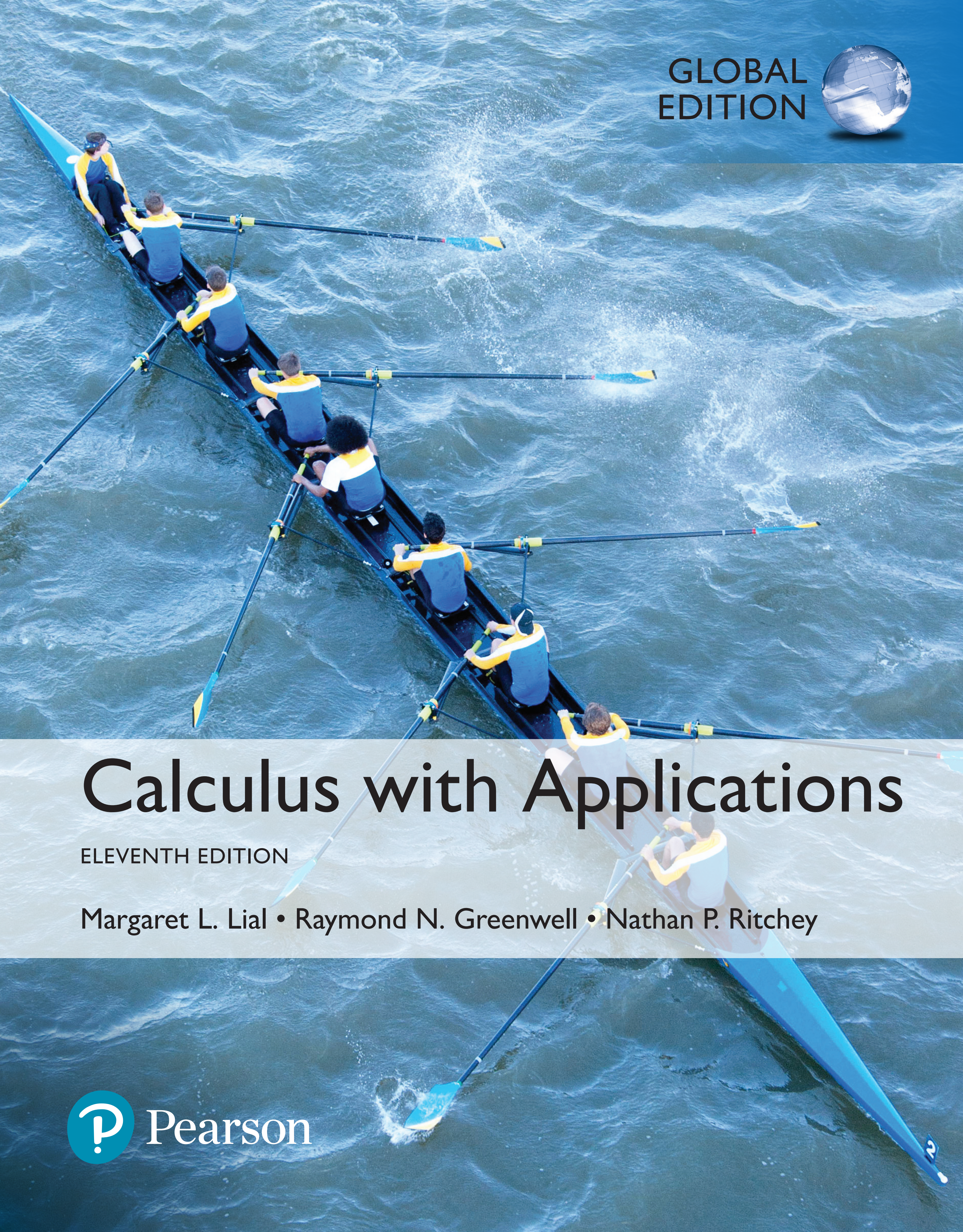

GLOBAL EDITION

Calculus with Applications

ELEVENTH EDITION

Margaret L. Lial • Raymond N. Greenwell • Nathan P. Ritchey

Pearson

Calculus with Applications

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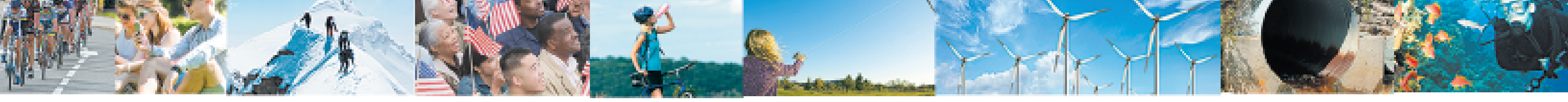

Preface

Calculus with Applications is a thorough, applications-oriented text for students majoring in business, management, economics, or the life or social sciences. In addition to its clear exposition, this text consistently connects the mathematics to career and everyday-life situations. A prerequisite of two years of high school algebra is assumed. A greatly enhanced MyMathLab course, new applications and exercises, and other new learning tools make this 11th edition a rich learning resource for students.

Our Approach

Our main goal is to present applied calculus in a concise and meaningful way so that students can understand the full picture of the concepts they are learning and apply them to real-life situations. This is done through a variety of means.

Focus on Applications Making this course meaningful to students is critical to their success. Applications of the mathematics are integrated throughout the text in the exposition, the examples, the exercise sets, and the supplementary resources. We are constantly on the lookout for novel applications, and the text reflects our efforts to infuse it with relevance. Our research is showcased in the Index of Applications at the back of the book and the extended list of sources of real-world data on www.pearsonglobaleditions.com/lial. *Calculus with Applications* presents students with myriad opportunities to relate what they're learning to career situations through the *Apply It* question at the beginning of sections, the applied examples and exercises, and the *Extended Application* at the end of each chapter.

Pedagogy to Support Students Students need careful explanations of the mathematics along with examples presented in a clear and consistent manner. Additionally, students and instructors should have a means to assess the basic prerequisite skills needed for the course content. This can be done with the *Prerequisite Skills Diagnostic Test,* located just prior to Chapter R. If the diagnostic test reveals gaps in basic skills, students can find help right within the text. Further, *Warm-Up Exercises* are now included at the beginning of many exercise sets. Within MyMathLab are additional diagnostic tests (one per chapter), and remediation is automatically personalized to meet student needs. Students will appreciate the many annotated examples within the text, the *Your Turn* exercises that follow examples, the *For Review* references, and the wealth of learning resources within MyMathLab.

Beyond the Textbook Students want resources at their fingertips and, for them, that means digital access. So Pearson has developed a robust MyMathLab course for *Calculus with Applications*. MyMathLab has a well-established and well-documented track record of helping students succeed in mathematics. The MyMathLab online course for this text contains over 2100 exercises to challenge students and provides help when they need it. Students who learn best through video can view (and review) section- and example-level videos within MyMathLab. These and other resources are available to students as a unified and reliable tool for their success.

New to the Eleventh Edition

Based on our experience in the classroom along with feedback from many instructors across the country, the focus of this revision is to improve the clarity of the presentation and provide students with more opportunities to learn, practice, and apply what they've learned on their own. We do this both in the presentation of the content and in the new features added to the text.

New Features

- *Warm-Up Exercises* were added to many exercise sets to provide an opportunity for students to refresh key prerequisite skills at "point of use."
- Graphing calculator screens have been updated to reflect the TI-84 Plus C, which features color and a higher screen resolution. Additionally, the graphing calculator notes have been updated throughout.
- We added more "help text" annotations to examples. These notes, set in small blue type, appear next to the steps within worked-out examples and provide an additional aid for students with weaker algebra skills.
- For many years this text has featured enormous amounts of real data used in examples and exercises. The 11th edition will not disappoint in this area. We have added or updated 157 (15.9%) of the application exercises throughout the text.
- We updated exercises and examples based on user feedback and other factors. Of the 3516 exercises within the sections, 397 (11.2%) are new or updated. Of the 413 examples in the text, 55 (13.3%) are new or updated.
- MyMathLab contains a wealth of new resources to help students learn and to help you as you teach. Some resources were added or revised based on student usage of the *previous* edition of the MyMathLab course. For example, more exercises were added to those chapters and sections that are more widely assigned.
    - Hundreds of new exercises were added to the course to provide you with more options for assignments, including:
        - More application exercises throughout the text
        - *Setup & Solve* exercises that require students to specify how to set up a problem as well as solve it
        - Exercises that take advantage of the enhanced graphing tool
    - The videos for the course have increased in number, type, and quality:
        - New videos feature more applications and more challenging examples.
        - In addition to full-length lecture videos, MyMathLab now includes assignable, shorter video clips that focus on a specific concept or example.
        - MathTalk Videos help motivate students by pointing out relevant connections to their majors—especially business. The videos feature Andrea Young from Ripon College (WI), a dynamic math professor (and actor!). The videos can be used as lecture starters or as part of homework assignments (in regular or flipped classes). Assignable exercises that accompany the videos help make these videos a part of homework assignments.
        - A Guide to Video-Based Instruction shows which exercises correspond to each video, making it easy to assess students after they watch an instructional video. This is perfect for flipped-classroom situations.
    - Learning Catalytics is a "bring your own device" student engagement, assessment, and classroom intelligence system. Students can use any web-enabled device—laptop, smartphone, or tablet—that they already have. Those with access to MyMathLab have instant access to Learning Catalytics and can log in using their MyMathLab username and password. With Learning Catalytics, you assess students in real time, using open-ended tasks to probe student understanding. It allows you to engage students by creating open-ended questions that ask for numerical, algebraic, textual, or graphical responses—or just simple multiple-choice. Learning Catalytics contains Pearson-created content for calculus so you can take advantage of this exciting technology immediately.

New and Revised Content

The chapters and sections in the text are in the same order as the previous edition, making it easy for users to transition to the new edition. In addition to revising exercises and examples throughout, updating and adding real-world data, we made the following changes:

Chapter R

- Added new *Your Turn* exercises to ensure that there is a student assessment for each major concept.
- Added more detail to R.2 on factoring perfect squares.

Chapter 1

- Rewrote the part of 1.1 involving graphing lines, emphasizing different methods for graphing.
- Rewrote 1.2 on supply, demand, break-even analysis, and equilibrium; giving formal definitions that match what students would see in business and economics courses. All of the business applications were revised, according to recommendations from reviewers, to be more in line with business texts. Also added a new Example 6 on finding a cost function.
- Added color for pedagogical reasons to make content easier to follow.

Chapter 2

- Updated the introduction to 2.1, rewriting it as an example to make it easier for students to reference the necessary skills to identify nonlinear functions, determine the domain and range, and estimate values from a graph.
- In 2.2, added another approach to graphing parabolas by splitting former Example 4 into two separate examples. The new Example 5 illustrates how to graph a parabola by first finding its characteristics (including orientation, intercepts, vertex, and axis of symmetry). The characteristics are highlighted in a box for easy reference.
- Added quadratic regression to 2.2. Example 9 includes a by-hand method and a method using technology.
- Rewrote Example 10 in 2.2, which illustrates translations and reflections of a graph, by breaking it into three parts. The first part is a basic transformation, and the ensuing parts build in complexity.
- Added the definition of a real root to 2.3 and added a Technology Note to illustrate how to use a graphing calculator to approximate the roots of higher degree polynomials.
- Added cubic regression to 2.3 (Example 5).

Chapter 3

- Added Caution note to 3.1 and added a new solution method to Example 9.
- Added new Example 2 to 3.3, using recent data.
- Updated Example 4 in 3.3 to use clearer wording.

Chapter 4

- Clarified the rules for differentiation in 4.1, 4.2, and 4.3 and added a new Example 8.
- Expanded Example 9 in 4.1 to include a new graph.
- Updated Example 10 in 4.1 and Example 4 in 4.5.

Chapter 5

- Added new examples to 5.2 (Example 3(c)) and 5.3 (Example 6(b)).
- Expanded Example 6(a) in 5.4 to show the inflection point.

Chapter 6

- Updated Example 3 in 6.1 to show an application of the concept.
- Modified examples in 6.2 (Example 3), 6.4 (Example 2), and 6.6 (Example 1).

Chapter 7

- Added annotations and comments to Example 10 in 7.1.
- Simplified Examples 1, 2, 3, and 6 in 7.2 and added annotations and comments.
- Added a “For Review” box to 7.3.
- Enlarged all small integral signs throughout the chapter for clarity.
- Updated Example 7 in 7.4 and Example 5 in 7.5.
- Added more explanation of the consumer surplus to 7.5.

Chapter 8

- Added annotations to several examples in 8.1 to denote steps in integration by parts.
- Revised the solutions to Examples 4 and 5 in 8.3, giving more detail and adding annotation to denote the steps in determining the accumulated amount of money flow.

Chapter 9

- Rewrote and expanded Exercise 8 in 9.1, on the Cobb-Douglas Production Function, emphasizing the interpretation of the solutions.
- Added three new exercises to 9.1 on exponential and logarithmic functions of several variables.
- Revised the solution to Example 4 in 9.3, giving more detail.
- Rewrote the solution to Example 3 in 9.4, illustrating how to find the extrema of a constrained function of one or more variables using a spreadsheet.

Chapter 10

- Revised the solution to Example 5 in 10.1, adding annotation to denote steps in separation of variables.
- In 10.1, added the definition of equilibrium point, explained how to determine the stability (stable, unstable, or semistable) of the equilibrium point, and added Example 8 on equilibrium points and stability.

Chapter 11

- Changed the introductory example in 11.1, which continues into 11.2, to avoid rounding issues.
- Added a new part (d) to Example 3 in 11.3, as well as Method 2 using a graphing calculator and Method 3 using a spreadsheet.
- Changed 11.3 so that graphing calculators are the primary method of calculating normal probabilities, and the normal table is the secondary method.

Chapter 12

- Revised Example 4 Method 1 (Graphing Calculator) in 12.1.
- Added clarification on the TVM Solver to Example 8 in 12.2.

Chapter 13

- In 13.1, revised coverage of translating graphs of sine and cosine functions. Also added a box to highlight the transformation of trigonometric functions.
- Added Example 8 to 13.2, which illustrates how to find the relative extrema for trigonometric functions.
- In 13.2, added new exercises (37–56), which use applications of the derivative applied to trigonometric functions. Applications include: critical numbers, intervals in which the function is increasing and decreasing, relative extrema, higher order derivatives, intervals in which the functions are concave upward and concave downward, inflection points, detailed graphs, absolute extrema, implicit differentiation, related rates, and differential approximation.

Features of *Calculus with Applications*

Chapter Opener

Each chapter opens with a quick introduction that relates to an application presented in the chapter.

Apply It

An Apply It question, typically at the start of a section, motivates the math content of the section by posing a real-world question that is then answered within the examples or exercises.

For Review

For Review boxes are provided in the margin as appropriate, giving students just-in-time help with skills they should already know but may have forgotten. For Review comments sometimes include an explanation, while others refer students back to earlier parts of the book for a more thorough review.

FOR REVIEW

Recall that $e^x > 0$ for all x, so there can never be a solution to $e^{g(x)} = 0$ for any function $g(x)$.

Caution

Caution notes provide students with a quick "heads up" to common difficulties and errors.

CAUTION Notice from Example 5(c) that $g(x + h)$ is *not* the same as $g(x) + h$, which equals $-x^2 + 4x - 5 + h$. There is a significant difference between applying a function to the quantity $x + h$ and applying a function to x and adding h afterward.

Your Turn Exercises

These exercises follow selected examples and provide students with an easy way to quickly stop and check their understanding. Answers are provided at the end of the section's exercises.

Technology Notes

Material on graphing calculators or Microsoft Excel is clearly labeled to make it easier for instructors to use this material (or not).

