof for $\varepsilon < 4$.

If $\varepsilon \geq 4$, then we take δ to be the distance from $x = 2$ to the nearer endpoint of the interval $\left(0, \sqrt{4 + \varepsilon}\right)$. In other words, take $\delta = \min\left\{2, \sqrt{4 + \varepsilon} - 2\right\}$. (See Figure 2.23.) ■

Using the Definition to Prove Theorems

We do not usually rely on the formal definition of limit to verify specific limits such as those in the preceding examples. Rather, we appeal to general theorems about limits, in particular the theorems of Section 2.2. The definition is used to prove these theorems (Appendix 6). As an example, we prove part 1 of Theorem 1, the Sum Rule.

EXAMPLE 6 Given that $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, prove that

$$\lim_{x\to c} (f(x) + g(x)) = L + M.$$

Solution Let $\varepsilon > 0$ be given. We want to find a positive number δ such that

$$|f(x) + g(x) - (L + M)| < \varepsilon \qquad \text{whenever} \qquad 0 < |x - c| < \delta.$$

Regrouping terms, we get

$$\begin{aligned} |f(x) + g(x) - (L + M)| &= |(f(x) - L) + (g(x) - M)| \\ &\leq |f(x) - L| + |g(x) - M|. \end{aligned}$$

Triangle Inequality: $|a + b| \leq |a| + |b|$

Since $\lim_{x\to c} f(x) = L$, there exists a number $\delta_1 > 0$ such that

$$|f(x) - L| < \varepsilon/2 \qquad \text{whenever} \qquad 0 < |x - c| < \delta_1.$$

Can find δ_1 since $\lim_{x\to c} f(x) = L$

Similarly, since $\lim_{x\to c} g(x) = M$, there exists a number $\delta_2 > 0$ such that

$$|g(x) - M| < \varepsilon/2 \qquad \text{whenever} \qquad 0 < |x - c| < \delta_2.$$

Can find δ_2 since $\lim_{x\to c} g(x) = M$

Let $\delta = \min\{\delta_1, \delta_2\}$, the smaller of δ_1 and δ_2. If $0 < |x - c| < \delta$ then $|x - c| < \delta_1$, so $|f(x) - L| < \varepsilon/2$, and $|x - c| < \delta_2$, so $|g(x) - M| < \varepsilon/2$. Therefore

$$|f(x) + g(x) - (L + M)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This shows that $\lim_{x\to c}(f(x) + g(x)) = L + M$. ■

EXERCISES 2.3

Centering Intervals About a Point

In Exercises 1–6, sketch the interval (a, b) on the x-axis with the point c inside. Then find a value of $\delta > 0$ such that $a < x < b$ whenever $0 < |x - c| < \delta$.

1. $a = 1, \quad b = 7, \quad c = 5$

2. $a = 1, \quad b = 7, \quad c = 2$

3. $a = -7/2, \quad b = -1/2, \quad c = -3$

4. $a = -7/2, \quad b = -1/2, \quad c = -3/2$

5. $a = 4/9, \quad b = 4/7, \quad c = 1/2$

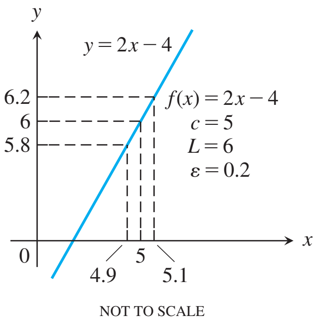
6. $a = 2.7591, \quad b = 3.2391, \quad c = 3$

Finding Deltas Graphically

In Exercises 7–14, use the graphs to find a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

7.



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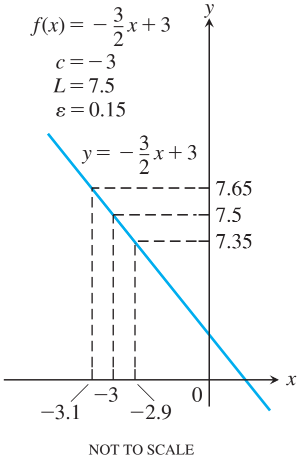

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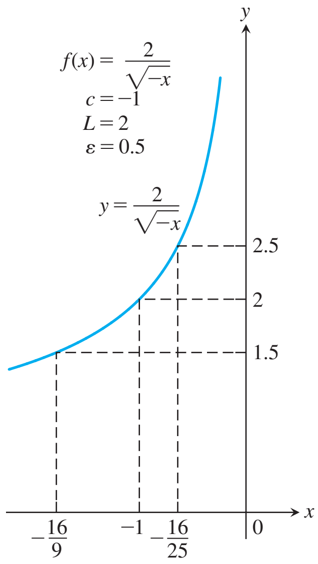

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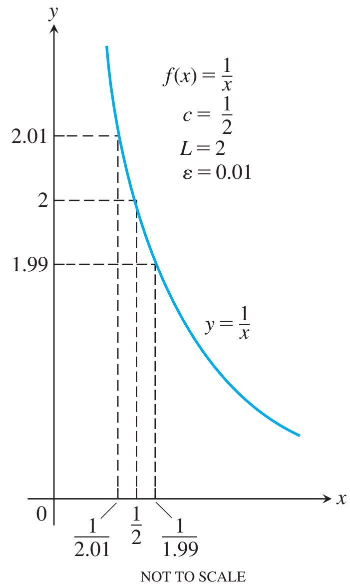

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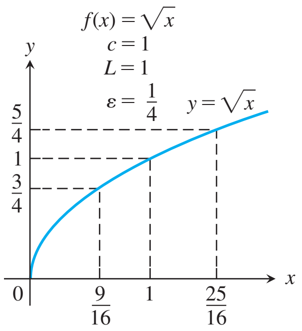

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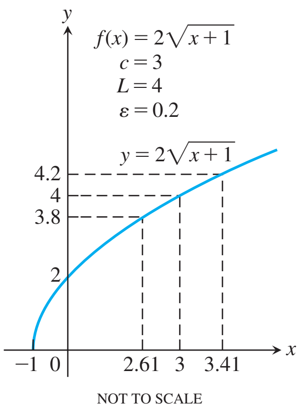

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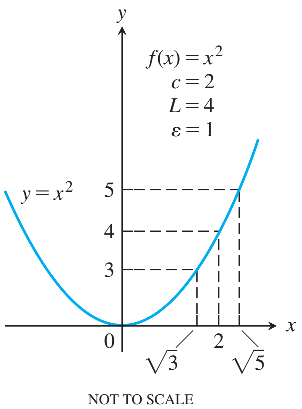

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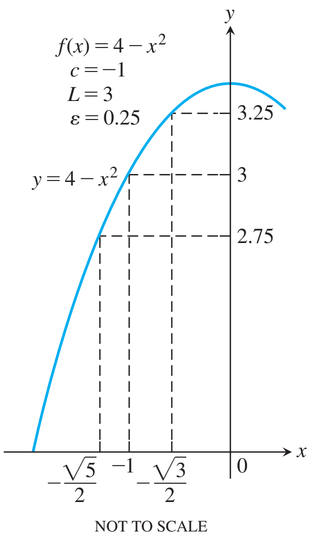


Finding Deltas Algebraically

Each of Exercises 15–30 gives a function $f(x)$ and numbers L, c, and $\varepsilon > 0$. In each case, find an open interval about c on which the inequality $|f(x) - L| < \varepsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x - c| < \delta$ the inequality $|f(x) - L| < \varepsilon$ holds.

15. $f(x) = x + 1, \quad L = 5, \quad c = 4, \quad \varepsilon = 0.01$

16. $f(x) = 2x - 2, \quad L = -6, \quad c = -2, \quad \varepsilon = 0.02$

17. $f(x) = \sqrt{x + 1}, \quad L = 1, \quad c = 0, \quad \varepsilon = 0.1$

18. $f(x) = \sqrt{x}, \quad L = 1/2, \quad c = 1/4, \quad \varepsilon = 0.1$

19. $f(x) = \sqrt{19 - x}, \quad L = 3, \quad c = 10, \quad \varepsilon = 1$

20. $f(x) = \sqrt{x - 7}, \quad L = 4, \quad c = 23, \quad \varepsilon = 1$

21. $f(x) = 1/x, \quad L = 1/4, \quad c = 4, \quad \varepsilon = 0.05$

22. $f(x) = x^2, \quad L = 3, \quad c = \sqrt{3}, \quad \varepsilon = 0.1$

23. $f(x) = x^2, \quad L = 4, \quad c = -2, \quad \varepsilon = 0.5$

24. $f(x) = 1/x, \quad L = -1, \quad c = -1, \quad \varepsilon = 0.1$

25. $f(x) = x^2 - 5, \quad L = 11, \quad c = 4, \quad \varepsilon = 1$

26. $f(x) = 120/x, \quad L = 5, \quad c = 24, \quad \varepsilon = 1$

27. $f(x) = mx, \quad m > 0, \quad L = 2m, \quad c = 2, \quad \varepsilon = 0.03$

28. $f(x) = mx, \quad m > 0, \quad L = 3m, \quad c = 3, \quad \varepsilon = c > 0$

29. $f(x) = mx + b, \quad m > 0, \quad L = (m/2) + b,$
$c = 1/2, \quad \varepsilon = c > 0$

30. $f(x) = mx + b, \quad m > 0, \quad L = m + b, \quad c = 1, \quad \varepsilon = 0.05$

Using the Formal Definition

Each of Exercises 31–36 gives a function $f(x)$, a point c, and a positive number ε. Find $L = \lim_{x \to c} f(x)$. Then find a number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

31. $f(x) = 3 - 2x, \quad c = 3, \quad \varepsilon = 0.02$

32. $f(x) = -3x - 2, \quad c = -1, \quad \varepsilon = 0.03$

33. $f(x) = \dfrac{x^2 - 4}{x - 2}, \quad c = 2, \quad \varepsilon = 0.05$

34. $f(x) = \dfrac{x^2 + 6x + 5}{x + 5}, \quad c = -5, \quad \varepsilon = 0.05$

35. $f(x) = \sqrt{1 - 5x}, \quad c = -3, \quad \varepsilon = 0.5$

36. $f(x) = 4/x, \quad c = 2, \quad \varepsilon = 0.4$

Prove the limit statements in Exercises 37–50.

37. $\lim_{x\to 4} (9 - x) = 5$

38. $\lim_{x\to 3} (3x - 7) = 2$

39. $\lim_{x\to 9} \sqrt{x - 5} = 2$

40. $\lim_{x\to 0} \sqrt{4 - x} = 2$

41. $\lim_{x\to 1} f(x) = 1 \quad \text{if} \quad f(x) = \begin{cases} x^2, & x \neq 1 \\ 2, & x = 1 \end{cases}$

42. $\lim_{x\to -2} f(x) = 4 \quad \text{if} \quad f(x) = \begin{cases} x^2, & x \neq -2 \\ 1, & x = -2 \end{cases}$

43. $\lim_{x\to 1} \dfrac{1}{x} = 1$

44. $\lim_{x\to\sqrt{3}} \dfrac{1}{x^2} = \dfrac{1}{3}$

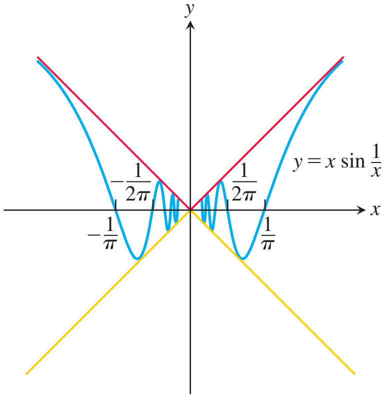
45. $\lim_{x\to -3} \dfrac{x^2 - 9}{x + 3} = -6$

46. $\lim_{x\to 1} \dfrac{x^2 - 1}{x - 1} = 2$

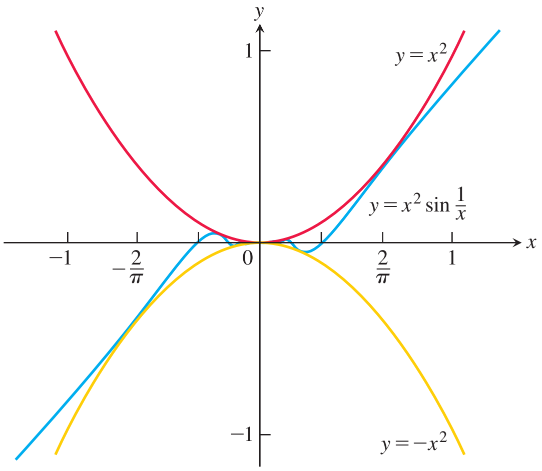
47. $\lim_{x\to 1} f(x) = 2 \quad \text{if} \quad f(x) = \begin{cases} 4 - 2x, & x < 1 \\ 6x - 4, & x \geq 1 \end{cases}$

48. $\lim_{x\to 0} f(x) = 0 \quad \text{if} \quad f(x) = \begin{cases} 2x, & x < 0 \\ x/2, & x \geq 0 \end{cases}$

49. $\lim_{x\to 0} x \sin \dfrac{1}{x} = 0$



50. $\lim_{x\to 0} x^2 \sin \dfrac{1}{x} = 0$



Theory and Examples

51. Define what it means to say that $\lim_{x\to 0} g(x) = k$.

52. Prove that $\lim_{x\to c} f(x) = L$ if and only if $\lim_{h\to 0} f(h + c) = L$.

53. A wrong statement about limits Show by example that the following statement is wrong.

> The number L is the limit of $f(x)$ as x approaches c if $f(x)$ gets closer to L as x approaches c.

Explain why the function in your example does not have the given value of L as a limit as $x \to c$.

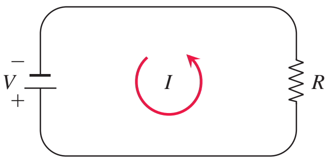
54. Another wrong statement about limits Show by example that the following statement is wrong.

> The number L is the limit of $f(x)$ as x approaches c if, given any $\varepsilon > 0$, there exists a value of x for which $|f(x) - L| < \varepsilon$.

Explain why the function in your example does not have the given value of L as a limit as $x \to c$.

T **55. Grinding engine cylinders** Before contracting to grind engine cylinders to a cross-sectional area of 60 cm^2, you need to know how much deviation from the ideal cylinder diameter of $c = 8.7404$ cm you can allow and still have the area come within 0.1 cm^2 of the required 60 cm^2. To find out, you let $A = \pi(x/2)^2$ and look for the interval in which you must hold x to make $|A - 60| \leq 0.1$. What interval do you find?

56. Manufacturing electrical resistors Ohm's law for electrical circuits like the one shown in the accompanying figure states that $V = RI$. In this equation, V is a constant voltage, I is the current in amperes, and R is the resistance in ohms. Your firm has been asked to supply the resistors for a circuit in which V will be 120 volts and I is to be 5 ± 0.1 amp. In what interval does R have to lie for I to be within 0.1 amp of the value $I_0 = 5$?



