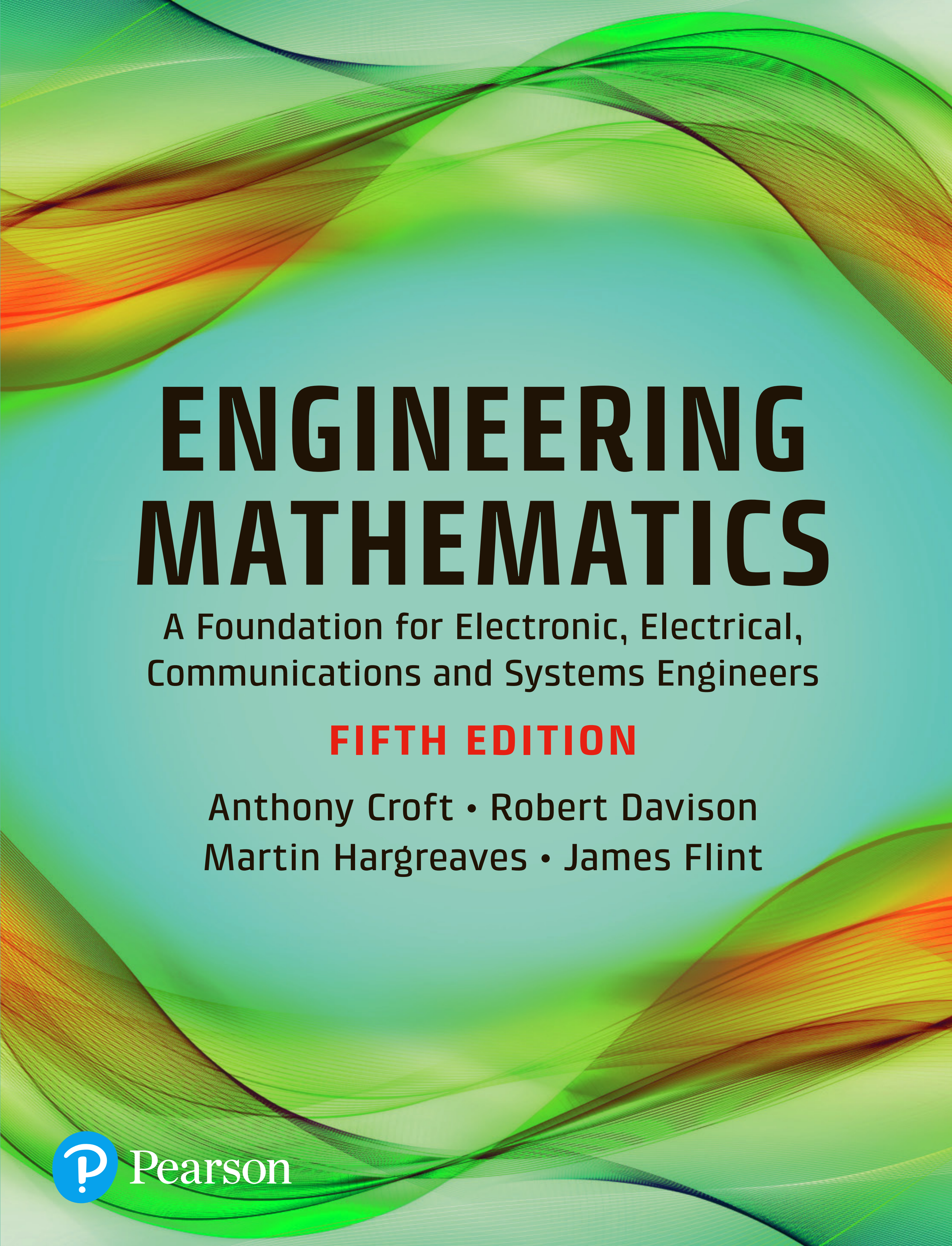


ENGINEERING MATHEMATICS

A Foundation for Electronic, Electrical, Communications and Systems Engineers

FIFTH EDITION

Anthony Croft • Robert Davison
Martin Hargreaves • James Flint

Pearson

Engineering Mathematics

Pearson

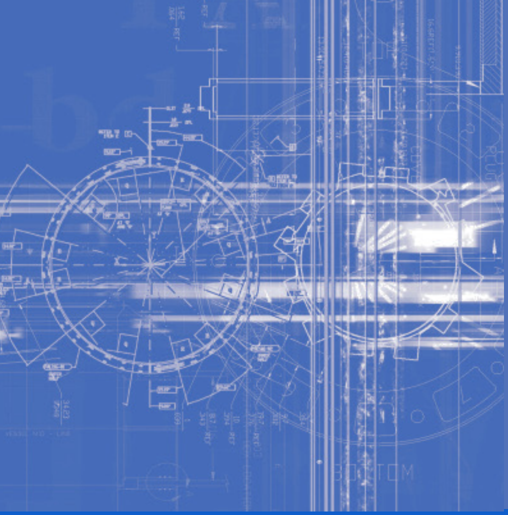
At Pearson, we have a simple mission: to help people make more of their lives through learning.

We combine innovative learning technology with trusted content and educational expertise to provide engaging and effective learning experiences that serve people wherever and whenever they are learning.

From classroom to boardroom, our curriculum materials, digital learning tools and testing programmes help to educate millions of people worldwide – more than any other private enterprise.

Every day our work helps learning flourish, and wherever learning flourishes, so do people.

To learn more, please visit us at **www.pearson.com/uk**



Fifth Edition

Engineering Mathematics

A Foundation for Electronic, Electrical, Communications and Systems Engineers

Anthony Croft
Loughborough University

Robert Davison

Martin Hargreaves
Chartered Physicist

James Flint
Loughborough University

Pearson

Harlow, England • London • New York • Boston • San Francisco • Toronto • Sydney
Dubai • Singapore • Hong Kong • Tokyo • Seoul • Taipei • New Delhi
Cape Town • São Paulo • Mexico City • Madrid • Amsterdam • Munich • Paris • Milan

PEARSON EDUCATION LIMITED
Edinburgh Gate
Harlow CM20 2JE
United Kingdom
Tel: +44 (0)1279 623623
Web: www.pearson.com/uk

First edition published under the Addison-Wesley imprint 1992 (print)
Second edition published under the Addison-Wesley imprint 1996 (print)
Third edition published under the Prentice Hall imprint 2001 (print)
Fourth edition published 2013 (print and electronic)
Fifth edition published 2017 (print and electronic)

© Addison-Wesley Publishers Limited 1992, 1996 (print)
© Pearson Education Limited 2001 (print)
© Pearson Education Limited 2013, 2017 (print and electronic)

The rights of Anthony Croft, Robert Davison, Martin Hargreaves and James Flint to be identified as authors of this work have been asserted by them in accordance with the Copyright, Designs and Patents Act 1988.

The print publication is protected by copyright. Prior to any prohibited reproduction, storage in a retrieval system, distribution or transmission in any form or by any means, electronic, mechanical, recording or otherwise, permission should be obtained from the publisher or, where applicable, a licence permitting restricted copying in the United Kingdom should be obtained from the Copyright Licensing Agency Ltd, Barnard's Inn, 86 Fetter Lane, London EC4A 1EN.

The ePublication is protected by copyright and must not be copied, reproduced, transferred, distributed, leased, licensed or publicly performed or used in any way except as specifically permitted in writing by the publishers, as allowed under the terms and conditions under which it was purchased, or as strictly permitted by applicable copyright law. Any unauthorised distribution or use of this text may be a direct infringement of the authors' and the publisher's rights and those responsible may be liable in law accordingly.

Pearson Education is not responsible for the content of third-party internet sites.

ISBN: 978-1-292-14665-2 (print)
978-1-292-14667-6 (PDF)
978-1-292-14666-9 (ePub)

British Library Cataloguing-in-Publication Data
A catalogue record for the print edition is available from the British Library

Library of Congress Cataloging-in-Publication Data
Names: Croft, Tony, 1957– author.
Title: Engineering mathematics : a foundation for electronic, electrical, communications and systems engineers / Anthony Croft, Loughborough University, Robert Davison, De Montfort University, Martin Hargreaves, De Montfort University, James Flint, Loughborough University.
Description: Fifth edition. | Harlow, England ; New York : Pearson, 2017. || Revised edition of: Engineering mathematics : a foundation for electronic, electrical, communications, and systems engineers / Anthony Croft, Robert Davison, Martin Hargreaves. 3rd editon. 2001. | Includes index.
Identifiers: LCCN 2017011081| ISBN 9781292146652 (Print) | ISBN 9781292146676 (PDF) | ISBN 9781292146669 (ePub)
Subjects: LCSH: Engineering mathematics. | Electrical engineering–Mathematics. | Electronics–Mathematics.
Classification: LCC TA330 .C76 2017 | DDC 510–dc23
LC record available at https://lccn.loc.gov/2017011081
A catalog record for the print edition is available from the Library of Congress

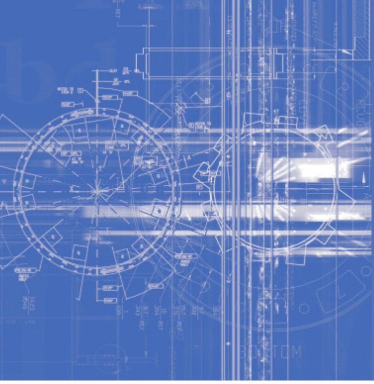
10 9 8 7 6 5 4 3 2 1
21 20 19 18 17

Print edition typeset in 10/12 Times Roman by iEnerziger Aptara®, Ltd.
Printed in Slovakia by Neografia

NOTE THAT ANY PAGE CROSS REFERENCES REFER TO THE PRINT EDITION

To Kate, Tom and Harvey – A.C.
To Kathy – R.D.
To my father and mother – M.H.
To Suzanne, Alexandra and Dominic – J.F.

This page intentionally left blank



Contents

Chapter 4 Coordinate systems 154

Chapter 5 Discrete mathematics 175

Chapter 6 Sequences and series 200

Chapter 7 Vectors 224

Chapter 8 Matrix algebra 257

Chapter 24 The Fourier transform 757

Chapter 25 Functions of several variables 823

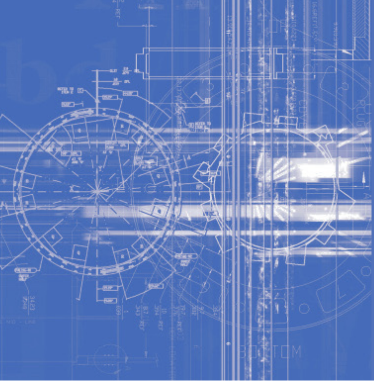
Chapter 26 Vector calculus 849

Lecturer Resources

ON THE WEBSITE

For password-protected online resources tailored to support the use of this textbook in teaching, please visit **www.pearsoned.co.uk/croft**

This page intentionally left blank



Preface

Audience

This book has been written to serve the mathematical needs of students engaged in a first course in engineering at degree level. It is primarily aimed at students of electronic, electrical, communications and systems engineering. Systems engineering typically encompasses areas such as manufacturing, control and production engineering. The textbook will also be useful for engineers who wish to engage in self-study and continuing education.

Motivation

Engineers are called upon to analyse a variety of engineering systems, which can be anything from a few electronic components connected together through to a complete factory. The analysis of these systems benefits from the intelligent application of mathematics. Indeed, many cannot be analysed without the use of mathematics. Mathematics is the language of engineering. It is essential to understand how mathematics works in order to master the complex relationships present in modern engineering systems and products.

Aims

There are two main aims of the book. Firstly, we wish to provide an accessible, readable introduction to engineering mathematics at degree level. The second aim is to encourage the integration of engineering and mathematics.

Content

The first three chapters include a review of some important functions and techniques that the reader may have met in previous courses. This material ensures that the book is self-contained and provides a convenient reference.

Traditional topics in algebra, trigonometry and calculus have been covered. Also included are chapters on set theory, sequences and series, Boolean algebra, logic, difference equations and the z transform. The importance of signal processing techniques is reflected by a thorough treatment of integral transform methods. Thus the Laplace, z and Fourier transforms have been given extensive coverage.

In the light of feedback from readers, new topics and new examples have been added in the fifth edition. Recognizing that motivation comes from seeing the applicability of mathematics we have focused mainly on the enhancement of the range of applied examples. These include topics on the discrete cosine transform, image processing, applications in music technology, communications engineering and frequency modulation.

Style

The style of the book is to develop and illustrate mathematical concepts through examples. We have tried throughout to adopt an informal approach and to describe mathematical processes using everyday language. Mathematical ideas are often developed by examples rather than by using abstract proof, which has been kept to a minimum. This reflects the authors' experience that engineering students learn better from practical examples, rather than from formal abstract development. We have included many engineering examples and have tried to make them as free-standing as possible to keep the necessary engineering prerequisites to a minimum. The engineering examples, which have been carefully selected to be relevant, informative and modern, range from short illustrative examples through to complete sections which can be regarded as case studies. A further benefit is the development of the link between mathematics and the physical world. An appreciation of this link is essential if engineers are to take full advantage of engineering mathematics. The engineering examples make the book more colourful and, more importantly, they help develop the ability to see an engineering problem and translate it into a mathematical form so that a solution can be obtained. This is one of the most difficult skills that an engineer needs to acquire. The ability to manipulate mathematical equations is by itself insufficient. It is sometimes necessary to derive the equations corresponding to an engineering problem. Interpretation of mathematical solutions in terms of the physical variables is also essential. Engineers cannot afford to get lost in mathematical symbolism.

Format

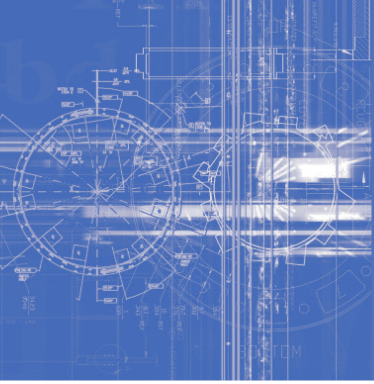
Important results are highlighted for easy reference. Exercises and solutions are provided at the end of most sections; it is essential to attempt these as the only way to develop competence and understanding is through practice. A further set of review exercises is provided at the end of each chapter. In addition some sections include exercises that are intended to be carried out on a computer using a technical computing language such as MATLAB®, GNU Octave, Mathematica or Python®. The MATLAB® command syntax is supported in several software packages as well as MATLAB® itself and will be used throughout the book.

Supplements

A comprehensive Solutions Manual is obtainable free of charge to lecturers using this textbook. It is also available for download via the web at *www.pearsoned.co.uk/croft*.

Finally we hope you will come to share our enthusiasm for engineering mathematics and enjoy the book.

Anthony Croft
Robert Davison
Martin Hargreaves
James Flint
March 2017



Acknowledgements

We are grateful to the following for permission to reproduce copyright material:

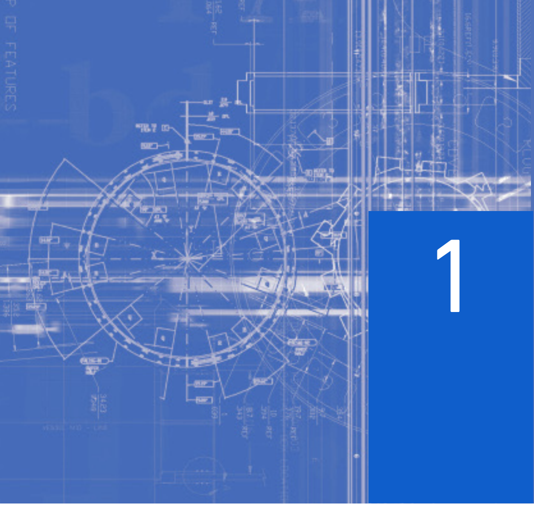
Tables

Table 29.7 from *Biometrika Tables for Statisticians*, Vol. 1, New York: Holt, Rinehart & Winston (Hays, W.L. and Winkler, R.L. 1970) Table 1, © Cambridge University Press.

Text

General Displayed Text on page xviii from https://www.mathworks.com/products/matlab.html, MATLAB® is a registered trademark of The MathWorks, Inc.; General Displayed Text xviii from Mathematica, https://www.wolfram.com/mathematica/, © Wolfram; General Displayed Text xviii from https://www.python.org/, Python® and the Python logos are trademarks or registered trademarks of the Python Software Foundation, used by Pearson Education Ltd with permission from the Foundation; General Displayed Text on page 291 from http://www.blu-raydisc.com/en/, Blu-ray Disc™ is a trademark owned by Blu-ray Disc Association (BDA); General Displayed Text on page 291 from http://wimaxforum.org/home, WiMAX® is a registered trademarks of the WiMAX Forum. This work is produced by Pearson Education and is not endorsed by any trademark owner referenced in this publication.

This page intentionally left blank



1

Review of algebraic techniques

Contents

1.1 INTRODUCTION

This chapter introduces some algebraic techniques which are commonly used in engineering mathematics. For some readers this may be revision. Section 1.2 examines the laws of indices. These laws are used throughout engineering mathematics. Section 1.3 looks at number bases. Section 1.4 looks at methods of solving polynomial equations. Section 1.5 examines algebraic fractions, while Section 1.6 examines the solution of inequalities. Section 1.7 looks at partial fractions. The chapter closes with a study of summation notation.

Computers are used extensively in all engineering disciplines to perform calculations. Some of the examples provided in this book make use of the technical computing language **MATLAB®**, which is commonly used in both an academic and industrial setting.

Because MATLAB® and many other similar languages are designed to compute not just with single numbers but with entire sequences of numbers at the same time, data is entered in the form of **arrays**. These are multi-dimensional objects. Two particular types of array are **vectors** and **matrices** which are studied in detail in Chapters 7 and 8.

Apart from being able to perform basic mathematical operations with vectors and matrices, MATLAB® has, in addition, a vast range of built-in computational functions which are straightforward to use but nevertheless are very powerful. Many of these high-level functions are accessible by passing data to them in the form of vectors and matrices. A small number of these special functions are used and explained in this text. However, to get the most out of a technical computing language it is necessary to develop a good understanding of what the software can do and to make regular reference to the manual.

1.2 LAWS OF INDICES

Consider the product $6 \times 6 \times 6 \times 6 \times 6$. This may be written more compactly as 6^5. We call 5 the **index** or **power**. The **base** is 6. Similarly, $y \times y \times y \times y$ may be written as y^4. Here the base is y and the index is 4.

Example 1.1 Write the following using index notation:

(a) $(-2)(-2)(-2)$ (b) $4.4.4.5.5$ (c) $\dfrac{yyy}{xxxx}$ (d) $\dfrac{aa(-a)(-a)}{bb(-b)}$

Solution (a) $(-2)(-2)(-2)$ may be written as $(-2)^3$.

(b) $4.4.4.5.5$ may be written as $4^3 5^2$.

(c) $\dfrac{yyy}{xxxx}$ may be written as $\dfrac{y^3}{x^4}$.

(d) Note that $(-a)(-a) = aa$ since the product of two negative quantities is positive. So $aa(-a)(-a) = aaaa = a^4$. Also $bb(-b) = -bbb = -b^3$. Hence

$$\frac{aa(-a)(-a)}{bb(-b)} = \frac{a^4}{-b^3} = -\frac{a^4}{b^3}$$

Example 1.2 Evaluate

(a) 7^3 (b) $(-3)^3$ (c) $2^3(-3)^4$

Solution (a) $7^3 = 7.7.7 = 343$

(b) $(-3)^3 = (-3)(-3)(-3) = -27$

(c) $2^3(-3)^4 = 8(81) = 648$

Most scientific calculators have an $\boldsymbol{x^y}$ button to enable easy calculation of expressions of a similar form to those in Example 1.2.

1.2.1 Multiplying expressions involving indices

Consider the product $(6^2)(6^3)$. We may write this as

$$(6^2)(6^3) = (6.6)(6.6.6) = 6^5$$

So

$$6^2 6^3 = 6^5$$

This illustrates the **first law of indices** which is

$$a^m a^n = a^{m+n}$$

When expressions with the same base are multiplied, the indices are added.

Example 1.3 Simplify each of the following expressions:

(a) $3^9 3^{10}$ (b) $4^3 4^4 4^6$ (c) $x^3 x^6$ (d) $y^4 y^2 y^3$

Solution

(a) $3^9 3^{10} = 3^{9+10} = 3^{19}$

(b) $4^3 4^4 4^6 = 4^{3+4+6} = 4^{13}$

(c) $x^3 x^6 = x^{3+6} = x^9$

(d) $y^4 y^2 y^3 = y^{4+2+3} = y^9$

Engineering application 1.1

Power dissipation in a resistor

The **resistor** is one of the three fundamental electronic components. The other two are the capacitor and the inductor, which we will meet later. The role of the resistor is to reduce the current flow within the branch of a circuit for a given voltage. As current flows through the resistor, electrical energy is converted into heat. Because the energy is lost from the circuit and is effectively wasted, it is termed **dissipated** energy. The rate of energy dissipation is known as the power, P, and is given by

$$P = I^2 R \qquad (1.1)$$

where I is the current flowing through the resistor and R is the resistance value. Note that the current is raised to the power 2. Note that power, P, is measured in watts; current, I, is measured in amps; and resistance, R, is measured in ohms.

There is an alternative formula for power dissipation in a resistor that uses the voltage, V, across the resistor. To obtain this alternative formula we need to use **Ohm's law**, which states that the voltage across a resistor, V, and the current passing through it, are related by the formula

$$V = IR \qquad (1.2)$$

From Equation (1.2) we see that

$$I = \frac{V}{R} \qquad (1.3)$$

Combining Equations (1.1) and (1.3) gives

$$P = \left(\frac{V}{R}\right)^2 R = \frac{V}{R} \cdot \frac{V}{R} \cdot R = \frac{V^2}{R}$$

→

Note that in this formula for P, the voltage is raised to the power 2. Note an important consequence of this formula is that doubling the voltage, while keeping the resistance fixed, results in the power dissipation increasing by a factor of 4, that is 2^2. Also trebling the voltage, for a fixed value of resistance, results in the power dissipation increasing by a factor of 9, that is 3^2.

Similar considerations can be applied to Equation 1.1. For a fixed value of resistance, doubling the current results in the power dissipation increasing by a factor of 4, and trebling the current results in the power dissipation increasing by a factor of 9.

Consider the product $3(3^3)$. Now

$$3(3^3) = 3(3.3.3) = 3^4$$

Also, using the first law of indices we see that $3^1 3^3 = 3^4$. This suggests that 3 is the same as 3^1. This illustrates the general rule:

$$\boxed{a = a^1}$$

Raising a number to the power 1 leaves the number unchanged.

Example 1.4 Simplify (a) $5^6 5$ (b) $x^3 x x^2$

Solution (a) $5^6 5 = 5^{6+1} = 5^7$ (b) $x^3 x x^2 = x^{3+1+2} = x^6$

1.2.2 Dividing expressions involving indices

Consider the expression $\dfrac{4^5}{4^3}$:

$$\begin{aligned} \frac{4^5}{4^3} &= \frac{4.4.4.4.4}{4.4.4} \\ &= 4.4 \qquad \text{by cancelling 4s} \\ &= 4^2 \end{aligned}$$

This serves to illustrate the **second law of indices** which is

$$\boxed{\frac{a^m}{a^n} = a^{m-n}}$$

When expressions with the same base are divided, the indices are subtracted.

Example 1.5 Simplify

(a) $\dfrac{5^9}{5^7}$ (b) $\dfrac{(-2)^{16}}{(-2)^{13}}$ (c) $\dfrac{x^9}{x^5}$ (d) $\dfrac{y^6}{y}$

Solution (a) $\dfrac{5^9}{5^7} = 5^{9-7} = 5^2$

(b) $\dfrac{(-2)^{16}}{(-2)^{13}} = (-2)^{16-13} = (-2)^3$

(c) $\dfrac{x^9}{x^5} = x^{9-5} = x^4$

(d) $\dfrac{y^6}{y} = y^{6-1} = y^5$

Consider the expression $\dfrac{2^3}{2^3}$. Using the second law of indices we may write

$$\frac{2^3}{2^3} = 2^{3-3} = 2^0$$

But, clearly, $\dfrac{2^3}{2^3} = 1$, and so $2^0 = 1$. This illustrates the general rule:

$$a^0 = 1$$

Any expression raised to the power 0 is 1.

1.2.3 Negative indices

Consider the expression $\dfrac{4^3}{4^5}$. We can write this as

$$\frac{4^3}{4^5} = \frac{4.4.4}{4.4.4.4.4} = \frac{1}{4.4} = \frac{1}{4^2}$$

Alternatively, using the second law of indices we have

$$\frac{4^3}{4^5} = 4^{3-5} = 4^{-2}$$

So we see that

$$4^{-2} = \frac{1}{4^2}$$

Thus we are able to interpret negative indices. The sign of an index changes when the expression is inverted. In general we can state

$$a^{-m} = \frac{1}{a^m} \qquad a^m = \frac{1}{a^{-m}}$$

Example 1.6 Evaluate the following:

(a) 3^{-2} (b) $\dfrac{2}{4^{-3}}$ (c) 3^{-1} (d) $(-3)^{-2}$ (e) $\dfrac{6^{-3}}{6^{-2}}$

Solution (a) $3^{-2} = \dfrac{1}{3^2} = \dfrac{1}{9}$

(b) $\dfrac{2}{4^{-3}} = 2(4^3) = 2(64) = 128$

(c) $3^{-1} = \dfrac{1}{3^1} = \dfrac{1}{3}$

(d) $(-3)^{-2} = \dfrac{1}{(-3)^2} = \dfrac{1}{9}$

(e) $\dfrac{6^{-3}}{6^{-2}} = 6^{-3-(-2)} = 6^{-1} = \dfrac{1}{6^1} = \dfrac{1}{6}$

Example 1.7 Write the following expressions using only positive indices:

(a) x^{-4} (b) $3x^{-4}$ (c) $\frac{x^{-2}}{y^{-2}}$ (d) $3x^{-2}y^{-3}$

Solution (a) $x^{-4} = \frac{1}{x^4}$

(b) $3x^{-4} = \frac{3}{x^4}$

(c) $\frac{x^{-2}}{y^{-2}} = x^{-2}y^2 = \frac{y^2}{x^2}$

(d) $3x^{-2}y^{-3} = \frac{3}{x^2y^3}$

Engineering application 1.2

Power density of a signal transmitted by a radio antenna

A **radio antenna** is a device that is used to convert electrical energy into electromagnetic radiation, which is then transmitted to distant points.