- New The figures depicting calculator screens now reflect the TI-84 Plus C, which features color and higher pixel counts.

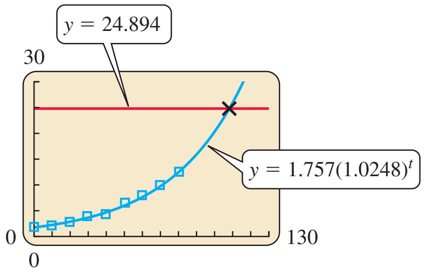

FIGURE 56

Exercise Sets

Basic exercises are followed by an Applications section, which is grouped by subheads such as "Business and Economics." Other types of exercises include the following:

- New **Warm-Up** exercises at the beginning of most sections provide a chance for students to refresh the key prerequisite skills needed for the section's exercises.
- **Connections** exercises integrate topics presented in different sections or chapters and are indicated with .

- **Writing** exercises, labeled with , provide students with an opportunity to explain important mathematical ideas.
- **Technology** exercises are labeled for graphing calculator and for spreadsheets.

Chapter Summary and Review

- The end-of-chapter **Summary** provides students with a quick summary of the key ideas of the chapter followed by a list of key definitions, terms, and examples.
- Chapter **Review Exercises** include Concept Check exercises and an ample set of Practice and Exploration exercises. This arrangement provides students with a comprehensive set of exercises to prepare for chapter exams.

Extended Applications

- Extended Applications are provided at the end of every chapter as in-depth applied exercises to help stimulate student interest. These activities can be completed individually or as a group project.

Supplements

FOR STUDENTS

Graphing Calculator Manual for Applied Mathematics (downloadable)

- Contains detailed instruction for using the TI-83/ TI-83+/ TI-84+C
- Instructions are organized by topic.
- Downloadable from within MyMathLab

Excel Spreadsheet Manual for Applied Mathematics (downloadable)

- Contains detailed instruction for using Excel 2013
- Instructions are organized by topic.
- Downloadable from within MyMathLab

FOR INSTRUCTORS

Instructor's Resource and Solutions Manual (Download Only)

- Provides complete solutions to all exercises, two versions of a pre-test and final exam, and teaching tips.
- Available to qualified instructors within MyMathLab or through the Pearson Instructor Resource Center (www.pearsonglobaleditions.com/lial).
- ISBN 1292109025 / 9781292109022

PowerPoint Presentations (Download Only)

- Includes lecture content and key graphics from the book.
- Available to qualified instructors within MyMathLab or through the Pearson Instructor Resource Center (www.pearsonglobaleditions.com/lial).
- ISBN 129210905X / 9781292109053

FOR STUDENTS	FOR INSTRUCTORS

TestGen Computerized Test Bank

- TestGen® (www.pearsoned.com/testgen) enables instructors to build, edit, print, and administer tests using a computerized bank of questions developed to cover all the objectives of the text.
- TestGen is algorithmically based, allowing instructors to create multiple but equivalent versions of the same question or test with the click of a button. Instructors can also modify test bank questions or add new questions.
- The software and testbank are available to qualified instructors within MyMathLab or through the Pearson Instructor Resource Center (www.pearsonglobaleditions.com/lial).
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MyMathLab delivers **proven results** in helping individual students succeed.

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MyMathLab provides **engaging experiences** that personalize, stimulate, and measure learning for each student.

- **Personalized Learning:** MyMathLab offers several features that support adaptive learning: personalized homework and the adaptive study plan. These features allow your students to work on just what they need to learn when it makes the most sense, maximizing their potential for understanding and success.
- **Exercises:** The homework and practice exercises in MyMathLab are correlated to the exercises in the textbook, and they regenerate algorithmically to give students unlimited opportunity for practice and mastery. The software provides helpful feedback when students enter incorrect answers and includes optional learning aids including guided solutions, sample problems, animations, videos, and eText.
- **Learning and Teaching Tools** include:
  - **Learning Catalytics**—a "bring your own device" student engagement, assessment, and classroom intelligence system, included within MyMathLab. Includes questions written specifically for this course.
  - **Instructional videos**—full-length lecture videos as well as shorter example-based videos.
  - **MathTalk videos**—connect the math to the real world (particularly business). Also include assignable exercises to gauge student understanding of video content.

- ◦ **Help for Gaps in Prerequisite Skills**—diagnostic quizzes tied to personalized assignments help address gaps in algebra skills that might otherwise impede success.
- ◦ **Excel Spreadsheet Manual**—specifically written for this course.
- ◦ **Graphing Calculator Manual**—specifically written for this course.
- ◦ **Interactive Figures**—illustrate key concepts and allow manipulation for use as teaching and learning tools. Includes assignable exercises that require use of the figures.

- **Complete eText** is available to students through MyMathLab courses for the lifetime of the edition, giving students unlimited access to the eBook within any course using that edition of the textbook.

MyMathLab comes from an **experienced partner** with educational expertise and an eye on the future.

- Knowing that you are using a Pearson product means knowing that you are using quality content. This means that our eTexts are accurate and our assessment tools work. It means we are committed to making MyMathLab as accessible as possible.
- Whether you are just getting started with MyMathLab, or have a question along the way, we're here to help you learn about our technologies and how to incorporate them into your course.
- To learn more about how MyMathLab combines proven learning applications with powerful assessment and continuously adaptive capabilities, visit www.mymathlab.com or contact your Pearson representative.

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Raymond N. Greenwell
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Prerequisite Skills Diagnostic Test

Below is a very brief test to help you recognize which, if any, prerequisite skills you may need to remediate in order to be successful in this course. After completing the test, check your answers in the back of the book. In addition to the answers, we have also provided the solutions to these problems in Appendix A. These solutions should help remind you how to solve the problems. For problems 5-26, the answers are followed by references to sections within Chapter R where you can find guidance on how to solve the problem and/or additional instruction. Addressing any weak prerequisite skills now will make a positive impact on your success as you progress through this course.

1. What percent of 50 is 10?

2. Simplify $\frac{13}{7} - \frac{2}{5}$.

3. Let x be the number of apples and y be the number of oranges. Write the following statement as an algebraic equation: "The total number of apples and oranges is 75."

4. Let s be the number of students and p be the number of professors. Write the following statement as an algebraic equation: "There are at least four times as many students as professors."

5. Solve for k: $7k + 8 = -4(3 - k)$.

6. Solve for x: $\frac{5}{8}x + \frac{1}{16}x = \frac{11}{16} + x$.

7. Write in interval notation: $-2 < x \leq 5$.

8. Using the variable x, write the following interval as an inequality: $(-\infty, -3]$.

9. Solve for y: $5(y - 2) + 1 \leq 7y + 8$.

10. Solve for p: $\frac{2}{3}(5p - 3) > \frac{3}{4}(2p + 1)$.

11. Carry out the operations and simplify: $(5y^2 - 6y - 4) - 2(3y^2 - 5y + 1)$.

12. Multiply out and simplify $(x^2 - 2x + 3)(x + 1)$.

13. Multiply out and simplify $(a - 2b)^2$.

14. Factor $3pq + 6p^2q + 9pq^2$.

15. Factor $3x^2 - x - 10$.

16. Perform the operation and simplify: $\frac{a^2 - 6a}{a^2 - 4} \cdot \frac{a - 2}{a}$.

17. Perform the operation and simplify: $\dfrac{x+3}{x^2-1} + \dfrac{2}{x^2+x}$.

18. Solve for x: $3x^2 + 4x = 1$.

19. Solve for z: $\dfrac{8z}{z+3} \leq 2$.

20. Simplify $\dfrac{4^{-1}(x^2y^3)^2}{x^{-2}y^5}$.

21. Simplify $\dfrac{4^{1/4}(p^{2/3}q^{-1/3})^{-1}}{4^{-1/4}p^{4/3}q^{4/3}}$.

22. Simplify as a single term without negative exponents: $k^{-1} - m^{-1}$.

23. Factor $(x^2+1)^{-1/2}(x+2) + 3(x^2+1)^{1/2}$.

24. Simplify $\sqrt[3]{64b^6}$.

25. Rationalize the denominator: $\dfrac{2}{4-\sqrt{10}}$.

26. Simplify $\sqrt{y^2 - 10y + 25}$.

R Algebra Reference

In this chapter, we will review the most important topics in algebra. Knowing algebra is a fundamental prerequisite to success in higher mathematics. This algebra reference is designed for self-study; study it all at once or refer to it when needed throughout the course. Since this is a review, answers to all exercises are given in the answer section at the back of the book.

R.1 Polynomials

An expression such as $9p^4$ is a **term**; the number 9 is the **coefficient**, p is the **variable**, and 4 is the **exponent**. The expression p^4 means $p \cdot p \cdot p \cdot p$, while p^2 means $p \cdot p$, and so on. Terms having the same variable and the same exponent, such as $9x^4$ and $-3x^4$, are **like terms**. Terms that do not have both the same variable and the same exponent, such as m^2 and m^4, are **unlike terms**.

A **polynomial** is a term or a finite sum of terms in which all variables have whole number exponents, and no variables appear in denominators. Examples of polynomials include

$$5x^4 + 2x^3 + 6x, \qquad 8m^3 + 9m^2n - 6mn^2 + 3n^3, \qquad 10p, \qquad \text{and} \qquad -9.$$

Order of Operations

Algebra is a language, and you must be familiar with its rules to correctly interpret algebraic statements. The following order of operations has been agreed upon through centuries of usage.

- Expressions in **parentheses** (or other grouping symbols) are calculated first, working from the inside out. The numerator and denominator of a fraction are treated as expressions in parentheses.
- **Powers** are performed next, going from left to right.
- **Multiplication** and **division** are performed next, going from left to right.
- **Addition** and **subtraction** are performed last, going from left to right.

For example, in the expression $[6(x + 1)^2 + 3x - 22]^2$, suppose x has the value of 2. We would evaluate this as follows:

$$[6(2 + 1)^2 + 3(2) - 22]^2 = [6(3)^2 + 3(2) - 22]^2$$ Evaluate the expression in the innermost parentheses.

$$= [6(9) + 3(2) - 22]^2$$ Evaluate 3 raised to a power.

$$= (54 + 6 - 22)^2$$ Perform the multiplications.

$$= (38)^2$$ Perform the addition and subtraction from left to right.

$$= 1444$$ Evaluate the power.

In the expression $\dfrac{x^2 + 3x + 6}{x + 6}$, suppose x has the value of 2. We would evaluate this as follows:

$$\frac{2^2 + 3(2) + 6}{2 + 6} = \frac{16}{8}$$ Evaluate the numerator and the denominator.

$$= 2$$ Simplify the fraction.

Adding and Subtracting Polynomials

The following properties of real numbers are useful for performing operations on polynomials.

Properties of Real Numbers

For all real numbers a, b, and c:

1. $a + b = b + a$; $ab = ba$; — **Commutative properties**
2. $(a + b) + c = a + (b + c)$; $(ab)c = a(bc)$; — **Associative properties**
3. $a(b + c) = ab + ac$. — **Distributive property**

EXAMPLE 1 Properties of Real Numbers

(a) $2 + x = x + 2$ — Commutative property of addition

(b) $x \cdot 3 = 3x$ — Commutative property of multiplication

(c) $(7x)x = 7(x \cdot x) = 7x^2$ — Associative property of multiplication

(d) $3(x + 4) = 3x + 12$ — Distributive property

One use of the distributive property is to add or subtract polynomials. Only like terms may be added or subtracted. For example,

$$12y^4 + 6y^4 = (12 + 6)y^4 = 18y^4,$$

and

$$-2m^2 + 8m^2 = (-2 + 8)m^2 = 6m^2,$$

but the polynomial $8y^4 + 2y^5$ cannot be further simplified. To subtract polynomials, we use the facts that $-(a + b) = -a - b$ and $-(a - b) = -a + b$. In the next example, we show how to add and subtract polynomials.

EXAMPLE 2 Adding and Subtracting Polynomials

Add or subtract as indicated.

(a) $(8x^3 - 4x^2 + 6x) + (3x^3 + 5x^2 - 9x + 8)$

SOLUTION Combine like terms.

$$\begin{aligned}&(8x^3 - 4x^2 + 6x) + (3x^3 + 5x^2 - 9x + 8)\\ &\quad = (8x^3 + 3x^3) + (-4x^2 + 5x^2) + (6x - 9x) + 8\\ &\quad = 11x^3 + x^2 - 3x + 8\end{aligned}$$

(b) $2(-4x^4 + 6x^3 - 9x^2 - 12) + 3(-3x^3 + 8x^2 - 11x + 7)$

SOLUTION Multiply each polynomial by the factor in front of the polynomial, and then combine terms as before.

$$\begin{aligned}&2(-4x^4 + 6x^3 - 9x^2 - 12) + 3(-3x^3 + 8x^2 - 11x + 7)\\ &\quad = -8x^4 + 12x^3 - 18x^2 - 24 - 9x^3 + 24x^2 - 33x + 21\\ &\quad = -8x^4 + 3x^3 + 6x^2 - 33x - 3\end{aligned}$$

(c) $(2x^2 - 11x + 8) - (7x^2 - 6x + 2)$

SOLUTION Distributing the minus sign and combining like terms yields

$$\begin{aligned}&(2x^2 - 11x + 8) + (-7x^2 + 6x - 2)\\ &\quad = -5x^2 - 5x + 6.\end{aligned}$$

TRY YOUR TURN 1

YOUR TURN 1 Perform the operation $3(x^2 - 4x - 5) - 4(3x^2 - 5x - 7)$.

Multiplying Polynomials

The distributive property is also used to multiply polynomials, along with the fact that $a^m \cdot a^n = a^{m+n}$. For example,

$$x \cdot x = x^1 \cdot x^1 = x^{1+1} = x^2 \qquad \text{and} \qquad x^2 \cdot x^5 = x^{2+5} = x^7.$$

EXAMPLE 3 Multiplying Polynomials

Multiply.

(a) $8x(6x - 4)$

SOLUTION Using the distributive property yields

$$\begin{aligned}8x(6x - 4) &= 8x(6x) - 8x(4)\\ &= 48x^2 - 32x.\end{aligned}$$

(b) $(3p - 2)(p^2 + 5p - 1)$

SOLUTION Using the distributive property yields

$$\begin{aligned}&(3p - 2)(p^2 + 5p - 1)\\ &= 3p(p^2 + 5p - 1) - 2(p^2 + 5p - 1)\\ &= 3p(p^2) + 3p(5p) + 3p(-1) - 2(p^2) - 2(5p) - 2(-1)\\ &= 3p^3 + 15p^2 - 3p - 2p^2 - 10p + 2\\ &= 3p^3 + 13p^2 - 13p + 2.\end{aligned}$$

(c) $(x + 2)(x + 3)(x - 4)$

SOLUTION Multiplying the first two polynomials and then multiplying their product by the third polynomial yields

$$\begin{aligned}&(x + 2)(x + 3)(x - 4)\\ &= [(x + 2)(x + 3)](x - 4)\\ &= (x^2 + 2x + 3x + 6)(x - 4)\\ &= (x^2 + 5x + 6)(x - 4)\\ &= x^3 - 4x^2 + 5x^2 - 20x + 6x - 24\\ &= x^3 + x^2 - 14x - 24.\end{aligned}$$

TRY YOUR TURN 2

YOUR TURN 2 Perform the operation $(3y + 2)(4y^2 - 2y - 5)$.

A **binomial** is a polynomial with exactly two terms, such as $2x + 1$ or $m + n$. When two binomials are multiplied, the FOIL method (First, Outer, Inner, Last) is used as a memory aid.

EXAMPLE 4 Multiplying Polynomials

Find $(2m - 5)(m + 4)$ using the FOIL method.