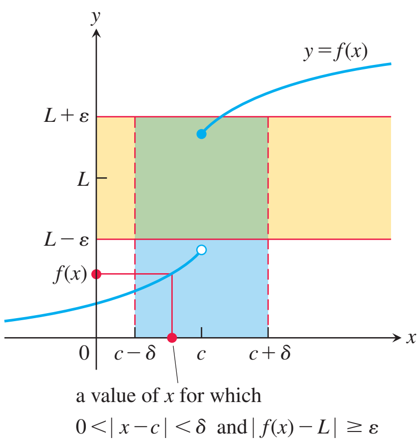
When Is a Number L Not the Limit of $f(x)$ as $x \to c$?

Showing L is not a limit We can prove that $\lim_{x\to c} f(x) \neq L$ by providing an $\varepsilon > 0$ such that no possible $\delta > 0$ satisfies the condition

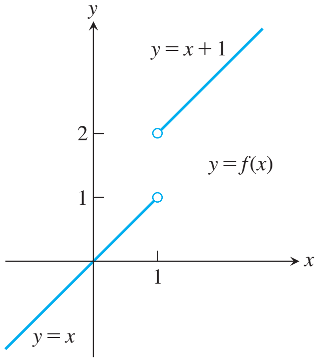
$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

We accomplish this for our candidate ε by showing that for each $\delta > 0$ there exists a value of x such that

$$0 < |x - c| < \delta \qquad \text{and} \qquad |f(x) - L| \geq \varepsilon.$$



57. Let $f(x) = \begin{cases} x, & x < 1 \\ x + 1, & x > 1. \end{cases}$



a. Let $\varepsilon = 1/2$. Show that no possible $\delta > 0$ satisfies the following condition:

$$|f(x) - 2| < 1/2 \quad \text{whenever} \quad 0 < |x - 1| < \delta.$$

That is, for each $\delta > 0$ show that there is a value of x such that

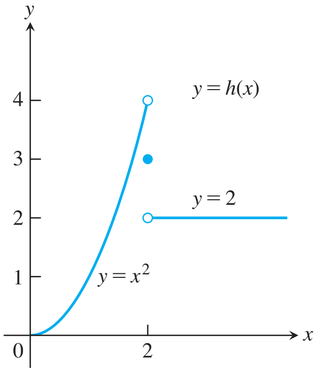
$$0 < |x - 1| < \delta \qquad \text{and} \qquad |f(x) - 2| \geq 1/2.$$

This will show that $\lim_{x \to 1} f(x) \neq 2$.

b. Show that $\lim_{x \to 1} f(x) \neq 1$.

c. Show that $\lim_{x \to 1} f(x) \neq 1.5$.

58. Let $h(x) = \begin{cases} x^2, & x < 2 \\ 3, & x = 2 \\ 2, & x > 2. \end{cases}$



Show that

a. $\lim_{x \to 2} h(x) \neq 4$

b. $\lim_{x \to 2} h(x) \neq 3$

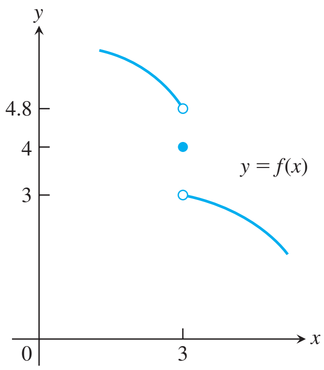
c. $\lim_{x \to 2} h(x) \neq 2$

59. For the function graphed here, explain why

a. $\lim_{x \to 3} f(x) \neq 4$

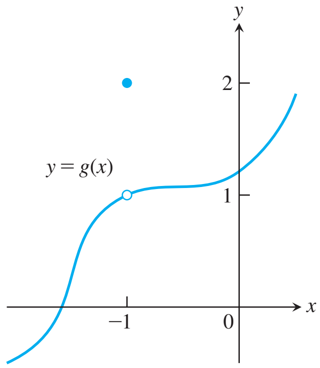
b. $\lim_{x \to 3} f(x) \neq 4.8$

c. $\lim_{x \to 3} f(x) \neq 3$



60. a. For the function graphed here, show that $\lim_{x \to -1} g(x) \neq 2$.

b. Does $\lim_{x \to -1} g(x)$ appear to exist? If so, what is the value of the limit? If not, why not?



COMPUTER EXPLORATIONS

In Exercises 61–66, you will further explore finding deltas graphically. Use a CAS to perform the following steps:

a. Plot the function $y = f(x)$ near the point c being approached.

b. Guess the value of the limit L and then evaluate the limit symbolically to see if you guessed correctly.

c. Using the value $\varepsilon = 0.2$, graph the banding lines $y_1 = L - \varepsilon$ and $y_2 = L + \varepsilon$ together with the function f near c.

d. From your graph in part (c), estimate a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

Test your estimate by plotting f, y_1, and y_2 over the interval $0 < |x - c| < \delta$. For your viewing window use $c - 2\delta \leq x \leq c + 2\delta$ and $L - 2\varepsilon \leq y \leq L + 2\varepsilon$. If any function values lie outside the interval $[L - \varepsilon, L + \varepsilon]$, your choice of δ was too large. Try again with a smaller estimate.

e. Repeat parts (c) and (d) successively for $\varepsilon = 0.1$, 0.05, and 0.001.

61. $f(x) = \dfrac{x^4 - 81}{x - 3}$, $c = 3$

62. $f(x) = \dfrac{5x^3 + 9x^2}{2x^5 + 3x^2}$, $c = 0$

63. $f(x) = \dfrac{\sin 2x}{3x}$, $c = 0$

64. $f(x) = \dfrac{x(1 - \cos x)}{x - \sin x}$, $c = 0$

65. $f(x) = \dfrac{\sqrt[3]{x} - 1}{x - 1}$, $c = 1$

66. $f(x) = \dfrac{3x^2 - (7x + 1)\sqrt{x} + 5}{x - 1}$, $c = 1$

2.4 One-Sided Limits

In this section we extend the limit concept to *one-sided limits*, which are limits as x approaches the number c from the left-hand side (where $x < c$) or the right-hand side $(x > c)$ only. These allow us to describe functions that have different limits at a point, depending on whether we approach the point from the left or from the right. One-sided limits also allow us to say what it means for a function to have a limit at an endpoint of an interval.

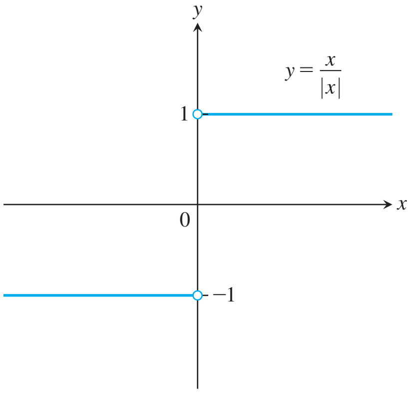


FIGURE 2.24 Different right-hand and left-hand limits at the origin.

Approaching a Limit from One Side

Suppose a function f is defined on an interval that extends to both sides of a number c. In order for f to have a limit L as x approaches c, the values of $f(x)$ must approach the value L as x approaches c from either side. Because of this, we sometimes say that the limit is **two-sided**.

If f fails to have a two-sided limit at c, it may still have a one-sided limit, that is, a limit if the approach is only from one side. If the approach is from the right, the limit is a **right-hand limit** or **limit from the right**. From the left, it is a **left-hand limit** or **limit from the left**.

The function $f(x) = x/|x|$ (Figure 2.24) has limit 1 as x approaches 0 from the right, and limit -1 as x approaches 0 from the left. Since these one-sided limit values are not the same, there is no single number that $f(x)$ approaches as x approaches 0. So $f(x)$ does not have a (two-sided) limit at 0.

Intuitively, if we only consider the values of $f(x)$ on an interval (c, b), where $c < b$, and the values of $f(x)$ become arbitrarily close to L as x approaches c from within that interval, then f has **right-hand limit** L at c. In this case we write

$$\lim_{x \to c^+} f(x) = L.$$

The notation "$x \to c^+$" means that we consider only values of $f(x)$ for x greater than c. We don't consider values of $f(x)$ for $x \le c$.

Similarly, if $f(x)$ is defined on an interval (a, c), where $a < c$ and $f(x)$ approaches arbitrarily close to M as x approaches c from within that interval, then f has **left-hand limit** M at c. We write

$$\lim_{x \to c^-} f(x) = M.$$

The symbol "$x \to c^-$" means that we consider the values of f only at x-values less than c.

These informal definitions of one-sided limits are illustrated in Figure 2.25. For the function $f(x) = x/|x|$ in Figure 2.24 we have

$$\lim_{x \to 0^+} f(x) = 1 \qquad \text{and} \qquad \lim_{x \to 0^-} f(x) = -1.$$

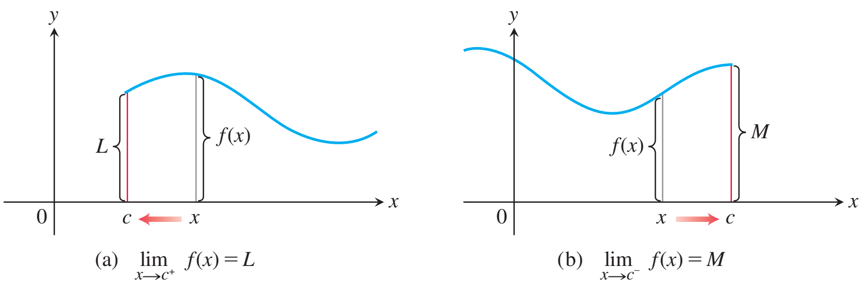


FIGURE 2.25 (a) Right-hand limit as x approaches c. (b) Left-hand limit as x approaches c.

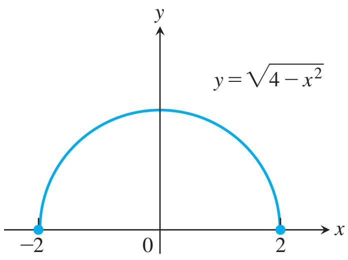


FIGURE 2.26 The function $f(x) = \sqrt{4 - x^2}$ has a right-hand limit 0 at $x = -2$ and a left-hand limit 0 at $x = 2$ (Example 1).

We now give the definition of the limit of a function at a boundary point of its domain. This definition is consistent with limits at boundary points of regions in the plane and in space, as we will see in Chapter 13. When the domain of f is an interval lying to the left of c, such as $(a, c]$ or (a, c), then we say that f has a limit at c if it has a left-hand limit at c. Similarly, if the domain of f is an interval lying to the right of c, such as $[c, b)$ or (c, b), then we say that f has a limit at c if it has a right-hand limit at c.

EXAMPLE 1 The domain of $f(x) = \sqrt{4 - x^2}$ is $[-2, 2]$; its graph is the semicircle in Figure 2.26. We have

$$\lim_{x\to -2^+} \sqrt{4 - x^2} = 0 \qquad \text{and} \qquad \lim_{x\to 2^-} \sqrt{4 - x^2} = 0.$$

This function has a two-sided limit at each point in $(-2, 2)$. It has a left-hand limit at $x = 2$ and a right-hand limit at $x = -2$. The function does not have a left-hand limit at $x = -2$ or a right-hand limit at $x = 2$. It does not have a two-sided limit at either -2 or 2 because f is not defined on both sides of these points. At the domain boundary points, where the domain is an interval on one side of the point, we have $\lim_{x\to -2} \sqrt{4 - x^2} = 0$ and $\lim_{x\to 2} \sqrt{4 - x^2} = 0$. The function f does have a limit at $x = -2$ and at $x = 2$. ■

One-sided limits have all the properties listed in Theorem 1 in Section 2.2. The right-hand limit of the sum of two functions is the sum of their right-hand limits, and so on. The theorems for limits of polynomials and rational functions hold with one-sided limits, as does the Sandwich Theorem. One-sided limits are related to limits at interior points in the following way.

THEOREM 6 Suppose that a function f is defined on an open interval containing c, except perhaps at c itself. Then $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x\to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x\to c^-} f(x) = L \quad \text{and} \quad \lim_{x\to c^+} f(x) = L.$$

Theorem 6 applies at interior points of a function's domain. At a boundary point of its domain, a function has a limit when it has an appropriate one-sided limit.

EXAMPLE 2 For the function graphed in Figure 2.27,

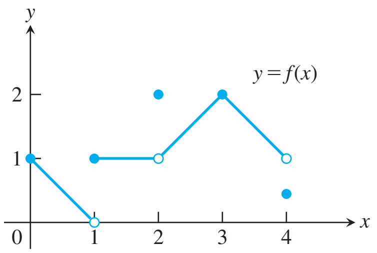


FIGURE 2.27 Graph of the function in Example 2.

At $x = 0$:	$\lim_{x\to 0^-} f(x)$ does not exist,	f is not defined to the left of $x = 0$.
	$\lim_{x\to 0^+} f(x) = 1$,	f has a right-hand limit at $x = 0$.
	$\lim_{x\to 0} f(x) = 1$.	f has a limit at domain endpoint $x = 0$.
At $x = 1$:	$\lim_{x\to 1^-} f(x) = 0$,	Even though $f(1) = 1$.
	$\lim_{x\to 1^+} f(x) = 1$,	
	$\lim_{x\to 1} f(x)$ does not exist.	Right- and left-hand limits are not equal.
At $x = 2$:	$\lim_{x\to 2^-} f(x) = 1$,	
	$\lim_{x\to 2^+} f(x) = 1$,	
	$\lim_{x\to 2} f(x) = 1$.	Even though $f(2) = 2$.
At $x = 3$:	$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = \lim_{x\to 3} f(x) = f(3) = 2$.	
At $x = 4$:	$\lim_{x\to 4^-} f(x) = 1$,	Even though $f(4) \neq 1$.
	$\lim_{x\to 4^+} f(x)$ does not exist,	f is not defined to the right of $x = 4$.
	$\lim_{x\to 4} f(x) = 1$.	f has a limit at domain endpoint $x = 4$.

At every other point c in $[0, 4]$, $f(x)$ has limit $f(c)$. ■

Precise Definitions of One-Sided Limits

The formal definition of the limit in Section 2.3 is readily modified for one-sided limits.

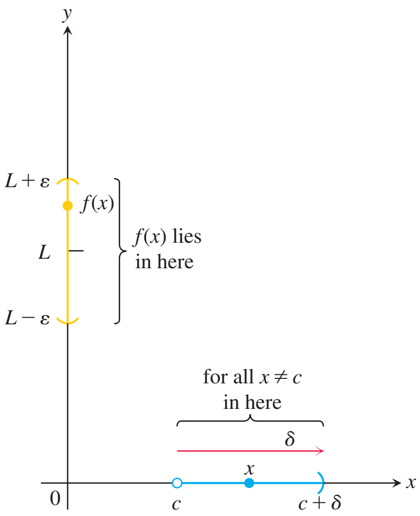


FIGURE 2.28 Intervals associated with the definition of right-hand limit.