An ideal theoretical point source radio antenna which radiates the same power in all directions is termed an **isotropic** antenna. When it transmits a radio wave, the wave spreads out equally in all directions, providing there are no obstacles to block the expansion of the wave. The power generated by the antenna is uniformly distributed on the surface of an expanding sphere of area, A, given by

$$A = 4\pi r^2$$

where r is the distance from the generating antenna to the wave front.

The **power density**, S, provides an indication of how much of the signal can potentially be received by another antenna placed at a distance r. The actual power received depends on the effective area or aperture of the antenna, which is usually expressed in units of m^2.

Electromagnetic field exposure limits for humans are sometimes specified in terms of a power density. The closer a person is to the transmitter, the higher the power density will be. So a safe distance needs to be determined.

The power density is the ratio of the power transmitted, P_t, to the area over which it is spread

$$S = \frac{\text{power transmitted}}{\text{area}} = \frac{P_\mathrm{t}}{4\pi r^2} = \frac{P_\mathrm{t}}{4\pi} r^{-2} \quad \mathrm{W\ m^{-2}}$$

Note that r in this equation has a **negative index**. This type of relationship is known as an **inverse square law** and is found commonly in science and engineering.

Note that if the distance, r, is doubled, then the area, A, increases by a factor of 4 (i.e. 2^2). If the distance is trebled, the area increases by a factor of 9 (i.e. 3^2) and so on. This means that as the distance from the antenna doubles, the power density, S, decreases to a quarter of its previous value; if the distance trebles then the power density is only a ninth of its previous value.

1.2.4 Multiple indices

Consider the expression $(4^3)^2$. This may be written as

$$(4^3)^2 = 4^3 . 4^3 = 4^{3+3} = 4^6$$

This illustrates the **third law of indices** which is

$$\boxed{(a^m)^n = a^{mn}}$$

Note that the indices m and n have been multiplied.

Example 1.8 Write the following expressions using a single index:

(a) $(3^2)^4$ (b) $(7^{-2})^3$ (c) $(x^2)^{-3}$ (d) $(x^{-2})^{-3}$

Solution (a) $(3^2)^4 = 3^{2\times 4} = 3^8$

(b) $(7^{-2})^3 = 7^{-2\times 3} = 7^{-6}$

(c) $(x^2)^{-3} = x^{2\times(-3)} = x^{-6}$

(d) $(x^{-2})^{-3} = x^{-2\times -3} = x^6$

Consider the expression $(2^4 5^2)^3$. We see that

$$\begin{aligned}(2^4 5^2)^3 &= (2^4 5^2)(2^4 5^2)(2^4 5^2)\\ &= 2^4 2^4 2^4 5^2 5^2 5^2\\ &= 2^{12} 5^6\end{aligned}$$

This illustrates a generalization of the third law of indices which is

$$\boxed{(a^m b^n)^k = a^{mk} b^{nk}}$$

Example 1.9 Remove the brackets from

(a) $(2x^2)^3$ (b) $(-3y^4)^2$ (c) $(x^{-2}y)^3$

Solution (a) $(2x^2)^3 = (2^1 x^2)^3 = 2^3 x^6 = 8x^6$

(b) $(-3y^4)^2 = (-3)^2 y^8 = 9y^8$

(c) $(x^{-2}y)^3 = x^{-6}y^3$

Engineering application 1.3

Radar scattering

It has already been shown in Engineering application 1.2 that the power density of an isotropic transmitter of radio waves is

$$S = \frac{P_t}{4\pi} r^{-2} \text{ W m}^{-2}$$

→

It is possible to use radio waves to detect distant objects. The technique involves transmitting a radio signal, which is then reflected back when it strikes a target. This weak reflected signal is then picked up by a receiving antenna, thus allowing a number of properties of the target to be deduced, such as its angular position and distance from the transmitter. This system is known as **radar**, which was originally an acronym standing for **RA**dio **D**etection **A**nd **R**anging.

When the wave hits the target it produces a quantity of reflected power. The power depends upon the object's **radar cross-section** (RCS), normally denoted by the Greek lower case letter sigma, σ, and having units of m^2. The power reflected at the object, P_r, is given by

$$P_\mathrm{r} = S\sigma = \frac{P_\mathrm{t}\sigma}{4\pi} r^{-2} \text{ W}$$

Some military aircraft use special techniques to minimize the RCS in order to reduce the amount of power they reflect and hence minimize the chance of being detected.

If the reflected power at the target is assumed to spread spherically, when it returns to the transmitter position it will have the power density, S_r, given by

$$S_\mathrm{r} = \frac{\text{power reflected at target}}{\text{area}} = \frac{P_\mathrm{r}}{4\pi} r^{-2} \text{ W m}^{-2}$$

Substituting for the reflected power, P_r, gives

$$S_\mathrm{r} = \frac{\text{power reflected at target}}{\text{area}} = \frac{\left(\dfrac{P_\mathrm{t}\sigma}{4\pi} r^{-2}\right)}{4\pi} r^{-2} = \frac{P_\mathrm{t}\sigma}{4\pi \times 4\pi} (r^{-2})^2$$

$$= \frac{P_\mathrm{t}\sigma}{(4\pi)^2} r^{-4} \text{ W m}^{-2}$$

Note that the product of the two r^{-2} terms has been calculated using the third law of indices.

This example illustrates one of the main challenges with radar design which is that the power density returned by a distant object is very much smaller than the transmitted power, even for targets with a large RCS. For theoretical isotropic antennas, the received power density depends upon the factor r^{-4}. This factor diminishes rapidly for large values of r, that is, as the object being detected gets further away.

In practice, the transmit antennas used are not isotropic but **directive** and often scan the area of interest. They also make use of receive antennas with a large effective area which can produce a viable signal from the small reflected power densities.

1.2.5 Fractional indices

The third law of indices states that $(a^m)^n = a^{mn}$. If we take $a = 2$, $m = \frac{1}{2}$ and $n = 2$ we obtain

$$(2^{1/2})^2 = 2^1 = 2$$

So when $2^{1/2}$ is squared, the result is 2. Thus, $2^{1/2}$ is a square root of 2. Each positive number has two square roots and so

$$2^{1/2} = \sqrt{2} = \pm 1.4142\ldots$$

Similarly

$$(2^{1/3})^3 = 2^1 = 2$$

so that $2^{1/3}$ is a cube root of 2:

$$2^{1/3} = \sqrt[3]{2} = 1.2599\ldots$$

In general $2^{1/n}$ is an nth root of 2. The general law states

$$x^{1/n} \text{ is an } n\text{th root of } x$$

Example 1.10 Write the following using a single positive index:

(a) $(3^{-2})^{1/4}$ (b) $x^{2/3}x^{5/3}$ (c) $yy^{-2/5}$ (d) $\sqrt{k^3}$

Solution

(a) $(3^{-2})^{1/4} = 3^{-2\times\frac{1}{4}} = 3^{-1/2} = \dfrac{1}{3^{1/2}}$

(b) $x^{2/3}x^{5/3} = x^{2/3+5/3} = x^{7/3}$

(c) $yy^{-2/5} = y^1y^{-2/5} = y^{1-2/5} = y^{3/5}$

(d) $\sqrt{k^3} = (k^3)^{1/2} = k^{3\times\frac{1}{2}} = k^{3/2}$

Example 1.11 Evaluate

(a) $8^{1/3}$ (b) $8^{2/3}$ (c) $8^{-1/3}$ (d) $8^{-2/3}$ (e) $8^{4/3}$

Solution We note that 8 may be written as 2^3.

(a) $8^{1/3} = (2^3)^{1/3} = 2^1 = 2$

(b) $8^{2/3} = (8^{1/3})^2 = 2^2 = 4$

(c) $8^{-1/3} = \dfrac{1}{8^{1/3}} = \dfrac{1}{2}$

(d) $8^{-2/3} = \dfrac{1}{8^{2/3}} = \dfrac{1}{4}$

(e) $8^{4/3} = (8^{1/3})^4 = 2^4 = 16$

Engineering application 1.4

Skin depth in a radial conductor

When an alternating current signal travels along a conductor, such as a copper wire, most of the current is found near the surface of the conductor. Nearer to the centre of the conductor, the current diminishes. The depth of penetration of the signal, termed the **skin depth**, into the conductor depends on the frequency of the signal. Skin depth, illustrated in Figure 1.1, is defined as the depth at which the current

→

density has decayed to approximately 37% of that at the edge. Skin depth is important because it affects the resistance of wires and other conductors: the smaller the skin depth, the higher the effective resistance and the greater the loss due to heating.

At low frequencies, such as those found in the domestic mains supply, the skin depth is so large that often it can be neglected; however, in very large-diameter conductors and smaller conductors at microwave frequencies it becomes important and has to be taken into account.

The skin depth, δ, is given by

$$\delta = \left(\frac{2}{\omega\mu\sigma}\right)^{1/2}$$

where μ is a material constant known as the **permeability** of the conductor, ω is the angular frequency of the signal and σ is the conductivity of the conductor.

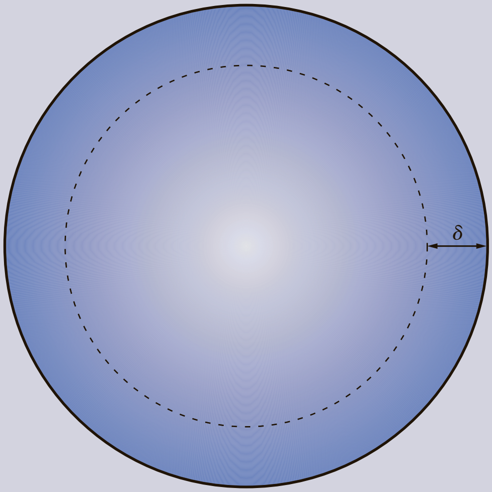


Figure 1.1
Cross-section of a radial conductor illustrating a skin depth δ.

EXERCISES 1.2

1 Evaluate

(a) 2^3 (b) 3^2 (c) $\frac{5^{13}}{5^{12}}$

(d) $\frac{19^{-11}}{19^{-13}}$ (e) $(2^{1/4})^8$ (f) $(-4)^{-2}$

(g) $4^{-1/2}$ (h) $(9^{1/3})^{3/2}$ (i) $\sqrt{32}\sqrt{2}$

(j) $\sqrt{0.01}$ (k) $81^{3/4}$

2 Use a scientific calculator to evaluate

(a) $10^{1.2}$ (b) $6^{-0.7}$ (c) $6^{2.5}$

(d) $(3^{-1}4^2)^{0.8}$

3 Express each of the following expressions using a single positive index:

(a) x^4x^7 (b) $x^2(-x)$

(c) $\frac{x^2}{x}$ (d) $\frac{x^{-2}}{x^{-1}}$

(e) $(x^{-2})^4$ (f) $(x^{-2.5}x^{-3.5})^2$

4 Simplify as much as possible

(a) $\frac{x^{1/2}}{x^{1/3}}$ (b) $(16x^4)^{0.25}$

(c) $\left(\frac{27}{y^3}\right)^{1/3}$ (d) $\frac{2xy^2}{(2xy)^2}$

(e) $\sqrt{a^2b^6c^4}$ (f) $(64t^3)^{2/3}$

Solutions

1 (a) 8 (b) 9 (c) 5 (d) 361
(e) 4 (f) $\frac{1}{16}$ (g) $\frac{1}{2}$ (h) 3
(i) 8 (j) 0.1 (k) 27

2 (a) 15.8489 (b) 0.2853
(c) 88.1816 (d) 3.8159

3 (a) x^{11} (b) $-x^3$ (c) x
(d) $\frac{1}{x}$ (e) $\frac{1}{x^8}$ (f) $\frac{1}{x^{12}}$

4 (a) $x^{1/6}$ (b) $2x$ (c) $\frac{3}{y}$
(d) $\frac{1}{2x}$ (e) ab^3c^2 (f) $16t^2$

1.3 NUMBER BASES

The **decimal** system of numbers in common use is based on the 10 digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9. However, other number systems have important applications in computer science and electronic engineering. In this section we remind the reader of what is meant by a number in the decimal system, and show how we can use powers or indices with bases of 2 and 16 to represent numbers in the **binary** and **hexadecimal** systems respectively. We follow this by an explanation of an alternative binary representation of a number known as **binary coded decimal**.

1.3.1 The decimal system

The numbers that we commonly use in everyday life are based on 10. For example, 253 can be written as

$$\begin{aligned} 253 &= 200 + 50 + 3 \\ &= 2(100) + 5(10) + 3(1) \\ &= 2(10^2) + 5(10^1) + 3(10^0) \end{aligned}$$

In this form it is clear why we refer to this as a ‘base 10’ number. When we use 10 as a base we say we are writing in the **decimal system**. Note that in the decimal system there are 10 digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. You may recall the phrase ‘hundreds, tens and units’ and as we have seen these are simply powers of 10. To avoid possible confusion with numbers using other bases, we denote numbers in base 10 with a small subscript, for example, 5192_{10}:

$$\begin{aligned} 5192_{10} &= 5000 + 100 + 90 + 2 \\ &= 5(1000) + 1(100) + 9(10) + 2(1) \\ &= 5(10^3) + 1(10^2) + 9(10^1) + 2(10^0) \end{aligned}$$

Note that, in the previous line, as we move from right to left, the powers of 10 increase.

1.3.2 The binary system

A **binary system** uses the number 2 for its base. A binary system has only two digits, 0 and 1, and these are called **binary digits** or simply **bits**. Binary numbers are based on powers of 2. In a computer, binary numbers are usually stored in groups of 8 bits which we call a **byte**.

Converting from binary to decimal

Consider the binary number 110101_2. As the base is 2 this means that as we move from right to left the position of each digit represents an increasing power of 2 as follows:

$$\begin{aligned}110101_2 &= 1(2^5) + 1(2^4) + 0(2^3) + 1(2^2) + 0(2^1) + 1(2^0)\\ &= 1(32) + 1(16) + 0(8) + 1(4) + 0(2) + 1(1)\\ &= 32 + 16 + 4 + 1\\ &= 53_{10}\end{aligned}$$

Hence 110101_2 and 53_{10} are equivalent.

Example 1.12 Convert the following to decimal: (a) 1111_2 (b) 101010_2

Solution (a)

$$\begin{aligned}1111_2 &= 1(2^3) + 1(2^2) + 1(2^1) + 1(2^0)\\ &= 1(8) + 1(4) + 1(2) + 1(1)\\ &= 8 + 4 + 2 + 1\\ &= 15_{10}\end{aligned}$$

(b)

$$\begin{aligned}101010_2 &= 1(2^5) + 0(2^4) + 1(2^3) + 0(2^2) + 1(2^1) + 0(2^0)\\ &= 1(32) + 0 + 1(8) + 0 + 1(2) + 0\\ &= 32 + 8 + 2\\ &= 42_{10}\end{aligned}$$

Converting decimal to binary

We now look at some examples of converting numbers in base 10 to numbers in base 2, that is from decimal to binary. We make use of Table 1.1, which shows various powers of 2, when converting from decimal to binary. Table 1.1 may be extended as necessary.

Table 1.1
Powers of 2.

2^0	1	2^4	16	2^8	256
2^1	2	2^5	32	2^9	512
2^2	4	2^6	64	2^{10}	1024
2^3	8	2^7	128	2^{11}	2048

Example 1.13 Convert 83_{10} to a binary number.

Solution We need to express 83_{10} as the sum of a set of numbers, each of which is a power of 2. From Table 1.1 we see that 64 is the highest number in the table that does not exceed the given number of 83. We write

$$83 = 64 + 19$$

We now focus on the 19. From Table 1.1, 16 is the highest number that does not exceed 19. So we write

$$19 = 16 + 3$$

giving

$$83 = 64 + 16 + 3$$

We now focus on the 3 and again using Table 1.1 we may write

$$\begin{aligned}83 &= 64 + 16 + 2 + 1\\ &= 2^6 + 2^4 + 2^1 + 2^0\\ &= 1(2^6) + 0(2^5) + 1(2^4) + 0(2^3) + 0(2^2) + 1(2^1) + 1(2^0)\\ &= 1010011_2\end{aligned}$$

Example 1.14 Express 200_{10} as a binary number.

Solution From Table 1.1 we note that 128 is the highest number that does not exceed 200 so we write

$$200 = 128 + 72$$

Using Table 1.1 repeatedly we may write

$$\begin{aligned}200 &= 128 + 72\\ &= 128 + 64 + 8\\ &= 2^7 + 2^6 + 2^3\\ &= 1(2^7) + 1(2^6) + 0(2^5) + 0(2^4) + 1(2^3) + 0(2^2) + 0(2^1) + 0(2^0)\\ &= 11001000_2\end{aligned}$$

Another way to convert decimal numbers to binary numbers is to divide by 2 repeatedly and note the remainder. We rework the previous two examples using this method.

Example 1.15 Convert the following decimal numbers to binary: (a) 83 (b) 200

Solution (a) We divide by 2 repeatedly and note the remainder.

	Remainder
$83 \div 2 = 41$ r 1	1
$41 \div 2 = 20$ r 1	1
$20 \div 2 = 10$ r 0	0
$10 \div 2 = 5$ r 0	0
$5 \div 2 = 2$ r 1	1
$2 \div 2 = 1$ r 0	0
$1 \div 2 = 0$ r 1	1

To obtain the binary number we write out the remainder, working from the bottom one to the top one. This gives

$$83_{10} = 1010011_2$$

as before.

(b) We repeat the process by repeatedly dividing 200 by 2 and noting the remainder.

	Remainder
$200 \div 2 = 100 \text{ r } 0$	0
$100 \div 2 = 50 \text{ r } 0$	0
$50 \div 2 = 25 \text{ r } 0$	0
$25 \div 2 = 12 \text{ r } 1$	1
$12 \div 2 = 6 \text{ r } 0$	0
$6 \div 2 = 3 \text{ r } 0$	0
$3 \div 2 = 1 \text{ r } 1$	1
$1 \div 2 = 0 \text{ r } 1$	1

Reading the remainder column from the bottom to the top gives the required binary number:

$$200_{10} = 11001000_2$$

1.3.3 Hexadecimal system

We now consider the number system which uses 16 as a base. This system is termed **hexadecimal** (or simply hex). There are 16 digits in the hexadecimal system: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, and F. Notice that conventional decimal digits are insufficient to represent hexadecimal numbers and so additional ‘digits’, A, B, C, D, E, and F, are included. Table 1.2 shows the equivalence between decimal and hexadecimal digits. Hexadecimal numbers are based on powers of 16.

Table 1.2
Hexadecimal numbers.

Decimal	*Hexadecimal*	*Decimal*	*Hexadecimal*
0	0	8	8
1	1	9	9
2	2	10	A
3	3	11	B
4	4	12	C
5	5	13	D
6	6	14	E
7	7	15	F

Converting from hexadecimal to decimal

The following example illustrates how to convert from hexadecimal to decimal. We use the fact that as we move from right to left, the position of each digit represents an increasing power of 16.

Example 1.16 Convert the following hexadecimal numbers to decimal numbers: (a) 93A (b) F9B3

Solution (a) Noting that hexadecimal numbers use base 16 we have

$$\begin{aligned}93\text{A}_{16} &= 9(16^2) + 3(16^1) + \text{A}(16^0)\\ &= 9(256) + 3(16) + 10(1)\\ &= 2362_{10}\end{aligned}$$

(b)

$$\begin{aligned}\text{F9B3}_{16} &= \text{F}(16^3) + 9(16^2) + \text{B}(16^1) + 3(16^0)\\ &= 15(4096) + 9(256) + 11(16) + 3(1)\\ &= 63\,923_{10}\end{aligned}$$

Converting from decimal to hexadecimal

Table 1.3 provides powers of 16 which help in the conversion from decimal to hexadecimal.

Table 1.3

16^0	1
16^1	16
16^2	256
16^3	4096
16^4	65 536

The following example illustrates how to convert from decimal to hexadecimal.

Example 1.17 Convert 14 397 to a hexadecimal number.

Solution We need to express 14397 as the sum of multiples of powers of 16. From Table 1.3 we see that the highest number that does not exceed 14397 is 4096. We express 14397 as a multiple of 4096 with an appropriate remainder. Dividing 14397 by 4096 we obtain 3 with a remainder of 2109. So we may write

$$14397 = 3(4096) + 2109$$

We now focus on 2109 and apply the same process as above. From Table 1.3 the highest number that does not exceed 2109 is 256:

$$2109 = 8(256) + 61$$

Finally, $61 = 3(16) + 13$. So we have

$$\begin{aligned}14\,397 &= 3(4096) + 8(256) + 3(16) + 13\\ &= 3(16^3) + 8(16^2) + 3(16^1) + 13(16^0)\end{aligned}$$

From Table 1.2 we see that 13_{10} is D in hexadecimal, so we have

$$14\,397_{10} = 383\text{D}_{16}$$

As with base 2 we can convert decimal numbers by repeated division and noting the remainder. The previous example is reworked to illustrate this.

Example 1.18 Convert 14 397 to hexadecimal.

Solution We divide repeatedly by 16, noting the remainder.

	Remainder
$14\,397 \div 16 = 899$ r 13	13
$899 \div 16 = 56$ r 3	3
$56 \div 16 = 3$ r 8	8
$3 \div 16 = 0$ r 3	3

Recall that 13 in hexadecimal is D. Reading up the Remainder column we have

$$14\,397_{10} = 383\text{D}_{16}$$

as before.

Electronic engineers need to be familiar with the decimal, binary and hexadecimal systems and be able to convert between them. The equivalent representations of the decimal numbers 0–15 are provided in Table 1.4.

Table 1.4

Decimal	*Binary*	*Hex*	*Decimal*	*Binary*	*Hex*
0	0000	0	8	1000	8
1	0001	1	9	1001	9
2	0010	2	10	1010	A
3	0011	3	11	1011	B
4	0100	4	12	1100	C
5	0101	5	13	1101	D
6	0110	6	14	1110	E
7	0111	7	15	1111	F

Converting from binary to hexadecimal

There is a straightforward way of converting a binary number into a hexadecimal number. The digits of the binary number are grouped into fours, or quartets, (from the right-hand side) and each quartet is converted to its hex equivalent using Table 1.4.

Example 1.19 Convert 1101011100111_2 into hexadecimal.

Solution Working from the right, the binary number is grouped into fours, with additional zeros being added as necessary to the final grouping.

$$0001\ 1010\ 1110\ 0111$$

Table 1.4 is used to express each group of four as its hex equivalent. For example, 0111 = 7_{16}, and continuing in this way we obtain

1AE7

Thus $110101110\,0111_2 = 1AE7_{16}$.

1.3.4 Binary coded decimal

We have seen in Section 1.3.2 that decimal numbers can be expressed in an equivalent binary form where the position of each binary digit, moving from the right to the left, represents an increasing power of 2. There is an alternative way of expressing numbers using the binary digits 1 and 0 that is often used in electronic engineering because for some applications it is more straightforward to build the necessary hardware. This system is called **binary coded decimal** (b.c.d.).

First of all, recall how the decimal digits 0, 1, 2, . . . , 9 are expressed in their usual binary form. Note that the largest decimal digit 9 is 1001 in binary, and so we need at most four digits to store the binary representations of 0, 1, . . . , 9. Expressing each decimal digit as a four-digit binary number we obtain Table 1.5.

Table 1.5
Decimal digits and their four-digit binary representations.

0	0000	5	0101
1	0001	6	0110
2	0010	7	0111
3	0011	8	1000
4	0100	9	1001

A four-digit binary number is referred to as a **nibble**. To express a multi-digit decimal number, such as 347, in b.c.d. each decimal digit in turn is converted into its binary representation as shown. Note that a nibble is used for each decimal digit.

$$\begin{array}{ccc} 3 & 4 & 7 \\ \downarrow & \downarrow & \downarrow \\ 0011 & 0100 & 0111 \end{array}$$

Recall from Section 1.3.2 that a byte is a group of 8 bits (binary digits). Computers usually store numbers in 8-bit bytes so there are two common ways of encoding b.c.d. The first is to use a whole byte for each nibble, with the first 4 bits always set to 0. So, for example, 347_{10} can be stored as

00000011 00000100 00000111

Alternatively, each byte can be used to store two nibbles, in which case 347_{10} would be stored as

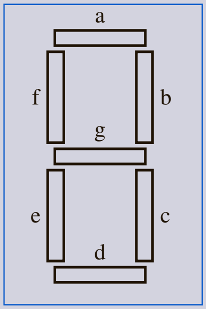
00000011 01000111

Rules have been developed for performing calculations in b.c.d. but these are beyond the scope of this book.

Engineering application 1.5

Seven-segment displays

The number displays found on music systems, video and other electronic equipment commonly employ one or more **seven-segment** indicators. A single seven-segment indicator is shown in Figure 1.2(a). The individual segments are typically illuminated with a **light-emitting diode** (LED) or similar optical device and are either on or off. The segments are illuminated according to the table shown in Figure 1.2(b), where 1 indicates that the segment is turned on and 0 indicates that it is turned off.



(a)

b.c.d. number	a	b	c	d	e	f	g
0000	1	1	1	1	1	1	0
0001	0	1	1	0	0	0	0
0010	1	1	0	1	1	0	1
0011	1	1	1	1	0	0	1
0100	0	1	1	0	0	1	1
0101	1	0	1	1	0	1	1
0110	1	0	1	1	1	1	1
0111	1	1	1	0	0	0	0
1000	1	1	1	1	1	1	1
1001	1	1	1	1	0	1	1

(b)

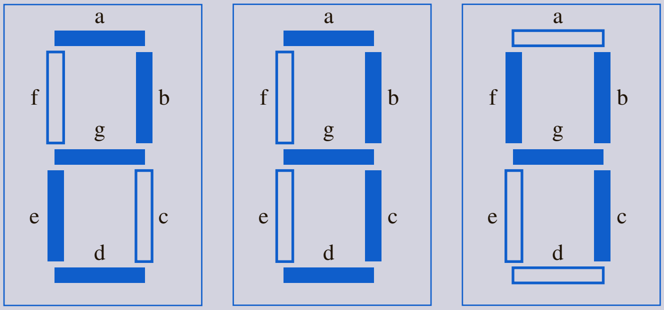
Figure 1.2
(a) Seven-segment LED display. (b) Seven-segment coding.