SOLUTION

$$\begin{aligned}(2m - 5)(m + 4) &= \overset{F}{(2m)(m)} + \overset{O}{(2m)(4)} + \overset{I}{(-5)(m)} + \overset{L}{(-5)(4)}\\ &= 2m^2 + 8m - 5m - 20\\ &= 2m^2 + 3m - 20\end{aligned}$$

TRY YOUR TURN 3

YOUR TURN 3 Find $(2x + 7)(3x - 1)$ using the FOIL method.

EXAMPLE 5 Multiplying Polynomials

Find $(2k - 5m)^3$.

SOLUTION Write $(2k - 5m)^3$ as $(2k - 5m)(2k - 5m)(2k - 5m)$. Then multiply the first two factors using FOIL.

$$\begin{aligned}(2k - 5m)(2k - 5m) &= 4k^2 - 10km - 10km + 25m^2\\ &= 4k^2 - 20km + 25m^2\end{aligned}$$

Now multiply this last result by $(2k - 5m)$ using the distributive property, as in Example 3(c).

$$\begin{aligned}&(4k^2 - 20km + 25m^2)(2k - 5m)\\ &= 4k^2(2k - 5m) - 20km(2k - 5m) + 25m^2(2k - 5m)\\ &= 8k^3 - 20k^2m - 40k^2m + 100km^2 + 50km^2 - 125m^3\\ &= 8k^3 - 60k^2m + 150km^2 - 125m^3 \quad \text{Combine like terms.}\end{aligned}$$

TRY YOUR TURN 4

YOUR TURN 4 Find $(3x + 2y)^3$.

Notice in the first part of Example 5, when we multiplied $(2k - 5m)$ by itself, that the product of the square of a binomial is the square of the first term, $(2k)^2$, plus twice the product of the two terms, $(2)(2k)(-5m)$, plus the square of the last term, $(-5k)^2$.

CAUTION Avoid the common error of writing $(x + y)^2 = x^2 + y^2$. As the first step of Example 5 shows, the square of a binomial has three terms, so

$$(x + y)^2 = x^2 + 2xy + y^2.$$

Furthermore, higher powers of a binomial also result in more than two terms. For example, verify by multiplication that

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3.$$

Remember, for any value of $n \neq 1$,

$$(x + y)^n \neq x^n + y^n.$$

R.1 EXERCISES

Perform the indicated operations.

1. $(2x^2 - 6x + 11) + (-3x^2 + 7x - 2)$
2. $(-4y^2 - 3y + 8) - (2y^2 - 6y - 2)$
3. $-6(2q^2 + 4q - 3) + 4(-q^2 + 7q - 3)$
4. $2(3r^2 + 4r + 2) - 3(-r^2 + 4r - 5)$
5. $(0.613x^2 - 4.215x + 0.892) - 0.47(2x^2 - 3x + 5)$
6. $0.5(5r^2 + 3.2r - 6) - (1.7r^2 - 2r - 1.5)$
7. $-9m(2m^2 + 3m - 1)$
8. $6x(-2x^3 + 5x + 6)$
9. $(3t - 2y)(3t + 5y)$
10. $(9k + q)(2k - q)$
11. $(2 - 3x)(2 + 3x)$
12. $(6m + 5)(6m - 5)$
13. $\left(\frac{2}{5}y + \frac{1}{8}z\right)\left(\frac{3}{5}y + \frac{1}{2}z\right)$
14. $\left(\frac{3}{4}r - \frac{2}{3}s\right)\left(\frac{5}{4}r + \frac{1}{3}s\right)$
15. $(3p - 1)(9p^2 + 3p + 1)$
16. $(3p + 2)(5p^2 + p - 4)$
17. $(2m + 1)(4m^2 - 2m + 1)$
18. $(k + 2)(12k^3 - 3k^2 + k + 1)$
19. $(x + y + z)(3x - 2y - z)$
20. $(r + 2s - 3t)(2r - 2s + t)$
21. $(x + 1)(x + 2)(x + 3)$
22. $(x - 1)(x + 2)(x - 3)$
23. $(x + 2)^2$
24. $(2a - 4b)^2$
25. $(x - 2y)^3$
26. $(3x + y)^3$

YOUR TURN ANSWERS

1. $-9x^2 + 8x + 13$
2. $12y^3 + 2y^2 - 19y - 10$
3. $6x^2 + 19x - 7$
4. $27x^3 + 54x^2y + 36xy^2 + 8y^3$

R.2 Factoring

Multiplication of polynomials relies on the distributive property. The reverse process, where a polynomial is written as a product of other polynomials, is called **factoring**. For example, one way to factor the number 18 is to write it as the product $9 \cdot 2$; both 9 and 2 are **factors** of 18. Usually, only integers are used as factors of integers. The number 18 can also be written with three integer factors as $2 \cdot 3 \cdot 3$.

The Greatest Common Factor

To factor the algebraic expression $15m + 45$, first note that both $15m$ and 45 are divisible by 15; $15m = 15 \cdot m$ and $45 = 15 \cdot 3$. By the distributive property,

$$15m + 45 = 15 \cdot m + 15 \cdot 3 = 15(m + 3).$$

Both 15 and $m + 3$ are factors of $15m + 45$. Since 15 divides into both terms of $15m + 45$ (and is the largest number that will do so), 15 is the **greatest common factor** for the polynomial $15m + 45$. The process of writing $15m + 45$ as $15(m + 3)$ is often called **factoring out** the greatest common factor.

EXAMPLE 1 Factoring

Factor out the greatest common factor.

(a) $12p - 18q$

SOLUTION Both $12p$ and $18q$ are divisible by 6. Therefore,

$$12p - 18q = 6 \cdot 2p - 6 \cdot 3q = 6(2p - 3q).$$

(b) $8x^3 - 9x^2 + 15x$

SOLUTION Each of these terms is divisible by x.

$$8x^3 - 9x^2 + 15x = (8x^2) \cdot x - (9x) \cdot x + 15 \cdot x$$
$$= x(8x^2 - 9x + 15) \quad \text{or} \quad (8x^2 - 9x + 15)x$$

TRY YOUR TURN 1

YOUR TURN 1 Factor $4z^4 + 4z^3 + 18z^2$.

One can always check factorization by finding the product of the factors and comparing it to the original expression.

CAUTION When factoring out the greatest common factor in an expression like $2x^2 + x$, be careful to remember the 1 in the second term.

$$2x^2 + x = 2x^2 + 1x = x(2x + 1), \qquad \text{not } x(2x).$$

Factoring Trinomials

A polynomial that has no greatest common factor (other than 1) may still be factorable. For example, the polynomial $x^2 + 5x + 6$ can be factored as $(x + 2)(x + 3)$. To see that this is correct, find the product $(x + 2)(x + 3)$; you should get $x^2 + 5x + 6$. A polynomial such as this with three terms is called a **trinomial**. To factor a trinomial of the form $x^2 + bx + c$, where the coefficient of x^2 is 1, use FOIL backwards. We look for two factors of c whose sum is b. When c is positive, its factors must have the same sign. Since b is the sum of these two factors, the factors will have the same sign as b. When c is negative, its factors have opposite signs. Again, since b is the sum of these two factors, the factor with the greater absolute value will have the same sign as b.

EXAMPLE 2 Factoring a Trinomial

Factor $y^2 + 8y + 15$.

SOLUTION Since the coefficient of y^2 is 1, factor by finding two numbers whose *product* is 15 and whose *sum* is 8. Because the constant and the middle term are positive, the numbers must both be positive. Begin by listing all pairs of positive integers having a product of 15. As you do this, also form the sum of each pair of numbers.

Products	**Sums**
$15 \cdot 1 = 15$	$15 + 1 = 16$
$5 \cdot 3 = 15$	$5 + 3 = 8$

The numbers 5 and 3 have a product of 15 and a sum of 8. Thus, $y^2 + 8y + 15$ factors as

$$y^2 + 8y + 15 = (y + 5)(y + 3).$$

The answer can also be written as $(y + 3)(y + 5)$.

TRY YOUR TURN 2

YOUR TURN 2 Factor $x^2 - 3x - 10$.

If the coefficient of the squared term is *not* 1, work as shown on the next page.

EXAMPLE 3 Factoring a Trinomial

Factor $4x^2 + 8xy - 5y^2$.

SOLUTION The possible factors of $4x^2$ are $4x$ and x or $2x$ and $2x$; the possible factors of $-5y^2$ are $-5y$ and y or $5y$ and $-y$. Try various combinations of these factors until one works (if, indeed, any work). For example, try the product $(x + 5y)(4x - y)$.

$$\begin{aligned}(x + 5y)(4x - y) &= 4x^2 - xy + 20xy - 5y^2 \\ &= 4x^2 + 19xy - 5y^2\end{aligned}$$

This product is not correct, so try another combination.

$$\begin{aligned}(2x - y)(2x + 5y) &= 4x^2 + 10xy - 2xy - 5y^2 \\ &= 4x^2 + 8xy - 5y^2\end{aligned}$$

YOUR TURN 3 Factor $6a^2 + 5ab - 4b^2$.

Since this combination gives the correct polynomial,

$$4x^2 + 8xy - 5y^2 = (2x - y)(2x + 5y).$$

TRY YOUR TURN 3

Special Factorizations

Four special factorizations occur so often that they are listed here for future reference.

Special Factorizations

$x^2 - y^2 = (x + y)(x - y)$	**Difference of two squares**
$x^2 + 2xy + y^2 = (x + y)^2$	**Perfect square**
$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$	**Difference of two cubes**
$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$	**Sum of two cubes**

A polynomial that cannot be factored is called a **prime polynomial**.

EXAMPLE 4 Factoring Polynomials

Factor each polynomial, if possible.

(a) $64p^2 - 49q^2 = (8p)^2 - (7q)^2 = (8p + 7q)(8p - 7q)$ — Difference of two squares

(b) $x^2 + 36$ is a prime polynomial.

(c) $x^2 + 12x + 36 = x^2 + 2(x)(6) + 6^2 = (x + 6)^2$ — Perfect square

(d) $9y^2 - 24yz + 16z^2 = (3y)^2 + 2(3y)(-4z) + (-4z)^2$ — Perfect square

$$= [3y + (-4z)]^2 = (3y - 4z)^2$$

(e) $y^3 - 8 = y^3 - 2^3 = (y - 2)(y^2 + 2y + 4)$ — Difference of two cubes

(f) $m^3 + 125 = m^3 + 5^3 = (m + 5)(m^2 - 5m + 25)$ — Sum of two cubes

(g) $8k^3 - 27z^3 = (2k)^3 - (3z)^3 = (2k - 3z)(4k^2 + 6kz + 9z^2)$ — Difference of two cubes

(h) $p^4 - 1 = (p^2 + 1)(p^2 - 1) = (p^2 + 1)(p + 1)(p - 1)$ — Difference of two squares

CAUTION In factoring, always look for a common factor first. Since $36x^2 - 4y^2$ has a common factor of 4,

$$36x^2 - 4y^2 = 4(9x^2 - y^2) = 4(3x + y)(3x - y).$$

It would be incomplete to factor it as

$$36x^2 - 4y^2 = (6x + 2y)(6x - 2y),$$

since each factor can be factored still further. To *factor* means to factor completely, so that each polynomial factor is prime.

R.2 EXERCISES

Factor each polynomial. If a polynomial cannot be factored, write *prime*. Factor out the greatest common factor as necessary.

1. $7a^3 + 14a^2$
2. $3y^3 + 24y^2 + 9y$
3. $13p^4q^2 - 39p^3q + 26p^2q^2$
4. $60m^4 - 120m^3n + 50m^2n^2$
5. $m^2 - 5m - 14$
6. $x^2 + 4x - 5$
7. $z^2 + 9z + 20$
8. $b^2 - 8b + 7$
9. $a^2 - 6ab + 5b^2$
10. $s^2 + 2st - 35t^2$
11. $y^2 - 4yz - 21z^2$
12. $3x^2 + 4x - 7$
13. $3a^2 + 10a + 7$
14. $15y^2 + y - 2$
15. $21m^2 + 13mn + 2n^2$
16. $6a^2 - 48a - 120$
17. $3m^3 + 12m^2 + 9m$
18. $4a^2 + 10a + 6$
19. $24a^4 + 10a^3b - 4a^2b^2$
20. $24x^4 + 36x^3y - 60x^2y^2$
21. $x^2 - 64$
22. $9m^2 - 25$
23. $10x^2 - 160$
24. $9x^2 + 64$
25. $z^2 + 14zy + 49y^2$
26. $s^2 - 10st + 25t^2$
27. $9p^2 - 24p + 16$
28. $a^3 - 216$
29. $27r^3 - 64s^3$
30. $3m^3 + 375$
31. $x^4 - y^4$
32. $16a^4 - 81b^4$

YOUR TURN ANSWERS

1. $2z^2(2z^2 + 2z + 9)$
2. $(x + 2)(x - 5)$
3. $(2a - b)(3a + 4b)$

R.3 Rational Expressions

Many algebraic fractions are **rational expressions**, which are quotients of polynomials with nonzero denominators. Examples include

$$\frac{8}{x-1}, \quad \frac{3x^2+4x}{5x-6}, \quad \text{and} \quad \frac{2y+1}{y^2}.$$

Next, we summarize properties for working with rational expressions.

Properties of Rational Expressions

For all mathematical expressions P, Q, R, and S, with $Q \neq 0$ and $S \neq 0$:

$\frac{P}{Q} = \frac{PS}{QS}$	**Fundamental property**
$\frac{P}{Q} + \frac{R}{Q} = \frac{P+R}{Q}$	**Addition**
$\frac{P}{Q} - \frac{R}{Q} = \frac{P-R}{Q}$	**Subtraction**
$\frac{P}{Q} \cdot \frac{R}{S} = \frac{PR}{QS}$	**Multiplication**
$\frac{P}{Q} \div \frac{R}{S} = \frac{P}{Q} \cdot \frac{S}{R} \quad (R \neq 0)$	**Division**

When writing a rational expression in lowest terms, we may need to use the fact that $\frac{a^m}{a^n} = a^{m-n}$. For example,

$$\frac{x^4}{3x} = \frac{1x^4}{3x} = \frac{1}{3} \cdot \frac{x^4}{x} = \frac{1}{3} \cdot x^{4-1} = \frac{1}{3}x^3 = \frac{x^3}{3}.$$

EXAMPLE 1 Reducing Rational Expressions

Write each rational expression in lowest terms, that is, reduce the expression as much as possible.

(a) $\dfrac{8x+16}{4} = \dfrac{8(x+2)}{4} = \dfrac{4 \cdot 2(x+2)}{4} = 2(x+2)$

Factor both the numerator and denominator in order to identify any common factors, which have a quotient of 1. The answer could also be written as $2x+4$.

(b) $\dfrac{k^2+7k+12}{k^2+2k-3} = \dfrac{(k+4)(k+3)}{(k-1)(k+3)} = \dfrac{k+4}{k-1}$

The answer cannot be further reduced.

TRY YOUR TURN 1

YOUR TURN 1 Write in lowest terms $\dfrac{z^2+5z+6}{2z^2+7z+3}$.

CAUTION One of the most common errors in algebra involves incorrect use of the fundamental property of rational expressions. Only common *factors* may be divided or "canceled." It is essential to factor rational expressions before writing them in lowest terms. In Example 1(b), for instance, it is not correct to "cancel" k^2 (or cancel k, or divide 12 by -3) because the additions and subtraction must be performed first. Here they cannot be performed, so it is not possible to divide. After factoring, however, the fundamental property can be used to write the expression in lowest terms.

EXAMPLE 2 Combining Rational Expressions

Perform each operation.