DEFINITIONS (a) Assume the domain of f contains an interval (c, d) to the right of c. We say that $f(x)$ has **right-hand limit L at c**, and write

$$\lim_{x \to c^+} f(x) = L$$

if for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad c < x < c + \delta.$$

(b) Assume the domain of f contains an interval (b, c) to the left of c. We say that f has **left-hand limit L at c**, and write

$$\lim_{x \to c^-} f(x) = L$$

if for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad c - \delta < x < c.$$

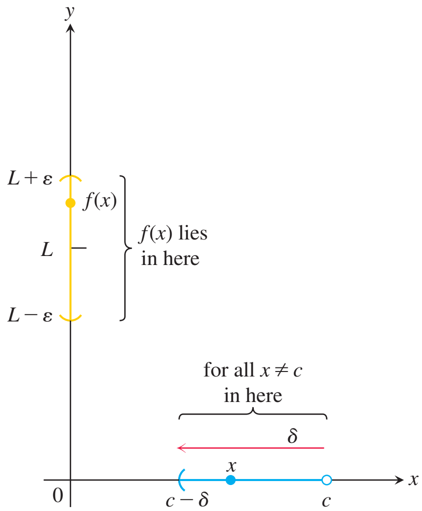


FIGURE 2.29 Intervals associated with the definition of left-hand limit.

The definitions are illustrated in Figures 2.28 and 2.29.

EXAMPLE 3 Prove that

$$\lim_{x \to 0^+} \sqrt{x} = 0.$$

Solution Let $\varepsilon > 0$ be given. Here $c = 0$ and $L = 0$, so we want to find a $\delta > 0$ such that

$$|\sqrt{x} - 0| < \varepsilon \qquad \text{whenever} \qquad 0 < x < \delta,$$

or

$$\sqrt{x} < \varepsilon \qquad \text{whenever} \qquad 0 < x < \delta. \qquad \sqrt{x} \gtrsim 0 \text{ so } |\sqrt{x}| = \sqrt{x}$$

Squaring both sides of this last inequality gives

$$x < \varepsilon^2 \qquad \text{if} \qquad 0 < x < \delta.$$

If we choose $\delta = \varepsilon^2$ we have

$$\sqrt{x} < \varepsilon \qquad \text{whenever} \qquad 0 < x < \delta = \varepsilon^2,$$

or

$$|\sqrt{x} - 0| < \varepsilon \qquad \text{whenever} \qquad 0 < x < \varepsilon^2.$$

According to the definition, this shows that $\lim_{x \to 0^+} \sqrt{x} = 0$ (Figure 2.30). ■

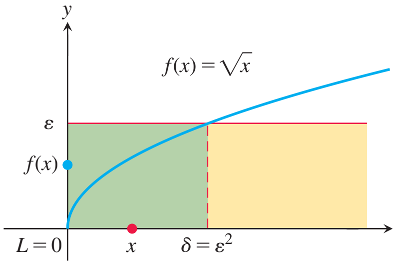


FIGURE 2.30 $\lim_{x \to 0^+} \sqrt{x} = 0$ in Example 3.

The functions examined so far have had some kind of limit at each point of interest. In general, that need not be the case.

EXAMPLE 4 Show that $y = \sin(1/x)$ has no limit as x approaches zero from either side (Figure 2.31).

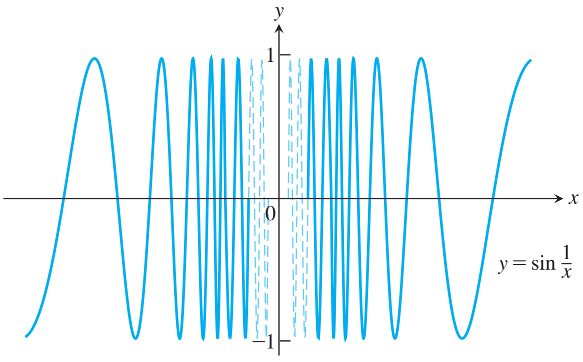


FIGURE 2.31 The function $y = \sin(1/x)$ has neither a right-hand nor a left-hand limit as x approaches zero (Example 4). The graph here omits values very near the y-axis.

Solution As x approaches zero, its reciprocal, $1/x$, grows without bound and the values of $\sin(1/x)$ cycle repeatedly from -1 to 1. There is no single number L that the function's values stay increasingly close to as x approaches zero. This is true even if we restrict x to positive values or to negative values. The function has neither a right-hand limit nor a left-hand limit at $x = 0$. ■

Limits Involving $(\sin \theta)/\theta$

A central fact about $(\sin \theta)/\theta$ is that in radian measure its limit as $\theta \to 0$ is 1. We can see this in Figure 2.32 and confirm it algebraically using the Sandwich Theorem. You will see the importance of this limit in Section 3.5, where instantaneous rates of change of the trigonometric functions are studied.

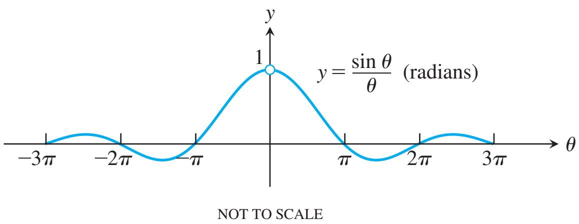


FIGURE 2.32 The graph of $f(\theta) = (\sin \theta)/\theta$ suggests that the right- and left-hand limits as θ approaches 0 are both 1.

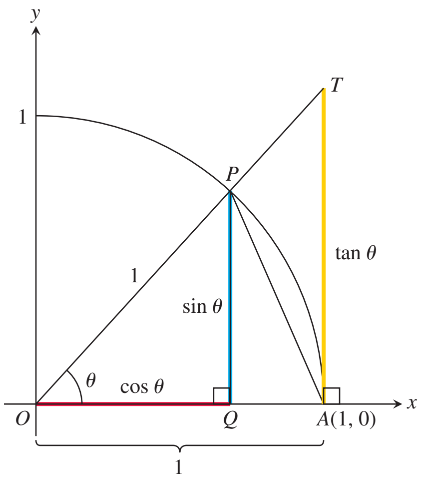


FIGURE 2.33 The ratio $TA/OA = \tan \theta$, and $OA = 1$, so $TA = \tan \theta$.

> **THEOREM 7—Limit of the Ratio $\sin \theta/\theta$ as $\theta \to 0$**
>
> $$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \qquad (\theta \text{ in radians}) \tag{1}$$

Proof The plan is to show that the right-hand and left-hand limits are both 1. Then we will know that the two-sided limit is 1 as well.

To show that the right-hand limit is 1, we begin with positive values of θ less than $\pi/2$ (Figure 2.33). Notice that

$$\text{Area } \Delta OAP < \text{ area sector } OAP < \text{ area } \Delta OAT.$$

We can express these areas in terms of θ as follows:

The use of radians to measure angles is essential in Equation (2): The area of sector OAP is $\theta/2$ only if θ is measured in radians.

$$\begin{aligned}\text{Area } \Delta OAP &= \frac{1}{2}\text{base} \times \text{height} = \frac{1}{2}(1)(\sin\theta) = \frac{1}{2}\sin\theta\\ \text{Area sector } OAP &= \frac{1}{2}r^2\theta = \frac{1}{2}(1)^2\theta = \frac{\theta}{2}\\ \text{Area } \Delta OAT &= \frac{1}{2}\text{base} \times \text{height} = \frac{1}{2}(1)(\tan\theta) = \frac{1}{2}\tan\theta.\end{aligned} \tag{2}$$

Thus,

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta.$$

This last inequality goes the same way if we divide all three terms by the number $(1/2)\sin\theta$, which is positive, since $0 < \theta < \pi/2$:

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

Taking reciprocals reverses the inequalities:

$$1 > \frac{\sin\theta}{\theta} > \cos\theta.$$

Since $\lim_{\theta\to 0^+}\cos\theta = 1$ (Example 11b, Section 2.2), the Sandwich Theorem gives

$$\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta} = 1.$$

To consider the left-hand limit, we recall that $\sin\theta$ and θ are both *odd functions* (Section 1.1). Therefore, $f(\theta) = (\sin\theta)/\theta$ is an *even function*, with a graph symmetric about the y-axis (see Figure 2.32). This symmetry implies that the left-hand limit at 0 exists and has the same value as the right-hand limit:

$$\lim_{\theta\to 0^-}\frac{\sin\theta}{\theta} = 1 = \lim_{\theta\to 0^+}\frac{\sin\theta}{\theta},$$

so $\lim_{\theta\to 0}(\sin\theta)/\theta = 1$ by Theorem 6. ■

EXAMPLE 5 Show that **(a)** $\displaystyle\lim_{y\to 0}\frac{\cos y - 1}{y} = 0$ and **(b)** $\displaystyle\lim_{x\to 0}\frac{\sin 2x}{5x} = \frac{2}{5}$.

Solution

(a) Using the half-angle formula $\cos y = 1 - 2\sin^2(y/2)$, we calculate

$$\begin{aligned}\lim_{y\to 0}\frac{\cos y - 1}{y} &= \lim_{y\to 0} -\frac{2\sin^2(y/2)}{y}\\ &= -\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\sin\theta && \text{Let } \theta = y/2.\\ &= -(1)(0) = 0. && \text{Eq. (1) and Example 11a in Section 2.2}\end{aligned}$$

(b) Equation (1) does not apply to the original fraction. We need a $2x$ in the denominator, not a $5x$. We produce it by multiplying numerator and denominator by $2/5$:

$$\begin{aligned}\lim_{x\to 0}\frac{\sin 2x}{5x} &= \lim_{x\to 0}\frac{(2/5)\cdot\sin 2x}{(2/5)\cdot 5x}\\ &= \frac{2}{5}\lim_{x\to 0}\frac{\sin 2x}{2x} && \text{Eq. (1) applies with } \theta = 2x.\\ &= \frac{2}{5}(1) = \frac{2}{5}.\end{aligned}$$

■

EXAMPLE 6 Find $\lim_{t \to 0} \frac{\tan t \sec 2t}{3t}$.

Solution From the definition of $\tan t$ and $\sec 2t$, we have

$$\begin{aligned} \lim_{t \to 0} \frac{\tan t \sec 2t}{3t} &= \lim_{t \to 0} \frac{1}{3} \cdot \frac{1}{t} \cdot \frac{\sin t}{\cos t} \cdot \frac{1}{\cos 2t} \\ &= \frac{1}{3} \lim_{t \to 0} \frac{\sin t}{t} \cdot \frac{1}{\cos t} \cdot \frac{1}{\cos 2t} \\ &= \frac{1}{3}(1)(1)(1) = \frac{1}{3}. \end{aligned}$$

Eq. (1) and Example 11b in Section 2.2

EXAMPLE 7 Show that for nonzero constants A and B.

$$\lim_{\theta \to 0} \frac{\sin A\theta}{\sin B\theta} = \frac{A}{B}.$$

Solution

$$\lim_{\theta \to 0} \frac{\sin A\theta}{\sin B\theta} = \lim_{\theta \to 0} \frac{\sin A\theta}{A\theta} A\theta \frac{B\theta}{\sin B\theta} \frac{1}{B\theta}$$

Multiply and divide by $A\theta$ and $B\theta$.

$$= \lim_{\theta \to 0} \frac{\sin A\theta}{A\theta} \frac{B\theta}{\sin B\theta} \frac{A}{B}$$

$\lim_{u \to 0} \frac{\sin u}{u} = 1$, with $u = A\theta$

$$= \lim_{\theta \to 0} (1)(1) \frac{A}{B}$$

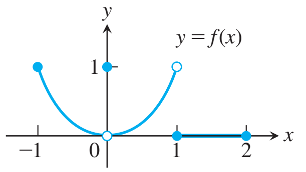
$\lim_{v \to 0} \frac{v}{\sin v} = 1$, with $v = B\theta$

$$= \frac{A}{B}.$$

EXERCISES 2.4

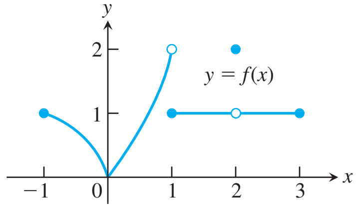
Finding Limits Graphically

1. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?



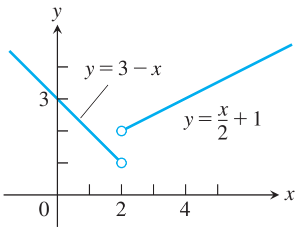
a. $\lim_{x \to -1^+} f(x) = 1$
b. $\lim_{x \to 0^-} f(x) = 0$
c. $\lim_{x \to 0^-} f(x) = 1$
d. $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$
e. $\lim_{x \to 0} f(x)$ exists.
f. $\lim_{x \to 0} f(x) = 0$
g. $\lim_{x \to 0} f(x) = 1$
h. $\lim_{x \to 1} f(x) = 1$
i. $\lim_{x \to 1} f(x) = 0$
j. $\lim_{x \to 2^-} f(x) = 2$
k. $\lim_{x \to -1^-} f(x)$ does not exist.
l. $\lim_{x \to 2^+} f(x) = 0$

2. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?



a. $\lim_{x \to -1^+} f(x) = 1$
b. $\lim_{x \to 2} f(x)$ does not exist.
c. $\lim_{x \to 2} f(x) = 2$
d. $\lim_{x \to 1^-} f(x) = 2$
e. $\lim_{x \to 1^+} f(x) = 1$
f. $\lim_{x \to 1} f(x)$ does not exist.
g. $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x)$
h. $\lim_{x \to c} f(x)$ exists at every c in the open interval $(-1, 1)$.
i. $\lim_{x \to c} f(x)$ exists at every c in the open interval $(1, 3)$.
j. $\lim_{x \to -1^-} f(x) = 0$
k. $\lim_{x \to 3^+} f(x)$ does not exist.

3. Let $f(x) = \begin{cases} 3 - x, & x < 2 \\ \dfrac{x}{2} + 1, & x > 2. \end{cases}$



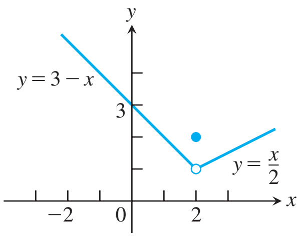
a. Find $\lim_{x\to 2^+} f(x)$ and $\lim_{x\to 2^-} f(x)$.

b. Does $\lim_{x\to 2} f(x)$ exist? If so, what is it? If not, why not?

c. Find $\lim_{x\to 4^-} f(x)$ and $\lim_{x\to 4^+} f(x)$.

d. Does $\lim_{x\to 4} f(x)$ exist? If so, what is it? If not, why not?

4. Let $f(x) = \begin{cases} 3 - x, & x < 2 \\ 2, & x = 2 \\ \dfrac{x}{2}, & x > 2. \end{cases}$



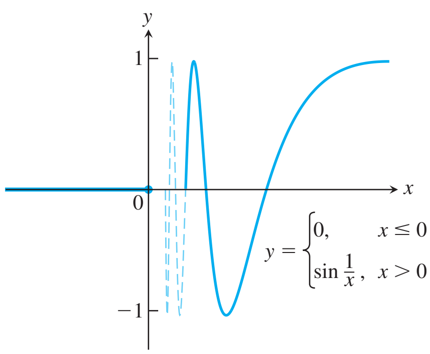
a. Find $\lim_{x\to 2^+} f(x)$, $\lim_{x\to 2^-} f(x)$, and $f(2)$.

b. Does $\lim_{x\to 2} f(x)$ exist? If so, what is it? If not, why not?

c. Find $\lim_{x\to -1^-} f(x)$ and $\lim_{x\to -1^+} f(x)$.

d. Does $\lim_{x\to -1} f(x)$ exist? If so, what is it? If not, why not?