The numbers in the microprocessor system driving the display are typically stored in binary format, known as, **binary coded decimal** (b.c.d.). As an example we consider displaying binary number 11101010_2 as a decimal number on seven-segment displays. This represents the decimal number 234, which requires three seven-segment displays.

The microprocessor first divides the input number by 100 and in this case obtains the result 2 with a remainder of 34. This can be done directly on the binary number itself via a series of operations within the assembly language of the microprocessor without first converting to a decimal number. The result $2 = 0010_2$ is then decoded using Figure 1.2(b), giving the bit pattern 1101101 which is passed to the 'hundreds' display.

The remainder of 34 is then divided by 10 giving 3 with a final remainder of 4. The number $3 = 0011_2$ and so this can be outputted to the 'tens' display as the pattern 1111001. Finally, $4 = 0100_2$, which is passed to the display as the pattern 0110011.

The display shows



Notice that prior to decoding for display, by successive division by 100 and 10 the number has been converted into separate b.c.d. digits. Integrated circuits are available which convert b.c.d. directly into the bit patterns for display. Hence the output bit pattern of the microprocessor may be chosen to be b.c.d. In this case it has the advantage that fewer pins are required on the microprocessor to operate the display.

EXERCISES 1.3

1 Convert the following decimal numbers to binary numbers: (a) 19 (b) 36 (c) 100 (d) 796 (e) 5000

2 Convert the following binary numbers to decimal numbers: (a) 111 (b) 10101 (c) 111001 (d) 1110001 (e) 11111111

3 What is the highest decimal number that can be written in binary form using a maximum of (a) 2 binary digits (b) 3 binary digits (c) 4 binary digits (d) 5 binary digits? Can you spot a pattern? (e) Write a formula for the highest decimal number that can be written using N binary digits.

4 Write the decimal number 0.5 in binary.

5 Convert the following hexadecimal numbers to decimal numbers: (a) 91 (b) 6C (c) A1B (d) F9D4 (e) ABCD

6 Convert the following decimal numbers to hexadecimal numbers: (a) 160 (b) 396 (c) 5010 (d) 25 000 (e) 1 000 000

7 Calculate the highest decimal number that can be represented by a hexadecimal number with (a) 1 digit (b) 2 digits (c) 3 digits (d) 4 digits (e) N digits

8 Express the decimal number 375 as both a pure binary number and a number in b.c.d.

9 Convert (a) 1111111_2 (b) 101010111_2 into hexadecimal.

Solutions

1 (a) $19_{10} = 10011_2$ (b) 100100 (c) 1100100 (d) 1100011100 (e) 1001110001000

2 (a) $111_2 = 7$ (b) 21 (c) 57 (d)113 (e) 255

3 (a) 3 (b) 7 (c) 15 (d) 31 (e) $2^N - 1$

4 The binary system is based on powers of 2. The examples in the text can be extended to the case of negative powers of 2 just as in the decimal system numbers after the decimal place represent negative powers of 10. So, for example, the binary number 11.101_2 is converted to decimal as follows:

$$\begin{aligned} 11.101_2 &= 1\times 2^1 + 1\times 2^0 + 1\times 2^{-1} \\ &\quad + 0\times 2^{-2} + 1\times 2^{-3} \\ &= 2 + 1 + \frac{1}{2} + \frac{1}{8} \\ &= 3\frac{5}{8} \end{aligned}$$

In the same way the binary equivalent of the decimal number 0.5 is 0.1.

5 (a) $91_{16} = 145_{10}$ (b) $6\text{C} = 108$ (c) 2587 (d) 63 956 (e) 43 981

6 (a) $160_{10} = \text{A0}$ (b) 18C (c) 1392 (d) 61A8 (e) F4240

7 (a) 15 (b) 255 (c) 4095 (d) 65 535 (e) $16^N - 1$

8 (a) 101110111_2 (b) $0011\,0111\,0101_{\text{bcd}}$

9 (a) 7F (b) 157

1.4 POLYNOMIAL EQUATIONS

A **polynomial equation** has the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0 = 0 \qquad (1.4)$$

where n is a positive whole number, $a_n, a_{n-1}, \ldots, a_0$ are constants and x is a variable. The constants $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ are called the **coefficients** of the polynomial.

The **roots** of an equation are those values of x which satisfy $P(x) = 0$. So if $x = x_1$ is a root then $P(x_1) = 0$.

Examples of polynomial equations are

$$7x^2 + 4x - 1 = 0 \qquad (1.5)$$

$$2x - 3 = 0 \qquad (1.6)$$

$$x^3 - 20 = 0 \qquad (1.7)$$

The **degree** of an equation is the value of the highest power. Equation (1.5) has degree 2, Equation (1.6) has degree 1 and Equation (1.7) has degree 3. A polynomial equation of degree n has n roots.

There are some special names for polynomial equations of low degree (see Table 1.6).

Table 1.6

Equation	*Degree*	*Name*
$ax + b = 0$	1	Linear
$ax^2 + bx + c = 0$	2	Quadratic
$ax^3 + bx^2 + cx + d = 0$	3	Cubic
$ax^4 + bx^3 + cx^2 + dx + e = 0$	4	Quartic

1.4.1 Quadratic equations

We now focus attention on quadratic equations. The standard form of a quadratic equation is $ax^2 + bx + c = 0$. We look at three methods of solving quadratic equations:

(1) factorization,

(2) use of a formula,

(3) completing the square.

Example 1.20 illustrates solution by factorization.

Example 1.20 Solve

$$6x^2 + 11x - 10 = 0$$

Solution The left-hand side (l.h.s.) is factorized:

$$(3x - 2)(2x + 5) = 0$$

So either

$$3x - 2 = 0 \qquad \text{or} \qquad 2x + 5 = 0$$

Hence

$$x = \frac{2}{3}, -\frac{5}{2}$$

When roots cannot be found by factorization we can make use of a formula.

> The formula for finding the roots of $ax^2 + bx + c = 0$ is
>
> $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 1.21 Use the quadratic formula to solve

$$3x^2 - x - 6 = 0$$

Solution Comparing $3x^2 - x - 6$ with $ax^2 + bx + c$ we see that $a = 3$, $b = -1$ and $c = -6$. So

$$\begin{aligned} x &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(3)(-6)}}{2(3)} \\ &= \frac{1 \pm \sqrt{73}}{6} \\ &= -1.2573, 1.5907 \end{aligned}$$

Engineering application 1.6

Current used by an electric vehicle

Personal transport systems that make use of electrical power are becoming increasingly common. One of the factors behind this change is that their use can reduce roadside pollution in an urban environment. Electrical vehicles have also become the base for self-driving cars when combined with electrical control and navigation systems.

The motor in an operational electric vehicle has to do work to overcome wind, inertia, friction, road resistance and in order to climb inclines. The energy supply in the form of electrical power comes from the on-board battery pack. Due to its internal construction the battery pack has a total internal resistance, R, which serves to reduce the power available to the motor.

→

A simplified circuit diagram of a vehicle is shown in Figure 1.3.

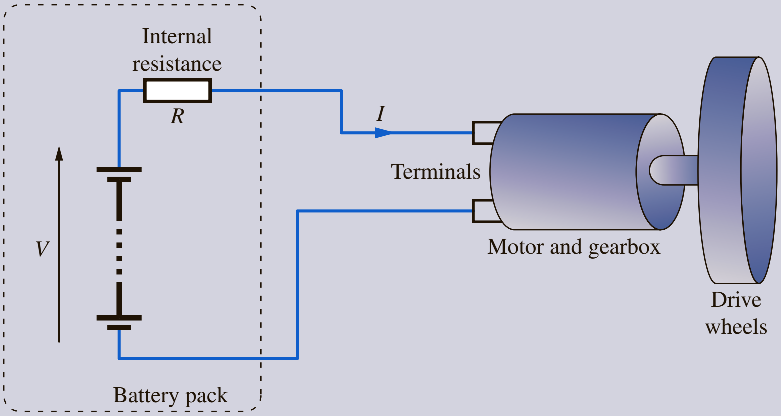


Figure 1.3
Electric vehicle wiring diagram.

The total power delivered by the battery pack is

$$\text{power} = \text{voltage} \times \text{current} = VI$$

This is shared between loss due to the internal resistance and the power, P, to the motor. The power loss due to the internal resistance is I^2R (see Engineering application 1.1). So the equation for the power in the circuit is

$$VI = I^2R + P$$

This can be rewritten into the form of a quadratic equation

$$RI^2 - VI + P = 0$$

which can be solved to calculate the current in the wire for a particular power delivered to the motor. It is important to know the current in order to specify the size of the fuses, the motor controller and the wire diameters used in the vehicle.

Consider the case where the power output is 2 kW. If the circuit parameters are $V = 150$ volts, $R = 1.6\ \Omega$, we have

$$1.6I^2 - 150I + 2000 = 0$$

The solutions to the quadratic equation are

$$I = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-150) \pm \sqrt{(-150)^2 - (4 \times 1.6 \times 2000)}}{2 \times 1.6}$$

$$= 77.7\text{ A},\ \ 16.1\text{ A}$$

The relevant solution depends on the electrical characteristics of the motor used in the circuit. In practice, the larger of the two currents would correspond to a substantial loss in the internal resistance and would be avoided by the correct choice of motor.

The technical computing language MATLAB® has the function `roots` which finds the solutions of a polynomial equation. In this example we would type `roots ([1.6 -150 2000])` at the command line to obtain the results calculated above.

We now introduce the method of **completing the square**. The idea behind completing the square is to absorb both the x^2 and the x term into a single squared term. Note that this is possible since

$$x^2 + 2kx + k^2 = (x + k)^2$$

and so

$$x^2 + 2kx = (x + k)^2 - k^2$$

and finally

$$x^2 + 2kx + A = (x + k)^2 + A - k^2$$

The x^2 and the x terms are both contained in the $(x + k)^2$ term. The coefficient of x on the l.h.s. is $2k$. The squared term on the right-hand side (r.h.s.) has the form $(x + k)^2$, that is $\left(x + \dfrac{\text{coefficient of } x}{2}\right)^2$. The following example illustrates the idea.

Example 1.22 Solve the following quadratic equations by completing the square:

(a) $x^2 + 8x + 2 = 0$

(b) $2x^2 - 4x + 1 = 0$

Solution (a) By comparing $x^2 + 8x + 2$ with $x^2 + 2kx + A$ we see $k = 4$. Thus the squared term must be $(x + 4)^2$. Now

$$(x + 4)^2 = x^2 + 8x + 16$$

and so

$$x^2 + 8x = (x + 4)^2 - 16$$

Therefore

$$\begin{aligned} x^2 + 8x + 2 &= (x + 4)^2 - 16 + 2 \\ &= (x + 4)^2 - 14 \end{aligned}$$

At this stage we have completed the square. Finally, solving $x^2 + 8x + 2 = 0$ we have

$$\begin{aligned} x^2 + 8x + 2 &= 0 \\ (x + 4)^2 - 14 &= 0 \\ (x + 4)^2 &= 14 \\ x + 4 &= \pm\sqrt{14} \\ x &= -4 \pm \sqrt{14} = -7.7417, -0.2583 \end{aligned}$$

(b) $2x^2 - 4x + 1 = 0$ may be expressed as $x^2 - 2x + 0.5 = 0$. Comparing $x^2 - 2x + 0.5$ with $x^2 + 2kx + A$ we see that $k = -1$. Thus the required squared term must be $(x - 1)^2$. Now

$$(x - 1)^2 = x^2 - 2x + 1$$

and so

$$x^2 - 2x = (x - 1)^2 - 1$$

and

$$\begin{aligned} x^2 - 2x + 0.5 &= (x-1)^2 - 1 + 0.5 \\ &= (x-1)^2 - 0.5 \end{aligned}$$

Finally, solving $x^2 - 2x + 0.5 = 0$ we have

$$\begin{aligned} (x-1)^2 - 0.5 &= 0 \\ (x-1)^2 &= 0.5 \\ x - 1 &= \pm\sqrt{0.5} \\ x &= 1 \pm \sqrt{0.5} = 0.2929, 1.7071 \end{aligned}$$

1.4.2 Polynomial equations of higher degree

Example 1.23 Verify that $x = 1$ and $x = 2$ are roots of

$$P(x) = x^4 - 2x^3 - x + 2 = 0$$

Solution

$$\begin{aligned} &P(x) = x^4 - 2x^3 - x + 2 \\ &P(1) = 1 - 2 - 1 + 2 = 0 \\ &P(2) = 2^4 - 2(2^3) - 2 + 2 = 16 - 16 - 2 + 2 = 0 \end{aligned}$$

Since $P(1) = 0$ and $P(2) = 0$, then $x = 1$ and $x = 2$ are roots of the given polynomial equation and are sometimes referred to as **real roots**. Further knowledge is required to find the two remaining roots, which are known as **complex roots**. This topic is covered in Chapter 9.

Example 1.24 Solve the equation

$$P(x) = x^3 + 2x^2 - 37x + 52 = 0$$

Solution As seen in Example 1.21 a formula can be used to solve quadratic equations. For higher degree polynomial equations such simple formulae do not always exist. However, if one of the roots can be found by inspection we can proceed as follows. By inspection $P(4) = 4^3 + 2(4)^2 - 37(4) + 52 = 0$ so that $x = 4$ is a root. Hence $x - 4$ is a factor of $P(x)$. Therefore $P(x)$ can be written as

$$P(x) = x^3 + 2x^2 - 37x + 52 = (x-4)(x^2 + \alpha x + \beta)$$

where α and β must now be found. Expanding the r.h.s. gives

$$P(x) = x^3 + \alpha x^2 + \beta x - 4x^2 - 4\alpha x - 4\beta$$

Hence

$$x^3 + 2x^2 - 37x + 52 = x^3 + (\alpha - 4)x^2 + (\beta - 4\alpha)x - 4\beta$$

By comparing constant terms on the l.h.s. and r.h.s. we see that

$$52 = -4\beta$$

so that

$$\beta = -13$$

By comparing coefficients of x^2 we see that

$$2 = \alpha - 4$$

Therefore,

$$\alpha = 6$$

Hence, $P(x) = (x-4)(x^2+6x-13)$. The quadratic equation $x^2+6x-13=0$ can be solved using the formula

$$\begin{aligned} x &= \frac{-6 \pm \sqrt{36-4(-13)}}{2} \\ &= \frac{-6 \pm \sqrt{88}}{2} \\ &= 1.690, -7.690 \end{aligned}$$

We conclude that $P(x) = 0$ has roots at $x = 4$, $x = 1.690$ and $x = -7.690$.

EXERCISES 1.4

1 Calculate the roots of the following linear equations:

(a) $4x - 12 = 0$
(b) $5t + 20 = 0$
(c) $t + 10 = 2t$
(d) $\frac{y}{2} - 1 = 3$
(e) $0.5t - 6 = 0$
(f) $2x + 3 = 5x - 6$
(g) $\frac{3x}{2} - 17 = 0$
(h) $\frac{x}{2} + \frac{x}{3} = 1$
(i) $2x - 1 = \frac{x}{2} + 2$
(j) $2(y+1) = 6$
(k) $3(2y-1) = 2(y+2)$
(l) $\frac{3}{2}(t+3) = \frac{2}{3}(4t-1)$

2 Solve the following quadratic equations by factorization:

(a) $t^2 - 5t + 6 = 0$
(b) $x^2 + x - 12 = 0$
(c) $t^2 = 10t - 25$
(d) $x^2 + 4x - 21 = 0$
(e) $x^2 - 9x + 18 = 0$
(f) $x^2 = 1$
(g) $y^2 - 10y + 9 = 0$
(h) $2z^2 - z - 1 = 0$
(i) $2x^2 + 3x - 2 = 0$
(j) $3t^2 + 4t + 1 = 0$
(k) $4y^2 + 12y + 5 = 0$
(l) $4r^2 - 9r + 2 = 0$
(m) $6d^2 - d - 2 = 0$
(n) $6x^2 - 13x + 2 = 0$

3 Complete the square for the following quadratic equations and hence find their roots:

(a) $x^2 + 2x - 8 = 0$
(b) $x^2 - 6x - 5 = 0$
(c) $x^2 + 4x - 6 = 0$
(d) $x^2 - 14x - 10 = 0$
(e) $x^2 + 5x - 49 = 0$

4 Solve the following quadratic equations using the quadratic formula:

(a) $x^2 + x - 1 = 0$
(b) $t^2 - 3t - 2 = 0$

(c) $h^2 + 5h + 1 = 0$

(d) $0.5x^2 + 3x - 2 = 0$

(e) $2k^2 - k - 3 = 0$

(f) $-y^2 + 3y + 1 = 0$

(g) $3r^2 = 7r + 2$

(h) $x^2 - 70 = 0$

(i) $4s^2 - 2 = s$

(j) $2t^2 + 5t + 2 = 0$

(k) $3x^2 = 50$

5 Calculate the roots of the following polynomial equations:

(a) $x^3 - 6x^2 + 11x - 6 = 0$ given $x = 1$ is a root

(b) $t^3 - 2t^2 - 5t + 6 = 0$ given $t = 3$ is a root

(c) $v^3 - v^2 - 30v + 72 = 0$ given $v = 4$ is a root

(d) $2y^3 + 3y^2 - 11y + 3 = 0$ given $y = 1.5$ is a root

(e) $2x^3 + 3x^2 - 7x - 5 = 0$ given $x = -\frac{5}{2}$ is a root.

6 Check that the given values are roots of the following polynomial equations:

(a) $x^2 + x - 2 = 0 \qquad x = -2, 1$

(b) $2t^3 - 3t^2 - 3t + 2 = 0 \qquad t = -1, 0.5$

(c) $y^3 + y^2 + y + 1 = 0 \qquad y = -1$

(d) $v^4 + 4v^3 + 6v^2 + 3v = 0 \qquad v = -1, 0$

Solutions

1 (a) 3 (b) -4 (c) 10 (d) 8

(e) 12 (f) 3 (g) $\frac{34}{3}$ (h) $\frac{6}{5}$

(i) 2 (j) 2 (k) $\frac{7}{4}$ (l) $\frac{31}{7}$

2 (a) 2, 3 (b) $-4, 3$ (c) 5

(d) $-7, 3$ (e) 3, 6 (f) $-1, 1$

(g) 1, 9 (h) $-0.5, 1$ (i) $-2, 0.5$

(j) $-1, -\frac{1}{3}$ (k) $-2.5, -0.5$ (l) 0.25, 2

(m) $-\frac{1}{2}, \frac{2}{3}$ (n) $\frac{1}{6}, 2$

3 (a) $(x + 1)^2 - 9 = 0, x = -4, 2$

(b) $(x - 3)^2 - 14 = 0, x = -0.7417, 6.7417$

(c) $(x + 2)^2 - 10 = 0, x = -5.1623, 1.1623$

(d) $(x - 7)^2 - 59 = 0, x = -0.6811, 14.6811$

(e) $\left(x + \frac{5}{2}\right)^2 - \frac{221}{4} = 0, x = -9.9330, 4.9330$

4 (a) $-1.6180, 0.6180$

(b) $-0.5616, 3.5616$

(c) $-4.7913, -0.2087$

(d) $-6.6056, 0.6056$

(e) $-1, 1.5$

(f) $-0.3028, 3.3028$

(g) $-0.2573, 2.5907$

(h) $-8.3666, 8.3666$

(i) $-0.5931, 0.8431$

(j) $-2, -0.5$

(k) $-4.0825, 4.0825$

5 (a) 1, 2, 3 (b) $-2, 1, 3$

(c) $-6, 3, 4$ (d) $-3.3028, 0.3028, 1.5$

(e) $-2.5, -0.6180, 1.6180$

1.5 ALGEBRAIC FRACTIONS

An algebraic fraction has the form

$$\text{algebraic fraction} = \frac{\text{numerator}}{\text{denominator}} = \frac{\text{polynomial expression}}{\text{polynomial expression}}$$

For example,

$$\frac{3t + 1}{t^2 + t + 4}, \qquad \frac{x^3}{x^2 + 1} \qquad \text{and} \qquad \frac{y^2 + 1}{y^2 + 2y + 3}$$

are all algebraic fractions.

1.5.1 Proper and improper fractions

When presented with a fraction, we can note the degree of the numerator, say n, and the degree of the denominator, say d.

A fraction is **proper** if $d > n$, that is the degree of the denominator is greater than the degree of the numerator. If $d \leqslant n$ then the fraction is **improper**.

Example 1.25 Classify the following fractions as either proper or improper. In each case, state the degree of both numerator and denominator.

(a) $\dfrac{x^2+9x-6}{3x^3+x^2+100}$

(b) $\dfrac{t^3+t^2+9t-6}{t^5+9}$

(c) $\dfrac{(v+1)(v-6)}{v^2+3v+6}$

(d) $\dfrac{(z+2)^3}{5z^2+10z+16}$

Solution (a) The degree of the numerator, n, is 2. The degree of the denominator, d, is 3. Since $d > n$ the fraction is proper.

(b) Here $n = 3$ and $d = 5$. The fraction is proper since $d > n$.

(c) Here $n = 2$ and $d = 2$, so $d = n$ and the fraction is improper.

(d) Here $n = 3$ and $d = 2$, so $d < n$ and the fraction is improper.

1.5.2 Equivalent fractions

Consider the numerical fractions $\frac{1}{2}$ and $\frac{2}{4}$. These fractions have the same value. Similarly, $\frac{2}{3}$, $\frac{6}{9}$ and $\frac{20}{30}$ all have the same value. The algebraic fractions $\frac{x}{y}$, $\frac{2x}{2y}$ and $\frac{xt}{yt}$ all have the same value. Fractions with the same value are called **equivalent fractions**.

The value of a fraction remains unchanged if both numerator and denominator are multiplied or divided by the same quantity. This fact can be used to write a fraction in many equivalent forms. Consider for example the fractions

$$\text{(a)}\ \frac{2}{x} \qquad \text{(b)}\ \frac{2(x+1)}{x(x+1)} \qquad \text{(c)}\ \frac{2xt}{x^2t}$$

These are all equivalent fractions. Fraction (b) can be obtained by multiplying both numerator and denominator of fraction (a) by $(x+1)$, so they are equivalent. Fraction (a) can be obtained by dividing numerator and denominator of fraction (c) by xt and so they are also equivalent.

Example 1.26 Show that

$$\frac{x+1}{x+7} \quad \text{and} \quad \frac{x^2+4x+3}{x^2+10x+21}$$

are equivalent.

Solution We factorize the numerator and denominator of the second fraction:

$$\frac{x^2+4x+3}{x^2+10x+21} = \frac{(x+1)(x+3)}{(x+7)(x+3)}$$

Dividing both numerator and denominator by $(x+3)$ results in $\frac{x+1}{x+7}$. So the two given fractions are equivalent.

Dividing both numerator and denominator by $x+3$ is often referred to as 'cancelling $x+3$'.

1.5.3 Expressing a fraction in its simplest form

Consider the numerical fraction $\frac{6}{10}$. To simplify this we factorize both numerator and denominator and then cancel any common factors. Thus

$$\frac{6}{10} = \frac{2\times 3}{2\times 5} = \frac{3}{5}$$

The fractions $\frac{6}{10}$ and $\frac{3}{5}$ have identical values but $\frac{3}{5}$ is in a simpler form than $\frac{6}{10}$. It is important to stress that only factors which are common to both numerator and denominator can be cancelled.

Example 1.27 Simplify

(a) $\frac{6x}{18x^2}$

(b) $\frac{12x^3y^2}{4x^2yz}$

Solution (a) Note that 18 can be factorized to 6×3 and so 6 is a factor common to both numerator and denominator. Also x^2 is $x\times x$ and so x is also a common factor. Cancelling the common factors, 6 and x, produces

$$\frac{6x}{18x^2} = \frac{6x}{(6)(3)(x)(x)} = \frac{1}{3x}$$

(b) The common factors are 4, x^2 and y. Cancelling these factors gives

$$\frac{12x^3y^2}{4x^2yz} = \frac{3xy}{z}$$

Example 1.28 Simplify (a) $\frac{4}{6x+4}$ (b) $\frac{6t^3+3t^2+6t}{3t^2+3t}$

Solution (a) Factorizing both numerator and denominator and cancelling common factors yields

$$\frac{4}{6x+4} = \frac{(2)(2)}{2(3x+2)} = \frac{2}{3x+2}$$

(b) Factorizing and cancelling common factors yields

$$\frac{6t^3+3t^2+6t}{3t^2+3t}=\frac{3t(2t^2+t+2)}{3t(t+1)}=\frac{2t^2+t+2}{t+1}$$

Note that the common factor, $3t$, has been cancelled.

Example 1.29 Simplify (a) $\dfrac{4t+8}{t^2+3t+2}$ (b) $\dfrac{2y^2-y-1}{y^2-2y+1}$

Solution The numerator and denominator are factorized and common factors are cancelled.

(a) $$\frac{4t+8}{t^2+3t+2}=\frac{4(t+2)}{(t+2)(t+1)}=\frac{4}{t+1}$$

The common factor, $t+2$, has been cancelled.

(b) $$\frac{2y^2-y-1}{y^2-2y+1}=\frac{(2y+1)(y-1)}{(y-1)^2}=\frac{2y+1}{y-1}$$

The common factor, $y-1$, has been cancelled.

1.5.4 Multiplication and division of algebraic fractions

To multiply two algebraic fractions together, we multiply their numerators together, and multiply their denominators together, that is

$$\frac{a}{b}\times\frac{c}{d}=\frac{a\times c}{b\times d}=\frac{ac}{bd}$$

Division is performed by inverting the second fraction and then multiplying, that is

$$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc}$$

Before multiplying or dividing fractions it is advisable to express each fraction in its simplest form.