(a) $\dfrac{3y+9}{6} \cdot \dfrac{18}{5y+15}$

SOLUTION Factor where possible, then multiply numerators and denominators and reduce to lowest terms.

$$\frac{3y+9}{6} \cdot \frac{18}{5y+15} = \frac{3(y+3)}{6} \cdot \frac{18}{5(y+3)} \qquad \text{Factor.}$$

$$= \frac{3 \cdot 18(y+3)}{6 \cdot 5(y+3)} \qquad \text{Multiply.}$$

$$= \frac{3 \cdot \cancel{6} \cdot 3\cancel{(y+3)}}{\cancel{6} \cdot 5\cancel{(y+3)}} = \frac{3 \cdot 3}{5} = \frac{9}{5} \qquad \text{Reduce to lowest terms.}$$

(b) $\dfrac{m^2+5m+6}{m+3} \cdot \dfrac{m}{m^2+3m+2}$

SOLUTION Factor where possible.

$$\frac{(m+2)(m+3)}{m+3} \cdot \frac{m}{(m+2)(m+1)} \qquad \text{Factor.}$$

$$= \frac{m\cancel{(m+2)}\cancel{(m+3)}}{\cancel{(m+3)}\cancel{(m+2)}(m+1)} = \frac{m}{m+1} \qquad \text{Reduce to lowest terms.}$$

(c) $\dfrac{9p-36}{12} \div \dfrac{5(p-4)}{18}$

SOLUTION Use the division property of rational expressions.

$$\frac{9p-36}{12} \div \frac{5(p-4)}{18} = \frac{9p-36}{12} \cdot \frac{18}{5(p-4)} \qquad \text{Invert and multiply.}$$

$$= \frac{9\cancel{(p-4)}}{\cancel{6} \cdot 2} \cdot \frac{\cancel{6} \cdot 3}{5\cancel{(p-4)}} = \frac{27}{10} \qquad \text{Factor and reduce to lowest terms.}$$

(d) $\frac{4}{5k} - \frac{11}{5k}$

SOLUTION As shown in the list of properties, to subtract two rational expressions that have the same denominators, subtract the numerators while keeping the same denominator.

$$\frac{4}{5k} - \frac{11}{5k} = \frac{4 - 11}{5k} = -\frac{7}{5k}$$

(e) $\frac{7}{p} + \frac{9}{2p} + \frac{1}{3p}$

SOLUTION These three fractions cannot be added until their denominators are the same. A **common denominator** into which p, $2p$, and $3p$ all divide is $6p$. Note that $12p$ is also a common denominator, but $6p$ is the **least common denominator**. Use the fundamental property to rewrite each rational expression with a denominator of $6p$.

$$\frac{7}{p} + \frac{9}{2p} + \frac{1}{3p} = \frac{6 \cdot 7}{6 \cdot p} + \frac{3 \cdot 9}{3 \cdot 2p} + \frac{2 \cdot 1}{2 \cdot 3p}$$ Rewrite with common denominator $6p$.

$$= \frac{42}{6p} + \frac{27}{6p} + \frac{2}{6p}$$

$$= \frac{42 + 27 + 2}{6p}$$

$$= \frac{71}{6p}$$

(f) $\frac{x + 1}{x^2 + 5x + 6} - \frac{5x - 1}{x^2 - x - 12}$

SOLUTION To find the least common denominator, we first factor each denominator. Then we change each fraction so they all have the same denominator, being careful to multiply only by quotients that equal 1.

$$\frac{x + 1}{x^2 + 5x + 6} - \frac{5x - 1}{x^2 - x - 12}$$

$$= \frac{x + 1}{(x + 2)(x + 3)} - \frac{5x - 1}{(x + 3)(x - 4)}$$ Factor denominators.

$$= \frac{x + 1}{(x + 2)(x + 3)} \cdot \frac{(x - 4)}{(x - 4)} - \frac{5x - 1}{(x + 3)(x - 4)} \cdot \frac{(x + 2)}{(x + 2)}$$ Rewrite with common denominators.

$$= \frac{(x^2 - 3x - 4) - (5x^2 + 9x - 2)}{(x + 2)(x + 3)(x - 4)}$$ Multiply numerators.

$$= \frac{-4x^2 - 12x - 2}{(x + 2)(x + 3)(x - 4)}$$ Subtract.

$$= \frac{-2(2x^2 + 6x + 1)}{(x + 2)(x + 3)(x - 4)}$$ Factor numerator.

Because the numerator cannot be factored further, we leave our answer in this form. We could also multiply out the denominator, but factored form is usually more useful.

TRY YOUR TURN 2

YOUR TURN 2 Perform each of the following operations.

(a) $\frac{z^2 + 5z + 6}{2z^2 - 5z - 3} \cdot \frac{2z^2 - z - 1}{z^2 + 2z - 3}$.

(b) $\frac{a - 3}{a^2 + 3a + 2} + \frac{5a}{a^2 - 4}$.

R.3 EXERCISES

Write each rational expression in lowest terms.

1. $\frac{5v^2}{35v}$

2. $\frac{25p^3}{10p^2}$

3. $\frac{8k + 16}{9k + 18}$

4. $\frac{2(t - 15)}{(t - 15)(t + 2)}$

5. $\frac{4x^3 - 8x^2}{4x^2}$

6. $\frac{36y^2 + 72y}{9y}$

7. $\frac{m^2 - 4m + 4}{m^2 + m - 6}$

8. $\frac{r^2 - r - 6}{r^2 + r - 12}$

9. $\frac{3x^2 + 3x - 6}{x^2 - 4}$

10. $\frac{z^2 - 5z + 6}{z^2 - 4}$

11. $\frac{m^4 - 16}{4m^2 - 16}$

12. $\frac{6y^2 + 11y + 4}{3y^2 + 7y + 4}$

Perform the indicated operations.

13. $\frac{9k^2}{25} \cdot \frac{5}{3k}$

14. $\frac{15p^3}{9p^2} \div \frac{6p}{10p^2}$

15. $\frac{3a + 3b}{4c} \cdot \frac{12}{5(a + b)}$

16. $\frac{a - 3}{16} \div \frac{a - 3}{32}$

17. $\frac{2k - 16}{6} \div \frac{4k - 32}{3}$

18. $\frac{9y - 18}{6y + 12} \cdot \frac{3y + 6}{15y - 30}$

19. $\frac{4a + 12}{2a - 10} \div \frac{a^2 - 9}{a^2 - a - 20}$

20. $\frac{6r - 18}{9r^2 + 6r - 24} \cdot \frac{12r - 16}{4r - 12}$

21. $\frac{k^2 + 4k - 12}{k^2 + 10k + 24} \cdot \frac{k^2 + k - 12}{k^2 - 9}$

22. $\frac{m^2 + 3m + 2}{m^2 + 5m + 4} \div \frac{m^2 + 5m + 6}{m^2 + 10m + 24}$

23. $\frac{2m^2 - 5m - 12}{m^2 - 10m + 24} \div \frac{4m^2 - 9}{m^2 - 9m + 18}$

24. $\frac{4n^2 + 4n - 3}{6n^2 - n - 15} \cdot \frac{8n^2 + 32n + 30}{4n^2 + 16n + 15}$

25. $\frac{a + 1}{2} - \frac{a - 1}{2}$

26. $\frac{3}{p} + \frac{1}{2}$

27. $\frac{6}{5y} - \frac{3}{2}$

28. $\frac{1}{6m} + \frac{2}{5m} + \frac{4}{m}$

29. $\frac{1}{m - 1} + \frac{2}{m}$

30. $\frac{5}{2r + 3} - \frac{2}{r}$

31. $\frac{8}{3(a - 1)} + \frac{2}{a - 1}$

32. $\frac{2}{5(k - 2)} + \frac{3}{4(k - 2)}$

33. $\frac{4}{x^2 + 4x + 3} + \frac{3}{x^2 - x - 2}$

34. $\frac{y}{y^2 + 2y - 3} - \frac{1}{y^2 + 4y + 3}$

35. $\frac{3k}{2k^2 + 3k - 2} - \frac{2k}{2k^2 - 7k + 3}$

36. $\frac{4m}{3m^2 + 7m - 6} - \frac{m}{3m^2 - 14m + 8}$

37. $\frac{2}{a + 2} + \frac{1}{a} + \frac{a - 1}{a^2 + 2a}$

38. $\frac{5x + 2}{x^2 - 1} + \frac{3}{x^2 + x} - \frac{1}{x^2 - x}$

YOUR TURN ANSWERS

1. $(z + 2)/(2z + 1)$
2. **(a)** $(z + 2)/(z - 3)$
   (b) $6(a^2 + 1)/[(a - 2)(a + 2)(a + 1)]$

R.4 Equations

Linear Equations

Equations that can be written in the form $ax + b = 0$, where a and b are real numbers, with $a \neq 0$, are **linear equations**. Examples of linear equations include $5y + 9 = 16$, $8x = 4$, and $-3p + 5 = -8$. Equations that are *not* linear include absolute value equations such as $|x| = 4$. The following properties are used to solve linear equations.

Properties of Equality

For all real numbers a, b, and c:

1. **If $a = b$, then $a + c = b + c$.** (The same number may be added to both sides of an equation.) — **Addition property of equality**
2. **If $a = b$, then $ac = bc$.** (Both sides of an equation may be multiplied by the same number.) — **Multiplication property of equality**

EXAMPLE 1 Solving Linear Equations

Solve the following equations.

(a) $x - 2 = 3$

SOLUTION The goal is to isolate the variable. Using the addition property of equality yields

$$x - 2 + 2 = 3 + 2, \quad \text{or} \quad x = 5.$$

(b) $\frac{x}{2} = 3$

SOLUTION Using the multiplication property of equality yields

$$2 \cdot \frac{x}{2} = 2 \cdot 3, \quad \text{or} \quad x = 6.$$

The following example shows how these properties are used to solve linear equations. The goal is to isolate the variable. The solutions should always be checked by substitution into the original equation.

EXAMPLE 2 Solving a Linear Equation

Solve $2x - 5 + 8 = 3x + 2(2 - 3x)$.

SOLUTION

$2x - 5 + 8 = 3x + 4 - 6x$	Distributive property
$2x + 3 = -3x + 4$	Combine like terms.
$5x + 3 = 4$	Add $3x$ to both sides.
$5x = 1$	Add -3 to both sides.
$x = \frac{1}{5}$	Multiply both sides by $\frac{1}{5}$.

Check by substituting into the original equation. The left side becomes $2(1/5) - 5 + 8$ and the right side becomes $3(1/5) + 2[2 - 3(1/5)]$. Verify that both of these expressions simplify to $17/5$.

TRY YOUR TURN 1

YOUR TURN 1 Solve $3x - 7 = 4(5x + 2) - 7x$.

Quadratic Equations

An equation with 2 as the greatest exponent of the variable is a *quadratic equation*. A **quadratic equation** has the form $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$. A quadratic equation written in the form $ax^2 + bx + c = 0$ is said to be in **standard form**.

The simplest way to solve a quadratic equation, but one that is not always applicable, is by factoring. This method depends on the **zero-factor property**.

Zero-Factor Property

If a and b are real numbers, with $ab = 0$, then either

$$a = 0 \quad \text{or} \quad b = 0 \quad \text{(or both).}$$

EXAMPLE 3 Solving a Quadratic Equation

Solve $6r^2 + 7r = 3$.

SOLUTION First write the equation in standard form.

$$6r^2 + 7r - 3 = 0$$

Now factor $6r^2 + 7r - 3$ to get

$$(3r - 1)(2r + 3) = 0.$$

By the zero-factor property, the product $(3r - 1)(2r + 3)$ can equal 0 if and only if

$$3r - 1 = 0 \quad \text{or} \quad 2r + 3 = 0.$$

YOUR TURN 2 Solve $2m^2 + 7m = 15$.

Solve each of these equations separately to find that the solutions are $1/3$ and $-3/2$. Check these solutions by substituting them into the original equation. **TRY YOUR TURN 2**

CAUTION Remember, the zero-factor property requires that the product of two (or more) factors be equal to *zero*, not some other quantity. It would be incorrect to use the zero-factor property with an equation in the form $(x + 3)(x - 1) = 4$, for example.

If a quadratic equation cannot be solved easily by factoring, use the *quadratic formula*. (The derivation of the quadratic formula is given in most algebra books.)

Quadratic Formula

The solutions of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

EXAMPLE 4 Quadratic Formula

Solve $x^2 - 4x - 5 = 0$ by the quadratic formula.

SOLUTION The equation is already in standard form (it has 0 alone on one side of the equal sign), so the values of a, b, and c from the quadratic formula are easily identified. The coefficient of the squared term gives the value of a; here, $a = 1$. Also, $b = -4$ and $c = -5$, where b is the coefficient of the linear term and c is the constant coefficient. (Be careful to use the correct signs.) Substitute these values into the quadratic formula.

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-5)}}{2(1)} \qquad a = 1, b = -4, c = -5$$

$$x = \frac{4 \pm \sqrt{16 + 20}}{2} \qquad (-4)^2 = (-4)(-4) = 16$$

$$x = \frac{4 \pm 6}{2} \qquad \sqrt{16 + 20} = \sqrt{36} = 6$$

The $\pm$ sign represents the two solutions of the equation. To find both of the solutions, first use $+$ and then use $-$.

$$x = \frac{4 + 6}{2} = \frac{10}{2} = 5 \quad \text{or} \quad x = \frac{4 - 6}{2} = \frac{-2}{2} = -1$$

The two solutions are 5 and -1.

CAUTION Notice in the quadratic formula that the square root is added to or subtracted from the value of $-b$ *before* dividing by $2a$.

EXAMPLE 5 Quadratic Formula

Solve $x^2 + 1 = 4x$.

SOLUTION First, add $-4x$ on both sides of the equal sign in order to get the equation in standard form.

$$x^2 - 4x + 1 = 0$$

Now identify the values of a, b, and c. Here $a = 1$, $b = -4$, and $c = 1$. Substitute these numbers into the quadratic formula.

$$\begin{aligned} x &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)} \qquad a = 1, b = -4, c = 1 \\ &= \frac{4 \pm \sqrt{16 - 4}}{2} \\ &= \frac{4 \pm \sqrt{12}}{2} \end{aligned}$$

Simplify the solutions by writing $\sqrt{12}$ as $\sqrt{4 \cdot 3} = \sqrt{4} \cdot \sqrt{3} = 2\sqrt{3}$. Substituting $2\sqrt{3}$ for $\sqrt{12}$ gives

$$\begin{aligned} x &= \frac{4 \pm 2\sqrt{3}}{2} \\ &= \frac{2(2 \pm \sqrt{3})}{2} \qquad \text{Factor } 4 \pm 2\sqrt{3}. \\ &= 2 \pm \sqrt{3}. \qquad \text{Reduce to lowest terms.} \end{aligned}$$

The two solutions are $2 + \sqrt{3}$ and $2 - \sqrt{3}$.

The exact values of the solutions are $2 + \sqrt{3}$ and $2 - \sqrt{3}$. The $\sqrt{\ }$ key on a calculator gives decimal approximations of these solutions (to the nearest thousandth):

$$2 + \sqrt{3} \approx 2 + 1.732 = 3.732^*$$
$$2 - \sqrt{3} \approx 2 - 1.732 = 0.268$$

TRY YOUR TURN 3

YOUR TURN 3 Solve $z^2 + 6 = 8z$.

NOTE Sometimes the quadratic formula will give a result with a negative number under the radical sign, such as $3 \pm \sqrt{-5}$. A solution of this type is a complex number. Since this text deals only with real numbers, such solutions cannot be used.