5. Let $f(x) = \begin{cases} 0, & x \le 0 \\ \sin\dfrac{1}{x}, & x > 0. \end{cases}$

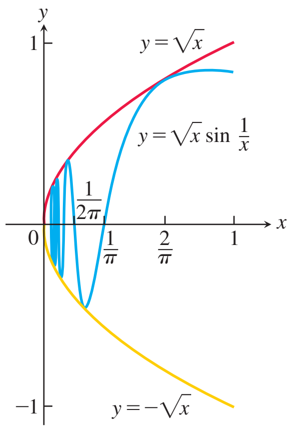


a. Does $\lim_{x\to 0^+} f(x)$ exist? If so, what is it? If not, why not?

b. Does $\lim_{x\to 0^-} f(x)$ exist? If so, what is it? If not, why not?

c. Does $\lim_{x\to 0} f(x)$ exist? If so, what is it? If not, why not?

6. Let $g(x) = \sqrt{x}\sin(1/x)$.



a. Does $\lim_{x\to 0^+} g(x)$ exist? If so, what is it? If not, why not?

b. Does $\lim_{x\to 0^-} g(x)$ exist? If so, what is it? If not, why not?

c. Does $\lim_{x\to 0} g(x)$ exist? If so, what is it? If not, why not?

7. a. Graph $f(x) = \begin{cases} x^3, & x \ne 1 \\ 0, & x = 1. \end{cases}$

b. Find $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$.

c. Does $\lim_{x\to 1} f(x)$ exist? If so, what is it? If not, why not?

8. a. Graph $f(x) = \begin{cases} 1 - x^2, & x \ne 1 \\ 2, & x = 1. \end{cases}$

b. Find $\lim_{x\to 1^+} f(x)$ and $\lim_{x\to 1^-} f(x)$.

c. Does $\lim_{x\to 1} f(x)$ exist? If so, what is it? If not, why not?

Graph the functions in Exercises 9 and 10. Then answer these questions.

a. What are the domain and range of f?

b. At what points c, if any, does $\lim_{x\to c} f(x)$ exist?

c. At what points does the left-hand limit exist but not the right-hand limit?

d. At what points does the right-hand limit exist but not the left-hand limit?

9. $f(x) = \begin{cases} \sqrt{1 - x^2}, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & x = 2 \end{cases}$

10. $f(x) = \begin{cases} x, & -1 \le x < 0, \text{ or } 0 < x \le 1 \\ 1, & x = 0 \\ 0, & x < -1 \text{ or } x > 1 \end{cases}$

Finding One-Sided Limits Algebraically

Find the limits in Exercises 11–20.

11. $\lim_{x\to -0.5^-} \sqrt{\dfrac{x + 2}{x + 1}}$

12. $\lim_{x\to 1^+} \sqrt{\dfrac{x - 1}{x + 2}}$

13. $\lim_{x\to -2^+} \left(\dfrac{x}{x + 1}\right)\left(\dfrac{2x + 5}{x^2 + x}\right)$

14. $\lim_{x\to -1^-} \left(\dfrac{1}{x+1}\right)\left(\dfrac{x+6}{x}\right)\left(\dfrac{3-x}{7}\right)$

15. $\lim_{h\to 0^+} \dfrac{\sqrt{h^2+4h+5}-\sqrt{5}}{h}$

16. $\lim_{h\to 0^-} \dfrac{\sqrt{6}-\sqrt{5h^2+11h+6}}{h}$

17. a. $\lim_{x\to -2^+} (x+3)\dfrac{|x+2|}{x+2}$ b. $\lim_{x\to -2^-} (x+3)\dfrac{|x+2|}{x+2}$

18. a. $\lim_{x\to 1^+} \dfrac{\sqrt{2x}(x-1)}{|x-1|}$ b. $\lim_{x\to 1^-} \dfrac{\sqrt{2x}(x-1)}{|x-1|}$

19. a. $\lim_{x\to 0^+} \dfrac{|\sin x|}{\sin x}$ b. $\lim_{x\to 0^-} \dfrac{|\sin x|}{\sin x}$

20. a. $\lim_{x\to 0^+} \dfrac{1-\cos x}{|\cos x - 1|}$ b. $\lim_{x\to 0^-} \dfrac{\cos x - 1}{|\cos x - 1|}$

Use the graph of the greatest integer function $y = \lfloor x \rfloor$, Figure 1.10 in Section 1.1, to help you find the limits in Exercises 21 and 22.

21. a. $\lim_{\theta\to 3^+} \dfrac{\lfloor \theta \rfloor}{\theta}$ b. $\lim_{\theta\to 3^-} \dfrac{\lfloor \theta \rfloor}{\theta}$

22. a. $\lim_{t\to 4^+} (t - \lfloor t \rfloor)$ b. $\lim_{t\to 4^-} (t - \lfloor t \rfloor)$

Using $\lim_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$

Find the limits in Exercises 23–46.

23. $\lim_{\theta\to 0} \dfrac{\sin\sqrt{2}\theta}{\sqrt{2}\theta}$

24. $\lim_{t\to 0} \dfrac{\sin kt}{t}$ (k constant)

25. $\lim_{y\to 0} \dfrac{\sin 3y}{4y}$

26. $\lim_{h\to 0^-} \dfrac{h}{\sin 3h}$

27. $\lim_{x\to 0} \dfrac{\tan 2x}{x}$

28. $\lim_{t\to 0} \dfrac{2t}{\tan t}$

29. $\lim_{x\to 0} \dfrac{x \csc 2x}{\cos 5x}$

30. $\lim_{x\to 0} 6x^2(\cot x)(\csc 2x)$

31. $\lim_{x\to 0} \dfrac{x + x\cos x}{\sin x \cos x}$

32. $\lim_{x\to 0} \dfrac{x^2 - x + \sin x}{2x}$

33. $\lim_{\theta\to 0} \dfrac{1-\cos\theta}{\sin 2\theta}$

34. $\lim_{x\to 0} \dfrac{x - x\cos x}{\sin^2 3x}$

35. $\lim_{t\to 0} \dfrac{\sin(1-\cos t)}{1-\cos t}$

36. $\lim_{h\to 0} \dfrac{\sin(\sin h)}{\sin h}$

37. $\lim_{\theta\to 0} \dfrac{\sin\theta}{\sin 2\theta}$

38. $\lim_{x\to 0} \dfrac{\sin 5x}{\sin 4x}$

39. $\lim_{\theta\to 0} \theta\cos\theta$

40. $\lim_{\theta\to 0} \sin\theta\cot 2\theta$

41. $\lim_{x\to 0} \dfrac{\tan 3x}{\sin 8x}$

42. $\lim_{y\to 0} \dfrac{\sin 3y \cot 5y}{y \cot 4y}$

43. $\lim_{\theta\to 0} \dfrac{\tan\theta}{\theta^2\cot 3\theta}$

44. $\lim_{\theta\to 0} \dfrac{\theta\cot 4\theta}{\sin^2\theta\cot^2 2\theta}$

45. $\lim_{x\to 0} \dfrac{1-\cos 3x}{2x}$

46. $\lim_{x\to 0} \dfrac{\cos^2 x - \cos x}{x^2}$

Theory and Examples

47. Once you know $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ at an interior point of the domain of f, do you then know $\lim_{x\to a} f(x)$? Give reasons for your answer.

48. If you know that $\lim_{x\to c} f(x)$ exists, can you find its value by calculating $\lim_{x\to c^+} f(x)$? Give reasons for your answer.

49. Suppose that f is an odd function of x. Does knowing that $\lim_{x\to 0^+} f(x) = 3$ tell you anything about $\lim_{x\to 0^-} f(x)$? Give reasons for your answer.

50. Suppose that f is an even function of x. Does knowing that $\lim_{x\to 2^-} f(x) = 7$ tell you anything about either $\lim_{x\to -2^-} f(x)$ or $\lim_{x\to -2^+} f(x)$? Give reasons for your answer.

Formal Definitions of One-Sided Limits

51. Given $\varepsilon > 0$, find an interval $I = (5, 5+\delta)$, $\delta > 0$, such that if x lies in I, then $\sqrt{x-5} < \varepsilon$. What limit is being verified and what is its value?

52. Given $\varepsilon > 0$, find an interval $I = (4-\delta, 4)$, $\delta > 0$, such that if x lies in I, then $\sqrt{4-x} < \varepsilon$. What limit is being verified and what is its value?

Use the definitions of right-hand and left-hand limits to prove the limit statements in Exercises 53 and 54.

53. $\lim_{x\to 0^-} \dfrac{x}{|x|} = -1$

54. $\lim_{x\to 2^+} \dfrac{x-2}{|x-2|} = 1$

55. **Greatest integer function** Find **(a)** $\lim_{x\to 400^+} \lfloor x \rfloor$ and **(b)** $\lim_{x\to 400^-} \lfloor x \rfloor$; then use limit definitions to verify your findings. **(c)** Based on your conclusions in parts (a) and (b), can you say anything about $\lim_{x\to 400} \lfloor x \rfloor$? Give reasons for your answer.

56. **One-sided limits** Let $f(x) = \begin{cases} x^2\sin(1/x), & x < 0 \\ \sqrt{x}, & x > 0. \end{cases}$

Find **(a)** $\lim_{x\to 0^+} f(x)$ and **(b)** $\lim_{x\to 0^-} f(x)$; then use limit definitions to verify your findings. **(c)** Based on your conclusions in parts (a) and (b), can you say anything about $\lim_{x\to 0} f(x)$? Give reasons for your answer.

2.5 Limits Involving Infinity; Asymptotes of Graphs

In this section we investigate the behavior of a function when the magnitude of the independent variable x becomes increasingly large, or $x \to \pm\infty$. We further extend the concept of limit to *infinite limits*. Infinite limits provide useful symbols and language for describing the behavior of functions whose values become arbitrarily large in magnitude. We use these ideas to analyze the graphs of functions having *horizontal* or *vertical asymptotes*.

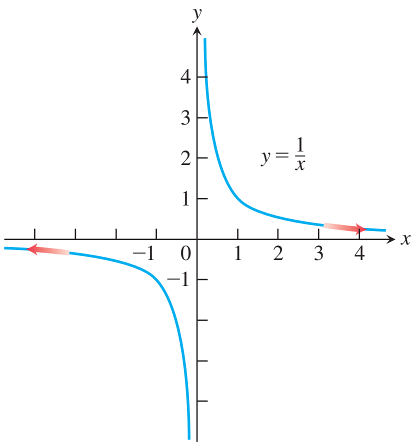


FIGURE 2.34 The graph of $y = 1/x$ approaches 0 as $x \to \infty$ or $x \to -\infty$.

Finite Limits as $x \to \pm\infty$

The symbol for infinity (∞) does not represent a real number. We use ∞ to describe the behavior of a function when the values in its domain or range outgrow all finite bounds. For example, the function $f(x) = 1/x$ is defined for all $x \neq 0$ (Figure 2.34). When x is positive and becomes increasingly large, $1/x$ becomes increasingly small. When x is negative and its magnitude becomes increasingly large, $1/x$ again becomes small. We summarize these observations by saying that $f(x) = 1/x$ has limit 0 as $x \to \infty$ or $x \to -\infty$, or that 0 is a *limit of* $f(x) = 1/x$ *at infinity and at negative infinity*. Here are precise definitions for the limit of a function whose domain contains positive or negative numbers of unbounded magnitude.

DEFINITIONS

1. We say that $f(x)$ has the **limit L as x approaches infinity** and write

$$\lim_{x\to\infty} f(x) = L$$

if, for every number $\varepsilon > 0$, there exists a corresponding number M such that for all x in the domain of f

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad x > M.$$

2. We say that $f(x)$ has the **limit L as x approaches negative infinity** and write

$$\lim_{x\to-\infty} f(x) = L$$

if, for every number $\varepsilon > 0$, there exists a corresponding number N such that for all x in the domain of f

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad x < N.$$

Intuitively, $\lim_{x\to\infty} f(x) = L$ if, as x moves increasingly far from the origin in the positive direction, $f(x)$ gets arbitrarily close to L. Similarly, $\lim_{x\to-\infty} f(x) = L$ if, as x moves increasingly far from the origin in the negative direction, $f(x)$ gets arbitrarily close to L.

The strategy for calculating limits of functions as $x \to +\infty$ or as $x \to -\infty$ is similar to the one for finite limits in Section 2.2. There we first found the limits of the constant and identity functions $y = k$ and $y = x$. We then extended these results to other functions by applying Theorem 1 on limits of algebraic combinations. Here we do the same thing, except that the starting functions are $y = k$ and $y = 1/x$ instead of $y = k$ and $y = x$.

The basic facts to be verified by applying the formal definition are

$$\lim_{x\to\pm\infty} k = k \quad \text{and} \quad \lim_{x\to\pm\infty} \frac{1}{x} = 0. \tag{1}$$

We prove the second result in Example 1, and leave the first to Exercises 93 and 94.

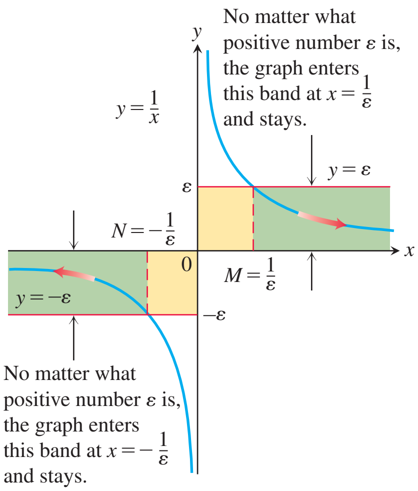


FIGURE 2.35 The geometry behind the argument in Example 1.

EXAMPLE 1 Show that

(a) $\lim_{x\to\infty} \frac{1}{x} = 0$ **(b)** $\lim_{x\to-\infty} \frac{1}{x} = 0.$

Solution

(a) Let $\varepsilon > 0$ be given. We must find a number M such that

$$\left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| < \varepsilon \quad \text{whenever} \quad x > M.$$

The implication will hold if $M = 1/\varepsilon$ or any larger positive number (Figure 2.35). This proves $\lim_{x\to\infty}(1/x) = 0$.

(b) Let $\varepsilon > 0$ be given. We must find a number N such that

$$\left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| < \varepsilon \qquad \text{whenever} \qquad x < N.$$

The implication will hold if $N = -1/\varepsilon$ or any number less than $-1/\varepsilon$ (Figure 2.35). This proves $\lim_{x\to-\infty}(1/x) = 0$. ■

Limits at infinity have properties similar to those of finite limits.

> **THEOREM 8** All the Limit Laws in Theorem 1 are true when we replace $\lim_{x\to c}$ by $\lim_{x\to\infty}$ or $\lim_{x\to-\infty}$. That is, the variable x may approach a finite number c or $\pm\infty$.

EXAMPLE 2 The properties in Theorem 8 are used to calculate limits in the same way as when x approaches a finite number c.