Example 1.30 Simplify

$$\frac{x^2+5x+6}{2x-2}\times\frac{x^2-x}{x^2+3x+2}$$

Solution Factorizing numerators and denominators produces

$$\frac{x^2+5x+6}{2x-2}\times\frac{x^2-x}{x^2+3x+2}=\frac{(x+2)(x+3)}{2(x-1)}\times\frac{x(x-1)}{(x+1)(x+2)}$$

$$=\frac{(x+2)(x+3)x(x-1)}{2(x-1)(x+1)(x+2)}$$

Common factors $(x+2)$ and $(x-1)$ can be cancelled from numerator and denominator to give

$$\frac{(x+2)(x+3)x(x-1)}{2(x-1)(x+1)(x+2)}=\frac{(x+3)x}{2(x+1)}$$

Hence

$$\frac{x^2+5x+6}{2x-2}\times\frac{x^2-x}{x^2+3x+2}=\frac{x(x+3)}{2(x+1)}$$

Example 1.31 Simplify

$$\frac{x^2+8x+7}{x^2-6x}\div\frac{x+7}{x^3+x^2}$$

Solution The second fraction is inverted to give

$$\frac{x^2+8x+7}{x^2-6x}\times\frac{x^3+x^2}{x+7}$$

Factorizing numerators and denominators yields

$$\frac{(x+1)(x+7)}{x(x-6)}\times\frac{x^2(x+1)}{(x+7)}=\frac{(x+1)(x+7)x^2(x+1)}{x(x-6)(x+7)}$$

Common factors of x and $(x+7)$ are cancelled leaving

$$\frac{(x+1)x(x+1)}{x-6}$$

which may be written as

$$\frac{x(x+1)^2}{x-6}$$

1.5.5 Addition and subtraction of algebraic fractions

The method of adding and subtracting algebraic fractions is identical to that for numerical fractions.

Each fraction is written in its simplest form. The denominators of the fractions are then examined and the **lowest common denominator** (l.c.d.) is found. This is the simplest expression that has the given denominators as factors. All fractions are then written in an equivalent form with the l.c.d. as denominator. Finally the numerators are added/subtracted and placed over the l.c.d. Consider the following examples.

Example 1.32 Express as a single fraction

$$\frac{2}{x+1}+\frac{4}{x+2}$$

Solution Both fractions are already in their simplest form. The l.c.d. of the denominators, $(x+1)$ and $(x+2)$, is found. This is $(x+1)(x+2)$. Note that this is the simplest expression that has both $x+1$ and $x+2$ as factors.

Each fraction is written in an equivalent form with the l.c.d. as denominator. So

$$\frac{2}{x+1}\text{ is written as }\frac{2(x+2)}{(x+1)(x+2)}$$

and

$$\frac{4}{x+2} \text{ is written as } \frac{4(x+1)}{(x+1)(x+2)}$$

Finally the numerators are added. Hence we have

$$\begin{aligned}\frac{2}{x+1}+\frac{4}{x+2} &= \frac{2(x+2)}{(x+1)(x+2)}+\frac{4(x+1)}{(x+1)(x+2)}\\ &= \frac{2(x+2)+4(x+1)}{(x+1)(x+2)}\\ &= \frac{6x+8}{(x+1)(x+2)}\\ &= \frac{6x+8}{x^2+3x+2}\end{aligned}$$

Example 1.33 Express as a single fraction

$$\frac{x^2+3x+2}{x^2-1}-\frac{2}{2x+6}$$

Solution Each fraction is written in its simplest form:

$$\frac{x^2+3x+2}{x^2-1}=\frac{(x+1)(x+2)}{(x-1)(x+1)}=\frac{x+2}{x-1}$$

$$\frac{2}{2x+6}=\frac{2}{2(x+3)}=\frac{1}{x+3}$$

The l.c.d. is $(x-1)(x+3)$. Each fraction is written in an equivalent form with l.c.d. as denominator:

$$\frac{x+2}{x-1}=\frac{(x+2)(x+3)}{(x-1)(x+3)}, \qquad \frac{1}{x+3}=\frac{x-1}{(x-1)(x+3)}$$

So

$$\begin{aligned}\frac{x^2+3x+2}{x^2-1}-\frac{2}{2x+6} &= \frac{x+2}{x-1}-\frac{1}{x+3}\\ &= \frac{(x+2)(x+3)}{(x-1)(x+3)}-\frac{(x-1)}{(x-1)(x+3)}\\ &= \frac{(x+2)(x+3)-(x-1)}{(x-1)(x+3)}\\ &= \frac{x^2+5x+6-x+1}{(x-1)(x+3)}\\ &= \frac{x^2+4x+7}{(x-1)(x+3)}\end{aligned}$$

Engineering application 1.7

Resistors in parallel

When carrying out circuit analysis it is often helpful to reduce the complexity of a circuit by calculating an equivalent single resistance for several resistors connected together in parallel. This simplified version of the original circuit then becomes much easier to understand. Figure 1.4 shows the simplest case of two resistors connected together in parallel.

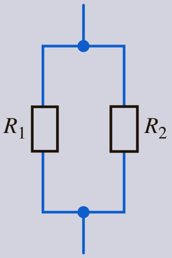


Figure 1.4
Two resistors in parallel.

The equivalent resistance, R_E, of this simple network is found from the formula

$$\frac{1}{R_E} = \frac{1}{R_1} + \frac{1}{R_2}$$

By combining the fractions on the r.h.s. we see

$$\frac{1}{R_E} = \frac{R_2 + R_1}{R_1 R_2}$$

and hence

$$R_E = \frac{R_1 R_2}{R_1 + R_2}$$

Consider the case when R_1 and R_2 are equal and have value R. The equivalent resistance then becomes

$$R_E = \frac{RR}{R + R} = \frac{R^2}{2R} = \frac{R}{2}$$

So

$$R_E = \frac{R}{2} = 0.5R$$

Therefore the effect of putting two equal resistors in parallel is to produce an overall equivalent resistance which is half that of a single resistor.

EXERCISES 1.5

1 Classify each fraction as either proper or improper.

(a) $\frac{x+2}{x^2+2}$ (b) $\frac{2}{x+2}$ (c) $\frac{2+x}{2}$

(d) $\frac{x^2+2}{x+2}$ (e) $\frac{x^2+2}{x^2+1}$ (f) $\frac{x^2+1}{x^2+2}$

2 Classify each of the following algebraic fractions as proper or improper.

(a) $\frac{3t+1}{t^2-1}$ (b) $\frac{10v^2+4v-6}{3v^2+v-1}$ (c) $\frac{6-4t+t^3}{6t^2+1}$

(d) $\frac{9t+1}{t+1}$ (e) $\frac{100f^2+1}{f^3-1}$ (f) $\frac{(x+1)(x+2)}{(x+3)^3}$

(g) $\dfrac{(y+1)(y+2)(y+3)}{(y+4)^3}$

(h) $\dfrac{(z+1)^{10}}{(2z+1)^{10}}$ (i) $\dfrac{(q+1)^{10}}{(q^2+1)^6}$

(j) $\dfrac{3k^2+2k-1}{k^3+k^2-4k+1}$

3 Express each fraction in its simplest form.

(a) $\dfrac{y^3+2y}{2y-y^2}$ (b) $\dfrac{5x^2+5}{10x-10}$

(c) $\dfrac{t^2+7t+12}{t^2+5t+4}$ (d) $\dfrac{x^2-1}{x^3-2x^2+x}$

(e) $\dfrac{x^2+2x+1}{x^2-2x+1}$

4 Simplify the following:

(a) $\dfrac{x+1}{x+3}\times\dfrac{x+3}{x+2}$

(b) $\dfrac{4}{x^2-1}\times\dfrac{x+1}{6}$

(c) $\dfrac{x^2+3x}{x^3+2x^2}\times\dfrac{x^2+4x+4}{4x}$

(d) $\dfrac{4xt+4t}{xt^2-t^2}\times\dfrac{4x^2-4}{8x+8}$

(e) $\dfrac{x^2+2x-15}{x^2+4x-5}\times\dfrac{x^2+3x-4}{x^2-4x+3}$

5 Express as a single fraction

(a) $\dfrac{3}{x+6}+\dfrac{2}{x+1}$

(b) $\dfrac{4}{x+2}-\dfrac{2}{(x+2)^2}$

(c) $\dfrac{2x+1}{x^2+x+1}+\dfrac{4}{x-1}$

(d) $\dfrac{x^2+3x-18}{x^2+7x+6}-\dfrac{2x^2+7x-4}{x^2+9x+20}$

(e) $\dfrac{3(x+1)}{x^2+4x+4}+\dfrac{2(x-1)}{x^2-4}$

Solutions

1 (a) proper (b) proper (c) improper
(d) improper (e) improper (f) improper

2 (a) proper (b) improper (c) improper
(d) improper (e) proper (f) proper
(g) improper (h) improper (i) proper
(j) proper

3 (a) $\dfrac{y^2+2}{2-y}$ (b) $\dfrac{x^2+1}{2x-2}$ (c) $\dfrac{t+3}{t+1}$

(d) $\dfrac{x+1}{x(x-1)}$ (e) $\dfrac{x^2+2x+1}{x^2-2x+1}$

4 (a) $\dfrac{x+1}{x+2}$ (b) $\dfrac{2}{3(x-1)}$

(c) $\dfrac{(x+2)(x+3)}{4x^2}$ (d) $\dfrac{2(x+1)}{t}$ (e) $\dfrac{x+4}{x-1}$

5 (a) $\dfrac{5x+15}{(x+1)(x+6)}$ (b) $\dfrac{4x+6}{(x+2)^2}$

(c) $\dfrac{6x^2+3x+3}{(x-1)(x^2+x+1)}$

(d) $\dfrac{-x^2+x-14}{(x+1)(x+5)}$ (e) $\dfrac{5x^2-x-10}{(x+2)^2(x-2)}$

1.6 SOLUTION OF INEQUALITIES

An **inequality** is any expression involving one of the symbols $>$, $\geqslant$, $<$, $\leqslant$.

$a > b$ means a is greater than b
$a < b$ means a is less than b
$a \geqslant b$ means a is greater than or equal to b
$a \leqslant b$ means a is less than or equal to b

Just as with an equation, when we add or subtract the same quantity to both sides of an inequality the inequality still remains. Mathematically we have

If $a > b$ then

$a + k > b + k$ adding k to both sides

$a - k > b - k$ subtracting k from both sides

We can make similar statements for $a \geqslant b$, $a < b$ and $a \leqslant b$.

When multiplying or dividing both sides of an inequality extra care must be taken. Suppose we wish to multiply or divide an inequality by a quantity k. If k is positive the inequality remains the same; if k is negative then the inequality is reversed.

If $a > b$ then

$$\left.\begin{aligned} ka &> kb \\ \frac{a}{k} &> \frac{b}{k} \end{aligned}\right\} \text{ if } k \text{ is positive} \qquad \left.\begin{aligned} ka &< kb \\ \frac{a}{k} &< \frac{b}{k} \end{aligned}\right\} \text{ if } k \text{ is negative}$$

Note that when k is negative the inequality changes from $>$ to $<$. Similar statements can be made for $a \geqslant b$, $a < b$ and $a \leqslant b$. When asked to solve an inequality we need to state all the values of the variable for which the inequality is true.

Example 1.34 Solve the following inequalities:

(a) $3t + 1 > t + 7$ (b) $2 - 3z \leqslant 6 + z$

Solution (a) $3t + 1 > t + 7$

$2t + 1 > 7$ subtracting t from both sides

$2t > 6$ subtracting 1 from both sides

$t > 3$ dividing both sides by 2

Hence all values of t greater than 3 satisfy the inequality.

(b) $2 - 3z \leqslant 6 + z$

$-3z \leqslant 4 + z$ subtracting 2 from both sides

$-4z \leqslant 4$ subtracting z from both sides

$z \geqslant -1$ dividing both sides by -4, remembering to reverse the inequality

Hence all values of z greater than or equal to -1 satisfy the inequality.

We often have inequalities of the form $\frac{\alpha}{\beta} > 0$, $\frac{\alpha}{\beta} < 0$, $\alpha\beta > 0$ and $\alpha\beta < 0$ to solve. It is useful to note that if

$\frac{\alpha}{\beta} > 0$ then either $\alpha > 0$ and $\beta > 0$ or $\alpha < 0$ and $\beta < 0$

$\frac{\alpha}{\beta} < 0$ then either $\alpha > 0$ and $\beta < 0$ or $\alpha < 0$ and $\beta > 0$

$\alpha\beta > 0$ then either $\alpha > 0$ and $\beta > 0$ or $\alpha < 0$ and $\beta < 0$

$\alpha\beta < 0$ then either $\alpha > 0$ and $\beta < 0$ or $\alpha < 0$ and $\beta > 0$

The following examples illustrate this.

Example 1.35 Solve the following inequalities:

(a) $\dfrac{x+1}{2x-6} > 0$ (b) $\dfrac{2t+3}{t+2} \leqslant 1$

Solution (a) Consider the fraction $\dfrac{x+1}{2x-6}$. For the fraction to be positive requires either of the following:

(i) $x+1 > 0$ and $2x-6 > 0$.

(ii) $x+1 < 0$ and $2x-6 < 0$.

We consider both cases.

Case (i)

$x+1 > 0$ and so $x > -1$.

$2x-6 > 0$ and so $x > 3$.

Both of these inequalities are true only when $x > 3$. Hence the fraction is positive when $x > 3$.

Case (ii)

$x+1 < 0$ and so $x < -1$.

$2x-6 < 0$ and so $x < 3$.

Both of these inequalities are true only when $x < -1$. Hence the fraction is positive when $x < -1$.

In summary, $\dfrac{x+1}{2x-6} > 0$ when $x > 3$ or $x < -1$.

(b)

$$\frac{2t+3}{t+2} \leqslant 1$$

$$\frac{2t+3}{t+2} - 1 \leqslant 0$$

$$\frac{t+1}{t+2} \leqslant 0$$

We now consider the fraction $\dfrac{t+1}{t+2}$. For the fraction to be negative or zero requires either of the following:

(i) $t+1 \leqslant 0$ and $t+2 > 0$.

(ii) $t+1 \geqslant 0$ and $t+2 < 0$.

We consider each case in turn.

Case (i)

$t+1 \leqslant 0$ and so $t \leqslant -1$.

$t+2 > 0$ and so $t > -2$.

Hence the inequality is true when t is greater than -2 and less than or equal to -1. We write this as $-2 < t \leqslant -1$.

Case (ii)

$t+1 \geqslant 0$ and so $t \geqslant -1$.

$t+2 < 0$ and so $t < -2$.

It is impossible to satisfy both $t \geqslant -1$ and $t < -2$ and so this case yields no values of t.

In summary, $\dfrac{2t+3}{t+2} \leqslant 1$ when $-2 < t \leqslant -1$.

Example 1.36 Solve the following inequalities:

(a) $x^2 > 4$ (b) $x^2 < 4$

Solution (a)

$$\begin{aligned} x^2 &> 4 \\ x^2 - 4 &> 0 \\ (x-2)(x+2) &> 0 \end{aligned}$$

For the product $(x-2)(x+2)$ to be positive requires either

(i) $x - 2 > 0$ and $x + 2 > 0$

or

(ii) $x - 2 < 0$ and $x + 2 < 0$.

We examine each case in turn.

Case (i)

$x - 2 > 0$ and so $x > 2$.

$x + 2 > 0$ and so $x > -2$.

Both of these are true only when $x > 2$.

Case (ii)

$x - 2 < 0$ and so $x < 2$.

$x + 2 < 0$ and so $x < -2$.

Both of these are true only when $x < -2$.
In summary, $x^2 > 4$ when $x > 2$ or $x < -2$.

(b)

$$\begin{aligned} x^2 &< 4 \\ x^2 - 4 &< 0 \\ (x-2)(x+2) &< 0 \end{aligned}$$

For the product $(x-2)(x+2)$ to be negative requires either

(i) $x - 2 > 0$ and $x + 2 < 0$

or

(ii) $x - 2 < 0$ and $x + 2 > 0$.

We examine each case in turn.

Case (i)

$x - 2 > 0$ and so $x > 2$.

$x + 2 < 0$ and so $x < -2$.

No values of x are possible.

Case (ii) $x - 2 < 0$ and so $x < 2$.

$x + 2 > 0$ and so $x > -2$.

Here we have $x < 2$ and $x > -2$. This is usually written as $-2 < x < 2$. Thus all values of x between -2 and 2 will ensure that $x^2 < 4$.
In summary, $x^2 < 4$ when $-2 < x < 2$.

The previous example illustrates a general rule.

If $x^2 > k$ then $x > \sqrt{k}$ or $x < -\sqrt{k}$.
If $x^2 < k$ then $-\sqrt{k} < x < \sqrt{k}$.

Example 1.37 Solve the following inequalities:

(a) $x^2 + x - 6 > 0$ (b) $x^2 + 8x + 1 < 0$

Solution (a)

$$x^2 + x - 6 > 0$$
$$(x-2)(x+3) > 0$$

For the product $(x-2)(x+3)$ to be positive requires either

(i) $x - 2 > 0$ and $x + 3 > 0$
or
(ii) $x - 2 < 0$ and $x + 3 < 0$.

Case (i) $x - 2 > 0$ and so $x > 2$.

$x + 3 > 0$ and so $x > -3$.

Both of these inequalities are satisfied only when $x > 2$.

Case (ii) $x - 2 < 0$ and so $x < 2$.

$x + 3 < 0$ and so $x < -3$.

Both of these inequalities are satisfied only when $x < -3$.
In summary, $x^2 + x - 6 > 0$ when either $x > 2$ or $x < -3$.

(b) The quadratic expression $x^2 + 8x + 1$ does not factorize and so the technique of completing the square is used.

$$x^2 + 8x + 1 = (x+4)^2 - 15$$

Hence

$$(x+4)^2 - 15 < 0$$
$$(x+4)^2 < 15$$

Using the result after Example 1.36 we may write

$$-\sqrt{15} < x+4 < \sqrt{15}$$
$$-\sqrt{15}-4 < x < \sqrt{15}-4$$
$$-7.873 < x < -0.127$$

EXERCISES 1.6

1 Solve the following inequalities:

(a) $2x > 6$

(b) $\frac{y}{4} > 0.6$

(c) $3t < 12$

(d) $z+1 \geqslant 4$

(e) $3v-2 \leqslant 4$

(f) $6-k \geqslant -1$

(g) $\frac{6-2v}{3} < 1$

(h) $m^2 \geqslant 2$

(i) $x^2 < 9$

(j) $v^2+1 \leqslant 10$

(k) $x^2+10 < 6$

(l) $2k^2-3 \geqslant 1$

(m) $10-2v^2 \leqslant 6$

(n) $5+4k^2 > 21$

(o) $(v-2)^2 \leqslant 25$

(p) $(3t+1)^2 > 16$

2 Solve the following inequalities:

(a) $x^2-6x+8 > 0$

(b) $x^2+6x+8 \leqslant 0$

(c) $2t^2+3t-2 < 0$

(d) $y^2-2y-24 \geqslant 0$

(e) $h^2+6h+9 \leqslant 1$

(f) $r^2+6r+7 \geqslant 0$

(g) $x^2+4x-6 < 0$

(h) $4t^2+4t+9 \leqslant 12$

(i) $\frac{x+4}{x-5} > 1$

(j) $\frac{2t-3}{t+6} \leqslant 6$

(k) $\frac{3v+12}{6-2v} \geqslant 0$

(l) $\frac{x^2}{x+1} > 0$

(m) $\frac{x}{x^2+1} < 0$

(n) $\frac{3y+1}{y-2} \leqslant 2$

(o) $k^3 > 0$

(p) $x^3 > 8$

(q) $\frac{t^2+6t+9}{t+5} < 0$

(r) $(x+1)(x-2)(x+3) > 0$

Solutions

1 (a) $x > 3$ (b) $y > 2.4$ (c) $t < 4$

(d) $z \geqslant 3$ (e) $v \leqslant 2$ (f) $k \leqslant 7$

(g) $v > \frac{3}{2}$ (h) $m \geqslant \sqrt{2}$ or $m \leqslant -\sqrt{2}$

(i) $-3 < x < 3$

(j) $-3 \leqslant v \leqslant 3$

(k) no solution

(l) $k \geqslant \sqrt{2}$ or $k \leqslant -\sqrt{2}$

(m) $v \geqslant \sqrt{2}$ or $v \leqslant -\sqrt{2}$

(n) $k > 2$ or $k < -2$

(o) $-3 \leqslant v \leqslant 7$

(p) $t > 1$ or $t < -\frac{5}{3}$

2 (a) $x > 4$ or $x < 2$

(b) $-4 \leqslant x \leqslant -2$

(c) $-2 < t < \frac{1}{2}$

(d) $y \geqslant 6$ or $y \leqslant -4$

(e) $-4 \leqslant h \leqslant -2$

(f) $r \geqslant \sqrt{2}-3$ or $r \leqslant -\sqrt{2}-3$

(g) $-\sqrt{10}-2 < x < \sqrt{10}-2$

(h) $-\frac{3}{2} \leqslant t \leqslant \frac{1}{2}$

(i) $x > 5$

(j) $t \leqslant -\frac{39}{4}$ or $t > -6$

(k) $-4 \leqslant v < 3$

(l) $x > -1$ with $x \neq 0$

(m) $x < 0$ (n) $-5 \leqslant y < 2$

(o) $k > 0$ (p) $x > 2$

(q) $t < -5$

(r) $x > 2$ or $-3 < x < -1$

1.7 PARTIAL FRACTIONS

Given a set of fractions, we can add them together to form a single fraction. For example, in Example 1.32 we saw

$$\begin{aligned}\frac{2}{x+1}+\frac{4}{x+2}&=\frac{2(x+2)+4(x+1)}{(x+1)(x+2)}\\&=\frac{6x+8}{x^2+3x+2}\end{aligned}$$

Alternatively, if we are given a single fraction, we can break it down into the sum of easier fractions. These simple fractions, which when added together form the given fraction, are called **partial fractions**. The partial fractions of $\frac{6x+8}{x^2+3x+2}$ are $\frac{2}{x+1}$ and $\frac{4}{x+2}$.

When expressing a given fraction as a sum of partial fractions it is important to classify the fraction as proper or improper. The denominator is then factorized into a product of factors which can be linear and/or quadratic. **Linear factors** are those of the form $ax+b$, for example $2x-1$, $\frac{x}{2}+6$. **Repeated linear factors** are those of the form $(ax+b)^2$, $(ax+b)^3$ and so on, for example $(3x-2)^2$ and $(2x+1)^3$ are repeated linear factors. **Quadratic factors** are those of the form ax^2+bx+c, for example $2x^2-6x+1$.

1.7.1 Linear factors

We can calculate the partial fractions of proper fractions whose denominator can be factorized into linear factors. The following steps are used:

(1) Factorize the denominator.

(2) Each factor of the denominator produces a partial fraction. A factor $ax+b$ produces a partial fraction of the form $\frac{A}{ax+b}$ where A is an unknown constant.

(3) Evaluate the unknown constants of the partial fractions. This is done by evaluation using a specific value of x or by equating coefficients.

A linear factor $ax+b$ in the denominator produces a partial fraction of the form $\frac{A}{ax+b}$.

Example 1.38 Express

$$\frac{6x+8}{x^2+3x+2}$$

as its partial fractions.

Solution The denominator is factorized as

$$x^2+3x+2=(x+1)(x+2)$$

The linear factor, $x+1$, produces a partial fraction of the form $\dfrac{A}{x+1}$. The linear factor, $x+2$, produces a partial fraction of the form $\dfrac{B}{x+2}$. A and B are unknown constants whose values have to be found. So we have

$$\frac{6x+8}{x^2+3x+2} = \frac{6x+8}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} \tag{1.8}$$

Multiplying both sides of Equation (1.8) by $(x+1)$ and $(x+2)$ we obtain

$$6x+8 = A(x+2) + B(x+1) \tag{1.9}$$

We now evaluate A and B. There are two techniques which enable us to do this: **evaluation using a specific value of x** and **equating coefficients**. Each is illustrated in turn.

Evaluation using a specific value of x

We examine Equation (1.9). We will substitute a specific value of x into this equation. Although any value can be substituted for x we will choose a value which simplifies the equation as much as possible. We note that substituting $x = -2$ will simplify the r.h.s. of the equation since the term $A(x+2)$ will then be zero. Similarly, substituting in $x = -1$ will simplify the r.h.s. because the term $B(x+1)$ will then be zero. So $x = -1$ and $x = -2$ are two convenient values to substitute into Equation (1.9). We substitute each in turn.

Evaluating Equation (1.9) with $x = -1$ gives

$$\begin{aligned} -6+8 &= A(-1+2) \\ 2 &= A \end{aligned}$$

Evaluating Equation (1.9) with $x = -2$ gives

$$\begin{aligned} -4 &= B(-1) \\ B &= 4 \end{aligned}$$

Substituting $A = 2$, $B = 4$ into Equation (1.8) yields

$$\frac{6x+8}{x^2+3x+2} = \frac{2}{x+1} + \frac{4}{x+2}$$

Thus the required partial fractions are $\dfrac{2}{x+1}$ and $\dfrac{4}{x+2}$.

The constants A and B could have been found by equating coefficients.

Equating coefficients

Equation (1.9) may be written as

$$6x+8 = (A+B)x + 2A + B$$

Equating the coefficients of x on both sides gives

$$6 = A + B$$

Equating the constant terms on both sides gives

$$8 = 2A + B$$

Thus we have two simultaneous equations in A and B, which may be solved to give $A = 2$ and $B = 4$ as before.

1.7.2 Repeated linear factor

We now examine proper fractions whose denominators factorize into linear factors, where one or more of the linear factors is repeated.

A repeated linear factor, $(ax + b)^2$, produces two partial fractions of the form

$$\frac{A}{ax+b} + \frac{B}{(ax+b)^2}$$

A repeated linear factor, $(ax + b)^2$, leads to partial fractions

$$\frac{A}{ax+b} + \frac{B}{(ax+b)^2}$$

Example 1.39 Express

$$\frac{2x+5}{x^2+2x+1}$$

as partial fractions.

Solution The denominator is factorized to give $(x + 1)^2$. Here we have a case of a repeated factor. This repeated factor generates partial fractions $\dfrac{A}{x+1} + \dfrac{B}{(x+1)^2}$. Thus

$$\frac{2x+5}{x^2+2x+1} = \frac{2x+5}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2}$$

Multiplying by $(x + 1)^2$ gives

$$2x + 5 = A(x + 1) + B = Ax + A + B$$

Equating coefficients of x gives $A = 2$. Evaluation with $x = -1$ gives $B = 3$. So

$$\frac{2x+5}{x^2+2x+1} = \frac{2}{x+1} + \frac{3}{(x+1)^2}$$

Example 1.40 Express

$$\frac{14x^2+13x}{(4x^2+4x+1)(x-1)}$$

as partial fractions.