Equations with Fractions

When an equation includes fractions, first eliminate all denominators by multiplying both sides of the equation by a common denominator, a number that can be divided (with no remainder) by each denominator in the equation. When an equation involves fractions with variable denominators, it is *necessary* to check all solutions in the original equation to be sure that no solution will lead to a zero denominator.

EXAMPLE 6 Solving Rational Equations

Solve each equation.

(a) $\dfrac{r}{10} - \dfrac{2}{15} = \dfrac{3r}{20} - \dfrac{1}{5}$

SOLUTION The denominators are 10, 15, 20, and 5. Each of these numbers can be divided into 60, so 60 is a common denominator. Multiply both sides of the equation by

*The symbol $\approx$ means "is approximately equal to."

60 and use the distributive property. (If a common denominator cannot be found easily, all the denominators in the problem can be multiplied together to produce one.)

$$\frac{r}{10} - \frac{2}{15} = \frac{3r}{20} - \frac{1}{5}$$

$$60\left(\frac{r}{10} - \frac{2}{15}\right) = 60\left(\frac{3r}{20} - \frac{1}{5}\right) \qquad \text{Multiply by the common denominator.}$$

$$60\left(\frac{r}{10}\right) - 60\left(\frac{2}{15}\right) = 60\left(\frac{3r}{20}\right) - 60\left(\frac{1}{5}\right) \qquad \text{Distributive property}$$

$$6r - 8 = 9r - 12$$

Add $-9r$ and 8 to both sides.

$$6r - 8 + (-9r) + 8 = 9r - 12 + (-9r) + 8$$

$$-3r = -4$$

$$r = \frac{4}{3} \qquad \text{Multiply each side by } -\tfrac{1}{3}.$$

Check by substituting into the original equation.

(b) $\dfrac{3}{x^2} - 12 = 0$

SOLUTION Begin by multiplying both sides of the equation by x^2 to get $3 - 12x^2 = 0$. This equation could be solved by using the quadratic formula with $a = -12$, $b = 0$, and $c = 3$. Another method that works well for the type of quadratic equation in which $b = 0$ is shown below.

$$3 - 12x^2 = 0$$

$$3 = 12x^2 \qquad \text{Add } 12x^2.$$

$$\frac{1}{4} = x^2 \qquad \text{Multiply by } \tfrac{1}{12}.$$

$$\pm\frac{1}{2} = x \qquad \text{Take square roots.}$$

Verify that there are two solutions, $-1/2$ and $1/2$.

(c) $\dfrac{2}{k} - \dfrac{3k}{k + 2} = \dfrac{k}{k^2 + 2k}$

SOLUTION Factor $k^2 + 2k$ as $k(k + 2)$. The least common denominator for all the fractions is $k(k + 2)$. Multiplying both sides by $k(k + 2)$ gives the following:

$$k(k + 2) \cdot \left(\frac{2}{k} - \frac{3k}{k + 2}\right) = k(k + 2) \cdot \frac{k}{k^2 + 2k}$$

$$2(k + 2) - 3k(k) = k$$

$$2k + 4 - 3k^2 = k \qquad \text{Distributive property}$$

$$-3k^2 + k + 4 = 0 \qquad \text{Add } -k\text{; rearrange terms.}$$

$$3k^2 - k - 4 = 0 \qquad \text{Multiply by } -1.$$

$$(3k - 4)(k + 1) = 0 \qquad \text{Factor.}$$

$$3k - 4 = 0 \quad \text{or} \quad k + 1 = 0$$

$$k = \frac{4}{3} \qquad k = -1$$

Verify that the solutions are $4/3$ and -1.

TRY YOUR TURN 4

YOUR TURN 4 Solve $\dfrac{1}{x^2 - 4} + \dfrac{2}{x - 2} = \dfrac{1}{x}$.

CAUTION It is possible to get, as a solution of a rational equation, a number that makes one or more of the denominators in the original equation equal to zero. That number is not a solution, so it is *necessary* to check all potential solutions of rational equations. These introduced solutions are called **extraneous solutions**.

EXAMPLE 7 Solving a Rational Equation

Solve $\dfrac{2}{x-3} + \dfrac{1}{x} = \dfrac{6}{x(x-3)}$.

SOLUTION The common denominator is $x(x-3)$. Multiply both sides by $x(x-3)$ and solve the resulting equation.

$$x(x-3)\cdot\left(\frac{2}{x-3}+\frac{1}{x}\right) = x(x-3)\cdot\left[\frac{6}{x(x-3)}\right]$$

$$2x + x - 3 = 6$$

$$3x = 9$$

$$x = 3$$

Checking this potential solution by substitution into the original equation shows that 3 makes two denominators 0. Thus, 3 cannot be a solution, so there is no solution for this equation.

R.4 EXERCISES

Solve each equation.

1. $2x + 8 = x - 4$
2. $5x + 2 = 8 - 3x$
3. $0.2m - 0.5 = 0.1m + 0.7$
4. $\frac{2}{3}k - k + \frac{3}{8} = \frac{1}{2}$
5. $3r + 2 - 5(r + 1) = 6r + 4$
6. $5(a + 3) + 4a - 5 = -(2a - 4)$
7. $2[3m - 2(3 - m) - 4] = 6m - 4$
8. $4[2p - (3 - p) + 5] = -7p - 2$

Solve each equation by factoring or by using the quadratic formula. If the solutions involve square roots, give both the exact solutions and the approximate solutions to three decimal places.

9. $x^2 + 5x + 6 = 0$
10. $x^2 = 3 + 2x$
11. $m^2 = 14m - 49$
12. $2k^2 - k = 10$
13. $12x^2 - 5x = 2$
14. $m(m - 7) = -10$
15. $4x^2 - 36 = 0$
16. $z(2z + 7) = 4$
17. $12y^2 - 48y = 0$
18. $3x^2 - 5x + 1 = 0$
19. $2m^2 - 4m = 3$
20. $p^2 + p - 1 = 0$
21. $k^2 - 10k = -20$
22. $5x^2 - 8x + 2 = 0$
23. $2r^2 - 7r + 5 = 0$
24. $2x^2 - 7x + 30 = 0$
25. $3k^2 + k = 6$
26. $5m^2 + 5m = 0$

Solve each equation.

27. $\dfrac{3x-2}{7} = \dfrac{x+2}{5}$
28. $\dfrac{x}{3} - 7 = 6 - \dfrac{3x}{4}$
29. $\dfrac{4}{x-3} - \dfrac{8}{2x+5} + \dfrac{3}{x-3} = 0$
30. $\dfrac{5}{p-2} - \dfrac{7}{p+2} = \dfrac{12}{p^2-4}$
31. $\dfrac{2m}{m-2} - \dfrac{6}{m} = \dfrac{12}{m^2-2m}$
32. $\dfrac{2y}{y-1} = \dfrac{5}{y} + \dfrac{10-8y}{y^2-y}$
33. $\dfrac{1}{x-2} - \dfrac{3x}{x-1} = \dfrac{2x+1}{x^2-3x+2}$
34. $\dfrac{5}{a} + \dfrac{-7}{a+1} = \dfrac{a^2-2a+4}{a^2+a}$
35. $\dfrac{5}{b+5} - \dfrac{4}{b^2+2b} = \dfrac{6}{b^2+7b+10}$
36. $\dfrac{2}{x^2-2x-3} + \dfrac{5}{x^2-x-6} = \dfrac{1}{x^2+3x+2}$
37. $\dfrac{4}{2x^2+3x-9} + \dfrac{2}{2x^2-x-3} = \dfrac{3}{x^2+4x+3}$

YOUR TURN ANSWERS

1. $-3/2$
2. $3/2, -5$
3. $4 \pm \sqrt{10}$
4. $-1, -4$

R.5 Inequalities

To write that one number is greater than or less than another number, we use the following symbols.

Inequality Symbols

$<$ means *is less than* $\le$ means *is less than or equal to*
$>$ means *is greater than* $\ge$ means *is greater than or equal to*

Linear Inequalities

An equation states that two expressions are equal; an **inequality** states that they are unequal. A **linear inequality** is an inequality that can be simplified to the form $ax < b$. (Properties introduced in this section are given only for $<$, but they are equally valid for $>$, $\le$, or $\ge$.) Linear inequalities are solved with the following properties.

Properties of Inequality

For all real numbers a, b, and c:

1. If $a < b$, then $a + c < b + c$.
2. If $a < b$ and if $c > 0$, then $ac < bc$.
3. If $a < b$ and if $c < 0$, then $ac < bc$.

Pay careful attention to property 3; it says that if both sides of an inequality are multiplied by a negative number, the direction of the inequality symbol must be reversed.

EXAMPLE 1 Solving a Linear Inequality

Solve $4 - 3y \le 7 + 2y$.

SOLUTION Use the properties of inequality.

$$4 - 3y + (-4) \le 7 + 2y + (-4) \quad \text{Add } -4 \text{ to both sides.}$$
$$-3y \le 3 + 2y$$

Remember that *adding* the same number to both sides never changes the direction of the inequality symbol.

$$-3y + (-2y) \le 3 + 2y + (-2y) \quad \text{Add } -2y \text{ to both sides.}$$
$$-5y \le 3$$

Multiply both sides by $-1/5$. Since $-1/5$ is negative, change the direction of the inequality symbol.

$$-\frac{1}{5}(-5y) \ge -\frac{1}{5}(3)$$
$$y \ge -\frac{3}{5}$$

TRY YOUR TURN 1

YOUR TURN 1 Solve $3z - 2 > 5z + 7$.

CAUTION It is a common error to forget to reverse the direction of the inequality sign when multiplying or dividing by a negative number. For example, to solve $-4x \le 12$, we must multiply by $-1/4$ on both sides *and* reverse the inequality symbol to get $x \ge -3$.

The solution $y \geq -3/5$ in Example 1 represents an interval on the number line. **Interval notation** often is used for writing intervals. With interval notation, $y \geq -3/5$ is written as $[-3/5, \infty)$. This is an example of a **half-open interval**, since one endpoint, $-3/5$, is included. The **open interval** $(2, 5)$ corresponds to $2 < x < 5$, with neither endpoint included. The **closed interval** $[2, 5]$ includes both endpoints and corresponds to $2 \leq x \leq 5$.

The **graph** of an interval shows all points on a number line that correspond to the numbers in the interval. To graph the interval $[-3/5, \infty)$, for example, use a solid circle at $-3/5$, since $-3/5$ is part of the solution. To show that the solution includes all real numbers greater than or equal to $-3/5$, draw a heavy arrow pointing to the right (the positive direction). See Figure 1.

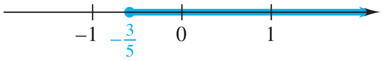


FIGURE 1

EXAMPLE 2 Graphing a Linear Inequality

Solve $-2 < 5 + 3m < 20$. Graph the solution.

SOLUTION The inequality $-2 < 5 + 3m < 20$ says that $5 + 3m$ is *between* -2 and 20. Solve this inequality with an extension of the properties given above. Work as follows, first adding -5 to each part.

$$-2 + (-5) < 5 + 3m + (-5) < 20 + (-5)$$
$$-7 < 3m < 15$$

Now multiply each part by $1/3$.

$$-\frac{7}{3} < m < 5$$

A graph of the solution is given in Figure 2; here open circles are used to show that $-7/3$ and 5 are *not* part of the graph.*

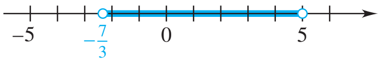


FIGURE 2

Quadratic Inequalities

A **quadratic inequality** has the form $ax^2 + bx + c > 0$ (or $<$, or $\leq$, or $\geq$). The greatest exponent is 2. The next few examples show how to solve quadratic inequalities.

EXAMPLE 3 Solving a Quadratic Inequality

Solve the quadratic inequality $x^2 - x < 12$.

SOLUTION Write the inequality with 0 on one side, as $x^2 - x - 12 < 0$. This inequality is solved with values of x that make $x^2 - x - 12$ negative (< 0). The quantity $x^2 - x - 12$ changes from positive to negative or from negative to positive at the points where it equals 0. For this reason, first solve the *equation* $x^2 - x - 12 = 0$.

$$x^2 - x - 12 = 0$$
$$(x - 4)(x + 3) = 0$$
$$x = 4 \quad \text{or} \quad x = -3$$

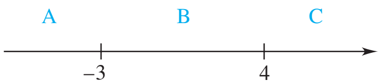


FIGURE 3

Locating -3 and 4 on a number line, as shown in Figure 3, determines three intervals A, B, and C. Decide which intervals include numbers that make $x^2 - x - 12$ negative by substituting any number from each interval into the polynomial. For example,

$$\text{choose } -4 \text{ from interval A: } (-4)^2 - (-4) - 12 = 8 > 0;$$
$$\text{choose } 0 \text{ from interval B: } 0^2 - 0 - 12 = -12 < 0;$$
$$\text{choose } 5 \text{ from interval C: } 5^2 - 5 - 12 = 8 > 0.$$

Only numbers in interval B satisfy the given inequality, so the solution is $(-3, 4)$. A graph of this solution is shown in Figure 4.

TRY YOUR TURN 2

FIGURE 4

YOUR TURN 2 Solve $3y^2 \leq 16y + 12$.

*Some textbooks use brackets in place of solid circles for the graph of a closed interval, and parentheses in place of open circles for the graph of an open interval.

EXAMPLE 4 Solving a Polynomial Inequality

Solve the inequality $x^3 + 2x^2 - 3x \geq 0$.

SOLUTION This is not a quadratic inequality because of the x^3 term, but we solve it in a similar way by first factoring the polynomial.

$$x^3 + 2x^2 - 3x = x(x^2 + 2x - 3) \quad \text{Factor out the common factor.}$$
$$= x(x - 1)(x + 3) \quad \text{Factor the quadratic.}$$

Now solve the corresponding equation.

$$x(x - 1)(x + 3) = 0$$
$$x = 0 \quad \text{or} \quad x - 1 = 0 \quad \text{or} \quad x + 3 = 0$$
$$x = 1 \qquad\qquad x = -3$$


FIGURE 5

These three solutions determine four intervals on the number line: $(-\infty, -3)$, $(-3, 0)$, $(0, 1)$, and $(1, \infty)$. Substitute a number from each interval into the original inequality to determine that the solution consists of the numbers between -3 and 0 (including the endpoints) and all numbers that are greater than or equal to 1. See Figure 5. In interval notation, the solution is

$$[-3, 0] \cup [1, \infty).*$$

Inequalities with Fractions

Inequalities with fractions are solved in a similar manner as quadratic inequalities.

EXAMPLE 5 Solving a Rational Inequality

Solve $\dfrac{2x - 3}{x} \geq 1$.

SOLUTION First solve the corresponding equation.

$$\frac{2x - 3}{x} = 1$$
$$2x - 3 = x \quad \text{Multiply both sides by } x.$$
$$x = 3 \quad \text{Solve for } x.$$

The solution, $x = 3$, determines the intervals on the number line where the fraction may change from greater than 1 to less than 1. This change also may occur on either side of a number that makes the denominator equal 0. Here, the x-value that makes the denominator 0 is $x = 0$. Test each of the three intervals determined by the numbers 0 and 3.

$$\text{For } (-\infty, 0), \text{ choose } -1\text{: } \frac{2(-1) - 3}{-1} = 5 \geq 1.$$
$$\text{For } (0, 3), \text{ choose } 1\text{: } \frac{2(1) - 3}{1} = -1 \not\geq 1.$$
$$\text{For } (3, \infty), \text{ choose } 4\text{: } \frac{2(4) - 3}{4} = \frac{5}{4} \geq 1.$$

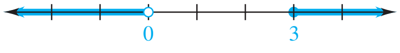


FIGURE 6

The symbol $\not\geq$ means "is *not* greater than or equal to." Testing the endpoints 0 and 3 shows that the solution is $(-\infty, 0) \cup [3, \infty)$, as shown in Figure 6.