(a)
$$\begin{aligned}\lim_{x\to\infty}\left(5 + \frac{1}{x}\right) &= \lim_{x\to\infty} 5 + \lim_{x\to\infty}\frac{1}{x} && \text{Sum Rule}\\ &= 5 + 0 = 5 && \text{Known limits}\end{aligned}$$

(b)
$$\begin{aligned}\lim_{x\to-\infty}\frac{\pi\sqrt{3}}{x^2} &= \lim_{x\to-\infty}\pi\sqrt{3}\cdot\frac{1}{x}\cdot\frac{1}{x}\\ &= \lim_{x\to-\infty}\pi\sqrt{3}\cdot\lim_{x\to-\infty}\frac{1}{x}\cdot\lim_{x\to-\infty}\frac{1}{x} && \text{Product Rule}\\ &= \pi\sqrt{3}\cdot 0\cdot 0 = 0 && \text{Known limits}\end{aligned}$$
■

Limits at Infinity of Rational Functions

To determine the limit of a rational function as $x \to \pm\infty$, we first divide the numerator and denominator by the highest power of x in the denominator. The result then depends on the degrees of the polynomials involved.

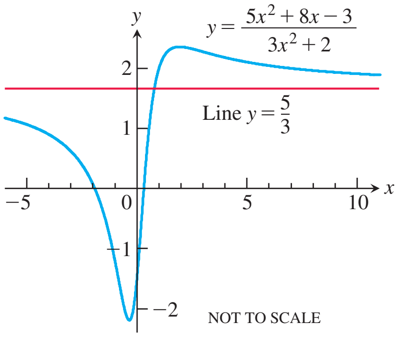


FIGURE 2.36 The graph of the function in Example 3a. The graph approaches the line $y = 5/3$ as $|x|$ increases.

EXAMPLE 3 These examples illustrate what happens when the degree of the numerator is less than or equal to the degree of the denominator.

(a)
$$\begin{aligned}\lim_{x\to\infty}\frac{5x^2 + 8x - 3}{3x^2 + 2} &= \lim_{x\to\infty}\frac{5 + (8/x) - (3/x^2)}{3 + (2/x^2)} && \text{Divide numerator and denominator by } x^2.\\ &= \frac{5 + 0 - 0}{3 + 0} = \frac{5}{3} && \text{See Fig. 2.36.}\end{aligned}$$

(b)
$$\begin{aligned}\lim_{x\to-\infty}\frac{11x + 2}{2x^3 - 1} &= \lim_{x\to-\infty}\frac{(11/x^2) + (2/x^3)}{2 - (1/x^3)} && \text{Divide numerator and denominator by } x^3.\\ &= \frac{0 + 0}{2 - 0} = 0 && \text{See Fig. 2.37.}\end{aligned}$$
■

Cases for which the degree of the numerator is greater than the degree of the denominator are illustrated in Examples 10 and 14.

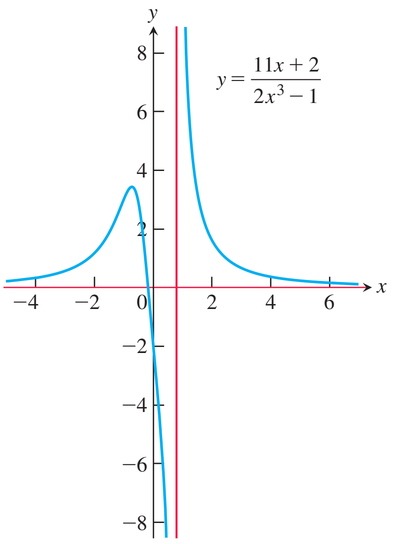


FIGURE 2.37 The graph of the function in Example 3b. The graph approaches the x-axis as $|x|$ increases.

Horizontal Asymptotes

If the distance between the graph of a function and some fixed line approaches zero as a point on the graph moves increasingly far from the origin, we say that the graph approaches the line asymptotically and that the line is an *asymptote* of the graph.

Looking at $f(x) = 1/x$ (see Figure 2.34), we observe that the x-axis is an asymptote of the curve on the right because

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

and on the left because

$$\lim_{x \to -\infty} \frac{1}{x} = 0.$$

We say that the x-axis is a *horizontal asymptote* of the graph of $f(x) = 1/x$.

DEFINITION A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = b \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = b.$$

The graph of a function can have zero, one, or two horizontal asymptotes, depending on whether the function has limits as $x \to \infty$ and as $x \to -\infty$.

The graph of the function

$$f(x) = \frac{5x^2 + 8x - 3}{3x^2 + 2}$$

sketched in Figure 2.36 (Example 3a) has the line $y = 5/3$ as a horizontal asymptote on both the right and the left because

$$\lim_{x \to \infty} f(x) = \frac{5}{3} \qquad \text{and} \qquad \lim_{x \to -\infty} f(x) = \frac{5}{3}.$$

EXAMPLE 4 Find the horizontal asymptotes of the graph of

$$f(x) = \frac{x^3 - 2}{|x|^3 + 1}.$$

Solution We calculate the limits as $x \to \pm\infty$.

For $x \geq 0$: $$\lim_{x \to \infty} \frac{x^3 - 2}{|x|^3 + 1} = \lim_{x \to \infty} \frac{x^3 - 2}{x^3 + 1} = \lim_{x \to \infty} \frac{1 - (2/x^3)}{1 + (1/x^3)} = 1.$$

For $x < 0$: $$\lim_{x \to -\infty} \frac{x^3 - 2}{|x|^3 + 1} = \lim_{x \to -\infty} \frac{x^3 - 2}{(-x)^3 + 1} = \lim_{x \to -\infty} \frac{1 - (2/x^3)}{-1 + (1/x^3)} = -1.$$

The horizontal asymptotes are $y = -1$ and $y = 1$. The graph is displayed in Figure 2.38. Notice that the graph crosses the horizontal asymptote $y = -1$ for a positive value of x. ■

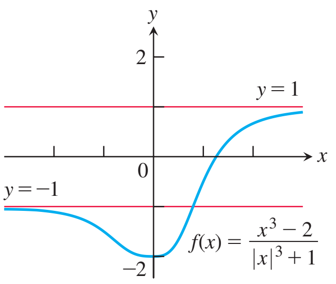


FIGURE 2.38 The graph of the function in Example 4 has two horizontal asymptotes.

EXAMPLE 5 The x-axis (the line $y = 0$) is a horizontal asymptote of the graph of $y = e^x$ because

$$\lim_{x\to-\infty} e^x = 0.$$

To see this, we use the definition of a limit as x approaches $-\infty$. So let $\varepsilon > 0$ be given, but arbitrary. We must find a constant N such that

$$|e^x - 0| < \varepsilon \quad \text{whenever} \quad x < N.$$

Now $|e^x - 0| = e^x$, so the condition that needs to be satisfied whenever $x < N$ is

$$e^x < \varepsilon.$$

Let $x = N$ be the number where $e^x = \varepsilon$. Since e^x is an increasing function, if $x < N$, then $e^x < \varepsilon$. We find N by taking the natural logarithm of both sides of the equation $e^N = \varepsilon$, so $N = \ln \varepsilon$ (see Figure 2.39). With this value of N the condition is satisfied, and we conclude that $\lim_{x\to-\infty} e^x = 0$. ■

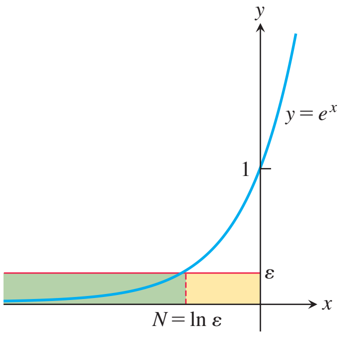


FIGURE 2.39 The graph of $y = e^x$ approaches the x-axis as $x \to -\infty$ (Example 5).

EXAMPLE 6 Find **(a)** $\lim_{x\to\infty} \sin(1/x)$ and **(b)** $\lim_{x\to\pm\infty} x\sin(1/x)$.

Solution

(a) We introduce the new variable $t = 1/x$. From Example 1, we know that $t \to 0^+$ as $x \to \infty$ (see Figure 2.34). Therefore,

$$\lim_{x\to\infty} \sin\frac{1}{x} = \lim_{t\to 0^+} \sin t = 0.$$

(b) We calculate the limits as $x \to \infty$ and $x \to -\infty$:

$$\lim_{x\to\infty} x\sin\frac{1}{x} = \lim_{t\to 0^+} \frac{\sin t}{t} = 1 \qquad \text{and} \qquad \lim_{x\to-\infty} x\sin\frac{1}{x} = \lim_{t\to 0^-} \frac{\sin t}{t} = 1.$$

The graph is shown in Figure 2.40, and we see that the line $y = 1$ is a horizontal asymptote. ■

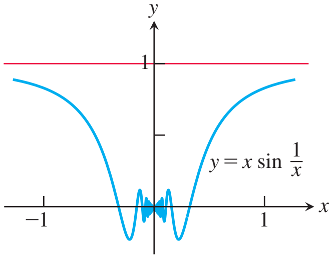


FIGURE 2.40 The line $y = 1$ is a horizontal asymptote of the function graphed here (Example 6b).

Similarly, we can investigate the behavior of $y = f(1/x)$ as $x \to 0$ by investigating $y = f(t)$ as $t \to \pm\infty$, where $t = 1/x$.

EXAMPLE 7 Find $\lim_{x\to 0^-} e^{1/x}$.

Solution We let $t = 1/x$. From Figure 2.34, we can see that $t \to -\infty$ as $x \to 0^-$. (We make this idea more precise further on.) Therefore,

$$\lim_{x\to 0^-} e^{1/x} = \lim_{t\to-\infty} e^t = 0 \qquad \text{Example 5}$$

(Figure 2.41). ■

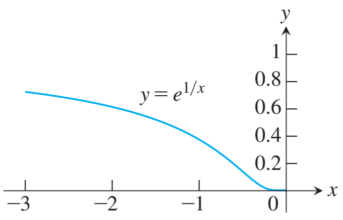


FIGURE 2.41 The graph of $y = e^{1/x}$ for $x < 0$ shows $\lim_{x\to 0^-} e^{1/x} = 0$ (Example 7).

The Sandwich Theorem also holds for limits as $x \to \pm\infty$. You must be sure, though, that the function whose limit you are trying to find stays between the bounding functions at very large values of x in magnitude consistent with whether $x \to \infty$ or $x \to -\infty$.

EXAMPLE 8 Using the Sandwich Theorem, find the horizontal asymptote of the curve

$$y = 2 + \frac{\sin x}{x}.$$

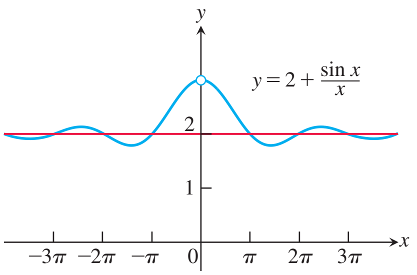


FIGURE 2.42 A curve may cross one of its asymptotes infinitely often (Example 8).

Solution We are interested in the behavior as $x \to \pm\infty$. Since

$$0 \le \left|\frac{\sin x}{x}\right| \le \left|\frac{1}{x}\right|$$

and $\lim_{x\to\pm\infty} |1/x| = 0$, we have $\lim_{x\to\pm\infty} (\sin x)/x = 0$ by the Sandwich Theorem. Hence,

$$\lim_{x\to\pm\infty}\left(2 + \frac{\sin x}{x}\right) = 2 + 0 = 2,$$

and the line $y = 2$ is a horizontal asymptote of the curve on both left and right (Figure 2.42).

This example illustrates that a curve may cross one of its horizontal asymptotes many times.

EXAMPLE 9 Find $\lim_{x\to\infty}\left(x - \sqrt{x^2 + 16}\right)$.

Solution Both of the terms x and $\sqrt{x^2 + 16}$ approach infinity as $x \to \infty$, so what happens to the difference in the limit is unclear (we cannot subtract ∞ from ∞ because the symbol does not represent a real number). In this situation we can multiply the numerator and the denominator by the conjugate radical expression to obtain an equivalent algebraic expression:

$$\lim_{x\to\infty}\left(x - \sqrt{x^2 + 16}\right) = \lim_{x\to\infty}\left(x - \sqrt{x^2 + 16}\right)\frac{x + \sqrt{x^2 + 16}}{x + \sqrt{x^2 + 16}}$$

Multiply and divide by the conjugate.

$$= \lim_{x\to\infty}\frac{x^2 - (x^2 + 16)}{x + \sqrt{x^2 + 16}} = \lim_{x\to\infty}\frac{-16}{x + \sqrt{x^2 + 16}}.$$

As $x \to \infty$, the denominator in this last expression becomes arbitrarily large, while the numerator remains constant, so we see that the limit is 0. We can also obtain this result by a direct calculation using the Limit Laws:

$$\lim_{x\to\infty}\frac{-16}{x + \sqrt{x^2 + 16}} = \lim_{x\to\infty}\frac{-\frac{16}{x}}{1 + \sqrt{\frac{x^2}{x^2} + \frac{16}{x^2}}} = \frac{0}{1 + \sqrt{1 + 0}} = 0.$$

Oblique Asymptotes

If the degree of the numerator of a rational function is 1 greater than the degree of the denominator, the graph has an **oblique** or **slant line asymptote**. We find an equation for the asymptote by dividing numerator by denominator to express f as a linear function plus a remainder that goes to zero as $x \to \pm\infty$.

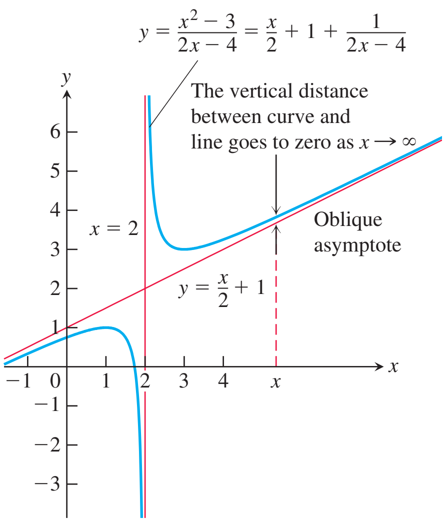


FIGURE 2.43 The graph of the function in Example 10 has an oblique asymptote.

EXAMPLE 10 Find the oblique asymptote of the graph of

$$f(x) = \frac{x^2 - 3}{2x - 4}$$

in Figure 2.43.

Solution We are interested in the behavior as $x \to \pm\infty$. We divide $(2x - 4)$ into $(x^2 - 3)$:

$$
\begin{array}{r}
\frac{x}{2} + 1 \\
2x - 4 \overline{\smash{)}\, x^2 + 0x - 3} \\
\underline{x^2 - 2x} \\
2x - 3 \\
\underline{2x - 4} \\
1
\end{array}
$$

This tells us that

$$f(x) = \frac{x^2 - 3}{2x - 4} = \underbrace{\left(\frac{x}{2} + 1\right)}_{\text{linear } g(x)} + \underbrace{\left(\frac{1}{2x - 4}\right)}_{\text{remainder}}.$$

As $x \to \pm\infty$, the remainder, whose magnitude gives the vertical distance between the graphs of f and g, goes to zero, making the slanted line

$$g(x) = \frac{x}{2} + 1$$

an asymptote of the graph of f (Figure 2.43). The line $y = g(x)$ is an asymptote both to the right and to the left. ■

Notice in Example 10 that if the degree of the numerator in a rational function is greater than the degree of the denominator, then the limit as $|x|$ becomes large is $+\infty$ or $-\infty$, depending on the signs assumed by the numerator and denominator.