Solution The denominator is factorized to $(2x+1)^2(x-1)$. The repeated factor, $(2x+1)^2$, produces partial fractions of the form

$$\frac{A}{2x+1} + \frac{B}{(2x+1)^2}$$

The factor, $(x-1)$, produces a partial fraction of the form $\dfrac{C}{x-1}$. So

$$\frac{14x^2+13x}{(4x^2+4x+1)(x-1)} = \frac{14x^2+13x}{(2x+1)^2(x-1)} = \frac{A}{2x+1} + \frac{B}{(2x+1)^2} + \frac{C}{x-1}$$

Multiplying both sides by $(2x+1)^2(x-1)$ gives

$$14x^2+13x = A(2x+1)(x-1) + B(x-1) + C(2x+1)^2 \qquad (1.10)$$

The unknown constants A, B and C can now be found.

Evaluating Equation (1.10) with $x = 1$ gives

$$27 = C(3)^2$$

from which

$$C = 3$$

Evaluating Equation (1.10) with $x = -0.5$ gives

$$14(-0.5)^2 + 13(-0.5) = B(-0.5-1)$$

from which

$$B = 2$$

Finally, comparing the coefficients of x^2 on both sides of Equation (1.10) we have

$$14 = 2A + 4C$$

Since we already have $C = 3$ then

$$A = 1$$

Hence we see that

$$\frac{14x^2+13x}{(4x^2+4x+1)(x-1)} = \frac{1}{2x+1} + \frac{2}{(2x+1)^2} + \frac{3}{x-1}$$

1.7.3 Quadratic factors

We now look at proper fractions whose denominator contains a quadratic factor, that is a factor of the form ax^2+bx+c.

A quadratic factor, ax^2+bx+c, produces a partial fraction of the form

$$\frac{Ax+B}{ax^2+bx+c}$$

Example 1.41 Noting that $x^3+2x^2-11x-52 = (x-4)(x^2+6x+13)$, express

$$\frac{3x^2+11x+14}{x^3+2x^2-11x-52}$$

as partial fractions.

Solution The denominator has already been factorized. The linear factor, $x-4$, produces a partial fraction of the form $\dfrac{A}{x-4}$.

The quadratic factor, $x^2+6x+13$, will not factorize further into two linear factors. Thus this factor generates a partial fraction of the form $\dfrac{Bx+C}{x^2+6x+13}$. Hence

$$\frac{3x^2+11x+14}{(x-4)(x^2+6x+13)}=\frac{A}{x-4}+\frac{Bx+C}{x^2+6x+13}$$

Multiplying by $(x-4)$ and $(x^2+6x+13)$ produces

$$3x^2+11x+14=A(x^2+6x+13)+(Bx+C)(x-4) \tag{1.11}$$

The constants A, B and C can now be found.

Putting $x=4$ into Equation (1.11) gives

$$\begin{aligned}106&=A(53)\\A&=2\end{aligned}$$

Equating the coefficients of x^2 gives

$$\begin{aligned}3&=A+B\\B&=1\end{aligned}$$

Equating the constant term on both sides gives

$$\begin{aligned}14&=A(13)-4C\\C&=3\end{aligned}$$

Hence

$$\frac{3x^2+11x+14}{x^3+2x^2-11x-52}=\frac{2}{x-4}+\frac{x+3}{x^2+6x+13}$$

1.7.4 Improper fractions

The techniques of calculating partial fractions in Sections 1.7.1 to 1.7.3 have all been applied to proper fractions. We now look at the calculation of partial fractions of improper fractions. The techniques described in Sections 1.7.1 to 1.7.3 are all applicable to improper fractions. However, when calculating the partial fractions of an improper fraction, an extra term needs to be included. The extra term is a polynomial of degree $n-d$, where n is the degree of the numerator and d is the degree of the denominator. A polynomial of degree 0 is a constant, a polynomial of degree 1 has the form $Ax+B$, a polynomial of degree 2 has the form Ax^2+Bx+C, and so on. For example, if the numerator has degree 3 and the denominator has degree 2, the partial fractions will include a polynomial of degree $n-d=3-2=1$, that is a term of the form $Ax+B$. If the numerator and denominator are of the same degree, the fraction is improper. The partial fractions will include a polynomial of degree $n-d=0$, that is a constant term.

> Let the degree of the numerator be n and the degree of the denominator be d. If $n \geqslant d$ then the fraction is improper. Improper fractions have partial fractions in addition to those generated by the factors of the denominator. These additional partial fractions take the form of a polynomial of degree $n - d$.

Example 1.42 Express as partial fractions

$$\frac{4x^3 + 10x + 4}{2x^2 + x}$$

Solution The degree of the numerator is 3, that is $n = 3$. The degree of the denominator is 2, that is $d = 2$. Thus, the fraction is improper.

Now $n - d = 1$ and this is a measure of the extent to which the fraction is improper. The partial fractions will include a polynomial of degree 1, that is $Ax + B$, in addition to the partial fractions generated by the factors of the denominator.

The denominator factorizes to $x(2x + 1)$. These factors generate partial fractions of the form $\frac{C}{x} + \frac{D}{2x+1}$. Hence

$$\frac{4x^3 + 10x + 4}{2x^2 + x} = \frac{4x^3 + 10x + 4}{x(2x+1)} = Ax + B + \frac{C}{x} + \frac{D}{2x+1}$$

Multiplying by x and $2x + 1$ yields

$$4x^3 + 10x + 4 = (Ax + B)x(2x + 1) + C(2x + 1) + Dx \qquad (1.12)$$

The constants A, B, C and D can now be evaluated.

Putting $x = 0$ into Equation (1.12) gives

$$4 = C$$

Putting $x = -0.5$ into Equation (1.12) gives

$$-1.5 = -\frac{D}{2}$$
$$D = 3$$

Equating coefficients of x^3 gives

$$4 = 2A$$
$$A = 2$$

Equating coefficients of x gives

$$10 = B + 2C + D$$
$$B = -1$$

Hence

$$\frac{4x^3 + 10x + 4}{2x^2 + x} = 2x - 1 + \frac{4}{x} + \frac{3}{2x+1}$$

EXERCISES 1.7

1 Calculate the partial fractions of the following fractions:

(a) $\dfrac{6x+14}{x^2+4x+3}$ (b) $\dfrac{7-2x}{x^2-x-2}$

(c) $\dfrac{3x+6}{2x^2+3x}$ (d) $\dfrac{8-x}{6x^2-x-1}$

(e) $\dfrac{13x^2+11x+2}{(x+1)(2x+1)(3x+1)}$

2 Calculate the partial fractions of the following fractions:

(a) $\dfrac{2x+7}{x^2+6x+9}$ (b) $\dfrac{4x-5}{x^2-2x+1}$

(c) $\dfrac{3x^2+8x+6}{(x^2+2x+1)(x+2)}$

(d) $\dfrac{3x^2-3x-2}{(x^2-1)(x-1)}$ (e) $\dfrac{3x^2+7x+6}{x^3+2x^2}$

3 Express the following as partial fractions:

(a) $\dfrac{x^2+x+2}{(x^2+1)(x+1)}$

(b) $\dfrac{5x^2+11x+5}{(2x+3)(x^2+5x+5)}$

(c) $\dfrac{4x^2+5}{(x^2+1)(x^2+2)}$

(d) $\dfrac{18x^2+7x+44}{(2x-3)(2x^2+5x+7)}$

(e) $\dfrac{2x}{(x^2-x+1)(x^2+x+1)}$

4 Express the following fractions as partial fractions:

(a) $\dfrac{x^2+7x+13}{x+4}$ (b) $\dfrac{12x-4}{2x-1}$

(c) $\dfrac{x^2+8x+2}{x^2+6x+1}$

(d) $\dfrac{x^3-2x^2+3x-3}{x^2-2x+1}$

(e) $\dfrac{2x^3+2x^2-2x-1}{x^2+x}$

Solutions

1 (a) $\dfrac{2}{x+3}+\dfrac{4}{x+1}$ (b) $\dfrac{1}{x-2}-\dfrac{3}{x+1}$

(c) $\dfrac{2}{x}-\dfrac{1}{2x+3}$ (d) $\dfrac{3}{2x-1}-\dfrac{5}{3x+1}$

(e) $\dfrac{2}{x+1}+\dfrac{1}{2x+1}-\dfrac{1}{3x+1}$

2 (a) $\dfrac{2}{x+3}+\dfrac{1}{(x+3)^2}$ (b) $\dfrac{4}{x-1}-\dfrac{1}{(x-1)^2}$

(c) $\dfrac{1}{x+1}+\dfrac{1}{(x+1)^2}+\dfrac{2}{x+2}$

(d) $\dfrac{2}{x-1}-\dfrac{1}{(x-1)^2}+\dfrac{1}{x+1}$

(e) $\dfrac{2}{x}+\dfrac{3}{x^2}+\dfrac{1}{x+2}$

3 (a) $\dfrac{1}{x+1}+\dfrac{1}{x^2+1}$

(b) $\dfrac{2x}{x^2+5x+5}+\dfrac{1}{2x+3}$

(c) $\dfrac{1}{x^2+1}+\dfrac{3}{x^2+2}$

(d) $\dfrac{5}{2x-3}+\dfrac{4x-3}{2x^2+5x+7}$

(e) $\dfrac{1}{x^2-x+1}-\dfrac{1}{x^2+x+1}$

4 (a) $x+3+\dfrac{1}{x+4}$ (b) $6+\dfrac{2}{2x-1}$

(c) $1+\dfrac{2x+1}{x^2+6x+1}$

(d) $x+\dfrac{2}{x-1}-\dfrac{1}{(x-1)^2}$

(e) $2x-\dfrac{1}{x}-\dfrac{1}{x+1}$

Technical Computing Exercises 1.7

1 Use a technical computing language such as MATLAB® to verify the solutions to the problems in Exercises 1.7. In MATLAB®, the function `residue` calculates the partial fraction expansion. For example, exercise 1(a) would be solved by typing the following:

```
b = [6 14];
a = [1 4 3];
[r,p,k] = residue(b, a)
```

Notice how the coefficients of the numerator are input in the form `b = [6 14];` this is known as a **row vector**. The concept of a vector will be discussed in later chapters. For now it is adequate to treat this as a horizontal list of numbers which are passed to MATLAB® in a specific order.

Similarly, the coefficients of the denominator are input by `a = [1 4 3].`

Each vector is arranged with the coefficient of the highest power of x first.

The result is:

```
r =

   4.0000
   2.0000

p =

  -1
  -3

k =
  []
```

Examining the solution we note that the output for both r and p is arranged as a vertical list. This way of representing the output is known as a **column vector**. We note that the numbers returned in column vector p have a negative sign. This is because the result calculated contains the **poles** of the partial fraction expansion. These are values of the variable which make the denominator of the fraction zero. The significance of this will become clear later in the text but for now it is adequate to note the difference in sign from what might have been expected.

1.8 SUMMATION NOTATION

In engineering we often want to measure the value of a variable, such as current, voltage or pressure.

Suppose we make three measurements of a variable x. We can label these measurements x_1, x_2 and x_3. In this context, the numbers 1, 2, 3 are called **subscripts**.

In mathematics, the Greek letter sigma, written $\sum$, stands for a 'sum'. For example, the sum $x_1 + x_2 + x_3$ is written

$$\sum_{k=1}^{3} x_k$$

Note that the subscript k ranges from 1 to 3. As k ranges from 1 to 3, x_k becomes x_1 then x_2 and then x_3 and the sigma sign tells us to add up these quantities.

In general,

$$\sum_{k=1}^{N} x_k = x_1 + x_2 + \cdots + x_N$$

This notation is often used to express some of the fundamental equations of electrical circuit analysis. Sometimes 'Summation Notation' is known as 'Sigma Notation'.

Engineering application 1.8

Kirchhoff's current law

Kirchhoff's current law, often abbreviated to KCL, provides one of the fundamental equations for analysing electrical circuits. The law states that the sum of the currents flowing out of any junction, or **node**, in a circuit must equal the sum of the currents flowing into it.

This principle is intuitive as it has a direct analogy with fluid flow in connected water pipes. Currents flowing into a junction are considered positive; those flowing out of a junction are negative. It is then valid to say that the sum of the currents at a junction is zero. If there are N currents at the junction, denoted $I_1, I_2, \ldots, I_N$, then

$$I_1 + I_2 + I_3 + \cdots + I_{N-1} + I_N = 0$$

This can be expressed using the summation notation as

$$\sum_{k=1}^{N} I_k = 0$$

Here I_k means 'the current, I, in branch k'. The first equation can be produced from the summation notation by first substituting $k = 1$, then $k = 2$, right up to $k = N$. The expression below the summation symbol tells you where to start and the variable to be substituted, and the number above the summation symbol indicates where to stop counting. Summation notation is a very compact and precise way of expressing KCL for any number of currents at a node.

Consider the node shown in Figure 1.5.

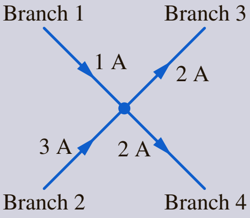


Figure 1.5
A circuit node with four separate branches. The currents are given in amperes (or amps, A).

It can be seen that the total current flowing into the node is $1 + 3 = 4$ amps. The current flowing out of the node is $2 + 2 = 4$ amps. Clearly,

Total current flowing into node = total current flowing out of node

Alternatively, using the summation form of KCL we have

$$\sum_{k=1}^{4} I_k = I_1 + I_2 + I_3 + I_4 = 0 = 1 + 3 - 2 - 2$$

Note that for currents flowing out of the node a negative sign is used and for currents flowing into the node a positive sign is used. This is equivalent to considering the currents separately as inward and outward flowing currents and equating the two.

Suppose for a moment that we did not know the current in branch 4 and, furthermore, it was not labelled with an arrow to show the direction of current flow.

→

This situation is likely to occur in a circuit problem in electronics. There are two options for labelling the current flow direction, and these are summarized in Figure 1.6.

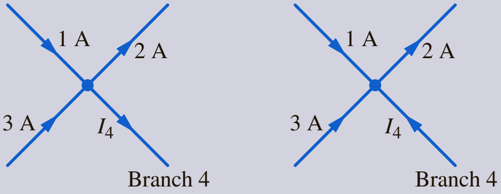


Figure 1.6
Two different ways of defining the current direction in Branch 4.

$$\begin{aligned} 1 + 3 - 2 - I_4 &= 0 \\ I_4 &= 2 \end{aligned}$$

$$\begin{aligned} 1 + 3 - 2 + I_4 &= 0 \\ I_4 &= -2 \end{aligned}$$

Note that the two solutions are both correct but $I_4 = -2$ has a negative sign, which simply indicates that the current flows in the opposite direction to the arrow drawn on the right-hand diagram. It does not matter which way round the arrow is marked, as long as we observe the sign.

Engineering application 1.9

Kirchhoff's voltage law

Kirchhoff's voltage law, often abbreviated to KVL, provides another of the fundamental equations for analysing electrical circuits. The law states that the sum of the voltages around a closed loop equals zero. It is often written down in the form of a summation, as follows:

$$\sum_{k=1}^{N} V_k = 0$$

For the circuit shown in Figure 1.7 there are three possible loops to which we could apply KVL.

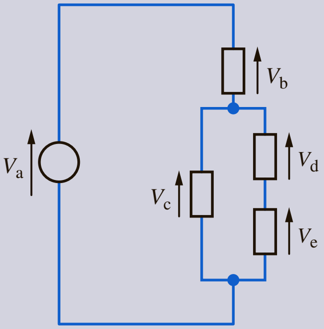
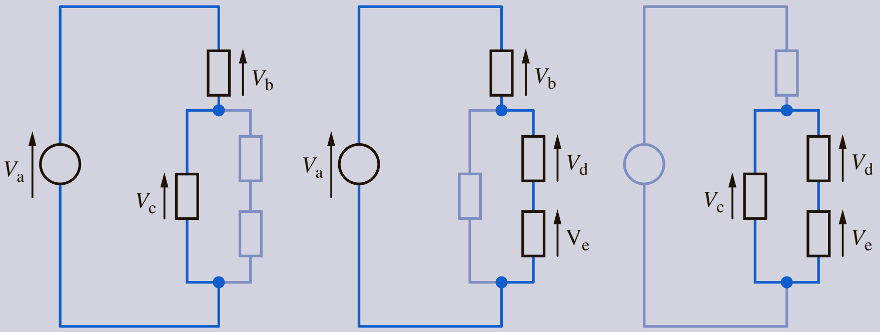


Figure 1.7
A simple circuit to illustrate Kirchhoff's voltage law.

In this example an ideal voltage source and resistors are used, although any components could be substituted as KVL applies universally. Note that we 'walk around' the circuit when writing down the equations. If the arrow is in the direction of travel then it is given a positive sign; if it opposes the direction of travel it is given a negative sign.

The equations are



$$V_a - V_b - V_c = 0 \qquad V_a - V_b - V_d - V_e = 0 \qquad V_c - V_d - V_e = 0$$

If the equations are solved and the voltage has a negative sign it indicates that the polarity is opposite to the direction of the voltage arrow drawn on the diagram. KVL and KCL are the fundamental circuit laws that allow networks of electronic components to be mathematically analysed. Although they are simple in concept they are very powerful techniques.

EXERCISES 1.8

1 Write out fully what is meant by each of the following expressions:

(a) $\sum_{k=1}^{4} x_k$ (b) $\sum_{i=1}^{4} x_i$

(c) $\sum_{k=1}^{7} x_k$ (d) $\sum_{k=1}^{3} x_k^2$

(e) $\sum_{j=1}^{4} (x_j - 2)^3$ (f) $\sum_{n=0}^{3} (2n+1)^2$

2 Write out fully

(a) $\sum_{k=1}^{4} (-1)^k k$ (b) $\sum_{k=1}^{5} (-1)^{k+1} k^2$

3 Write the following sums more concisely by using sigma notation:

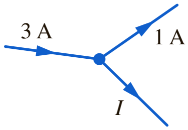
(a) $1^3 + 2^3 + 3^3 + \cdots + 10^3$

(b) $\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{12}$

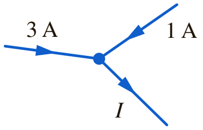
(c) $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7}$

4 Determine the current, I, at each of the following circuit nodes:

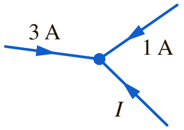
(a)



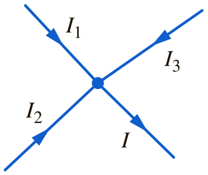
(b)



(c)

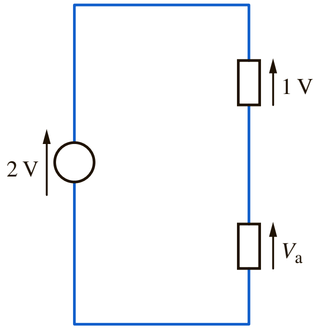


(d)

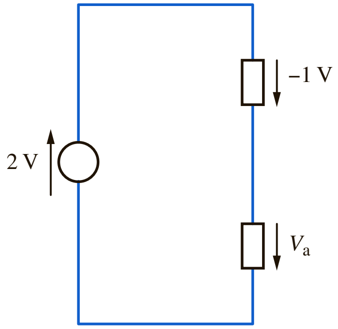


5 Find V_a in each of the following circuits:

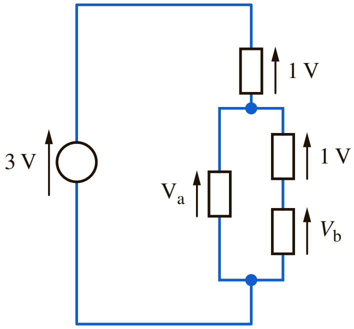
(a)



(b)



6 Find V_a and V_b using KVL.



Solutions

1 (a) $x_1 + x_2 + x_3 + x_4$ (b) $x_1 + x_2 + x_3 + x_4$

(c) $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$

(d) $x_1^2 + x_2^2 + x_3^2$

(e) $(x_1 - 2)^3 + (x_2 - 2)^3 + (x_3 - 2)^3 + (x_4 - 2)^3$

(f) $1 + 3^2 + 5^2 + 7^2$

2 (a) $-1 + 2 - 3 + 4$

(b) $1 - 4 + 9 - 16 + 25$

3 (a) $\sum_{k=1}^{10} k^3$

(b) $\sum_{k=1}^{12} \frac{(-1)^{k+1}}{k}$

(c) $\sum_{n=0}^{3} \frac{1}{2n+1}$ or $\sum_{n=1}^{4} \frac{1}{2n-1}$

4 All solved using KCL

(a) $3 - 1 - I = 0 \therefore I = 2$

(b) $3 + 1 - I = 0 \therefore I = 4$

(c) $3 + 1 + I = 0 \therefore I = -4$

(d) $I_1 + I_2 + I_3 - I = 0, \ \therefore I = I_1 + I_2 + I_3$
or $I = \sum_{k=1}^{3} I_k$

5 Both solved using KVL

(a) $2 - 1 - V_a = 0 \therefore V_a = 1$

(b) $2 + (-1) + V_a = 0 \therefore V_a = -1$

6 $3 - 1 - V_a = 0 \therefore V_a = 2$
$3 - 1 - 1 - V_b = 0 \therefore V_b = 1$
or $V_a - 1 - V_b = 0 \therefore$ by substitution for V_a,
$V_b = 1$

REVIEW EXERCISES 1

1 Simplify each of the following as far as possible:

(a) $7^6 7^4$ (b) $\frac{6^3}{6^{-2}}$ (c) $(3^4)^{-2}$

(d) $\sqrt{3^4 6^2}$ (e) $(3^{2/3} 4^{1/3})^6$ (f) $\frac{10^{-3}}{10^{-4}}$

2 Simplify as far as possible:

(a) $x^7 x^{-3}$ (b) $(x^2)^4$ (c) $\left(\frac{\sqrt{x}}{x}\right)^{-1}$

(d) $(y^{-2})^{-1}$ (e) $y^{1/3} y y^2$

3 Remove the brackets and simplify:

(a) $(2x^2y)^3$ (b) $(6a^2b^3\sqrt{c})^2$ (c) $\left(\dfrac{y^{-2}}{2}\right)^{-1}$

(d) $(x^2y^{-1})^{0.5}$ (e) $\left(\dfrac{3^{-1}x^{-2}}{y^{-3}}\right)^{-2}$

4 Express the following as their partial fractions:

(a) $\dfrac{3x+11}{(x-3)(x+7)}$ (b) $\dfrac{-3-x}{x^2-x}$

(c) $\dfrac{6x^2-2}{2x^2-x}$ (d) $\dfrac{2x^2-x-7}{(x+1)(x-1)(x+2)}$

(e) $\dfrac{4x-11}{2x^2+15x+7}$

5 Convert the following into a single fraction:

(a) $\dfrac{1}{x}+\dfrac{3}{x+2}+\dfrac{6}{8x+4}$

(b) $\dfrac{1}{s}+\dfrac{2}{s^2}+\dfrac{3s+4}{8s+6}$

(c) $\dfrac{6}{s}+\dfrac{10}{s^2}-\dfrac{s+1}{(s+2)(s+3)}+\dfrac{s-1}{(s+4)(s+3)}$

6 Express the following as partial fractions:

(a) $\dfrac{5x}{(x+1)(2x-3)}$ (b) $\dfrac{3x+2}{x^2+5x+6}$

(c) $\dfrac{y+3}{y^2+3y+2}$ (d) $\dfrac{1}{t^2+3t+2}$

(e) $\dfrac{2z^2+15z+30}{(z+2)(z+3)(z+6)}$

(f) $\dfrac{24x^2+33x+11}{(2x+1)(3x+2)(4x+3)}$

(g) $\dfrac{s+3}{(s+1)^2}$ (h) $\dfrac{2k^2+k+1}{k^3-k}$

(i) $\dfrac{x^3}{x^2+1}$ (j) $\dfrac{t+5}{(t+3)^2}$

(k) $\dfrac{s^2}{s^2+1}$ (l) $\dfrac{8x-15}{4x^2-12x+9}$

(m) $\dfrac{6d^2+15d+8}{(d+1)^2(d+2)}$ (n) $\dfrac{2x^2+x+3}{x^2+2x+1}$

(o) $\dfrac{-y-1}{(y^2+1)(y-1)}$ (p) $\dfrac{s^2-8s-5}{(s^2+s+1)(s-4)}$

(q) $\dfrac{t^2+t-2}{(t-2)^2(t+1)}$ (r) $\dfrac{2s^3+3s^2-s-4}{s^2+s-1}$

(s) $\dfrac{x^3+4x^2+7x+5}{x^2+3x+2}$

7 Solve the following quadratic equations using the quadratic formula:

(a) $x^2+10x+2=0$

(b) $y^2-6y-3=0$

(c) $2t^2+2t-9=0$

(d) $3z^2-9z-1=0$

(e) $5v^2+v-6=0$

8 Solve the quadratic equations in Question 7 by completing the square.

9 Solve

$$x^3-4x^2-25x+28=0$$

given $x=7$ is a root.