CAUTION A common error is to try to solve the inequality in Example 5 by multiplying both sides by x. The reason this is wrong is that we don't know in the beginning whether x is positive, negative, or 0. If x is negative, the $\geq$ would change to $\leq$ according to the third property of inequality listed at the beginning of this section.

*The symbol $\cup$ indicates the *union* of two sets, which includes all elements in either set.

EXAMPLE 6 Solving a Rational Inequality

Solve $\dfrac{(x-1)(x+1)}{x} \le 0.$

SOLUTION We first solve the corresponding equation.

$$\frac{(x-1)(x+1)}{x} = 0$$

$$(x-1)(x+1) = 0 \qquad \text{Multiply both sides by } x.$$

$$x = 1 \quad \text{or} \quad x = -1 \qquad \text{Use the zero-factor property.}$$

Setting the denominator equal to 0 gives $x = 0$, so the intervals of interest are $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, and $(1, \infty)$. Testing a number from each region in the original inequality and checking the endpoints, we find the solution is

$$(-\infty, -1] \cup (0, 1],$$

as shown in Figure 7.

FIGURE 7

CAUTION Remember to solve the equation formed by setting the *denominator* equal to zero. Any number that makes the denominator zero always creates two intervals on the number line. For instance, in Example 6, substituting $x = 0$ makes the denominator of the rational inequality equal to 0, so we know that there may be a sign change from one side of 0 to the other (as was indeed the case).

EXAMPLE 7 Solving a Rational Inequality

Solve $\dfrac{x^2 - 3x}{x^2 - 9} < 4.$

SOLUTION Solve the corresponding equation.

$$\frac{x^2-3x}{x^2-9} = 4$$

$$x^2 - 3x = 4x^2 - 36 \qquad \text{Multiply by } x^2 - 9.$$

$$0 = 3x^2 + 3x - 36 \qquad \text{Get 0 on one side.}$$

$$0 = x^2 + x - 12 \qquad \text{Multiply by } \tfrac{1}{3}.$$

$$0 = (x+4)(x-3) \qquad \text{Factor.}$$

$$x = -4 \quad \text{or} \quad x = 3$$

Now set the denominator equal to 0 and solve that equation.

$$x^2 - 9 = 0$$

$$(x-3)(x+3) = 0$$

$$x = 3 \quad \text{or} \quad x = -3$$

FIGURE 8

The intervals determined by the three (different) solutions are $(-\infty, -4)$, $(-4, -3)$, $(-3, 3)$, and $(3, \infty)$. Testing a number from each interval in the given inequality shows that the solution is

$$(-\infty, -4) \cup (-3, 3) \cup (3, \infty),$$

as shown in Figure 8. For this example, none of the endpoints are part of the solution because $x = 3$ and $x = -3$ make the denominator zero and $x = -4$ produces an equality.

TRY YOUR TURN 3

YOUR TURN 3 Solve $\dfrac{k^2 - 35}{k} \ge 2.$

R.5 EXERCISES

Write each expression in interval notation. Graph each interval.

1. $x < 4$

2. $x \geq -3$

3. $1 \leq x < 2$

4. $-2 \leq x \leq 3$

5. $-9 > x$

6. $6 \leq x$

Using the variable *x*, write each interval as an inequality.

7. $[-7, -3]$

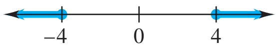
8. $[4, 10)$

9. $(-\infty, -1]$

10. $(3, \infty)$

11.


12.


13.


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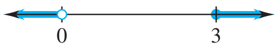

Solve each inequality and graph the solution.

15. $6p + 7 \leq 19$

16. $6k - 4 < 3k - 1$

17. $m - (3m - 2) + 6 < 7m - 19$

18. $-2(3y - 8) \geq 5(4y - 2)$

19. $3p - 1 < 6p + 2(p - 1)$

20. $x + 5(x + 1) > 4(2 - x) + x$

21. $-11 < y - 7 < -1$

22. $8 \leq 3r + 1 \leq 13$

23. $-2 < \dfrac{1 - 3k}{4} \leq 4$

24. $-1 \leq \dfrac{5y + 2}{3} \leq 4$

25. $\dfrac{3}{5}(2p + 3) \geq \dfrac{1}{10}(5p + 1)$

26. $\dfrac{8}{3}(z - 4) \leq \dfrac{2}{9}(3z + 2)$

Solve each inequality. Graph each solution.

27. $(m - 3)(m + 5) < 0$

28. $(t + 6)(t - 1) \geq 0$

29. $y^2 - 3y + 2 < 0$

30. $2k^2 + 7k - 4 > 0$

31. $x^2 - 16 > 0$

32. $2k^2 - 7k - 15 \leq 0$

33. $x^2 - 4x \geq 5$

34. $10r^2 + r \leq 2$

35. $3x^2 + 2x > 1$

36. $3a^2 + a > 10$

37. $9 - x^2 \leq 0$

38. $p^2 - 16p > 0$

39. $x^3 - 4x \geq 0$

40. $x^3 + 7x^2 + 12x \leq 0$

41. $2x^3 - 14x^2 + 12x < 0$

42. $3x^3 - 9x^2 - 12x > 0$

Solve each inequality.

43. $\dfrac{m - 3}{m + 5} \leq 0$

44. $\dfrac{r + 1}{r - 1} > 0$

45. $\dfrac{k - 1}{k + 2} > 1$

46. $\dfrac{a - 5}{a + 2} < -1$

47. $\dfrac{2y + 3}{y - 5} \leq 1$

48. $\dfrac{a + 2}{3 + 2a} \leq 5$

49. $\dfrac{2k}{k - 3} \leq \dfrac{4}{k - 3}$

50. $\dfrac{5}{p + 1} > \dfrac{12}{p + 1}$

51. $\dfrac{2x}{x^2 - x - 6} \geq 0$

52. $\dfrac{8}{p^2 + 2p} > 1$

53. $\dfrac{z^2 + z}{z^2 - 1} \geq 3$

54. $\dfrac{a^2 + 2a}{a^2 - 4} \leq 2$

YOUR TURN ANSWERS

1. $z < -9/2$ **2.** $[-2/3, 6]$ **3.** $[-5, 0) \cup [7, \infty)$

R.6 Exponents

Integer Exponents

Recall that $a^2 = a \cdot a$, while $a^3 = a \cdot a \cdot a$, and so on. In this section, a more general meaning is given to the symbol a^n.

> **Definition of Exponent**
>
> If n is a natural number, then
>
> $$a^n = a \cdot a \cdot a \cdot \cdots \cdot a,$$
>
> where a appears as a factor n times.

In the expression a^n, the power n is the **exponent** and a is the **base**. This definition can be extended by defining a^n for zero and negative integer values of n.

Zero and Negative Exponents

If a is any nonzero real number, and if n is a positive integer, then

$$\mathbf{a^0 = 1} \quad \text{and} \quad \mathbf{a^{-n} = \frac{1}{a^n}}.$$

(The symbol 0^0 is meaningless.)

EXAMPLE 1 Exponents

YOUR TURN 1 Find $\left(\frac{2}{3}\right)^{-3}$.

(a) $6^0 = 1$

(b) $(-9)^0 = 1$

(c) $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$

(d) $9^{-1} = \frac{1}{9^1} = \frac{1}{9}$

(e) $\left(\frac{3}{4}\right)^{-1} = \frac{1}{(3/4)^1} = \frac{1}{3/4} = \frac{4}{3}$

TRY YOUR TURN 1

The following properties follow from the definitions of exponents given above.

Properties of Exponents

For any integers m and n, and any real numbers a and b for which the following exist:

1. $a^m \cdot a^n = a^{m+n}$
2. $\frac{a^m}{a^n} = a^{m-n}$
3. $(a^m)^n = a^{mn}$
4. $(ab)^m = a^m \cdot b^m$
5. $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

Note that $(-a)^n = a^n$ if n is an even integer, but $(-a)^n = -a^n$ if n is an odd integer.

EXAMPLE 2 Simplifying Exponential Expressions

Use the properties of exponents to simplify each expression. Leave answers with positive exponents. Assume that all variables represent positive real numbers.

(a) $7^4 \cdot 7^6 = 7^{4+6} = 7^{10}$ (or 282,475,249) Property 1

(b) $\frac{9^{14}}{9^6} = 9^{14-6} = 9^8$ (or 43,046,721) Property 2

(c) $\frac{r^9}{r^{17}} = r^{9-17} = r^{-8} = \frac{1}{r^8}$ Property 2

(d) $(2m^3)^4 = 2^4 \cdot (m^3)^4 = 16m^{12}$ Properties 3 and 4

(e) $(3x)^4 = 3^4 \cdot x^4 = 81x^4$ Property 4

(f) $\left(\frac{x^2}{y^3}\right)^6 = \frac{(x^2)^6}{(y^3)^6} = \frac{x^{2\cdot 6}}{y^{3\cdot 6}} = \frac{x^{12}}{y^{18}}$ Properties 3 and 5

(g) $\frac{a^{-3}b^5}{a^4b^{-7}} = \frac{b^{5-(-7)}}{a^{4-(-3)}} = \frac{b^{5+7}}{a^{4+3}} = \frac{b^{12}}{a^7}$ Property 2

(h) $p^{-1} + q^{-1} = \frac{1}{p} + \frac{1}{q}$ — Definition of a^{-n}.

$= \frac{1}{p} \cdot \frac{q}{q} + \frac{1}{q} \cdot \frac{p}{p}$ — Get common denominator.

$= \frac{q}{pq} + \frac{p}{pq} = \frac{p + q}{pq}$ — Add.

(i) $\frac{x^{-2} - y^{-2}}{x^{-1} - y^{-1}} = \frac{\frac{1}{x^2} - \frac{1}{y^2}}{\frac{1}{x} - \frac{1}{y}}$ — Definition of a^{-n}

$= \frac{\frac{y^2 - x^2}{x^2y^2}}{\frac{y - x}{xy}}$ — Get common denominators and combine terms.

$= \frac{y^2 - x^2}{x^2y^2} \cdot \frac{xy}{y - x}$ — Invert and multiply.

$= \frac{(y - x)(y + x)}{x^2y^2} \cdot \frac{xy}{y - x}$ — Factor.

$= \frac{x + y}{xy}$ — Simplify.

TRY YOUR TURN 2

YOUR TURN 2 Simplify $\left(\frac{y^2z^{-4}}{y^{-3}z^4}\right)^{-2}$.

CAUTION If Example 2(e) were written $3x^4$, the properties of exponents would not apply. When no parentheses are used, the exponent refers only to the factor closest to it. Also notice in Examples 2(c), 2(g), 2(h), and 2(i) that a negative exponent does *not* indicate a negative number.

Roots

For *even* values of n and nonnegative values of a, the expression $a^{1/n}$ is defined to be the **positive *n*th root** of a or the **principal *n*th root** of a. For example, $a^{1/2}$ denotes the positive second root, or **square root**, of a, while $a^{1/4}$ is the positive fourth root of a. When n is *odd*, there is only one nth root, which has the same sign as a. For example, $a^{1/3}$, the **cube root** of a, has the same sign as a. By definition, if $b = a^{1/n}$, then $b^n = a$. On a calculator, a number is raised to a power using a key labeled x^y, y^x, or $\wedge$. For example, to take the fourth root of 6 on a TI-84 Plus C calculator, enter $6 \wedge (1/4)$ to get the result 1.56508458.

EXAMPLE 3 Calculations with Exponents

(a) $121^{1/2} = 11$, since 11 is positive and $11^2 = 121$.

(b) $625^{1/4} = 5$, since $5^4 = 625$.

(c) $256^{1/4} = 4$

(d) $64^{1/6} = 2$

(e) $27^{1/3} = 3$

(f) $(-32)^{1/5} = -2$

(g) $128^{1/7} = 2$

(h) $(-49)^{1/2}$ is not a real number.

TRY YOUR TURN 3

YOUR TURN 3 Find $125^{1/3}$.

Rational Exponents

In the following definition, the domain of an exponent is extended to include all rational numbers.

Definition of $a^{m/n}$

For all real numbers a for which the indicated roots exist, and for any rational number m/n,

$$a^{m/n} = (a^{1/n})^m.$$

EXAMPLE 4 Calculations with Exponents

YOUR TURN 4 Find $16^{-3/4}$.

(a) $27^{2/3} = (27^{1/3})^2 = 3^2 = 9$

(b) $32^{2/5} = (32^{1/5})^2 = 2^2 = 4$

(c) $64^{4/3} = (64^{1/3})^4 = 4^4 = 256$

(d) $25^{3/2} = (25^{1/2})^3 = 5^3 = 125$

TRY YOUR TURN 4

NOTE $27^{2/3}$ could also be evaluated as $(27^2)^{1/3}$, but this is more difficult to perform without a calculator because it involves squaring 27 and then taking the cube root of this large number. On the other hand, when we evaluate it as $(27^{1/3})^2$, we know that the cube root of 27 is 3 without using a calculator, and squaring 3 is easy.

All the properties for integer exponents given in this section also apply to any rational exponent on a nonnegative real-number base.

EXAMPLE 5 Simplifying Exponential Expressions

YOUR TURN 5 Simplify $\left(\dfrac{x^{1/2}x^4}{x^{3/2}}\right)^{1/3}$.

(a) $$\frac{y^{1/3}y^{5/3}}{y^3} = \frac{y^{1/3+5/3}}{y^3} = \frac{y^2}{y^3} = y^{2-3} = y^{-1} = \frac{1}{y}$$

(b) $$m^{2/3}(m^{7/3} + 2m^{1/3}) = m^{2/3+7/3} + 2m^{2/3+1/3} = m^3 + 2m$$

(c) $$\left(\frac{m^7n^{-2}}{m^{-5}n^2}\right)^{1/4} = \left(\frac{m^{7-(-5)}}{n^{2-(-2)}}\right)^{1/4} = \left(\frac{m^{12}}{n^4}\right)^{1/4} = \frac{(m^{12})^{1/4}}{(n^4)^{1/4}} = \frac{m^{12/4}}{n^{4/4}} = \frac{m^3}{n}$$

TRY YOUR TURN 5

In calculus, it is often necessary to factor expressions involving fractional exponents.

EXAMPLE 6 Simplifying Exponential Expressions

Factor out the smallest power of the variable, assuming all variables represent positive real numbers.

(a) $4m^{1/2} + 3m^{3/2}$

SOLUTION The smallest exponent is $1/2$. Factoring out $m^{1/2}$ yields

$$\begin{aligned} 4m^{1/2} + 3m^{3/2} &= m^{1/2}(4m^{1/2-1/2} + 3m^{3/2-1/2}) \\ &= m^{1/2}(4 + 3m). \end{aligned}$$

Check this result by multiplying $m^{1/2}$ by $4 + 3m$.

(b) $9x^{-2} - 6x^{-3}$

SOLUTION The smallest exponent here is -3. Since 3 is a common numerical factor, factor out $3x^{-3}$.

$$9x^{-2} - 6x^{-3} = 3x^{-3}(3x^{-2-(-3)} - 2x^{-3-(-3)}) = 3x^{-3}(3x - 2)$$

Check by multiplying. The factored form can be written without negative exponents as

$$\frac{3(3x-2)}{x^3}.$$

(c) $(x^2 + 5)(3x - 1)^{-1/2}(2) + (3x - 1)^{1/2}(2x)$

SOLUTION There is a common factor of 2. Also, $(3x - 1)^{-1/2}$ and $(3x - 1)^{1/2}$ have a common factor. Always factor out the quantity to the *smallest* exponent. Here $-1/2 < 1/2$, so the common factor is $2(3x - 1)^{-1/2}$ and the factored form is

$$2(3x - 1)^{-1/2}[(x^2 + 5) + (3x - 1)x] = 2(3x - 1)^{-1/2}(4x^2 - x + 5).$$

TRY YOUR TURN 6

YOUR TURN 6 Factor $5z^{1/3} + 4z^{-2/3}$.