Infinite Limits

Let us look again at the function $f(x) = 1/x$. As $x \to 0^+$, the values of f grow without bound, eventually reaching and surpassing every positive real number. That is, given any positive real number B, however large, the values of f become larger still (Figure 2.44).

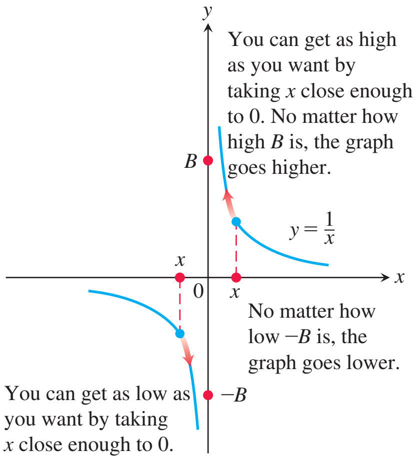


FIGURE 2.44 One-sided infinite limits:

$$\lim_{x \to 0^+} \frac{1}{x} = \infty \quad \text{and} \quad \lim_{x \to 0^-} \frac{1}{x} = -\infty.$$

Thus, f has no limit as $x \to 0^+$. It is nevertheless convenient to describe the behavior of f by saying that $f(x)$ approaches ∞ as $x \to 0^+$. We write

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{1}{x} = \infty.$$

In writing this equation, we are *not* saying that the limit exists. Nor are we saying that there is a real number ∞, for there is no such number. Rather, this expression is just a concise way of saying that $\lim_{x \to 0^+} (1/x)$ *does not exist because* $1/x$ *becomes arbitrarily large and positive as* $x \to 0^+$.

As $x \to 0^-$, the values of $f(x) = 1/x$ become arbitrarily large and negative. Given any negative real number $-B$, the values of f eventually lie below $-B$. (See Figure 2.44.) We write

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{1}{x} = -\infty.$$

Again, we are not saying that the limit exists and equals the number $-\infty$. There *is* no real number $-\infty$. We are describing the behavior of a function whose limit as $x \to 0^-$ *does not exist because its values become arbitrarily large and negative*.

EXAMPLE 11 Find $\lim_{x \to 1^+} \frac{1}{x - 1}$ and $\lim_{x \to 1^-} \frac{1}{x - 1}$.

Geometric Solution The graph of $y = 1/(x - 1)$ is the graph of $y = 1/x$ shifted 1 unit to the right (Figure 2.45). Therefore, $y = 1/(x - 1)$ behaves near 1 exactly the way $y = 1/x$ behaves near 0:

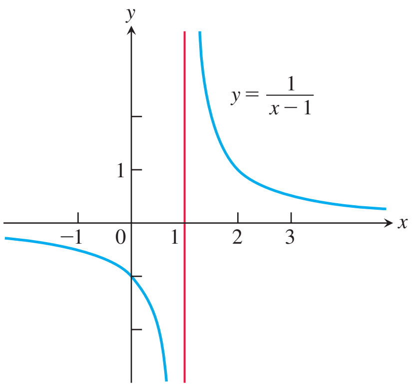


FIGURE 2.45 Near $x = 1$, the function $y = 1/(x - 1)$ behaves the way the function $y = 1/x$ behaves near $x = 0$. Its graph is the graph of $y = 1/x$ shifted 1 unit to the right (Example 11).

$$\lim_{x\to 1^+} \frac{1}{x-1} = \infty \qquad \text{and} \qquad \lim_{x\to 1^-} \frac{1}{x-1} = -\infty.$$

Analytic Solution Think about the number $x - 1$ and its reciprocal. As $x \to 1^+$, we have $(x - 1) \to 0^+$ and $1/(x - 1) \to \infty$. As $x \to 1^-$, we have $(x - 1) \to 0^-$ and $1/(x - 1) \to -\infty$. ■

EXAMPLE 12 Discuss the behavior of

$$f(x) = \frac{1}{x^2} \qquad \text{as} \qquad x \to 0.$$

Solution As x approaches zero from either side, the values of $1/x^2$ are positive and become arbitrarily large (Figure 2.46). This means that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1}{x^2} = \infty.$$

The function $y = 1/x$ shows no consistent behavior as $x \to 0$. We have $1/x \to \infty$ if $x \to 0^+$, but $1/x \to -\infty$ if $x \to 0^-$. All we can say about $\lim_{x\to 0} (1/x)$ is that it does not exist. The function $y = 1/x^2$ is different. Its values approach infinity as x approaches zero from either side, so we can say that $\lim_{x\to 0} (1/x^2) = \infty$. ■

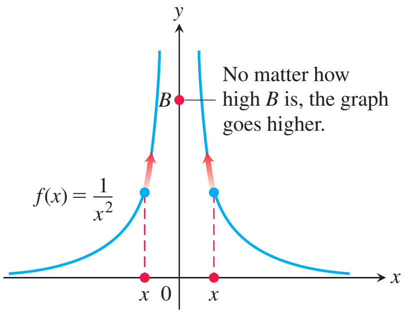


FIGURE 2.46 The graph of $f(x)$ in Example 12 approaches infinity as $x \to 0$.

EXAMPLE 13 These examples illustrate that rational functions can behave in various ways near zeros of the denominator.

(a) $$\lim_{x\to 2} \frac{(x-2)^2}{x^2-4} = \lim_{x\to 2} \frac{(x-2)^2}{(x-2)(x+2)} = \lim_{x\to 2} \frac{x-2}{x+2} = 0$$

Can substitute 2 for x after algebraic manipulation eliminates division by 0.

(b) $$\lim_{x\to 2} \frac{x-2}{x^2-4} = \lim_{x\to 2} \frac{x-2}{(x-2)(x+2)} = \lim_{x\to 2} \frac{1}{x+2} = \frac{1}{4}$$

Again substitute 2 for x after algebraic manipulation eliminates division by 0.

(c) $$\lim_{x\to 2^+} \frac{x-3}{x^2-4} = \lim_{x\to 2^+} \frac{x-3}{(x-2)(x+2)} = -\infty$$

The values are negative for $x > 2$, x near 2.

(d) $$\lim_{x\to 2^-} \frac{x-3}{x^2-4} = \lim_{x\to 2^-} \frac{x-3}{(x-2)(x+2)} = \infty$$

The values are positive for $x < 2$, x near 2.

(e) $$\lim_{x\to 2} \frac{x-3}{x^2-4} = \lim_{x\to 2} \frac{x-3}{(x-2)(x+2)} \text{ does not exist.}$$

Limits from left and from right differ.

(f) $$\lim_{x\to 2} \frac{2-x}{(x-2)^3} = \lim_{x\to 2} \frac{-(x-2)}{(x-2)^3} = \lim_{x\to 2} \frac{-1}{(x-2)^2} = -\infty$$

Denominator is positive, so values are negative near $x = 2$.

In parts (a) and (b) the effect of the zero in the denominator at $x = 2$ is canceled because the numerator is zero there also. Thus a finite limit exists. This is not true in part (f), where cancellation still leaves a zero factor in the denominator. ■

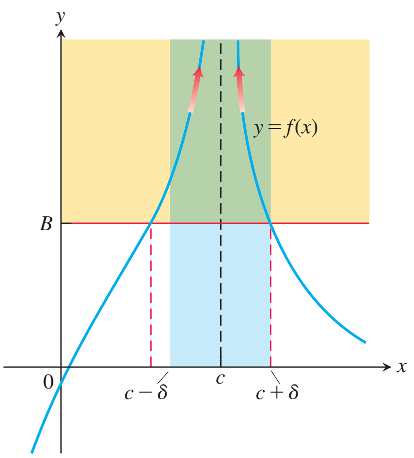


FIGURE 2.47 For $c - \delta < x < c + \delta$, the graph of $f(x)$ lies above the line $y = B$.

EXAMPLE 14 Find $\displaystyle\lim_{x\to -\infty} \frac{2x^5 - 6x^4 + 1}{3x^2 + x - 7}$.

Solution We are asked to find the limit of a rational function as $x \to -\infty$, so we divide the numerator and denominator by x^2, the highest power of x in the denominator:

$$\lim_{x\to -\infty} \frac{2x^5 - 6x^4 + 1}{3x^2 + x - 7} = \lim_{x\to -\infty} \frac{2x^3 - 6x^2 + x^{-2}}{3 + x^{-1} - 7x^{-2}}$$

$$= \lim_{x \to -\infty} \frac{2x^2(x - 3) + x^{-2}}{3 + x^{-1} - 7x^{-2}}$$

$$= -\infty, \qquad x^{-n} \to 0,\ x - 3 \to -\infty$$

because the numerator tends to $-\infty$ while the denominator approaches 3 as $x \to -\infty$. ■

Precise Definitions of Infinite Limits

Instead of requiring $f(x)$ to lie arbitrarily close to a finite number L for all x sufficiently close to c, the definitions of infinite limits require $f(x)$ to lie arbitrarily far from zero. Except for this change, the language is very similar to what we have seen before. Figures 2.47 and 2.48 accompany these definitions.

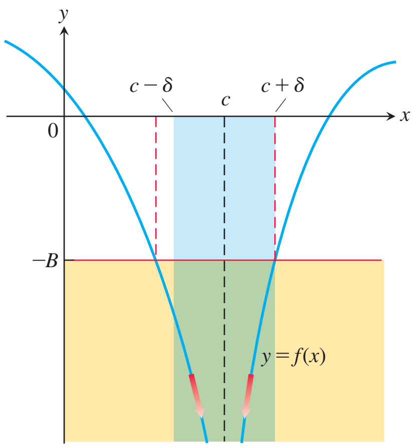


FIGURE 2.48 For $c - \delta < x < c + \delta$, the graph of $f(x)$ lies below the line $y = -B$.

DEFINITIONS

1. We say that **$f(x)$ approaches infinity as x approaches c**, and write

$$\lim_{x \to c} f(x) = \infty,$$

if for every positive real number B there exists a corresponding $\delta > 0$ such that

$$f(x) > B \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

2. We say that **$f(x)$ approaches negative infinity as x approaches c**, and write

$$\lim_{x \to c} f(x) = -\infty,$$

if for every negative real number $-B$ there exists a corresponding $\delta > 0$ such that

$$f(x) < -B \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

The precise definitions of one-sided infinite limits at c are similar and are stated in the exercises.

EXAMPLE 15 Prove that $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

Solution Given $B > 0$, we want to find $\delta > 0$ such that

$$0 < |x - 0| < \delta \quad \text{implies} \quad \frac{1}{x^2} > B.$$

Now,

$$\frac{1}{x^2} > B \qquad \text{if and only if} \qquad x^2 < \frac{1}{B}$$

or, equivalently,

$$|x| < \frac{1}{\sqrt{B}}.$$

Thus, choosing $\delta = 1/\sqrt{B}$ (or any smaller positive number), we see that

$$|x| < \delta \quad \text{implies} \quad \frac{1}{x^2} > \frac{1}{\delta^2} \geq B.$$

Therefore, by definition,

$$\lim_{x \to 0} \frac{1}{x^2} = \infty.$$

■

Vertical Asymptotes

Notice that the distance between a point on the graph of $f(x) = 1/x$ and the y-axis approaches zero as the point moves vertically along the graph and away from the origin (Figure 2.49). The function $f(x) = 1/x$ is unbounded as x approaches 0 because

$$\lim_{x\to 0^+} \frac{1}{x} = \infty \quad \text{and} \quad \lim_{x\to 0^-} \frac{1}{x} = -\infty.$$

We say that the line $x = 0$ (the y-axis) is a *vertical asymptote* of the graph of $f(x) = 1/x$. Observe that the denominator is zero at $x = 0$ and the function is undefined there.

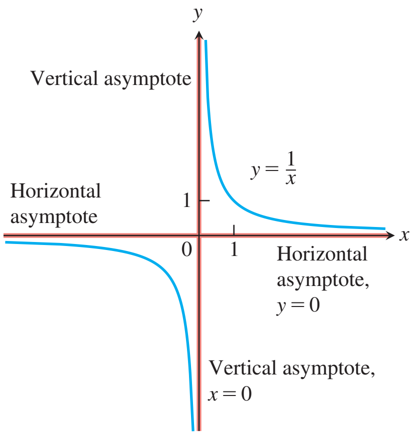


FIGURE 2.49 The coordinate axes are asymptotes of both branches of the hyperbola $y = 1/x$.

DEFINITION A line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = \pm\infty.$$

EXAMPLE 16 Find the horizontal and vertical asymptotes of the curve

$$y = \frac{x+3}{x+2}.$$

Solution We are interested in the behavior as $x \to \pm\infty$ and the behavior as $x \to -2$, where the denominator is zero.

The asymptotes are revealed if we recast the rational function as a polynomial with a remainder, by dividing $(x + 2)$ into $(x + 3)$:

$$\begin{array}{r} 1 \\ x+2\overline{)\,x+3} \\ \underline{x+2} \\ 1 \end{array}$$

This result enables us to rewrite y as:

$$y = 1 + \frac{1}{x+2}.$$

As $x \to \pm\infty$, the curve approaches the horizontal asymptote $y = 1$; as $x \to -2$, the curve approaches the vertical asymptote $x = -2$. We see that the curve in question is the graph of $f(x) = 1/x$ shifted 1 unit up and 2 units left (Figure 2.50). The asymptotes, instead of being the coordinate axes, are now the lines $y = 1$ and $x = -2$. ■

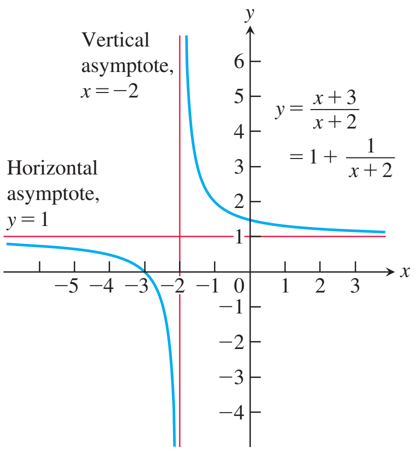


FIGURE 2.50 The lines $y = 1$ and $x = -2$ are asymptotes of the curve in Example 16.

EXAMPLE 17 Find the horizontal and vertical asymptotes of the graph of

$$f(x) = -\frac{8}{x^2 - 4}.$$

Solution We are interested in the behavior as $x \to \pm\infty$ and as $x \to \pm 2$, where the denominator is zero. Notice that f is an even function of x, so its graph is symmetric with respect to the y-axis.

(a) *The behavior as* $x \to \pm\infty$. Since $\lim_{x\to\infty} f(x) = 0$, the line $y = 0$ is a horizontal asymptote of the graph to the right. By symmetry it is an asymptote to the left as well (Figure 2.51). Notice that the curve approaches the x-axis from only the negative side (or from below). Also, $f(0) = 2$.