10 Solve the following inequalities:

(a) $6t-1\leqslant 4$ (b) $-6\leqslant 3r\leqslant 6$

(c) $1-2v<v+4$ (d) $2\leqslant\dfrac{x-2}{3}$

(e) $(x-2)^2\geqslant 36$ (f) $x^2-2x-3<0$

(g) $\dfrac{x-3}{x+1}\geqslant 0$ (h) $x^2-8x+5\leqslant 0$

(i) $\dfrac{x}{2}\leqslant\dfrac{3}{x}$ (j) $\dfrac{x^2-2x-3}{x-5}>0$

11 Express each fraction in its simplest form.

(a) $\dfrac{3ab^2}{12ab}$ (b) $\dfrac{6x^2y^2z}{3xy^3z}$ (c) $\dfrac{9t+6}{12-3t}$

(d) $\dfrac{3x^2+3x}{6x^2-3x}$ (e) $\dfrac{xyz-2x^2y^2z}{x^2y^2-2x^3y^3}$

12 Express each fraction in its simplest form.

(a) $\dfrac{x^2+2x-15}{x^2-2x-3}$ (b) $\dfrac{y^2+4y-12}{y^2+13y+42}$

(c) $\dfrac{2x^2+7x-4}{2x^2-3x+1}$ (d) $\dfrac{3x^2t+3xt-3t}{4x^2z+4xz-4z}$

(e) $\dfrac{x^3-2x^2+x-2}{x^3+x^2+x+1}$

13 Express as a single fraction in its simplest form.

(a) $\dfrac{x+1}{x+6}\times\dfrac{x+6}{x+2}$

(b) $\dfrac{3x-6}{xy+2y}\times\dfrac{xy+3y}{4x-8}$

(c) $\dfrac{x^2-1}{4}\div\dfrac{x-1}{6}$

(d) $\dfrac{x^2-9x}{x+1}\div\dfrac{x-9}{x^3+x^2}$

(e) $\dfrac{x^2-5x-6}{x^2+x-42}\div\dfrac{x^2-1}{x^2+6x-7}$

14 Express as a single fraction:

(a) $\dfrac{5}{x+6}+\dfrac{3}{x+1}$

(b) $\dfrac{3x}{2x-1}-\dfrac{4}{x+5}$

(c) $\dfrac{x+1}{x^2-5x-6}+\dfrac{5x}{x+3}$

(d) $x+1+\dfrac{2}{x-3}$

(e) $2x-3+\dfrac{1}{x+1}-\dfrac{x}{x^2+1}$

Solutions

1 (a) 7^{10} (b) 6^5 (c) 3^{-8}
(d) $3^2 6$ (e) $3^4 4^2$ (f) 10

2 (a) x^4 (b) x^8 (c) $\sqrt{x}$
(d) y^2 (e) $y^{10/3}$

3 (a) $8x^6y^3$ (b) $36a^4b^6c$ (c) $2y^2$
(d) $xy^{-0.5}$ (e) $\dfrac{9x^4}{y^6}$

4 (a) $\dfrac{2}{x-3}+\dfrac{1}{x+7}$

(b) $\dfrac{3}{x}-\dfrac{4}{x-1}$

(c) $3+\dfrac{2}{x}-\dfrac{1}{2x-1}$

(d) $\dfrac{2}{x+1}-\dfrac{1}{x-1}+\dfrac{1}{x+2}$

(e) $\dfrac{3}{x+7}-\dfrac{2}{2x+1}$

5 (a) $\dfrac{19x^2+22x+4}{2x(x+2)(2x+1)}$

(b) $\dfrac{3s^3+12s^2+22s+12}{2s^2(4s+3)}$

(c) $\dfrac{2(3s^4+30s^3+120s^2+202s+120)}{s^2(s+2)(s+3)(s+4)}$

6 (a) $\dfrac{1}{x+1}+\dfrac{3}{2x-3}$

(b) $\dfrac{7}{x+3}-\dfrac{4}{x+2}$

(c) $\dfrac{2}{y+1}-\dfrac{1}{y+2}$

(d) $\dfrac{1}{t+1}-\dfrac{1}{t+2}$

(e) $\dfrac{2}{z+2}-\dfrac{1}{z+3}+\dfrac{1}{z+6}$

(f) $\dfrac{1}{2x+1}+\dfrac{3}{3x+2}-\dfrac{2}{4x+3}$

(g) $\dfrac{1}{s+1}+\dfrac{2}{(s+1)^2}$

(h) $\dfrac{1}{k+1}+\dfrac{2}{k-1}-\dfrac{1}{k}$

(i) $x-\dfrac{x}{x^2+1}$

(j) $\dfrac{1}{t+3}+\dfrac{2}{(t+3)^2}$

(k) $1-\dfrac{1}{s^2+1}$

(l) $\dfrac{4}{2x-3}-\dfrac{3}{(2x-3)^2}$

(m) $\dfrac{4}{d+1}-\dfrac{1}{(d+1)^2}+\dfrac{2}{d+2}$

(n) $2-\dfrac{3}{x+1}+\dfrac{4}{(x+1)^2}$

(o) $\dfrac{y}{y^2+1}-\dfrac{1}{y-1}$

(p) $\dfrac{2s+1}{s^2+s+1}-\dfrac{1}{s-4}$

(q) $\dfrac{11}{9(t-2)}+\dfrac{4}{3(t-2)^2}-\dfrac{2}{9(t+1)}$

(r) $2s+1-\dfrac{3}{s^2+s-1}$

(s) $x+1+\dfrac{1}{x+1}+\dfrac{1}{x+2}$

7 (a) $-9.7958, -0.2042$
(b) $-0.4641, 6.4641$
(c) $-2.6794, 1.6794$
(d) $-0.1073, 3.1073$
(e) $-1.2, 1$

8 (a) $(x+5)^2-23=0$
(b) $(y-3)^2-12=0$
(c) $2\left[\left(t+\dfrac{1}{2}\right)^2-\dfrac{19}{4}\right]=0$

(d) $3\left[\left(z-\frac{3}{2}\right)^2-\frac{31}{12}\right]=0$

(e) $5\left[\left(v+\frac{1}{10}\right)^2-\frac{121}{100}\right]=0$

Solutions same as for Question 7

9 $x=-4, 1, 7$

10 (a) $t \leqslant \frac{5}{6}$ (b) $-2 \leqslant r \leqslant 2$

(c) $v > -1$ (d) $x \geqslant 8$

(e) $x \leqslant -4$ or $x \geqslant 8$

(f) $-1 < x < 3$

(g) $x < -1$ or $x \geqslant 3$

(h) $4-\sqrt{11} \leqslant x \leqslant 4+\sqrt{11}$

(i) $0 < x \leqslant \sqrt{6}, x \leqslant -\sqrt{6}$

(j) $x > 5$ or $-1 < x < 3$

11 (a) $\frac{b}{4}$ (b) $\frac{2x}{y}$ (c) $\frac{3t+2}{4-t}$

(d) $\frac{x+1}{2x-1}$ (e) $\frac{z}{xy}$

12 (a) $\frac{x+5}{x+1}$ (b) $\frac{y-2}{y+7}$ (c) $\frac{x+4}{x-1}$

(d) $\frac{3t}{4z}$ (e) $\frac{x-2}{x+1}$

13 (a) $\frac{x+1}{x+2}$ (b) $\frac{3(x+3)}{4(x+2)}$

(c) $\frac{3(x+1)}{2}$ (d) x^3 (e) 1

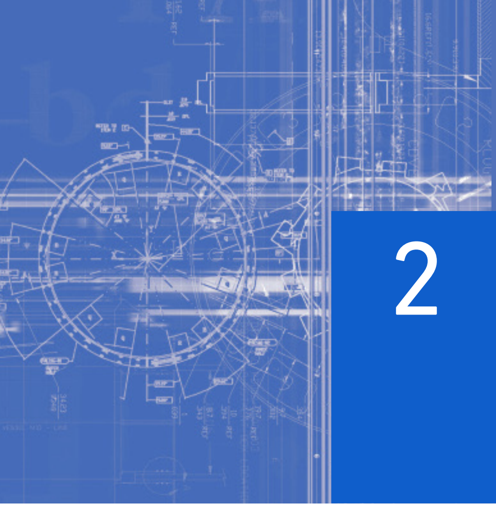
14 (a) $\frac{8x+23}{(x+1)(x+6)}$

(b) $\frac{3x^2+7x+4}{(2x-1)(x+5)}$

(c) $\frac{5x^2-29x+3}{(x-6)(x+3)}$

(d) $\frac{x^2-2x-1}{x-3}$

(e) $\frac{2x^4-x^3-x^2-2x-2}{(x+1)(x^2+1)}$



2 Engineering functions

Contents

2.1 INTRODUCTION

The study of functions is central to engineering mathematics. Functions can be used to describe the way quantities change: for example, the variation in the voltage across an electronic component with time, the variation in position of an electric motor with time and the variation in the strength of a signal with both position and time.

In this chapter we introduce several concepts associated with functions before going on to catalogue a number of engineering functions in Section 2.4. Much of the material of Section 2.4 will already be familiar to the reader and so this section should be treated as a reference section to be dipped into whenever necessary. A number of mathematical methods are also included in Section 2.4, most of which will be familiar but they have been collected together in order to make the book complete.

When trying to understand a mathematical function it is always useful to sketch a graph in order to obtain an idea of its behaviour. The reader is encouraged to sketch such graphs whenever a new function is met. Graphics calculators are now readily available and they make this task relatively easy. If you possess such a calculator then it would be useful to make use of it whenever a new function is introduced. Software packages are also available to allow such plots to be carried out on a computer. These can be useful for plotting more complicated functions and ones that depend on more than one variable. We examine functions of more than one variable in Chapter 25.

Throughout the book we make use of the term **mathematical model**. When doing so we mean an idealization of an engineering system or a physical situation so that it can be described by mathematical equations. To reflect an engineering system very accurately, a sophisticated model, consisting of many interrelated equations, may be needed. Although accurate, such a model may be cumbersome to use. Accuracy can be sacrificed in order to achieve a simple, easy-to-use model. A judgement is made as to when the right blend of accuracy and conciseness is achieved. For example, the most common mathematical model for a resistor uses Ohm's law which states that the voltage across a resistor equals the current through the resistor multiplied by the resistance value of the resistor, that is $V = IR$. However, this model is based on a number of simplifications. It ignores any variation in current density across the cross-section of the resistor and assumes a single current value is acceptable. It also ignores the fact that if a large enough voltage is placed across the resistor then the resistor will break down. In most cases it is worth accepting these simplifications in order to obtain a concise model.

Having obtained a mathematical model, it is then used to predict the effect of changing elements or conditions within the actual system. Using the model to examine these effects is often cheaper, safer and more convenient than using the actual system.

2.2 NUMBERS AND INTERVALS

Numbers can be grouped into various classes, or **sets**. The **integers** are the set of numbers

$$\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

denoted by $\mathbb{Z}$. The **natural numbers** are $\{0, 1, 2, 3, \ldots\}$ and this set is denoted by $\mathbb{N}$. The **positive integers**, denoted by $\mathbb{N}^+$, are given by $\{1, 2, 3, \ldots\}$. Note that some numbers occur in more than one set, that is the sets overlap.

A **rational number** has the form p/q, where p and q are integers with $q \neq 0$. For example, $5/2$, $7/118$, $-1/9$ and $3/1$ are all rational numbers. The set of rational numbers is denoted by $\mathbb{Q}$. When rational numbers are expressed as a decimal fraction they either terminate or recur infinitely.

$$\left.\begin{array}{l}\frac{5}{2}\text{ can be expressed as }2.5\\ \frac{1}{8}\text{ can be expressed as }0.125\end{array}\right\}\qquad\begin{array}{l}\text{These decimal fractions terminate,}\\ \text{that is they are of finite length.}\end{array}$$

$$\left.\begin{array}{l}\frac{1}{9}\text{ can be expressed as }0.111\,111\ldots\\ \frac{1}{11}\text{ can be expressed as }0.090\,909\ldots\end{array}\right\}\qquad\begin{array}{l}\text{These are infinitely}\\ \text{recurring decimal fractions.}\end{array}$$

A number which cannot be expressed in the form p/q is called **irrational**. When written as a decimal fraction, an irrational number is infinite in length and non-recurring. The numbers π and $\sqrt{2}$ are both irrational.

It is useful to introduce the **factorial notation**. We write 3! to represent the product $3 \times 2 \times 1$. The expression 3! is read as 'factorial 3'. Similarly 4! is a shorthand way of writing $4 \times 3 \times 2 \times 1$. In general, for any positive integer, n, we can write

$$n! = n(n-1)(n-2)(n-3)\ldots(3)(2)(1)$$

It is useful to represent numbers by points on the **real line**. Figure 2.1 illustrates some rational and irrational numbers marked on the real line. Numbers which can be represented by points on the real line are known as **real numbers**. The set of real numbers is denoted by $\mathbb{R}$. This set comprises all the rational and all the irrational numbers. In Chapter 9 we shall meet complex numbers which cannot be represented as points on the

Figure 2.1
Both rational and irrational numbers are represented on the real line.

Figure 2.2
The intervals $(-6, -4)$, $[-1, 2]$, $(3, 4]$ depicted on the real line.

real line. The real line extends indefinitely to the left and to the right so that any real number can be represented.

Sometimes we are interested in only a small section, or **interval**, of the real line. We write $[1, 3]$ to denote all the real numbers between 1 and 3 inclusive, that is 1 and 3 are included in the interval. Thus the interval $[1, 3]$ consists of all real numbers x, such that $1 \leqslant x \leqslant 3$. The square brackets, [ ], are used to denote that the end-points are included in the interval and such an interval is said to be **closed**. The interval $(1, 3)$ consists of all real numbers x, such that $1 < x < 3$. In this case the end-points are not included and the interval is said to be **open**. Brackets, (), denote open intervals. An interval may be open at one end and closed at the other. For example, $(1, 3]$ is open at the left and closed at the right. It consists of all real numbers x, such that $1 < x \leqslant 3$, and is known as a **semi-open** interval. Open and closed intervals can be represented on the real line. A closed end-point is denoted by •; an open end-point is denoted by ○. The intervals $(-6, -4)$, $[-1, 2]$ and $(3, 4]$ are illustrated in Figure 2.2.

An **upper bound** of a set of numbers is any number which is greater than or equal to every number in the given set. So, for example, 7 is an upper bound for the set $[3, 6]$. Clearly, 7 is greater than every number in the interval $[3, 6]$.

A **lower bound** of a set of numbers is any number which is less than or equal to every number in the given set. For example, 3 is a lower bound for the set $(3.7, 5)$.

Note that upper and lower bounds are not unique. Both 3 and 10 are upper bounds for $(1, 2)$. Both -1 and -3 are lower bounds for $[0, 6]$.

Technical computing languages such as MATLAB® usually have functions that automatically generate a set of numbers within a particular interval. In MATLAB® we could generate a set of time values, t, by typing:

```
t= 0:0.1:1
```

This generates a set of real numbers from the interval `[0,1]` stored in a row vector t, each individual number being separated by an increment of `0.1`. The values of t generated are:

```
     0   0.1000   0.2000   0.3000   0.4000   0.5000   0.6000   0.7000
0.8000   0.9000   1.0000
```

2.3 BASIC CONCEPTS OF FUNCTIONS

Loosely speaking, we can think of a function as a rule which, when given an input, produces a single output. If more than one output is produced, the rule is not a function. Consider the function given by the rule: ‘double the input’. If 3 is the input then 6 is the output. If x is the input then $2x$ is the output, as shown in Figure 2.3.

If the doubling function has the symbol f we write

$$f : x \rightarrow 2x$$

or more compactly,

$$f(x) = 2x$$

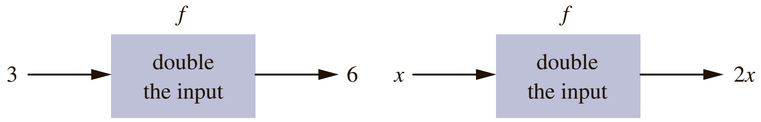


Figure 2.3
The function: 'double the input'.

The last form is often written simply as $f = 2x$. If $f(x)$ is a function of x, then the value of the function when $x = 3$, for example, is written as $f(x = 3)$ or simply as $f(3)$.

Example 2.1 Given $f(x) = 2x + 1$ find

(a) $f(3)$ (b) $f(0)$
(c) $f(-1)$ (d) $f(\alpha)$
(e) $f(2\alpha)$ (f) $f(t)$
(g) $f(t+1)$

Solution

(a) $f(3) = 2(3) + 1 = 7$
(b) $f(0) = 2(0) + 1 = 1$
(c) $f(-1) = 2(-1) + 1 = -1$
(d) $f(\alpha)$ is the value of $f(x)$ when x has a value of α, hence $f(\alpha) = 2\alpha + 1$
(e) $f(2\alpha) = 2(2\alpha) + 1 = 4\alpha + 1$
(f) $f(t) = 2t + 1$
(g) $f(t+1) = 2(t+1) + 1 = 2t + 3$

Observe from Example 2.1 that it is the rule that is important and not the letter being used. Both $f(t) = 2t + 1$ and $f(x) = 2x + 1$ instruct us to double the input and then add 1.

2.3.1 Argument of a function

The input to a function is often called the **argument**. In Example 2.1(d) the argument is α, while in Example 2.1(e) the argument is 2α.

Example 2.2 Given $f(x) = \dfrac{x}{5}$, write down

(a) $f(5x)$ (b) $f(-x)$
(c) $f(x+2)$ (d) $f(x^2)$

Solution

(a) $f(5x) = \dfrac{5x}{5} = x$ (b) $f(-x) = -\dfrac{x}{5}$
(c) $f(x+2) = \dfrac{x+2}{5}$ (d) $f(x^2) = \dfrac{x^2}{5}$

Example 2.3 Given $y(t) = t^2 + t$, write down

(a) $y(t+2)$ (b) $y\left(\dfrac{t}{2}\right)$

Solution (a) $y(t+2) = (t+2)^2 + (t+2) = t^2 + 5t + 6$

(b) $y\left(\frac{t}{2}\right) = \left(\frac{t}{2}\right)^2 + \left(\frac{t}{2}\right) = \frac{t^2}{4} + \frac{t}{2}$

2.3.2 Graph of a function

A function may be represented in graphical form. The function $f(x) = 2x$ is shown in Figure 2.4. Note that the function values are plotted vertically and the x values horizontally. The horizontal axis is then called the x axis. The vertical axis is commonly referred to as the y axis, so that we often write

$$y = f(x) = 2x$$

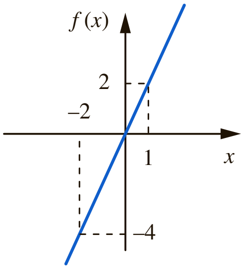


Figure 2.4
The function: $f(x) = 2x$.

Since x and y can have a number of possible values, they are called **variables**: x is the **independent variable** and y is the **dependent variable**. Knowing a value of the independent variable, x, allows us to calculate the corresponding value of the dependent variable, y. To show this dependence we often write $y(x)$. The set of values that x is allowed to take is called the **domain** of the function. A domain is often an interval on the x axis. For example, if

$$f(x) = 3x + 1 \qquad -5 \leqslant x \leqslant 10 \tag{2.1}$$

the domain of the function, f, is the closed interval $[-5, 10]$. If the domain of a function is not explicitly given it is taken to be the largest set possible. For example,

$$g(x) = x^2 - 4 \tag{2.2}$$

has a domain of $(-\infty, \infty)$ since g is defined for every value of x and the domain has not been given otherwise. The set of values that the function takes on is called the **range**. The range of $f(x)$ in Equation (2.1) is $[-14, 31]$; the range of $g(x)$ in Equation (2.2) is $[-4, \infty)$.

We now consider plotting the function $f(t) = t^2$ for $0 \leqslant t < 100$ in a technical computing language. First we generate a number set as shown in Section 2.2.

```
t= 0:1:100
```

Then we produce a graph of the function by using the MATLAB® `plot` command to give the following

```
plot(t, t.^2)
```

Example 2.4 Consider the function, f, given by the rule: ‘square the input’. This can be written as

$$f(x) = x^2$$

The rule and the graph of f are shown in Figure 2.5. The domain of f is $(-\infty, \infty)$ and the range is $[0, \infty)$.

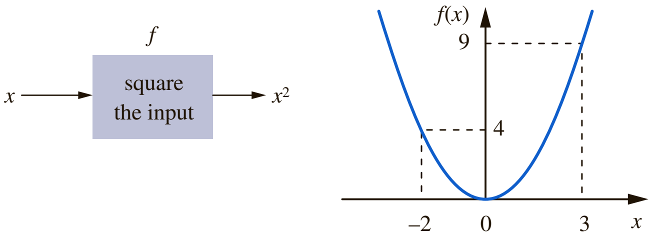


Figure 2.5
The function: 'square the input'.

Many variables of interest to engineers, for example voltage, resistance and current, can be related by means of functions. We try to choose an appropriate letter for a particular variable; so, for example, t is used for time and P for power.

Engineering application 2.1

Function to model the power dissipation in a resistor

Recall from Engineering application 1.1 that the power, P, dissipated by a resistor depends on the current, I, flowing through the resistance, R. The relationship is given by

$$P = I^2R$$

The power dissipated in the resistor depends on the square of the current passing through it. In this case I is the independent variable and P is the dependent variable, assuming R remains constant. The function is given by the rule: 'square the input and multiply by the constant R', and the input to the function is I. The output from the function is P. This is illustrated in Figure 2.6, for the cases $R = 4$ and $R = 2$.

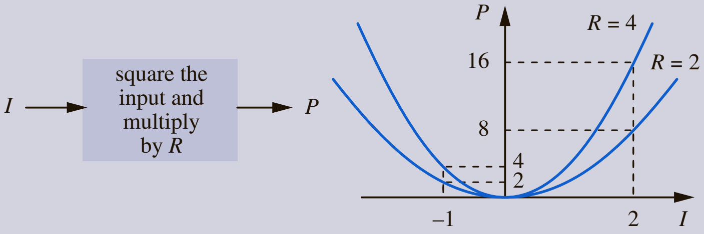


Figure 2.6
The function: $P = I^2R$.

This model for a resistor only approximates the behaviour of the device. In practice, changes in the temperature of the resistor lead to slight changes in the resistance value. If the current through the resistor is excessively high then the resistor overheats and is permanently damaged. It no longer has the correct resistance value. The amount of power that a resistor can handle depends on the materials that have been used in its construction. A good circuit designer would calculate the amount of power to be dissipated and then allow a suitable safety margin to ensure that the resistor cannot be overloaded.

2.3.3 One-to-many

Some rules relating input to output are not functions. Consider the rule: 'take plus or minus the square root of the input', that is

$$x \to \pm\sqrt{x}$$

Now, for example, if 4 is the input, the output is $\pm\sqrt{4}$ which can be 2 or -2. Thus a single input has produced more than one output. The rule is said to be **one-to-many**, meaning that one input has produced many outputs. Rules with this property are not functions. For a rule to be a function there must be a single output for any given input.

By defining a rule more specifically, it may become a function. For example, consider the rule: 'take the positive square root of the input'. This rule is a function because there is a single output for a given input. Note that the domain of this function is $[0, \infty)$ and the range is also $[0, \infty)$.

2.3.4 Many-to-one and one-to-one functions

Consider again the function $f(x) = x^2$ given in Example 2.4. The inputs 2 and -2 both produce the same output, 4, and the function is said to be **many-to-one**. This means that many inputs produce the same output. A many-to-one function can be recognized from its graph. If a horizontal line intersects the graph in more than one place, the function is many-to-one. Figure 2.7 illustrates a many-to-one function, $g(x)$. The inputs x_1, x_2, x_3 and x_4 all produce the same output.

A function is **one-to-one** if different inputs always produce different outputs. A horizontal line will intersect the graph of a one-to-one function in only one place. Figure 2.8 illustrates a one-to-one function, $h(x)$.

Both one-to-one functions and many-to-one functions are supported in technical computing languages. For example, in MATLAB® the function $f(x) = x^2$ can be defined by using the command:

```
f = @(x) x^2;
```

It is now possible to type:

```
f(3)
```

or

```
f(-3)
```

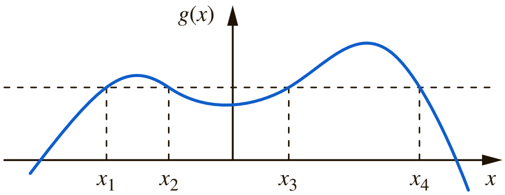


Figure 2.7
The inputs x_1, x_2, x_3 and x_4 all produce the same output, therefore $g(x)$ is a many-to-one function.

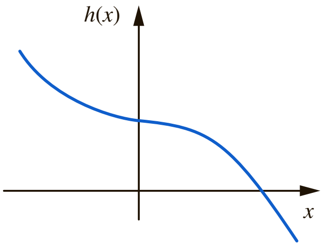


Figure 2.8
Each input produces a different output and so $h(x)$ is a one-to-one function.

both of which have the same result:

```
ans = 9
```

affirming that $f(x) = x^2$ is a many-to-one function.

Notice that the variable used by the function is defined in brackets after the @ sign. This indicates that the input to the function is x and the command creates a **function handle**, `f`. Giving the function a handle enables it to be used elsewhere in the program.

More complicated functions are usually created in a separate file and saved on the computer's internal storage devices. They can be easily reused to create sophisticated programs. In MATLAB® these files are saved with the file name extension `.m` and are often termed **m-files**.

2.3.5 Parametric definition of a function

Functions are often expressed in the form $y(x)$. For every value of x the corresponding value of y can be found and the point with coordinates (x, y) can then be plotted. Sometimes it is useful to express x and y coordinates in terms of a third variable known as a **parameter**. Commonly we use t or θ to denote a parameter. Thus the coordinates (x, y) of the points on a curve can be expressed in the form

$$x = f(t) \qquad y = g(t)$$

For example, given the parametric equations

$$x = t^2 \qquad y = 2t \qquad 0 \leqslant t \leqslant 5$$

we can calculate x and y for various values of the parameter t. Plotting the points (x, y) produces part of a curve known as a parabola.

2.3.6 Composition of functions

Consider the function $y(x) = 2x^2$. We can think of $y(x)$ as being composed of two functions. One function is described by the rule: 'square the input', while the other function is described by the rule: 'double the input'. This is shown in Figure 2.9.

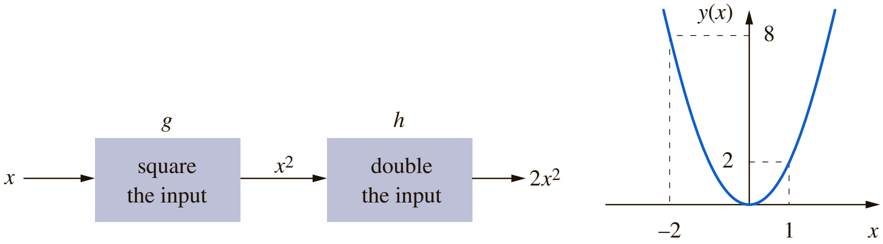


Figure 2.9
The function: $y(x) = h(g(x))$.

Mathematically, if $h(x) = 2x$ and $g(x) = x^2$ then

$$y(x) = 2x^2 = 2(g(x)) = h(g(x))$$

The form $h(g(x))$ is known as a **composition** of the functions h and g. Note that the composition $h(g(x))$ is different from $g(h(x))$ as Example 2.5 illustrates.