R.6 EXERCISES

Evaluate each expression. Write all answers without exponents.

1. 8^{-2} **2.** 3^{-4}

3. 5^0 **4.** $\left(-\frac{3}{4}\right)^0$

5. $-(-3)^{-2}$ **6.** $-(-3^{-2})$

7. $\left(\frac{1}{6}\right)^{-2}$ **8.** $\left(\frac{4}{3}\right)^{-3}$

Simplify each expression. Assume that all variables represent positive real numbers. Write answers with only positive exponents.

9. $\frac{4^{-2}}{4}$ **10.** $\frac{8^9 \cdot 8^{-7}}{8^{-3}}$

11. $\frac{10^8 \cdot 10^{-10}}{10^4 \cdot 10^2}$ **12.** $\left(\frac{7^{-12} \cdot 7^3}{7^{-8}}\right)^{-1}$

13. $\frac{x^4 \cdot x^3}{x^5}$ **14.** $\frac{y^{10} \cdot y^{-4}}{y^6}$

15. $\frac{(4k^{-1})^2}{2k^{-5}}$ **16.** $\frac{(3z^2)^{-1}}{z^5}$

17. $\frac{3^{-1} \cdot x \cdot y^2}{x^{-4} \cdot y^5}$ **18.** $\frac{5^{-2}m^2y^{-2}}{5^2m^{-1}y^{-2}}$

19. $\left(\frac{a^{-1}}{b^2}\right)^{-3}$ **20.** $\left(\frac{c^3}{7d^{-2}}\right)^{-2}$

Simplify each expression, writing the answer as a single term without negative exponents.

21. $a^{-1} + b^{-1}$ **22.** $b^{-2} - a$

23. $\frac{2n^{-1} - 2m^{-1}}{m + n^2}$ **24.** $\left(\frac{m}{3}\right)^{-1} + \left(\frac{n}{2}\right)^{-2}$

25. $(x^{-1} - y^{-1})^{-1}$ **26.** $(x \cdot y^{-1} - y^{-2})^{-2}$

Write each number without exponents.

27. $121^{1/2}$ **28.** $27^{1/3}$

29. $32^{2/5}$ **30.** $-125^{2/3}$

31. $\left(\frac{36}{144}\right)^{1/2}$ **32.** $\left(\frac{64}{27}\right)^{1/3}$

33. $8^{-4/3}$ **34.** $625^{-1/4}$

35. $\left(\frac{27}{64}\right)^{-1/3}$ **36.** $\left(\frac{121}{100}\right)^{-3/2}$

Simplify each expression. Write all answers with only positive exponents. Assume that all variables represent positive real numbers.

37. $3^{2/3} \cdot 3^{4/3}$ **38.** $27^{2/3} \cdot 27^{-1/3}$

39. $\frac{4^{9/4} \cdot 4^{-7/4}}{4^{-10/4}}$ **40.** $\frac{3^{-5/2} \cdot 3^{3/2}}{3^{7/2} \cdot 3^{-9/2}}$

41. $\left(\frac{x^6y^{-3}}{x^{-2}y^5}\right)^{1/2}$ **42.** $\left(\frac{a^{-7}b^{-1}}{b^{-4}a^2}\right)^{1/3}$

43. $\frac{7^{-1/3} \cdot 7r^{-3}}{7^{2/3} \cdot (r^{-2})^2}$ **44.** $\frac{12^{3/4} \cdot 12^{5/4} \cdot y^{-2}}{12^{-1} \cdot (y^{-3})^{-2}}$

45. $\frac{3k^2 \cdot (4k^{-3})^{-1}}{4^{1/2} \cdot k^{7/2}}$ **46.** $\frac{8p^{-3} \cdot (4p^2)^{-2}}{p^{-5}}$

47. $\frac{a^{4/3} \cdot b^{1/2}}{a^{2/3} \cdot b^{-3/2}}$ **48.** $\frac{x^{3/2} \cdot y^{4/5} \cdot z^{-3/4}}{x^{5/3} \cdot y^{-6/5} \cdot z^{1/2}}$

49. $\frac{k^{-3/5} \cdot h^{-1/3} \cdot t^{2/5}}{k^{-1/5} \cdot h^{-2/3} \cdot t^{1/5}}$ **50.** $\frac{m^{7/3} \cdot n^{-2/5} \cdot p^{3/8}}{m^{-2/3} \cdot n^{3/5} \cdot p^{-5/8}}$

Factor each expression.

51. $3x^3(x^2 + 3x)^2 - 15x(x^2 + 3x)^2$

52. $6x(x^3 + 7)^2 - 6x^2(3x^2 + 5)(x^3 + 7)$

53. $10x^3(x^2 - 1)^{-1/2} - 5x(x^2 - 1)^{1/2}$

54. $9(6x + 2)^{1/2} + 3(9x - 1)(6x + 2)^{-1/2}$

55. $x(2x + 5)^2(x^2 - 4)^{-1/2} + 2(x^2 - 4)^{1/2}(2x + 5)$

56. $(4x^2 + 1)^2(2x - 1)^{-1/2} + 16x(4x^2 + 1)(2x - 1)^{1/2}$

YOUR TURN ANSWERS

1. $27/8$ **2.** z^{16}/y^{10} **3.** 5

4. $1/8$ **5.** x **6.** $z^{-2/3}(5z + 4)$

R.7 Radicals

We have defined $a^{1/n}$ as the positive or principal nth root of a for appropriate values of a and n. An alternative notation for $a^{1/n}$ uses radicals.

Radicals

If n is an even natural number and $a > 0$, or n is an odd natural number, then

$$a^{1/n} = \sqrt[n]{a}.$$

The symbol $\sqrt[n]{}$ is a **radical sign**, the number a is the **radicand**, and n is the **index** of the radical. The familiar symbol $\sqrt{a}$ is used instead of $\sqrt[2]{a}$.

EXAMPLE 1 Radical Calculations

(a) $\sqrt[4]{16} = 16^{1/4} = 2$

(b) $\sqrt[5]{-32} = -2$

(c) $\sqrt[3]{1000} = 10$

(d) $\sqrt[6]{\dfrac{64}{729}} = \dfrac{2}{3}$

With $a^{1/n}$ written as $\sqrt[n]{a}$, the expression $a^{m/n}$ also can be written using radicals.

$$a^{m/n} = (\sqrt[n]{a})^m \qquad \text{or} \qquad a^{m/n} = \sqrt[n]{a^m}$$

The following properties of radicals depend on the definitions and properties of exponents.

Properties of Radicals

For all real numbers a and b and natural numbers m and n such that $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers:

1. $(\sqrt[n]{a})^n = a$

2. $\sqrt[n]{a^n} = \begin{cases} |a| & \text{if } n \text{ is even} \\ a & \text{if } n \text{ is odd} \end{cases}$

3. $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$

4. $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}} \quad (b \neq 0)$

5. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$

Property 3 can be used to simplify certain radicals. For example, since $48 = 16 \cdot 3$,

$$\sqrt{48} = \sqrt{16 \cdot 3} = \sqrt{16} \cdot \sqrt{3} = 4\sqrt{3}.$$

To some extent, simplification is in the eye of the beholder, and $\sqrt{48}$ might be considered as simple as $4\sqrt{3}$. In this textbook, we will consider an expression to be simpler when we have removed as many factors as possible from under the radical.

EXAMPLE 2 Radical Calculations

(a) $\sqrt{1000} = \sqrt{100 \cdot 10} = \sqrt{100} \cdot \sqrt{10} = 10\sqrt{10}$

(b) $\sqrt{128} = \sqrt{64 \cdot 2} = 8\sqrt{2}$

(c) $\sqrt{2} \cdot \sqrt{18} = \sqrt{2 \cdot 18} = \sqrt{36} = 6$

(d) $\sqrt[3]{54} = \sqrt[3]{27 \cdot 2} = \sqrt[3]{27} \cdot \sqrt[3]{2} = 3\sqrt[3]{2}$

(e) $\sqrt{288m^5} = \sqrt{144 \cdot m^4 \cdot 2m} = 12m^2\sqrt{2m}$

(f)
$$\begin{aligned} 2\sqrt{18} - 5\sqrt{32} &= 2\sqrt{9 \cdot 2} - 5\sqrt{16 \cdot 2} \\ &= 2\sqrt{9} \cdot \sqrt{2} - 5\sqrt{16} \cdot \sqrt{2} \\ &= 2(3)\sqrt{2} - 5(4)\sqrt{2} = -14\sqrt{2} \end{aligned}$$

(g) $\sqrt{x^5} \cdot \sqrt[3]{x^5} = x^{5/2} \cdot x^{5/3} = x^{5/2+5/3} = x^{25/6} = \sqrt[6]{x^{25}} = x^4\sqrt[6]{x}$

YOUR TURN 1 Simplify $\sqrt{28x^9y^5}$.

TRY YOUR TURN 1

When simplifying a square root, keep in mind that $\sqrt{x}$ is nonnegative by definition. Also, $\sqrt{x^2}$ is not x, but $|x|$, the **absolute value of** $\boldsymbol{x}$, defined as

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

For example, $\sqrt{(-5)^2} = |-5| = 5$. It is correct, however, to simplify $\sqrt{x^4} = x^2$. We need not write $|x^2|$ because x^2 is always nonnegative.

EXAMPLE 3 Simplifying by Factoring

Simplify $\sqrt{m^2 - 4m + 4}$.

SOLUTION Factor the polynomial as $m^2 - 4m + 4 = (m - 2)^2$. Then by property 2 of radicals and the definition of absolute value,

$$\sqrt{(m-2)^2} = |m - 2| = \begin{cases} m - 2 & \text{if } m - 2 \geq 0 \\ -(m - 2) = 2 - m & \text{if } m - 2 < 0. \end{cases}$$

CAUTION Avoid the common error of writing $\sqrt{a^2 + b^2}$ as $\sqrt{a^2} + \sqrt{b^2}$. We must add a^2 and b^2 *before* taking the square root. For example, $\sqrt{16 + 9} = \sqrt{25} = 5$, *not* $\sqrt{16} + \sqrt{9} = 4 + 3 = 7$. This idea applies as well to higher roots. For example, in general,

$$\sqrt[3]{a^3 + b^3} \neq \sqrt[3]{a^3} + \sqrt[3]{b^3},$$

$$\sqrt[4]{a^4 + b^4} \neq \sqrt[4]{a^4} + \sqrt[4]{b^4}.$$

Also, $$\sqrt{a + b} \neq \sqrt{a} + \sqrt{b}.$$

Rationalizing Denominators The next example shows how to *rationalize* (remove all radicals from) the denominator in an expression containing radicals.

EXAMPLE 4 Rationalizing Denominators

Simplify each expression by rationalizing the denominator.

(a) $\dfrac{4}{\sqrt{3}}$

SOLUTION To rationalize the denominator, multiply by $\sqrt{3}/\sqrt{3}$ (or 1) so the denominator of the product is a rational number.

$$\frac{4}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{4\sqrt{3}}{3} \qquad \sqrt{3} \cdot \sqrt{3} = \sqrt{9} = 3$$

(b) $\dfrac{2}{\sqrt[3]{x}}$

SOLUTION Here, we need a perfect cube under the radical sign to rationalize the denominator. Multiplying by $\sqrt[3]{x^2}/\sqrt[3]{x^2}$ gives

$$\frac{2}{\sqrt[3]{x}} \cdot \frac{\sqrt[3]{x^2}}{\sqrt[3]{x^2}} = \frac{2\sqrt[3]{x^2}}{\sqrt[3]{x^3}} = \frac{2\sqrt[3]{x^2}}{x}.$$

(c) $\dfrac{1}{1 - \sqrt{2}}$

SOLUTION The best approach here is to multiply both numerator and denominator by the number $1 + \sqrt{2}$. The expressions $1 + \sqrt{2}$ and $1 - \sqrt{2}$ are conjugates,* and their product is $1^2 - (\sqrt{2})^2 = 1 - 2 = -1$. Thus,

$$\frac{1}{1 - \sqrt{2}} = \frac{1(1 + \sqrt{2})}{(1 - \sqrt{2})(1 + \sqrt{2})} = \frac{1 + \sqrt{2}}{1 - 2} = -1 - \sqrt{2}.$$

TRY YOUR TURN 2

YOUR TURN 2 Rationalize the denominator in $\dfrac{5}{\sqrt{x} - \sqrt{y}}$.

Sometimes it is advantageous to rationalize the *numerator* of a rational expression. The following example arises in calculus when evaluating a *limit*.

EXAMPLE 5 Rationalizing Numerators

Rationalize each numerator.

(a) $\dfrac{\sqrt{x} - 3}{x - 9}$.

SOLUTION Multiply the numerator and denominator by the conjugate of the numerator, $\sqrt{x} + 3$.

$$\frac{\sqrt{x} - 3}{x - 9} \cdot \frac{\sqrt{x} + 3}{\sqrt{x} + 3} = \frac{(\sqrt{x})^2 - 3^2}{(x - 9)(\sqrt{x} + 3)} \qquad (a - b)(a + b) = a^2 - b^2$$

$$= \frac{x - 9}{(x - 9)(\sqrt{x} + 3)}$$

$$= \frac{1}{\sqrt{x} + 3}$$

*If a and b are real numbers, the *conjugate* of $a + b$ is $a - b$.

(b) $\dfrac{\sqrt{3}+\sqrt{x+3}}{\sqrt{3}-\sqrt{x+3}}$

SOLUTION Multiply the numerator and denominator by the conjugate of the numerator, $\sqrt{3}-\sqrt{x+3}$.

$$\frac{\sqrt{3}+\sqrt{x+3}}{\sqrt{3}-\sqrt{x+3}}\cdot\frac{\sqrt{3}-\sqrt{x+3}}{\sqrt{3}-\sqrt{x+3}}=\frac{3-(x+3)}{3-2\sqrt{3}\sqrt{x+3}+(x+3)}$$

$$=\frac{-x}{6+x-2\sqrt{3(x+3)}}$$

TRY YOUR TURN 3

YOUR TURN 3 Rationalize the numerator in $\dfrac{4+\sqrt{x}}{16-x}$.

R.7 EXERCISES

Simplify each expression by removing as many factors as possible from under the radical. Assume that all variables represent positive real numbers.

1. $\sqrt[3]{125}$
2. $\sqrt[4]{1296}$
3. $\sqrt[5]{-3125}$
4. $\sqrt{50}$
5. $\sqrt{2000}$
6. $\sqrt{32y^5}$
7. $\sqrt{27}\cdot\sqrt{3}$
8. $\sqrt{2}\cdot\sqrt{32}$
9. $7\sqrt{2}-8\sqrt{18}+4\sqrt{72}$
10. $4\sqrt{3}-5\sqrt{12}+3\sqrt{75}$
11. $4\sqrt{7}-\sqrt{28}+\sqrt{343}$
12. $3\sqrt{28}-4\sqrt{63}+\sqrt{112}$
13. $\sqrt[3]{2}-\sqrt[3]{16}+2\sqrt[3]{54}$
14. $2\sqrt[3]{5}-4\sqrt[3]{40}+3\sqrt[3]{135}$
15. $\sqrt{2x^3y^2z^4}$
16. $\sqrt{160r^7s^9t^{12}}$
17. $\sqrt[3]{128x^3y^8z^9}$
18. $\sqrt[4]{x^8y^7z^{11}}$
19. $\sqrt{a^3b^5}-2\sqrt{a^7b^3}+\sqrt{a^3b^9}$
20. $\sqrt{p^7q^3}-\sqrt{p^5q^9}+\sqrt{p^9q}$
21. $\sqrt{a}\cdot\sqrt[3]{a}$
22. $\sqrt{b^3}\cdot\sqrt[4]{b^3}$

Simplify each root, if possible.