(b) *The behavior as* $x \to \pm 2$. Since

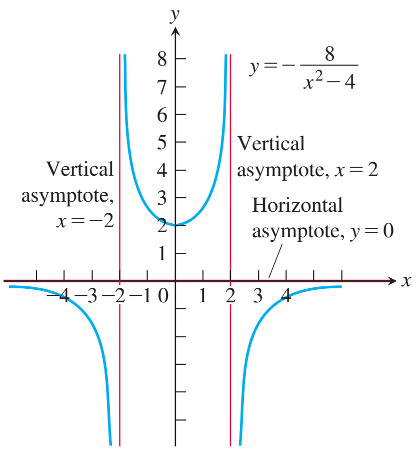


FIGURE 2.51 Graph of the function in Example 17. Notice that the curve approaches the x-axis from only one side. Asymptotes do not have to be two-sided.

$$\lim_{x\to 2^+} f(x) = -\infty \qquad \text{and} \qquad \lim_{x\to 2^-} f(x) = \infty,$$

the line $x = 2$ is a vertical asymptote both from the right and from the left. By symmetry, the line $x = -2$ is also a vertical asymptote.

There are no other asymptotes because f has a finite limit at all other points. ■

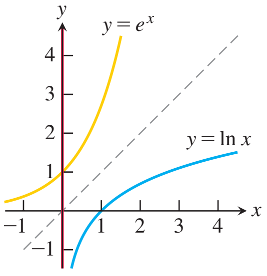


FIGURE 2.52 The line $x = 0$ is a vertical asymptote of the natural logarithm function (Example 18).

EXAMPLE 18 The graph of the natural logarithm function has the y-axis (the line $x = 0$) as a vertical asymptote. We see this from the graph sketched in Figure 2.52 (which is the reflection of the graph of the natural exponential function across the line $y = x$) and the fact that the x-axis is a horizontal asymptote of $y = e^x$ (Example 5). Thus,

$$\lim_{x\to 0^+} \ln x = -\infty.$$

The same result is true for $y = \log_a x$ whenever $a > 1$. ■

EXAMPLE 19 The curves

$$y = \sec x = \frac{1}{\cos x} \qquad \text{and} \qquad y = \tan x = \frac{\sin x}{\cos x}$$

both have vertical asymptotes at odd-integer multiples of $\pi/2$, where $\cos x = 0$ (Figure 2.53).

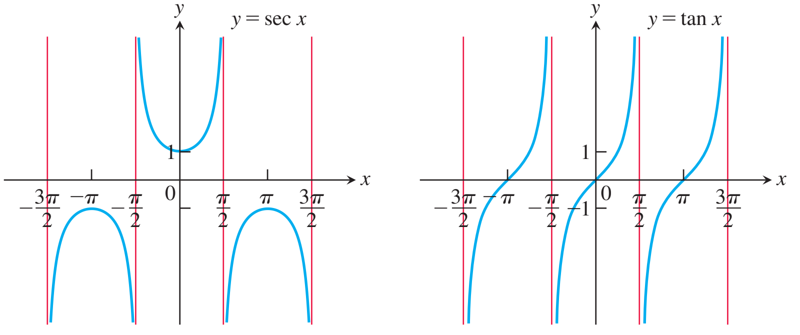


FIGURE 2.53 The graphs of $\sec x$ and $\tan x$ have infinitely many vertical asymptotes (Example 19). ■

Dominant Terms

In Example 10 we saw that by using long division, we can rewrite the function

$$f(x) = \frac{x^2 - 3}{2x - 4}$$

as a linear function plus a remainder term:

$$f(x) = \left(\frac{x}{2} + 1\right) + \left(\frac{1}{2x - 4}\right).$$

This tells us immediately that

$$f(x) \approx \frac{x}{2} + 1 \qquad \text{For } |x| \text{ large, } \frac{1}{2x-4} \text{ is near 0.}$$

$$f(x) \approx \frac{1}{2x - 4} \qquad \text{For } x \text{ near 2, this term is very large in absolute value.}$$

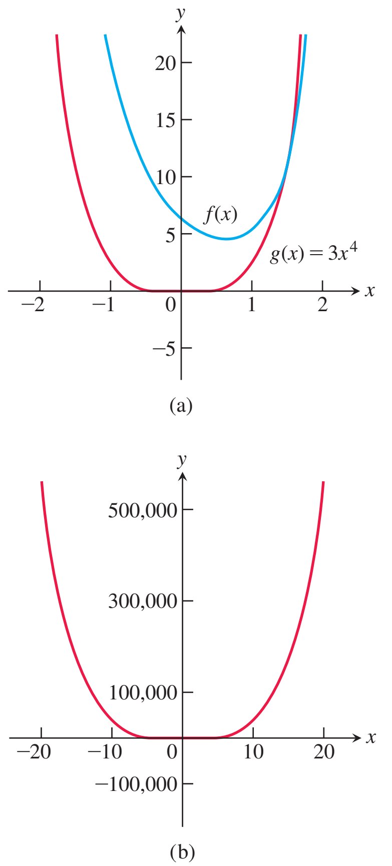


FIGURE 2.54 The graphs of f and g are (a) distinct for $|x|$ small, and (b) nearly identical for $|x|$ large (Example 20).

If we want to know how f behaves, this is the way to find out. It behaves like $y = (x/2) + 1$ when $|x|$ is large and the contribution of $1/(2x - 4)$ to the total value of f is insignificant. It behaves like $1/(2x - 4)$ when x is so close to 2 that $1/(2x - 4)$ makes the dominant contribution.

We say that $(x/2) + 1$ **dominates** when x approaches ∞ or $-\infty$, and we say that $1/(2x - 4)$ dominates when x approaches 2. **Dominant terms** like these help us predict a function's behavior.

EXAMPLE 20 Let $f(x) = 3x^4 - 2x^3 + 3x^2 - 5x + 6$ and $g(x) = 3x^4$. Show that although f and g are quite different for numerically small values of x, they behave similarly for $|x|$ very large, in the sense that their ratios approach 1 as $x \to \infty$ or $x \to -\infty$.

Solution The graphs of f and g behave quite differently near the origin (Figure 2.54a), but appear as virtually identical on a larger scale (Figure 2.54b).

We can test that the term $3x^4$ in f, represented graphically by g, dominates the polynomial f for numerically large values of x by examining the ratio of the two functions as $x \to \pm\infty$. We find that

$$\begin{aligned}\lim_{x\to\pm\infty}\frac{f(x)}{g(x)} &= \lim_{x\to\pm\infty}\frac{3x^4 - 2x^3 + 3x^2 - 5x + 6}{3x^4}\\ &= \lim_{x\to\pm\infty}\left(1 - \frac{2}{3x} + \frac{1}{x^2} - \frac{5}{3x^3} + \frac{2}{x^4}\right)\\ &= 1,\end{aligned}$$

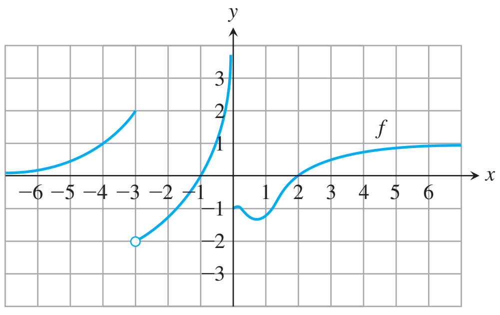
which means that f and g appear nearly identical when $|x|$ is large. ■

EXERCISES 2.5

Finding Limits

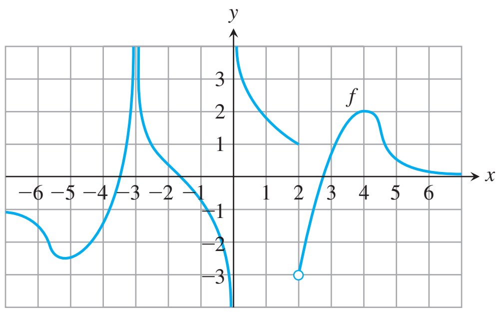
1. For the function f whose graph is given, determine the following limits.

   a. $\lim_{x\to 2} f(x)$ b. $\lim_{x\to -3^+} f(x)$
   c. $\lim_{x\to -3^-} f(x)$ d. $\lim_{x\to -3} f(x)$
   e. $\lim_{x\to 0^+} f(x)$ f. $\lim_{x\to 0^-} f(x)$
   g. $\lim_{x\to 0} f(x)$ h. $\lim_{x\to \infty} f(x)$
   i. $\lim_{x\to -\infty} f(x)$



2. For the function f whose graph is given, determine the following limits.

   a. $\lim_{x\to 4} f(x)$ b. $\lim_{x\to 2^+} f(x)$ c. $\lim_{x\to 2^-} f(x)$
   d. $\lim_{x\to 2} f(x)$ e. $\lim_{x\to -3^+} f(x)$ f. $\lim_{x\to -3^-} f(x)$
   g. $\lim_{x\to -3} f(x)$ h. $\lim_{x\to 0^+} f(x)$ i. $\lim_{x\to 0^-} f(x)$
   j. $\lim_{x\to 0} f(x)$ k. $\lim_{x\to \infty} f(x)$ l. $\lim_{x\to -\infty} f(x)$



In Exercises 3–8, find the limit of each function **(a)** as $x \to \infty$ and **(b)** as $x \to -\infty$. (You may wish to visualize your answer with a graphing calculator or computer.)

3. $f(x) = \dfrac{2}{x} - 3$

4. $f(x) = \pi - \dfrac{2}{x^2}$

5. $g(x) = \dfrac{1}{2 + (1/x)}$

6. $g(x) = \dfrac{1}{8 - (5/x^2)}$

7. $h(x) = \dfrac{-5 + (7/x)}{3 - (1/x^2)}$

8. $h(x) = \dfrac{3 - (2/x)}{4 + (\sqrt{2}/x^2)}$

Find the limits in Exercises 9–12.

9. $\displaystyle\lim_{x\to\infty} \frac{\sin 2x}{x}$

10. $\displaystyle\lim_{\theta\to-\infty} \frac{\cos\theta}{3\theta}$

11. $\displaystyle\lim_{t\to-\infty} \frac{2 - t + \sin t}{t + \cos t}$

12. $\displaystyle\lim_{r\to\infty} \frac{r + \sin r}{2r + 7 - 5\sin r}$

Limits of Rational Functions

In Exercises 13–22, find the limit of each rational function **(a)** as $x \to \infty$ and **(b)** as $x \to -\infty$.

13. $f(x) = \dfrac{2x + 3}{5x + 7}$

14. $f(x) = \dfrac{2x^3 + 7}{x^3 - x^2 + x + 7}$

15. $f(x) = \dfrac{x + 1}{x^2 + 3}$

16. $f(x) = \dfrac{3x + 7}{x^2 - 2}$

17. $h(x) = \dfrac{7x^3}{x^3 - 3x^2 + 6x}$

18. $h(x) = \dfrac{9x^4 + x}{2x^4 + 5x^2 - x + 6}$

19. $g(x) = \dfrac{10x^5 + x^4 + 31}{x^6}$

20. $g(x) = \dfrac{x^3 + 7x^2 - 2}{x^2 - x + 1}$

21. $f(x) = \dfrac{3x^7 + 5x^2 - 1}{6x^3 - 7x + 3}$

22. $h(x) = \dfrac{5x^8 - 2x^3 + 9}{3 + x - 4x^5}$

Limits as $x \to \infty$ or $x \to -\infty$

The process by which we determine limits of rational functions applies equally well to ratios containing noninteger or negative powers of x: Divide numerator and denominator by the highest power of x in the denominator and proceed from there. Find the limits in Exercises 23–36.

23. $\displaystyle\lim_{x\to\infty} \sqrt{\frac{8x^2 - 3}{2x^2 + x}}$

24. $\displaystyle\lim_{x\to-\infty} \left(\frac{x^2 + x - 1}{8x^2 - 3}\right)^{1/3}$

25. $\displaystyle\lim_{x\to-\infty} \left(\frac{1 - x^3}{x^2 + 7x}\right)^5$

26. $\displaystyle\lim_{x\to\infty} \sqrt{\frac{x^2 - 5x}{x^3 + x - 2}}$

27. $\displaystyle\lim_{x\to\infty} \frac{2\sqrt{x} + x^{-1}}{3x - 7}$

28. $\displaystyle\lim_{x\to\infty} \frac{2 + \sqrt{x}}{2 - \sqrt{x}}$

29. $\displaystyle\lim_{x\to-\infty} \frac{\sqrt[3]{x} - \sqrt[5]{x}}{\sqrt[3]{x} + \sqrt[5]{x}}$

30. $\displaystyle\lim_{x\to\infty} \frac{x^{-1} + x^{-4}}{x^{-2} - x^{-3}}$

31. $\displaystyle\lim_{x\to\infty} \frac{2x^{5/3} - x^{1/3} + 7}{x^{8/5} + 3x + \sqrt{x}}$

32. $\displaystyle\lim_{x\to-\infty} \frac{\sqrt[3]{x} - 5x + 3}{2x + x^{2/3} - 4}$

33. $\displaystyle\lim_{x\to\infty} \frac{\sqrt{x^2 + 1}}{x + 1}$

34. $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{x^2 + 1}}{x + 1}$

35. $\displaystyle\lim_{x\to\infty} \frac{x - 3}{\sqrt{4x^2 + 25}}$

36. $\displaystyle\lim_{x\to-\infty} \frac{4 - 3x^3}{\sqrt{x^6 + 9}}$

Infinite Limits

Find the limits in Exercises 37–48.

37. $\displaystyle\lim_{x\to0^+} \frac{1}{3x}$

38. $\displaystyle\lim_{x\to0^-} \frac{5}{2x}$

39. $\displaystyle\lim_{x\to2^-} \frac{3}{x - 2}$

40. $\displaystyle\lim_{x\to3^+} \frac{1}{x - 3}$

41. $\displaystyle\lim_{x\to-8^+} \frac{2x}{x + 8}$

42. $\displaystyle\lim_{x\to-5^-} \frac{3x}{2x + 10}$

43. $\displaystyle\lim_{x\to7} \frac{4}{(x - 7)^2}$

44. $\displaystyle\lim_{x\to0} \frac{-1}{x^2(x + 1)}$

45. **a.** $\displaystyle\lim_{x\to0^+} \frac{2}{3x^{1/3}}$ **b.** $\displaystyle\lim_{x\to0^-} \frac{2}{3x^{1/3}}$

46. **a.** $\displaystyle\lim_{x\to0^+} \frac{2}{x^{1/5}}$ **b.** $\displaystyle\lim_{x\to0^-} \frac{2}{x^{1/5}}$

47. $\displaystyle\lim_{x\to0} \frac{4}{x^{2/5}}$

48. $\displaystyle\lim_{x\to0} \frac{1}{x^{2/3}}$

Find the limits in Exercises 49–52.

49. $\displaystyle\lim_{x\to(\pi/2)^-} \tan x$

50. $\displaystyle\lim_{x\to(-\pi/2)^+} \sec x$

51. $\displaystyle\lim_{\theta\to0^-} (1 + \csc\theta)$

52. $\displaystyle\lim_{\theta\to0} (2 - \cot\theta)$

Find the limits in Exercises 53–58.