Example 2.5 If $f(t) = 2t + 3$ and $g(t) = \frac{t+1}{2}$ write expressions for the compositions

(a) $f(g(t))$

(b) $g(f(t))$

Solution (a) $f(g(t)) = f\left(\frac{t+1}{2}\right)$

The rule describing the function f is: 'double the input and then add 3'. Hence,

$$f\left(\frac{t+1}{2}\right) = 2\left(\frac{t+1}{2}\right) + 3 = t + 4$$

So

$$f(g(t)) = t + 4$$

(b) $g(f(t)) = g(2t + 3)$

The rule for g is: 'add 1 to the input and then divide everything by 2'. So,

$$g(2t + 3) = \frac{2t + 3 + 1}{2} = t + 2$$

Hence

$$g(f(t)) = t + 2$$

Clearly $f(g(t)) \neq g(f(t))$.

2.3.7 Inverse of a function

Consider a function $f(x)$. It can be thought of as accepting an input x, and producing an output $f(x)$. Suppose now that this output becomes the input to the function $g(x)$, and the output from $g(x)$ is x, that is

$$g(f(x)) = x$$

We can think of $g(x)$ as undoing the work of $f(x)$. Figure 2.10 illustrates this situation. Then $g(x)$ is the **inverse** of f, and is written as $f^{-1}(x)$. Since $f^{-1}(x)$ undoes the work of $f(x)$ we have

$$f(f^{-1}(x)) = f^{-1}(f(x)) = x$$

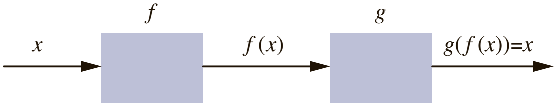


Figure 2.10
The function g is the inverse of f.

Example 2.6 If $f(x) = 5x$ verify that the inverse of f is given by $f^{-1}(x) = \dfrac{x}{5}$.

Solution The function f receives an input of x, and produces an output of $5x$. Hence when the inverse function, f^{-1}, receives an input of $5x$, it produces an output of x, that is

$$f^{-1}(5x) = x$$

We introduce a new variable, z, given by

$$z = 5x$$

so

$$x = \frac{z}{5}$$

Then

$$f^{-1}(z) = x = \frac{z}{5}$$

Writing f^{-1} with x as the argument gives

$$f^{-1}(x) = \frac{x}{5}$$

Example 2.7 If $f(x) = 2x + 1$, find $f^{-1}(x)$.

Solution The function f receives an input of x and produces an output of $2x + 1$. So when the inverse function, f^{-1}, receives an input of $2x + 1$ it produces an output of x, that is

$$f^{-1}(2x + 1) = x$$

We introduce a new variable, z, defined by

$$z = 2x + 1$$

Rearranging gives

$$x = \frac{z - 1}{2}$$

So

$$f^{-1}(z) = x = \frac{z - 1}{2}$$

Writing f^{-1} with x as the argument gives

$$f^{-1}(x) = \frac{x - 1}{2}$$

Example 2.8 Given $g(x) = \dfrac{x-1}{2}$ find the inverse of g.

Solution We know $g(x) = \dfrac{x-1}{2}$, and so $g^{-1}\left(\dfrac{x-1}{2}\right) = x$. Let $y = \dfrac{x-1}{2}$ so that

$$g^{-1}(y) = x$$

But,

$$x = 2y + 1$$

and so

$$g^{-1}(y) = 2y + 1$$

Using the same independent variable as for the function g, we obtain

$$g^{-1}(x) = 2x + 1$$

We note that the inverses of the functions in Examples 2.7 and 2.8 are themselves functions. They are called **inverse functions**. The inverse of $f(x) = 2x + 1$ is $f^{-1}(x) = \dfrac{x-1}{2}$, and the inverse of $g(x) = \dfrac{x-1}{2}$ is $g^{-1}(x) = 2x + 1$. This illustrates the important point that if $f(x)$ and $g(x)$ are two functions and $f(x)$ is the inverse of $g(x)$, then $g(x)$ is the inverse of $f(x)$. It is important to point out that not all functions possess an inverse function. Consider $f(x) = x^2$, for $-\infty < x < \infty$.

The function, f, is given by the rule: 'square the input'. Since both a positive and negative value of x will yield the output x^2, the inverse rule is given by: 'take plus or minus the square root of the input'. As discussed earlier, this is a one-to-many rule and so is not a function. Clearly not all functions have an inverse function. In fact, only one-to-one functions have an inverse function. Suppose we restrict the domain of $f(x) = x^2$ such that $x \geqslant 0$. Then f is a one-to-one function and so there is an inverse function. The inverse function is $f^{-1}(x)$ given by

$$f^{-1}(x) = +\sqrt{x}$$

Clearly,

$$f^{-1}(f(x)) = f^{-1}(x^2) = x$$

where x is the positive square root of x^2. Restricting the domain of a many-to-one function so that a one-to-one function results is a common technique of ensuring an inverse function can be found.

2.3.8 Continuous and piecewise continuous functions

We now introduce in an informal way the concept of continuous and piecewise continuous functions. A more rigorous treatment follows in Chapter 10 after we have discussed limits. Figure 2.11 shows a graph of $f(x) = \dfrac{1}{x}$. Note that there is a break, or discontinuity, in the graph at $x = 0$. The function $f(x) = \dfrac{1}{x}$ is said to be **discontinuous** at $x = 0$.

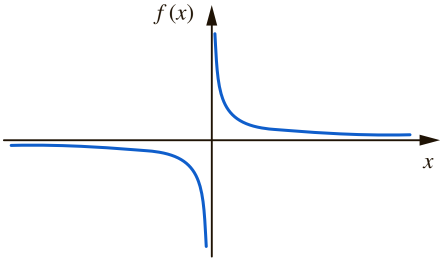


Figure 2.11
The function $f(x) = \frac{1}{x}$ has a discontinuity at $x = 0$.

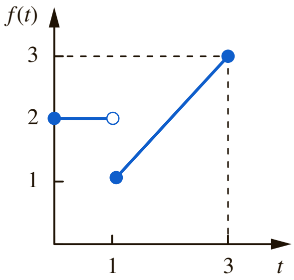


Figure 2.12
The function $f(t)$ is a piecewise continuous function with a discontinuity at $t = 1$.

If the graph of a function, $f(x)$, contains a break, then $f(x)$ is discontinuous.

A function whose graph has no breaks is a **continuous** function.

Sometimes a function is defined by different rules on different intervals of the domain. For example, consider

$$f(t) = \begin{cases} 2 & 0 \leqslant t < 1 \\ t & 1 \leqslant t \leqslant 3 \end{cases}$$

The domain is [0, 3] but the rule on [0, 1) is different to that on [1, 3]. The graph of $f(t)$ is shown in Figure 2.12. Recall the convention of using • to denote that the end-point is included and ○ to denote the end-point is excluded. Note that $f(t)$ has a discontinuity at $t = 1$. Each component, or piece, of the graph is continuous and $f(t)$ is said to be **piecewise continuous**.

A piecewise continuous function has a finite number of discontinuities in any given interval.

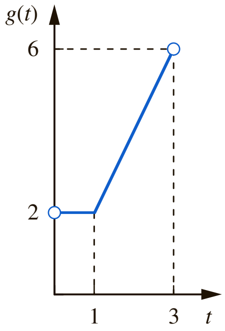


Figure 2.13
The function $g(t)$ is a continuous function on (0, 3).

Not all functions defined differently on different intervals are discontinuous. For example,

$$g(t) = \begin{cases} 2 & 0 < t < 1 \\ 2t & 1 \leqslant t < 3 \end{cases}$$

is a continuous function on the interval (0, 3), as shown in Figure 2.13.

2.3.9 Periodic functions

A **periodic function** is a function which has a definite pattern which is repeated at regular intervals. More formally we say a function, $f(t)$, is periodic if

$$f(t) = f(t + T)$$

for all values of t. The constant, T, is known as the **period** of the function.

Example 2.9 Figure 2.14 illustrates a periodic waveform. It is often referred to as a **triangular** waveform because of its shape. The form of the function is repeated every two seconds, that is

$$f(t) = f(t + 2)$$

and so the function is periodic. The period is 2 seconds, that is $T = 2$. Note that this function is continuous.

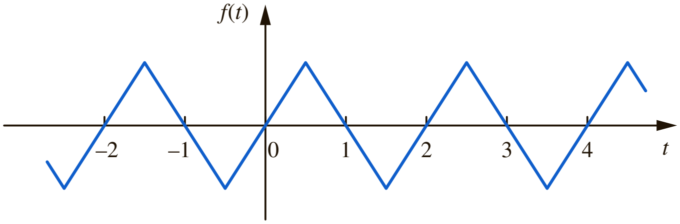


Figure 2.14
The triangular waveform is a periodic function.

Engineering application 2.2

Saw-tooth waveform

Figure 2.15 illustrates a **saw-tooth** voltage waveform. It is called a saw-tooth waveform because its shape is similar to that of the teeth on a saw. It has many uses in electronic engineering. One use would be to provide a signal to sweep a beam of electrons across a cathode ray tube in a uniform way and then quickly move the beam back to the start again. This technique is used in an analogue oscilloscope and forms a signal for the time base.

The form of the function is repeated every three seconds, that is

$$v(t) = v(t + 3)$$

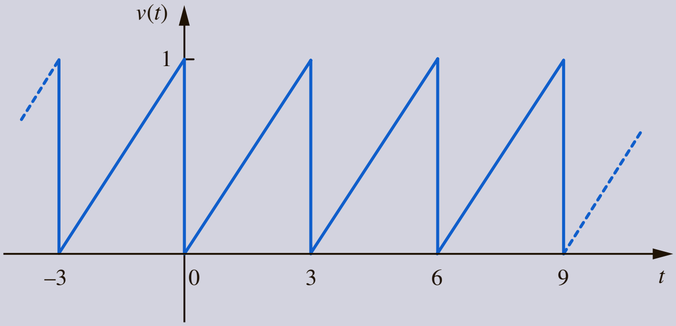


Figure 2.15
The saw-tooth waveform is a periodic function.

Technical computing languages often have a range of built-in functions for producing waveforms. Sometimes specialist functions are provided in a separate software package. In MATLAB®, these software packages are known as **toolboxes**. The signal processing toolbox has a function for generating saw-tooth waves. This can be accessed by typing, for example:

```
t=(-2*pi:0.1:2*pi);
plot(t, sawtooth(t));
```

This will plot two periods of a saw-tooth wave. The first line generates a set of time values $-2\pi \le t < 2\pi$ in a vector form with a spacing of `0.1` between each point. The second line plots t against the result of passing the vector t to the `sawtooth` function. The `sawtooth` function always produces a wave with a period of 2π. It highlights the need to read the manual pages carefully before using a function to understand how it will behave.

Engineering application 2.3

Square waveform

Periodic functions may be piecewise continuous. Consider the function $g(t)$ defined by

$$g(t) = \begin{cases} 1 & 0 \leqslant t < 1 \\ 0 & 1 \leqslant t < 2 \end{cases} \qquad \text{period} = 2$$

The function $g(t)$ is periodic with period 2. A graph of $g(t)$ is shown in Figure 2.16. This function is commonly referred to as a **square waveform** by engineers. In Figure 2.16 the open and closed end-points have been shown for mathematical correctness. Note, however, that engineers tend to omit these when sketching functions with discontinuities and usually they use a vertical line to show the discontinuity. This reflects the fact that no practical waveform can ever change its level instantaneously: even very fast rising waveforms still have a finite **rise time**. The function has discontinuities at $t = \ldots, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots$.

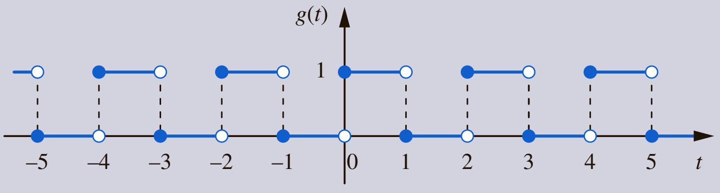


Figure 2.16
The function $g(t)$ is both piecewise continuous and periodic.

The square waveform is often used in electronic engineering, particularly in digital electronic systems. One example is the clock signal that is generated to ensure that all of the digital electronic circuits switch around the same time and so remain in synchronisation.

EXERCISES 2.3

1 Represent the following intervals on the real line:

(a) $[1, 3]$ (b) $[2, 4)$
(c) $(0, 3.5)$ (d) $[-2, 0)$
(e) $(-1, 1]$ (f) $2 \leqslant x < 4$
(g) $0 < x < 2$ (h) $-3 \leqslant x \leqslant -1$
(i) $0 \leqslant x < 3$

2 Describe the rule associated with the following functions, sketch their graphs and state their domains and ranges:

(a) $f(x) = 2x^2$
(b) $f(x) = x^2 - 1 \quad 0 \leqslant x$
(c) $g(t) = 3t - 4 \quad 0 \leqslant t$
(d) $y(x) = x^3$
(e) $f(t) = 0.5t + 2 \quad -2 \leqslant t \leqslant 10$
(f) $z(x) = 3x - 2 \quad 3 \leqslant x \leqslant 8$

3 If $f(x) = 5x + 4$, find

(a) $f(3)$
(b) $f(-3)$
(c) $f(\alpha)$
(d) $f(x + 1)$
(e) $f(3\alpha)$
(f) $f(x^2)$

4 If $g(t) = 5t^2 - 4$, find

(a) $g(0)$
(b) $g(2)$
(c) $g(-3)$
(d) $g(x)$
(e) $g(2t - 1)$

5 The reactance, X_C, offered by a capacitor is given by $X_C = \dfrac{1}{2\pi fC}$, where f is the frequency of the applied alternating current, and C is the capacitance of the capacitor. If $C = 10^{-6}$ F, find X_C when $f = 50$ Hz.

6 Classify the functions in Question 2 as one-to-one or many-to-one.

7 Find the inverse of the following functions:

(a) $f(x) = x + 4$
(b) $g(t) = 3t + 1$
(c) $y(x) = x^3$
(d) $h(t) = \dfrac{t - 8}{3}$
(e) $f(t) = \dfrac{t - 1}{3}$
(f) $h(x) = x^3 - 1$
(g) $k(v) = 7 - v$
(h) $m(n) = \frac{1}{3}(1 - 2n)$

8 Given $f(t) = 2t$, $g(t) = t - 1$ and $h(t) = t^2$ write expressions for

(a) $f(g(t))$ (b) $f(h(t))$
(c) $g(h(t))$ (d) $g(f(t))$
(e) $h(g(t))$ (f) $h(f(t))$
(g) $f(f(t))$ (h) $g(g(t))$
(i) $h(h(t))$ (j) $f(g(h(t)))$
(k) $g(f(h(t)))$ (l) $h(g(f(t)))$

9 Given $f(t) = t^2 + 1$, $g(t) = 3t + 2$ and $h(t) = \dfrac{1}{t}$, write expressions for

(a) $f(g(t))$ (b) $f(h(t))$
(c) $g(h(t))$ (d) $h(f(t))$
(e) $f(g(h(t)))$

10 Given $f(t) = 2t$, $g(t) = 2t + 1$, $h(t) = 1 - 3t$, write expressions for the following:

(a) $f^{-1}(t)$ (b) $g^{-1}(t)$ (c) $h^{-1}(t)$

11 Given $a(x) = 3x - 2$, $b(x) = \dfrac{2}{x}$, $c(x) = 1 + \dfrac{1}{x}$ write expressions for

(a) $a^{-1}(x)$ (b) $b^{-1}(x)$ (c) $c^{-1}(x)$

12 Given $f(t) = 2t + 3$, $g(t) = 3t$ and $h(t) = f(g(t))$ write expressions for

(a) $h(t)$
(b) $f^{-1}(t)$
(c) $g^{-1}(t)$
(d) $h^{-1}(t)$
(e) $g^{-1}(f^{-1}(t))$

What do you notice about (d) and (e)?

13 Sketch the following functions:

(a) $f(t) = \begin{cases} t & 0 \leqslant t \leqslant 3 \\ 3 & 3 < t \leqslant 4 \end{cases}$

(b) $g(x) = \begin{cases} 2 - x & 0 \leqslant x < 1 \\ 2 & 1 \leqslant x \leqslant 3 \end{cases}$

(c) $a(t) = \begin{cases} 1 - t & 0 \leqslant t \leqslant 1 \\ t - 1 & 1 < t \leqslant 2 \end{cases}$

(d) $b(x) = \begin{cases} 2 & 0 \leqslant x \leqslant 1 \\ 1 & 1 < x \leqslant 2 \\ 3 - x & 2 < x \leqslant 3 \end{cases}$

14 Sketch

$$f(t) = \begin{cases} t & 0 \leqslant t < 2 \\ 5 - 2t & 2 \leqslant t < 3 \end{cases}$$

Is the function piecewise continuous or continuous? State, if they exist, the position of any discontinuities.

15 The function $h(t)$ is defined by

$$h(t) = \begin{cases} 2 - t & 0 \leqslant t < 2 \\ 2t - 4 & 2 \leqslant t \leqslant 3 \end{cases}$$

and $h(t)$ has period 3. Sketch $h(t)$ on the interval [0, 6].

16 The function $g(t)$ is defined by

$$g(t) = \begin{cases} 1 & 0 \leqslant t \leqslant 1 \\ 2 - t & 1 < t < 2 \end{cases}$$

and $g(t)$ has period 2. Sketch $g(t)$ on the interval $[-1, 4]$. State any points of discontinuity.

Solutions

2 (a) Square the input and then multiply by 2; domain $(-\infty, \infty)$, range $[0, \infty)$

(b) Square the input, then subtract 1; domain $[0, \infty)$, range $[-1, \infty)$

(c) Multiply input by 3 and subtract 4; domain $[0, \infty)$, range $[-4, \infty)$

(d) Cube the input; domain $(-\infty, \infty)$, range $(-\infty, \infty)$

(e) Multiply input by 0.5 and then add 2; domain $[-2, 10]$, range $[1, 7]$

(f) Multiply input by 3 and then subtract 2; domain $[3, 8]$, range $[7, 22]$

3 (a) 19 (b) -11 (c) $5\alpha + 4$
(d) $5x + 9$ (e) $15\alpha + 4$ (f) $5x^2 + 4$

4 (a) -4 (b) 16 (c) 41
(d) $5x^2 - 4$ (e) $20t^2 - 20t + 1$

5 3183 ohms

6 (a) many-to-one (b) one-to-one
(c) one-to-one (d) one-to-one
(e) one-to-one (f) one-to-one

7 (a) $f^{-1}(x) = x - 4$

(b) $g^{-1}(t) = \dfrac{t-1}{3}$

(c) $y^{-1}(x) = x^{1/3}$

(d) $h^{-1}(t) = 3t + 8$

(e) $f^{-1}(t) = 3t + 1$

(f) $h^{-1}(x) = (x + 1)^{1/3}$

(g) $k^{-1}(v) = 7 - v$

(h) $m^{-1}(n) = \dfrac{1 - 3n}{2}$

8 (a) $2(t - 1)$ (b) $2t^2$ (c) $t^2 - 1$
(d) $2t - 1$ (e) $(t - 1)^2$ (f) $4t^2$
(g) $4t$ (h) $t - 2$ (i) t^4
(j) $2(t^2 - 1)$ (k) $2t^2 - 1$ (l) $(2t - 1)^2$

9 (a) $9t^2 + 12t + 5$ (b) $\dfrac{1}{t^2} + 1$
(c) $\dfrac{3}{t} + 2$ (d) $\dfrac{1}{t^2 + 1}$
(e) $\dfrac{9}{t^2} + \dfrac{12}{t} + 5$

10 (a) $\dfrac{t}{2}$ (b) $\dfrac{t-1}{2}$ (c) $\dfrac{1-t}{3}$

11 (a) $\dfrac{x+2}{3}$ (b) $\dfrac{2}{x}$ (c) $\dfrac{1}{x-1}$

12 (a) $6t + 3$ (b) $\dfrac{t-3}{2}$ (c) $\dfrac{t}{3}$
(d) $\dfrac{t-3}{6}$ (e) $\dfrac{t-3}{6}$

13 See Figure S.1.

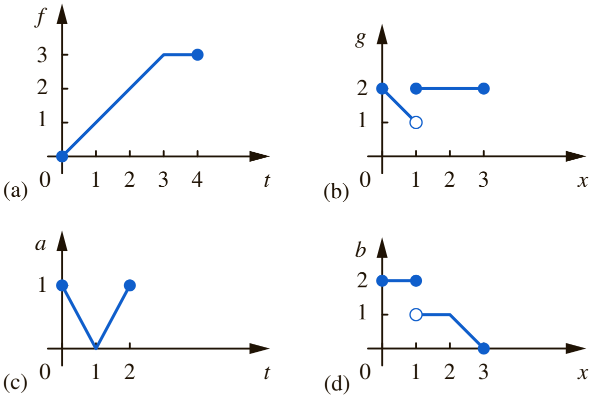


Figure S.1

14 Piecewise continuous; discontinuity at $t = 2$. See Figure S.2.

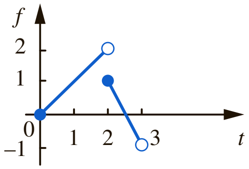


Figure S.2

15 See Figure S.3.

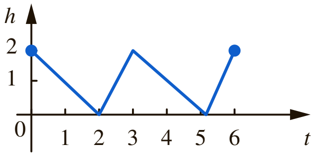


Figure S.3

16 Discontinuities at $t = 0, 2$. See Figure S.4.

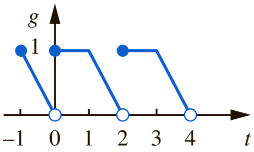


Figure S.4

2.4 REVIEW OF SOME COMMON ENGINEERING FUNCTIONS AND TECHNIQUES

This section provides a catalogue of the more common engineering functions. The important properties and definitions are included together with some techniques. It is intended that readers will refer to this section for revision purposes and as the need arises throughout the rest of the book.

2.4.1 Polynomial functions

A **polynomial expression** has the form

$$a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0$$

where n is a non-negative integer, $a_n, a_{n-1}, \ldots, a_1, a_0$ are constants and x is a variable. A **polynomial function**, $P(x)$, has the form

$$P(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0 \tag{2.3}$$

Examples of polynomial functions include

$$P_1(x) = 3x^2 - x + 2 \tag{2.4}$$

$$P_2(z) = 7z^4 + z^2 - 1 \tag{2.5}$$

$$P_3(t) = 3t + 9 \tag{2.6}$$

$$P_4(t) = 6 \tag{2.7}$$

where x, z and t are independent variables. It is common practice to contract the term polynomial expression to **polynomial**. By convention, a polynomial is usually written with the powers either increasing or decreasing. For example,

$$3x + 9x^2 - x^3 + 2$$

would be written as either

$$-x^3 + 9x^2 + 3x + 2 \qquad \text{or} \qquad 2 + 3x + 9x^2 - x^3$$

The **degree** of a polynomial or polynomial function is the value of the highest power. Equation (2.4) has degree 2, Equation (2.5) has degree 4, Equation (2.6) has degree 1 and Equation (2.7) has degree 0. Equation (2.3) has degree n. Polynomials with low degrees have special names (see Table 2.1).

Table 2.1

Polynomial	*Degree*	*Name*
$ax^4 + bx^3 + cx^2 + dx + e$	4	Quartic
$ax^3 + bx^2 + cx + d$	3	Cubic
$ax^2 + bx + c$	2	Quadratic
$ax + b$	1	Linear
a	0	Constant

Typical graphs of some polynomial functions are shown in Figure 2.17.

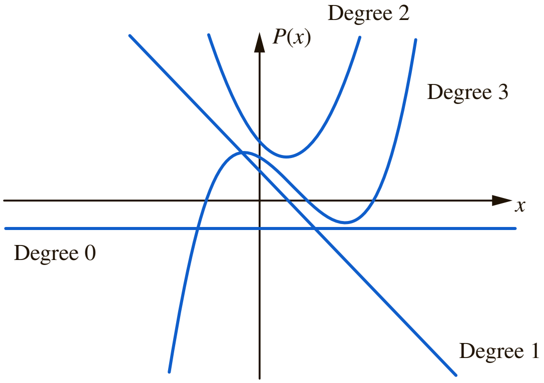


Figure 2.17
Some typical polynomials.

Engineering application 2.4

Ohm's law

Recall from Engineering application 1.1 that the current flowing through a resistor is related to the voltage applied across it by Ohm's law. The equation is

$$V = IR$$

where V = voltage across the resistor;
I = current through the resistor;
R = resistance value of the resistor, which is a constant for a given temperature.

→

Note that the voltage is a linear polynomial function with I as the independent variable.

This equation is only valid for a finite range of currents. If too much voltage is applied to the resistor, then the current flowing through the resistor becomes sufficient for the resistor to overheat and breakdown.

Engineering application 2.5

A non-ideal voltage source

An ideal voltage source has zero internal resistance and its output voltage, V, is independent of the load applied to it; that is, V remains constant, independent of the current it supplies. It is called an ideal voltage source because it is difficult to create such a source in practice; it is in effect an abstraction that is useful when developing engineering models of real electronic systems. A **non-ideal voltage source** has an internal resistance. Due to this internal resistance, the output voltage from such a source is reduced when current is drawn from the source. The voltage reduction increases as more current is drawn. Figure 2.18 shows a non-ideal voltage source. It is modelled as an ideal voltage source in series with an internal resistor with resistance R_s. The output voltage of the non-ideal voltage source is v_o while v_R is the voltage drop across the internal resistor and i is the load current. Using Kirchhoff's voltage law,

$$V = v_R + v_o$$

and hence by Ohm's law,

$$V = iR_s + v_o$$

$$v_o = V - iR_s$$

Note that V and R_s are constants and so the output voltage is a linear polynomial function with independent variable i. The equation gives the output voltage across the load as a function of the current through the load. The output characteristic for the non-ideal voltage source is obtained by varying the load resistor R_L and is plotted in Figure 2.19. Notice that the output voltage of the non-ideal voltage source decreases as the load current increases and is equal in value to the ideal voltage source only when there is no load current.

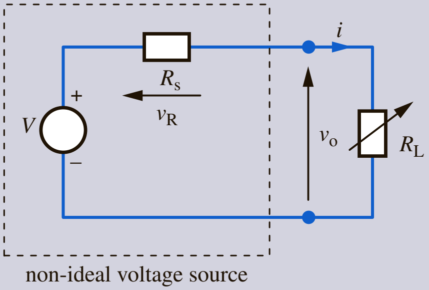


Figure 2.18
A non-ideal voltage source connected to a load resistor, R_L.