53. $\displaystyle\lim \frac{1}{x^2 - 4}$ as

a. $x \to 2^+$ **b.** $x \to 2^-$

c. $x \to -2^+$ **d.** $x \to -2^-$

54. $\displaystyle\lim \frac{x}{x^2 - 1}$ as

a. $x \to 1^+$ **b.** $x \to 1^-$

c. $x \to -1^+$ **d.** $x \to -1^-$

55. $\displaystyle\lim \left(\frac{x^2}{2} - \frac{1}{x}\right)$ as

a. $x \to 0^+$ **b.** $x \to 0^-$

c. $x \to \sqrt[3]{2}$ **d.** $x \to -1$

56. $\displaystyle\lim \frac{x^2 - 1}{2x + 4}$ as

a. $x \to -2^+$ **b.** $x \to -2^-$

c. $x \to 1^+$ **d.** $x \to 0^-$

57. $\lim \dfrac{x^2 - 3x + 2}{x^3 - 2x^2}$ as

a. $x \to 0^+$ **b.** $x \to 2^+$

c. $x \to 2^-$ **d.** $x \to 2$

e. What, if anything, can be said about the limit as $x \to 0$?

58. $\lim \dfrac{x^2 - 3x + 2}{x^3 - 4x}$ as

a. $x \to 2^+$ **b.** $x \to -2^+$

c. $x \to 0^-$ **d.** $x \to 1^+$

e. What, if anything, can be said about the limit as $x \to 0$?

Find the limits in Exercises 59–62.

59. $\lim \left(2 - \dfrac{3}{t^{1/3}}\right)$ as

a. $t \to 0^+$ **b.** $t \to 0^-$

60. $\lim \left(\dfrac{1}{t^{3/5}} + 7\right)$ as

a. $t \to 0^+$ **b.** $t \to 0^-$

61. $\lim \left(\dfrac{1}{x^{2/3}} + \dfrac{2}{(x - 1)^{2/3}}\right)$ as

a. $x \to 0^+$ **b.** $x \to 0^-$

c. $x \to 1^+$ **d.** $x \to 1^-$

62. $\lim \left(\dfrac{1}{x^{1/3}} - \dfrac{1}{(x - 1)^{4/3}}\right)$ as

a. $x \to 0^+$ **b.** $x \to 0^-$

c. $x \to 1^+$ **d.** $x \to 1^-$

Graphing Simple Rational Functions

Graph the rational functions in Exercises 63–68. Include the graphs and equations of the asymptotes and dominant terms.

63. $y = \dfrac{1}{x - 1}$ **64.** $y = \dfrac{1}{x + 1}$

65. $y = \dfrac{1}{2x + 4}$ **66.** $y = \dfrac{-3}{x - 3}$

67. $y = \dfrac{x + 3}{x + 2}$ **68.** $y = \dfrac{2x}{x + 1}$

Domains, Ranges, and Asymptotes

Determine the domain and range of each function. Use various limits to find the asymptotes and the ranges.

69. $y = 4 + \dfrac{3x^2}{x^2 + 1}$ **70.** $y = \dfrac{2x}{x^2 - 1}$

71. $y = \dfrac{8 - e^x}{2 + e^x}$ **72.** $y = \dfrac{4e^x + e^{2x}}{e^x + e^{2x}}$

73. $y = \dfrac{\sqrt{x^2 + 4}}{x}$ **74.** $y = \dfrac{x^3}{x^3 - 8}$

Inventing Graphs and Functions

In Exercises 75–78, sketch the graph of a function $y = f(x)$ that satisfies the given conditions. No formulas are required—just label the coordinate axes and sketch an appropriate graph. (The answers are not unique, so your graphs may not be exactly like those in the answer section.)

75. $f(0) = 0$, $f(1) = 2$, $f(-1) = -2$, $\lim_{x \to -\infty} f(x) = -1$, and $\lim_{x \to \infty} f(x) = 1$

76. $f(0) = 0$, $\lim_{x \to \pm\infty} f(x) = 0$, $\lim_{x \to 0^+} f(x) = 2$, and $\lim_{x \to 0^-} f(x) = -2$

77. $f(0) = 0$, $\lim_{x \to \pm\infty} f(x) = 0$, $\lim_{x \to 1^-} f(x) = \lim_{x \to -1^+} f(x) = \infty$, $\lim_{x \to 1^+} f(x) = -\infty$, and $\lim_{x \to -1^-} f(x) = -\infty$

78. $f(2) = 1$, $f(-1) = 0$, $\lim_{x \to \infty} f(x) = 0$, $\lim_{x \to 0^+} f(x) = \infty$, $\lim_{x \to 0^-} f(x) = -\infty$, and $\lim_{x \to -\infty} f(x) = 1$

In Exercises 79–82, find a function that satisfies the given conditions and sketch its graph. (The answers here are not unique. Any function that satisfies the conditions is acceptable. Feel free to use formulas defined in pieces if that will help.)

79. $\lim_{x \to \pm\infty} f(x) = 0$, $\lim_{x \to 2^-} f(x) = \infty$, and $\lim_{x \to 2^+} f(x) = \infty$

80. $\lim_{x \to \pm\infty} g(x) = 0$, $\lim_{x \to 3^-} g(x) = -\infty$, and $\lim_{x \to 3^+} g(x) = \infty$

81. $\lim_{x \to -\infty} h(x) = -1$, $\lim_{x \to \infty} h(x) = 1$, $\lim_{x \to 0^-} h(x) = -1$, and $\lim_{x \to 0^+} h(x) = 1$

82. $\lim_{x \to \pm\infty} k(x) = 1$, $\lim_{x \to 1^-} k(x) = \infty$, and $\lim_{x \to 1^+} k(x) = -\infty$

83. Suppose that $f(x)$ and $g(x)$ are polynomials in x and that $\lim_{x \to \infty} (f(x)/g(x)) = 2$. Can you conclude anything about $\lim_{x \to -\infty} (f(x)/g(x))$? Give reasons for your answer.

84. Suppose that $f(x)$ and $g(x)$ are polynomials in x. Can the graph of $f(x)/g(x)$ have an asymptote if $g(x)$ is never zero? Give reasons for your answer.

85. How many horizontal asymptotes can the graph of a given rational function have? Give reasons for your answer.

Finding Limits of Differences When $x \to \pm\infty$

Find the limits in Exercises 86–92. (*Hint*: Try multiplying and dividing by the conjugate.)

86. $\lim_{x \to \infty} (\sqrt{x + 9} - \sqrt{x + 4})$

87. $\lim_{x \to \infty} (\sqrt{x^2 + 25} - \sqrt{x^2 - 1})$

88. $\lim_{x \to -\infty} (\sqrt{x^2 + 3} + x)$

89. $\lim_{x \to -\infty} (2x + \sqrt{4x^2 + 3x - 2})$

90. $\lim_{x \to \infty} (\sqrt{9x^2 - x} - 3x)$

91. $\lim_{x \to \infty} (\sqrt{x^2 + 3x} - \sqrt{x^2 - 2x})$

92. $\lim_{x \to \infty} (\sqrt{x^2 + x} - \sqrt{x^2 - x})$

Using the Formal Definitions

Use the formal definitions of limits as $x \to \pm\infty$ to establish the limits in Exercises 93 and 94.

93. If f has the constant value $f(x) = k$, then $\lim_{x\to\infty} f(x) = k$.

94. If f has the constant value $f(x) = k$, then $\lim_{x\to-\infty} f(x) = k$.

Use formal definitions to prove the limit statements in Exercises 95–98.

95. $\lim_{x\to 0} \frac{-1}{x^2} = -\infty$

96. $\lim_{x\to 0} \frac{1}{|x|} = \infty$

97. $\lim_{x\to 3} \frac{-2}{(x-3)^2} = -\infty$

98. $\lim_{x\to -5} \frac{1}{(x+5)^2} = \infty$

99. Here is the definition of **infinite right-hand limit**.

> Suppose that an interval (c, d) lies in the domain of f. We say that $f(x)$ approaches infinity as x approaches c from the right, and write
>
> $$\lim_{x\to c^+} f(x) = \infty,$$
>
> if, for every positive real number B, there exists a corresponding number $\delta > 0$ such that
>
> $$f(x) > B \quad \text{whenever} \quad c < x < c + \delta.$$

Modify the definition to cover the following cases.

a. $\lim_{x\to c^-} f(x) = \infty$

b. $\lim_{x\to c^+} f(x) = -\infty$

c. $\lim_{x\to c^-} f(x) = -\infty$

Use the formal definitions from Exercise 99 to prove the limit statements in Exercises 100–104.

100. $\lim_{x\to 0^+} \frac{1}{x} = \infty$

101. $\lim_{x\to 0^-} \frac{1}{x} = -\infty$

102. $\lim_{x\to 2^-} \frac{1}{x-2} = -\infty$

103. $\lim_{x\to 2^+} \frac{1}{x-2} = \infty$

104. $\lim_{x\to 1^-} \frac{1}{1-x^2} = \infty$

Oblique Asymptotes

Graph the rational functions in Exercises 105–110. Include the graphs and equations of the asymptotes.

105. $y = \frac{x^2}{x-1}$

106. $y = \frac{x^2+1}{x-1}$

107. $y = \frac{x^2-4}{x-1}$

108. $y = \frac{x^2-1}{2x+4}$

109. $y = \frac{x^2-1}{x}$

110. $y = \frac{x^3+1}{x^2}$

Additional Graphing Exercises

T Graph the curves in Exercises 111–114. Explain the relationship between the curve's formula and what you see.

111. $y = \frac{x}{\sqrt{4-x^2}}$

112. $y = \frac{-1}{\sqrt{4-x^2}}$

113. $y = x^{2/3} + \frac{1}{x^{1/3}}$

114. $y = \sin\left(\frac{\pi}{x^2+1}\right)$

T Graph the functions in Exercises 115 and 116. Then answer the following questions.

a. How does the graph behave as $x \to 0^+$?

b. How does the graph behave as $x \to \pm\infty$?

c. How does the graph behave near $x = 1$ and $x = -1$?

Give reasons for your answers.

115. $y = \frac{3}{2}\left(x - \frac{1}{x}\right)^{2/3}$

116. $y = \frac{3}{2}\left(\frac{x}{x-1}\right)^{2/3}$

2.6 Continuity

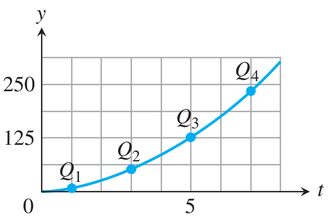


FIGURE 2.55 Connecting plotted points.

When we plot function values generated in a laboratory or collected in the field, we often connect the plotted points with an unbroken curve to show what the function's values are likely to have been at the points we did not measure (Figure 2.55). In doing so, we are assuming that we are working with a *continuous function*, so its outputs vary regularly and consistently with the inputs, and do not jump abruptly from one value to another without taking on the values in between. Intuitively, any function $y = f(x)$ whose graph can be sketched over its domain in one unbroken motion is an example of a continuous function. Such functions play an important role in the study of calculus and its applications.

Continuity at a Point

To understand continuity, it helps to consider a function like that in Figure 2.56, whose limits we investigated in Example 2 in Section 2.4.

EXAMPLE 1 At which numbers does the function f in Figure 2.56 appear to be not continuous? Explain why. What occurs at other numbers in the domain?

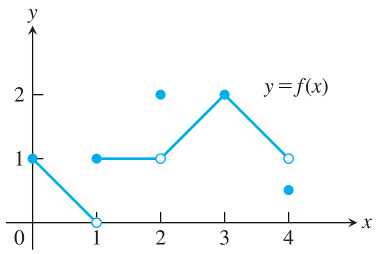


FIGURE 2.56 The function is not continuous at $x = 1, x = 2$, and $x = 4$ (Example 1).

Solution First we observe that the domain of the function is the closed interval $[0, 4]$, so we will be considering the numbers x within that interval. From the figure, we notice right away that there are breaks in the graph at the numbers $x = 1$, $x = 2$, and $x = 4$. The break at $x = 1$ appears as a jump, which we identify later as a "jump discontinuity." The break at $x = 2$ is called a "removable discontinuity" since by changing the function definition at that one point, we can create a new function that is continuous at $x = 2$. Similarly $x = 4$ is a removable discontinuity.

Numbers at which the graph of f has breaks:

At the interior point $x = 1$, the function fails to have a limit. It does have both a left-hand limit, $\lim_{x \to 1^-} f(x) = 0$, as well as a right-hand limit, $\lim_{x \to 1^+} f(x) = 1$, but the limit values are different, resulting in a jump in the graph. The function is not continuous at $x = 1$. However the function value $f(1) = 1$ is equal to the limit from the right, so the function *is* continuous from the right at $x = 1$.

At $x = 2$, the function does have a limit, $\lim_{x \to 2} f(x) = 1$, but the value of the function is $f(2) = 2$. The limit and function values are not the same, so there is a break in the graph and f is not continuous at $x = 2$.

At $x = 4$, the function does have a left-hand limit at this right endpoint, $\lim_{x \to 4^-} f(x) = 1$, but again the value of the function $f(4) = \frac{1}{2}$ differs from the value of the limit. We see again a break in the graph of the function at this endpoint and the function is not continuous from the left.

Numbers at which the graph of f has no breaks:

At $x = 3$, the function has a limit, $\lim_{x \to 3} f(x) = 2$. Moreover, the limit is the same value as the function there, $f(3) = 2$. The function is continuous at $x = 3$.

At $x = 0$, the function has a right-hand limit at this left endpoint, $\lim_{x \to 0^+} f(x) = 1$, and the value of the function is the same, $f(0) = 1$. The function is continuous from the right at $x = 0$. Because $x = 0$ is a left endpoint of the function's domain, we have that $\lim_{x \to 0} f(x) = 1$ and so f is continuous at $x = 0$.

At all other numbers $x = c$ in the domain, the function has a limit equal to the value of the function, so $\lim_{x \to c} f(x) = f(c)$. For example, $\lim_{x \to 5/2} f(x) = f\left(\frac{5}{2}\right) = \frac{3}{2}$. No breaks appear in the graph of the function at any of these numbers and the function is continuous at each of them. ■

The following definitions capture the continuity ideas we observed in Example 1.

> **DEFINITIONS** Let c be a real number that is either an interior point or an endpoint of an interval in the domain of f.
>
> The function f is **continuous at c** if
>
> $$\lim_{x \to c} f(x) = f(c).$$
>
> The function f is **right-continuous at c (or continuous from the right)** if
>
> $$\lim_{x \to c^+} f(x) = f(c).$$
>
> The function f is **left-continuous at c (or continuous from the left)** if
>
> $$\lim_{x \to c^-} f(x) = f(c).$$

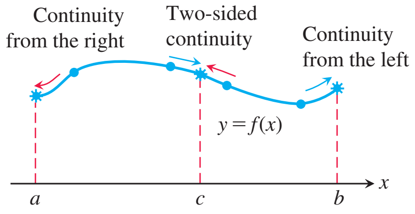


FIGURE 2.57 Continuity at points a, b, and c.

The function f in Example 1 is continuous at every x in $[0, 4]$ except $x = 1$, 2, and 4. It is right-continuous but not left-continuous at $x = 1$, neither right- nor left-continuous at $x = 2$, and not left-continuous at $x = 4$.

From Theorem 6, it follows immediately that a function f is continuous at an *interior* point c of an interval in its domain if and only if it is both right-continuous and left-continuous at c (Figure 2.57). We say that a function is **continuous over a closed interval** $[a, b]$ if it is right-continuous at a, left-continuous at b, and continuous at all interior