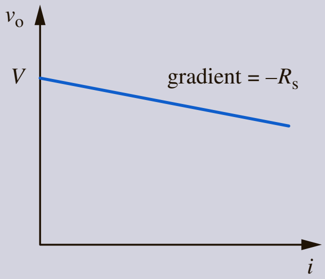


Figure 2.19
Output characteristic of a non-ideal voltage source.

This is why it is called a non-ideal voltage source. Engineers would prefer to have a source that maintained a constant voltage no matter how much current was drawn but it is not possible to build such a source.

Engineering application 2.6

Wind power turbines

Wind turbines are an important source of electrical power. The most common type, and the ones which are usually found in offshore installations, resemble a desktop fan and are called **horizontal axis turbines**. The wind driving a turbine blade consists of many molecules of air, each having a tiny amount of mass. This mass passing the blade area each second carries kinetic energy, which is the source of the wind power. The wind power, P, can be calculated using the formula

$$P = \frac{1}{2}Mv^2$$

where M is the total mass of air per second passing the blade in kg s^{-1} and v is the velocity of the air in m s^{-1}.

The mass per second can be calculated by considering the area swept out by the blade, A, the density of the air, ρ, and the velocity:

$$M = \rho A v$$

This equation can be substituted in the power equation

$$P = \frac{1}{2}(\rho A v)\,v^2 = \frac{1}{2}\rho A v^3 \tag{2.8}$$

The available wind power therefore increases with the cube of the velocity. Note that the power is a cubic polynomial function of the independent variable, v.

At 20 °C the air density is approximately 1.204 kg m^{-3}. Consider the case of an offshore turbine that has a swept area of 6362 m^2 and a rated wind speed of 15 m s^{-1}. The maximum theoretical power at the rated speed is therefore

$$P = \frac{1}{2}\rho A v^3 = \frac{1}{2} \times 1.204 \times 6362 \times 15^3 = 12.93 \text{ MW}$$

The actual rated power of the device is approximately 3 MW because other physical processes and losses have to be accounted for, yet Equation (2.8) remains one of the most fundamental in the study of wind power.

Many excellent computer software packages exist for plotting graphs and these, as well as graphics calculators, may be used to solve polynomial equations. The real roots of the equation $P(x) = 0$ are given by the values of the intercepts of the function $y = P(x)$ and the x axis, because on the x axis y is zero.

Figure 2.20 shows a graph of $y = P(x)$. The graph intersects the x axis at $x = x_1$, $x = x_2$ and $x = x_3$, and so the equation $P(x) = 0$ has real roots x_1, x_2 and x_3, that is $P(x_1) = P(x_2) = P(x_3) = 0$.

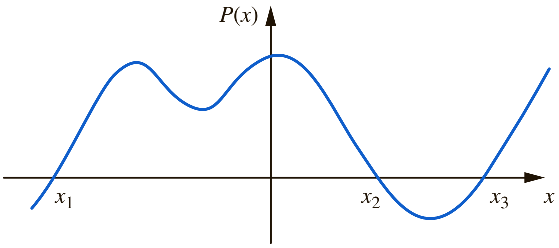


Figure 2.20
A polynomial function which cuts the x axis at points x_1, x_2 and x_3.

EXERCISES 2.4.1

1 State the degree of the following polynomial expressions:

(a) $z^3 + 2z^2 - 8 + 13z$
(b) $t^2 - 5t^5 + 2 - 8t^3$
(c) $3w - 5w^2 + 12w^4$
(d) $7x - x^2$
(e) $3(2t^2 - 9t + 1)$
(f) $2z(2z + 1)(2z - 1)$

Solutions

1 (a) 3 (b) 5 (c) 4 (d) 2 (e) 2 (f) 3

Technical Computing Exercises 2.4.1

Use a technical computing language such as MATLAB® to do the following exercises.

1 (a) Plot $y = x^3$ and $y = 4 - 2x$ in the interval [-3, 3]. Be aware that in MATLAB®, to carry out operations on individual elements of a vector, special notation is used. For example to multiply each element in vector `a` by the corresponding element in vector `b` (having the same dimension as vector `a`) you would type `a.*b`. Other functions such as raising to a power also require a dot prefix if they are to be carried out on each individual element, rather than the whole matrix.

Note the x coordinate of the point of intersection.

(b) Draw $y = x^3 + 2x - 4$. Note the coordinate of the point where the curve cuts the x axis. Compare your answer with that from (a). Explain your findings.

2 Plot the following functions:

(a) $y = 3x^3 - x^2 + 2x + 1 \qquad -2 \leqslant x \leqslant 2$

(b) $y = x^4 + \dfrac{x^3}{3} - \dfrac{5x^2}{2} + x - 1 \qquad -3 \leqslant x \leqslant 2$

(c) $y = x^5 - x^2 + 2 \qquad -2 \leqslant x \leqslant 2$

Hence estimate the real roots of

$$0 = 3x^3 - x^2 + 2x + 1 \qquad -2 \leqslant x \leqslant 2$$

$$0 = x^4 + \frac{x^3}{3} - \frac{5x^2}{2} + x - 1 \qquad -3 \leqslant x \leqslant 2$$

$$0 = x^5 - x^2 + 2 \qquad -2 \leqslant x \leqslant 2$$

3 Use the `roots` function in MATLAB® or equivalent to calculate a more accurate value for the real roots estimated in question 2 and confirm your answers are correct.

4 (a) Draw $y = 2x^2$ and $y = x^3 + 6$ using the same axes. Use your graphs to find approximate solutions to $x^3 - 2x^2 + 6 = 0$.

(b) Add the line $y = -3x + 5$ to your graph. State approximate solutions to
(i) $x^3 + 3x + 1 = 0$
(ii) $2x^2 + 3x - 5 = 0$

2.4.2 Rational functions

A **rational function**, $R(x)$, has the form

$$R(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomial functions; P is the **numerator** and Q is the **denominator**.

The functions

$$R_1(x) = \frac{x+6}{x^2+1} \qquad R_2(t) = \frac{t^3-1}{2t+3} \qquad R_3(z) = \frac{2z^2+z-1}{z^2+3z-2}$$

are all rational. When sketching the graph of a rational function, $y = f(x)$, it is usual to draw up a table of x and y values. Indeed this has been common practice when sketching any graph although the use of graphics calculators is now replacing this custom. It is still useful to answer questions such as:

'How does the function behave as x becomes large positively?'
'How does the function behave as x becomes large negatively?'
'What is the value of the function when $x = 0$?'
'At what values of x is the denominator zero?'

Figure 2.21 shows a graph of the function $y = \dfrac{1+2x}{x} = \dfrac{1}{x} + 2$. As x increases, the value of y approaches 2. We write this as

$$y \to 2 \qquad \text{as} \qquad x \to \infty$$

and say 'y tends to 2 as x tends to infinity'. Also from Figure 2.21, we see that

$$y \to \pm\infty \qquad \text{as} \qquad x \to 0$$

As $x \to \infty$, the graph gets nearer and nearer to the straight line $y = 2$. We say that $y = 2$ is an **asymptote** of the graph. Similarly, $x = 0$, that is the y axis, is an asymptote since the graph approaches the line $x = 0$ as $x \to 0$.

If the graph of any function gets closer and closer to a straight line then that line is called an asymptote. Figure 2.22 illustrates some rational functions with their asymptotes indicated by dashed lines. In Figure 2.22(a) the asymptotes are the horizontal line $y = 3$

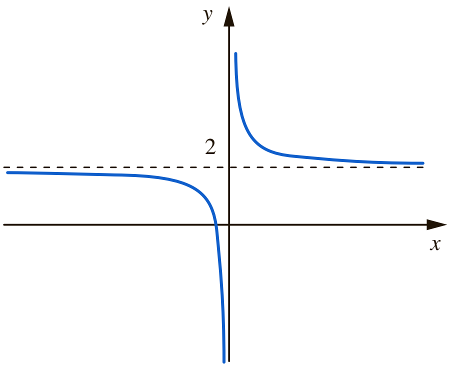


Figure 2.21
The function: $y = \dfrac{1+2x}{x} = \dfrac{1}{x} + 2$.

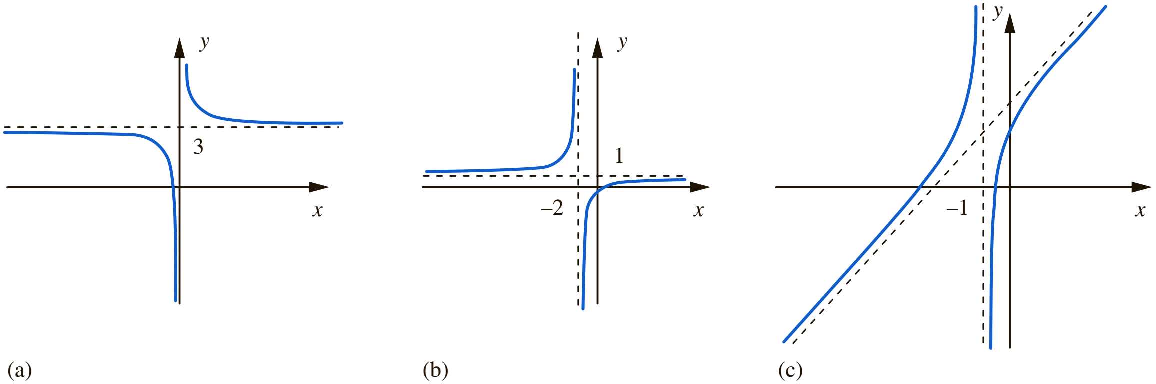


Figure 2.22
Some examples of functions with their asymptotes:
(a) $y = \dfrac{3x+1}{x} = 3 + \dfrac{1}{x}$; (b) $y = \dfrac{x-1}{x+2}$; (c) $y = \dfrac{x^2+4x+2}{x+1} = x + 3 - \dfrac{1}{x+1}$.

and the y axis, that is $x = 0$. In Figure 2.22(b) the asymptotes are the horizontal line $y = 1$ and the vertical line $x = -2$; in Figure 2.22(c) they are $y = x + 3$ and the vertical line $x = -1$. The asymptote $y = x + 3$, being neither horizontal nor vertical, is called an **oblique asymptote**. Oblique asymptotes occur only when the degree of the numerator exceeds the degree of the denominator by one.

We see that the vertical asymptotes occur at values of x which make the denominator zero. These values are particularly important to engineers and are known as the **poles** of the function. The function shown in Figure 2.22(a) has a pole at $x = 0$; the function shown in Figure 2.22(b) has a pole at $x = -2$; and the function shown in Figure 2.22(c) has a pole at $x = -1$.

If the graph of a function approaches a straight line, the line is known as an asymptote. Asymptotes may be horizontal, vertical or oblique.

Values of the independent variable where the denominator is zero are called poles of the function.

Example 2.10 Sketch the rational function $y = \dfrac{x}{x^2 + x - 2}$.

Solution For large values of x, the x^2 term in the denominator has a much greater value than the x in the numerator. Hence,

$$y \to 0 \quad \text{as} \quad x \to \infty$$
$$y \to 0 \quad \text{as} \quad x \to -\infty$$

Therefore the x axis, that is $y = 0$, is an asymptote. Writing y as

$$y = \frac{x}{(x-1)(x+2)}$$

we see the function has poles at $x = 1$ and $x = -2$; that is, there are vertical asymptotes at $x = 1$ and $x = -2$. Substitution into the function of a number of values of x allows a table to be drawn up:

x	-3	-2.5	-2.1	-1.9	-1.5	-1	0	0.5	0.9	1.1	1.5	2	3
y	-0.75	-1.43	-6.77	6.55	1.20	0.50	0	-0.40	-3.10	3.55	0.86	0.50	0.30

The graph of the function can then be sketched as shown in Figure 2.23.

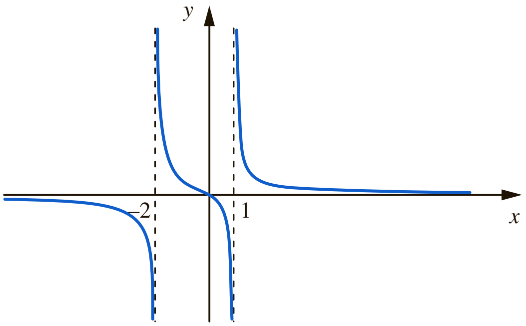


Figure 2.23
The function: $y = \dfrac{x}{x^2 + x - 2}$.

Engineering application 2.7

Equivalent resistance

Recall from Engineering application 1.2 that the formula for the equivalent resistance of two resistors in parallel is given by:

$$\frac{1}{R_E} = \frac{1}{R_1} + \frac{1}{R_2}$$

Consider a circuit consisting of two resistors in parallel as shown in Figure 2.24. One has a known resistance of 1 Ω and the other has a variable resistance, R Ω. The equivalent resistance, R_E Ω, satisfies

$$\frac{1}{R_E} = \frac{1}{R} + \frac{1}{1} = \frac{1+R}{R}$$

Hence,

$$R_E = \frac{R}{1+R}$$

Thus the equivalent resistance is a rational function of R, with domain $R \geqslant 0$. The graph of this function is shown in Figure 2.25. When $R = 0$ we note that $R_E = 0$, corresponding to a short circuit. As the value of R increases, that is $R \to \infty$, the equivalent resistance R_E approaches 1 so that $R_E = 1$ is an asymptote.

→

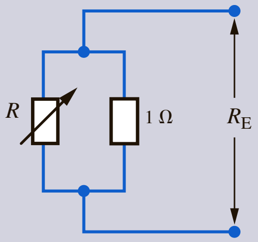


Figure 2.24
Two resistors in parallel.

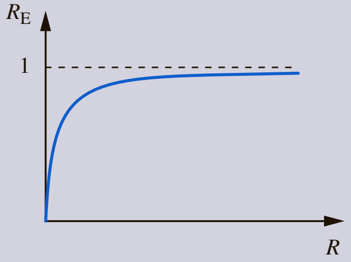


Figure 2.25
The equivalent resistance, R_E, increases as R increases.

EXERCISES 2.4.2

1 State the poles of the following rational functions:

(a) $y(x) = \dfrac{x+3}{x-2}$ (b) $y(x) = \dfrac{2x+1}{x+7}$

(c) $y(t) = \dfrac{t^2+t+1}{2t+3}$ (d) $X(s) = \dfrac{s+1}{s^2-4}$

(e) $H(s) = \dfrac{3}{s^2+6s+5}$

(f) $G(s) = \dfrac{2s+7}{s^2+3s-18}$

(g) $x(t) = \dfrac{9}{t^3-t}$

(h) $p(t) = \dfrac{2t-6}{t^2+10t+25}$

2 Describe the horizontal asymptote of each of the following functions:

(a) $y(x) = 6 + \dfrac{1}{x}$ (b) $h(t) = \dfrac{2}{t} - 1$

(c) $y(r) = 3 - \dfrac{2}{5r}$ (d) $v(t) = \dfrac{6+t}{t}$

(e) $r(v) = \dfrac{2v-5}{3v}$

(f) $a(t) = \dfrac{t^2+2t+1}{t^2}$

(g) $m(s) = \dfrac{10-2s-3s^2}{2s^2}$

3 Describe the vertical asymptotes of each of the following functions:

(a) $y(x) = \dfrac{3x+1}{x-2}$ (b) $y(t) = \dfrac{6t-3}{4t+4}$

(c) $h(s) = \dfrac{9}{(s+2)(s-1)}$

(d) $G(t) = \dfrac{1}{t^2-1}$ (e) $H(s) = \dfrac{s}{s^2-1}$

(f) $y(x) = \dfrac{2x}{x^2-1}$

(g) $w(f) = \dfrac{f+2}{f^2+f-6}$

(h) $P(t) = \dfrac{t^2+t+1}{t^2+6t+9}$

(i) $T(x) = \dfrac{x^3}{2x-1}$

(j) $Q(r) = \dfrac{6+r}{r^2-r-12}$

4 Describe the oblique asymptote of each of the following functions:

(a) $y(x) = x + 3 + \dfrac{1}{x-1}$

(b) $y(x) = 2x - 1 + \dfrac{3}{x+2}$

(c) $y(x) = \dfrac{x}{2} - \dfrac{3}{4} + \dfrac{1}{2x+7}$

(d) $y(x) = 3x - 1 + \dfrac{5}{2x+2}$

(e) $y(x) = 2x - 1 + \dfrac{x+2}{x^2-1}$

(f) $y(x) = 3x - 2 + \dfrac{4}{2x-1}$

(g) $y(x) = 4 - 2x + \dfrac{3}{2x+3}$

5 Show that

$$I(x) = \frac{2x}{3} - \frac{7}{9} + \frac{23}{9(3x+2)}$$

can be expressed in the equivalent form

$$\frac{2x^2 - x + 1}{3x + 2}$$

Sketch the rational function $I(x)$ and state any asymptotes.

6 Show that

$$p(x) = 2x + \frac{1}{2} + \frac{9}{2(2x+3)}$$

can be written in the equivalent form

$$\frac{4x^2 + 7x + 6}{2x + 3}$$

Sketch the rational function $p(x)$ and state any asymptotes.

7 Show that the function

$$y(x) = x + \frac{7 - x}{x^2 + 3}$$

can be expressed in the equivalent form

$$\frac{x^3 + 2x + 7}{x^2 + 3}$$

Sketch the rational function $y(x)$ and state any asymptotes.

Solutions

1 (a) 2 (b) -7 (c) $-\frac{3}{2}$

(d) $-2, 2$ (e) $-5, -1$ (f) $-6, 3$

(g) $-1, 0, 1$ (h) -5

2 (a) $y = 6$ (b) $h = -1$

(c) $y = 3$ (d) $v = 1$

(e) $r = \frac{2}{3}$ (f) $a = 1$

(g) $m = -\frac{3}{2}$

3 (a) $x = 2$ (b) $t = -1$

(c) $s = -2, 1$ (d) $t = -1, 1$

(e) $s = -1, 1$ (f) $x = -1, 1$

(g) $f = -3, 2$ (h) $t = -3$

(i) $x = 0.5$ (j) $r = -3, 4$

4 (a) $y = x + 3$ (b) $y = 2x - 1$

(c) $y = \frac{x}{2} - \frac{3}{4}$ (d) $y = 3x - 1$

(e) $y = 2x - 1$ (f) $y = 3x - 2$

(g) $y = 4 - 2x$

5 $x = -\frac{2}{3}, I = \frac{2x}{3} - \frac{7}{9}$

6 $x = -\frac{3}{2}, p = 2x + \frac{1}{2}$

7 $y = x$

Technical Computing Exercises 2.4.2

1 Use a technical computing language to plot the following rational functions. State any asymptotes.

(a) $f(x) = \dfrac{(2x+1)}{(x-3)} \quad -4 \leqslant x \leqslant 4$

(b) $g(s) = \dfrac{s}{(s+1)} \quad -3 \leqslant s \leqslant 3$

(c) $h(z) = \dfrac{z}{(z^2+1)} \quad -3 \leqslant z \leqslant 3$

(d) $y(x) = \dfrac{(x+1)}{x} \quad -3 \leqslant x \leqslant 3$

(e) $r(x) = \dfrac{2x}{(x-1)(x-2)} \quad -3 \leqslant x \leqslant 3$

2 Plot the functions given in Question 4 in Exercises 2.4.2 for $-10 \leqslant x \leqslant 10$.

2.4.3 Exponential functions

An **exponent** is another name for a power or index. Expressions involving exponents are called **exponential expressions**, for example 3^4, a^b, and m^n. In the exponential expression a^x, a is called the **base**; x is the exponent. Exponential expressions can be simplified and manipulated using the laws of indices. These laws are summarized here.

$$a^m a^n = a^{m+n} \qquad \frac{a^m}{a^n} = a^{m-n} \qquad a^0 = 1 \qquad a^{-m} = \frac{1}{a^m} \qquad (a^m)^n = a^{mn}$$

Example 2.11 Simplify

(a) $\dfrac{a^{3x}a^{2x}}{a^{4x}}$ (b) $a^{2t}(1-a^t)+a^{3t}$ (c) $\dfrac{(a^y)^2}{2a^y}$ (d) $\dfrac{a^{-6z}}{a^{-2z}}$

(e) $\dfrac{(2a^{3r})^2a^{2r}}{3a^{-5r}}$ (f) $\dfrac{a^{x+y}a^y}{a^{2x}}$ (g) $\dfrac{3a^{(x/y)}a^x}{a^y}$

Solution (a) $\dfrac{a^{3x}a^{2x}}{a^{4x}} = \dfrac{a^{5x}}{a^{4x}} = a^x$

(b) $a^{2t}(1-a^t)+a^{3t} = a^{2t} - a^{3t} + a^{3t} = a^{2t}$

(c) $\dfrac{(a^y)^2}{2a^y} = \dfrac{a^{2y}}{2a^y} = \dfrac{a^y}{2}$

(d) $\dfrac{a^{-6z}}{a^{-2z}} = a^{-6z-(-2z)} = a^{-4z}$

(e) $\dfrac{(2a^{3r})^2a^{2r}}{3a^{-5r}} = \dfrac{4a^{6r}a^{2r}}{3a^{-5r}} = \dfrac{4a^{8r}}{3a^{-5r}} = \dfrac{4a^{13r}}{3}$

(f) $\dfrac{a^{x+y}a^y}{a^{2x}} = a^{x+2y-2x} = a^{2y-x}$

(g) $\dfrac{3a^{(x/y)}a^x}{a^y} = 3a^{(x/y)+x-y}$

Exponential functions

An **exponential function**, $f(x)$, has the form

$$f(x) = a^x$$

where a is a positive constant called the base.

Some typical exponential functions are tabulated in Table 2.2 and are shown in Figure 2.26. Note from the graphs that these are one-to-one functions.

An exponential function is not a polynomial function. The powers of a polynomial function are constants; the power of an exponential function, that is the exponent, is the variable x.

Table 2.2
Values of a^x for $a = 0.5$, 2 and 3.

x	0.5^x	2^x	3^x
−3	8	0.125	0.037
−2	4	0.25	0.111
−1	2	0.5	0.333
0	1	1	1
1	0.5	2	3
2	0.25	4	9
3	0.125	8	27

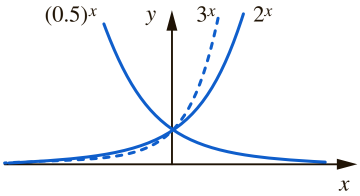


Figure 2.26
Some typical exponential functions.

The most widely used exponential function, commonly called **the** exponential function, is

$$f(x) = \mathrm{e}^x$$

where e is an irrational constant (e = 2.718 281 828 . . .) commonly called the **exponential constant**.

Most scientific calculators have values of e^x available. The function is tabulated in Table 2.3. The graph is shown in Figure 2.27. This particular exponential function so dominates engineering applications that whenever an engineer refers to the exponential function it almost invariably means this one. We will see later why it is so important.

As x increases positively, e^x increases very rapidly; that is, as $x \to \infty$, $\mathrm{e}^x \to \infty$. This situation is known as **exponential growth**. As x increases negatively, e^x approaches zero; that is, as $x \to -\infty$, $\mathrm{e}^x \to 0$. Thus $y = 0$ is an asymptote. Note that the exponential function is never negative.

Figure 2.28 shows a graph of e^{-x}. As x increases positively, e^{-x} decreases to zero; that is, as $x \to \infty$, $\mathrm{e}^{-x} \to 0$. This is known as **exponential decay**. The function is tabulated in Table 2.4.

Table 2.3
The values of the exponential function $f(x) = \mathrm{e}^x$ for various values of x.

x	e^x
−3	0.050
−2	0.135
−1	0.368
0	1
1	2.718
2	7.389
3	20.086

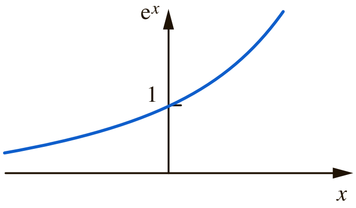


Figure 2.27
Graph of $y = \mathrm{e}^x$ showing exponential growth.

Table 2.4
The values of the exponential function $f(x) = e^{-x}$ for various values of x.

x	e^{-x}
-3	20.086
-2	7.389
-1	2.718
0	1
1	0.368
2	0.135
3	0.050

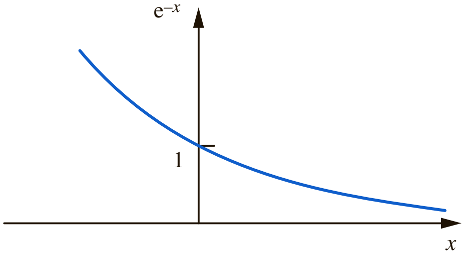


Figure 2.28
Graph of $y = e^{-x}$ showing exponential decay.

Engineering application 2.8

Discharge of a capacitor

The **capacitor** is another of the three fundamental electronic components. It is a device that is used to store electrical charge. It consists of two parallel conducting plates separated by an insulating material, known as a **dielectric**. A build up of a net positive charge on one plate and a net negative charge on the other plate creates an electric field across the dielectric, allowing electrical energy to be stored that has the potential to do useful work. The symbol for a capacitor is two parallel lines that are perpendicular to the conductors in the circuit.

Consider the circuit of Figure 2.29. Before the switch is closed, the capacitor has a voltage V across it. Suppose the switch is closed at time $t = 0$. A current then flows in the circuit and the voltage, v, across the capacitor decays with time. The voltage across the capacitor is given by

$$v = \begin{cases} V & t < 0 \\ Ve^{-t/(RC)} & t \geqslant 0 \end{cases}$$

The quantity RC is known as the **time constant** of the circuit and is usually denoted by τ. So

$$v = \begin{cases} V & t < 0 \\ Ve^{-t/\tau} & t \geqslant 0 \end{cases}$$

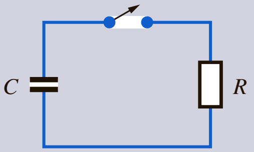


Figure 2.29
Circuit to discharge a capacitor.

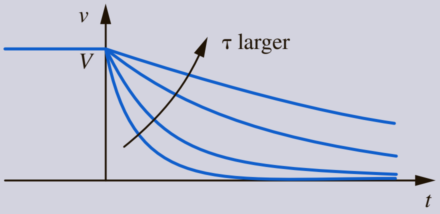


Figure 2.30
The capacitor takes longer to discharge for a larger circuit time constant, τ.

If τ is small, then the capacitor voltage decays more quickly than if τ is large. This is illustrated in Figure 2.30.