23. $\sqrt{16-8x+x^2}$
24. $\sqrt{9y^2+30y+25}$
25. $\sqrt{4-25z^2}$
26. $\sqrt{9k^2+h^2}$

Rationalize each denominator. Assume that all radicands represent positive real numbers.

27. $\dfrac{5}{\sqrt{7}}$
28. $\dfrac{5}{\sqrt{10}}$
29. $\dfrac{-3}{\sqrt{12}}$
30. $\dfrac{4}{\sqrt{8}}$
31. $\dfrac{3}{1-\sqrt{2}}$
32. $\dfrac{5}{2-\sqrt{6}}$
33. $\dfrac{6}{2+\sqrt{2}}$
34. $\dfrac{\sqrt{5}}{\sqrt{5}+\sqrt{2}}$
35. $\dfrac{1}{\sqrt{r}-\sqrt{3}}$
36. $\dfrac{5}{\sqrt{m}-\sqrt{5}}$
37. $\dfrac{y-5}{\sqrt{y}-\sqrt{5}}$
38. $\dfrac{\sqrt{z}-1}{\sqrt{z}-\sqrt{5}}$
39. $\dfrac{\sqrt{x}+\sqrt{x+1}}{\sqrt{x}-\sqrt{x+1}}$
40. $\dfrac{\sqrt{p}+\sqrt{p^2-1}}{\sqrt{p}-\sqrt{p^2-1}}$

Rationalize each numerator. Assume that all radicands represent positive real numbers.

41. $\dfrac{1+\sqrt{2}}{2}$
42. $\dfrac{3-\sqrt{3}}{6}$
43. $\dfrac{\sqrt{x}+\sqrt{x+1}}{\sqrt{x}-\sqrt{x+1}}$
44. $\dfrac{\sqrt{p}-\sqrt{p-2}}{\sqrt{p}}$

YOUR TURN ANSWERS

1. $2x^4y^2\sqrt{7xy}$ **2.** $5(\sqrt{x}+\sqrt{y})/(x-y)$ **3.** $1/(4-\sqrt{x})$

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Linear Functions

Over short time intervals, many changes in the economy are well modeled by linear functions. In an exercise in the first section of this chapter, we will examine a linear model that predicts the number of cellular telephone users in the United States. Such predictions are important tools for cellular telephone company executives and planners.

Before using mathematics to solve a real-world problem, we must usually set up a **mathematical model**, a mathematical description of the situation. In this chapter we look at some mathematics of *linear* models, which are used for data whose graphs can be approximated by straight lines. Linear models have an immense number of applications, because even when the underlying phenomenon is not linear, a linear model often provides an approximation that is sufficiently accurate and much simpler to use.

1.1 Slopes and Equations of Lines

APPLY IT **How fast has tuition at public colleges been increasing in recent years, and how well can we predict tuition in the future?**

In Example 13 of this section, we will answer these questions using the equation of a line.

There are many everyday situations in which two quantities are related. For example, if a bank account pays 6% simple interest per year, then the interest I that a deposit of P dollars would earn in one year is given by

$$I = 0.06 \cdot P, \qquad \text{or} \qquad I = 0.06P.$$

The formula $I = 0.06P$ describes the relationship between interest and the amount of money deposited.

Using this formula, we see, for example, that if $P = \$100$, then $I = \$6$, and if $P = \$200$, then $I = \$12$. These corresponding pairs of numbers can be written as **ordered pairs**, $(100, 6)$ and $(200, 12)$, whose order is important. The first number denotes the value of P and the second number the value of I.

Ordered pairs are graphed with the perpendicular number lines of a **Cartesian coordinate system**, shown in Figure 1.* The horizontal number line, or ***x*-axis**, represents the first components of the ordered pairs, while the vertical or ***y*-axis** represents the second components. The point where the number lines cross is the zero point on both lines; this point is called the **origin**.

Each point on the xy-plane corresponds to an ordered pair of numbers, where the x-value is written first. From now on, we will refer to the point corresponding to the ordered pair (x, y) as "the point (x, y)."

Locate the point $(-2, 4)$ on the coordinate system by starting at the origin and counting 2 units to the left on the horizontal axis and 4 units upward, parallel to the vertical axis. This point is shown in Figure 1, along with several other sample points. The number -2 is the ***x*-coordinate** and the number 4 is the ***y*-coordinate** of the point $(-2, 4)$.

The x-axis and y-axis divide the plane into four parts, or **quadrants**. For example, quadrant I includes all those points whose x- and y-coordinates are both positive. The quadrants are numbered as shown in Figure 1. The points on the axes themselves belong to no quadrant. The set of points corresponding to the ordered pairs of an equation is the **graph** of the equation.

The x- and y-values of the points where the graph of an equation crosses the axes are called the ***x*-intercept** and ***y*-intercept**, respectively.** See Figure 2.

*The name "Cartesian" honors René Descartes (1596–1650), one of the greatest mathematicians of the 17th century. According to legend, Descartes was lying in bed when he noticed an insect crawling on the ceiling and realized that if he could determine the distance from the bug to each of two perpendicular walls, he could describe its position at any given moment. The same idea can be used to locate a point in a plane.

**Some people prefer to define the intercepts as ordered pairs, rather than as numbers.

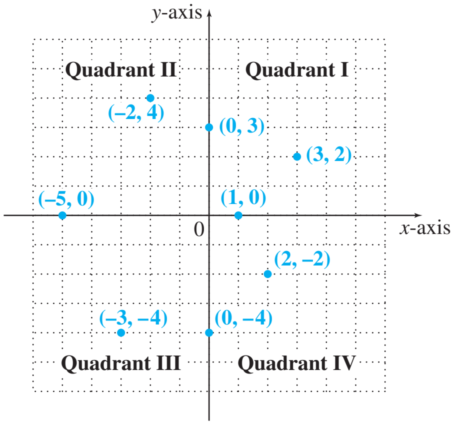


FIGURE 1

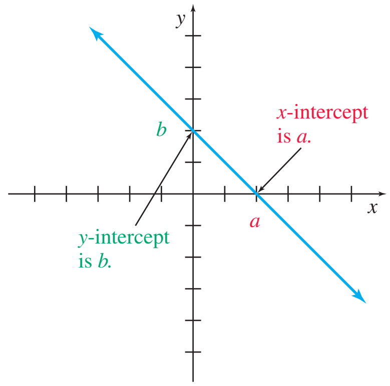


FIGURE 2

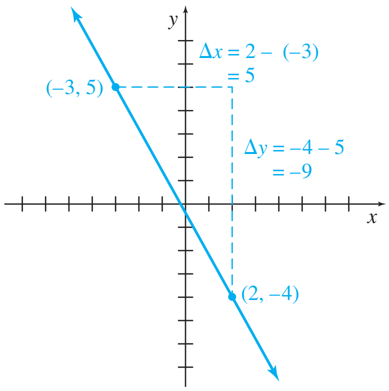


FIGURE 3

Slope of a Line

An important characteristic of a straight line is its *slope*, a number that represents the "steepness" of the line. To see how slope is defined, look at the line in Figure 3. The line passes through the points $(x_1, y_1) = (-3, 5)$ and $(x_2, y_2) = (2, -4)$. The difference in the two x-values,

$$x_2 - x_1 = 2 - (-3) = 5$$

in this example, is called the **change in x**. The symbol Δx (read "delta x") is used to represent the change in x. In the same way, Δy represents the **change in y**. In our example,

$$\begin{aligned}\Delta y &= y_2 - y_1\\ &= -4 - 5\\ &= -9.\end{aligned}$$

These symbols, Δx and Δy, are used in the following definition of slope.

Slope of a Nonvertical Line

The **slope** of a nonvertical line is defined as the vertical change (the "rise") over the horizontal change (the "run") as one travels along the line. In symbols, taking two different points (x_1, y_1) and (x_2, y_2) on the line, the slope is

$$m = \frac{\text{Change in } y}{\text{Change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1},$$

where $x_1 \neq x_2$.

By this definition, the slope of the line in Figure 3 is

$$m = \frac{\Delta y}{\Delta x} = \frac{-4 - 5}{2 - (-3)} = -\frac{9}{5}.$$

The slope of a line tells how fast y changes for each unit of change in x.

NOTE Using similar triangles, it can be shown that the slope of a line is independent of the choice of points on the line. That is, the same slope will be obtained for *any* choice of two different points on the line.

EXAMPLE 1 Slope

Find the slope of the line through each pair of points.

(a) $(7, 6)$ and $(-4, 5)$

SOLUTION Let $(x_1, y_1) = (7, 6)$ and $(x_2, y_2) = (-4, 5)$. Use the definition of slope.

$$m = \frac{\Delta y}{\Delta x} = \frac{5 - 6}{-4 - 7} = \frac{-1}{-11} = \frac{1}{11}$$

(b) $(5, -3)$ and $(-2, -3)$

SOLUTION Let $(x_1, y_1) = (5, -3)$ and $(x_2, y_2) = (-2, -3)$. Then

$$m = \frac{-3 - (-3)}{-2 - 5} = \frac{0}{-7} = 0.$$

Lines with zero slope are horizontal (parallel to the x-axis).

(c) $(2, -4)$ and $(2, 3)$

SOLUTION Let $(x_1, y_1) = (2, -4)$ and $(x_2, y_2) = (2, 3)$. Then

$$m = \frac{3 - (-4)}{2 - 2} = \frac{7}{0},$$

which is undefined. This happens when the line is vertical (parallel to the y-axis).

YOUR TURN 1 Find the slope of the line through $(1, 5)$ and $(4, 6)$.

TRY YOUR TURN 1

CAUTION The phrase "no slope" should be avoided; specify instead whether the slope is zero or undefined.

In finding the slope of the line in Example 1(a), we could have let $(x_1, y_1) = (-4, 5)$ and $(x_2, y_2) = (7, 6)$. In that case,

$$m = \frac{6 - 5}{7 - (-4)} = \frac{1}{11},$$

the same answer as before. The order in which coordinates are subtracted does not matter, as long as it is done consistently.

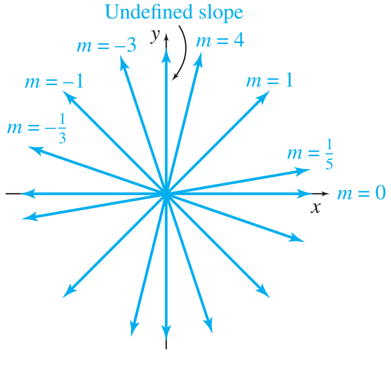


FIGURE 4

Figure 4 shows examples of lines with different slopes. Lines with positive slopes rise from left to right, while lines with negative slopes fall from left to right.

It might help you to compare slope with the percent grade of a hill. If a sign says a hill has a 10% grade uphill, this means the slope is 0.10, or 1/10, so the hill rises 1 foot for every 10 feet horizontally. A 15% grade downhill means the slope is −0.15.

Equations of a Line

An equation in two first-degree variables, such as $4x + 7y = 20$, has a line as its graph, so it is called a **linear equation**. In the rest of this section, we consider various forms of the equation of a line.

Suppose a line has a slope m and y-intercept b. This means that it passes through the point $(0, b)$. If (x, y) is any other point on the line, then the definition of slope tells us that

$$m = \frac{y - b}{x - 0}.$$

FOR REVIEW For review on solving a linear equation, see Section R.4.

We can simplify this equation by multiplying both sides by x and adding b to both sides. The result is

$$y = mx + b,$$

which we call the *slope-intercept* form of a line. This is the most common form for writing the equation of a line.

Slope-Intercept Form

If a line has slope m and y-intercept b, then the equation of the line in **slope-intercept form** is

$$y = mx + b.$$

When $b = 0$, we say that y is **proportional** to x.

EXAMPLE 2 Equation of a Line

Find an equation in slope-intercept form for each line.

(a) Through $(0, -3)$ with slope $3/4$

SOLUTION We recognize $(0, -3)$ as the y-intercept because it's the point with 0 as its x-coordinate, so $b = -3$. The slope is $3/4$, so $m = 3/4$. Substituting these values into $y = mx + b$ gives

$$y = \frac{3}{4}x + (-3) = \frac{3}{4}x - 3.$$

(b) With x-intercept 7 and y-intercept 2

SOLUTION Notice that $b = 2$. To find m, use the definition of slope after writing the x-intercept as $(7, 0)$ (because the y-coordinate is 0 where the line crosses the x-axis) and the y-intercept as $(0, 2)$.

$$m = \frac{0 - 2}{7 - 0} = -\frac{2}{7}$$

Substituting these values into $y = mx + b$, we have

$$y = -\frac{2}{7}x + 2.$$

TRY YOUR TURN 2

YOUR TURN 2 Find the equation of the line with x-intercept -4 and y-intercept 6.

EXAMPLE 3 Finding the Slope

Find the slope of the line whose equation is $3x - 4y = 12$.

SOLUTION To find the slope, solve the equation for y.

$$3x - 4y = 12$$

$$-4y = -3x + 12 \quad \text{Subtract } 3x \text{ from both sides.}$$

$$y = \frac{3}{4}x - 3 \quad \text{Divide both sides by } -4.$$

The coefficient of x is $3/4$, which is the slope of the line. Notice that this is the same line as in Example 2(a).

TRY YOUR TURN 3

YOUR TURN 3 Find the slope of the line whose equation is $8x + 3y = 5$.

The slope-intercept form of the equation of a line involves the slope and the y-intercept. Sometimes, however, the slope of a line is known, together with one point (perhaps *not* the y-intercept) that the line passes through. The *point-slope form* of the equation of a line is used to find the equation in this case. Let (x_1, y_1) be any fixed point on the line, and let (x, y) represent any other point on the line. If m is the slope of the line, then by the definition of slope,

$$\frac{y - y_1}{x - x_1} = m,$$

or

$$y - y_1 = m(x - x_1). \quad \text{Multiply both sides by } x - x_1.$$

Point-Slope Form

If a line has slope m and passes through the point (x_1, y_1), then an equation of the line is given by

$$y - y_1 = m(x - x_1),$$

the **point-slope form** of the equation of a line.

EXAMPLE 4 Point-Slope Form

Find an equation of the line that passes through the point $(3, -7)$ and has slope $m = 5/4$.

SOLUTION Use the point-slope form.

$$y - y_1 = m(x - x_1)$$

$$y - (-7) = \frac{5}{4}(x - 3) \qquad y_1 = -7, m = \tfrac{5}{4}, x_1 = 3$$

$$y + 7 = \frac{5}{4}(x - 3)$$

$$4y + 28 = 5(x - 3) \qquad \text{Multiply both sides by 4.}$$

$$4y + 28 = 5x - 15 \qquad \text{Distribute.}$$

$$4y = 5x - 43 \qquad \text{Subtract 28 from both sides.}$$

$$y = \frac{5}{4}x - \frac{43}{4} \qquad \text{Divide both sides by 4.}$$

FOR REVIEW

See Section R.4 for details on eliminating denominators in an equation.

The point-slope form also can be useful to find an equation of a line if we know two different points that the line goes through, as in the next example.

EXAMPLE 5 Point-Slope Form with Two Points

Find an equation of the line through $(5, 4)$ and $(-10, -2)$.

SOLUTION Begin by using the definition of slope to find the slope of the line that passes through the given points.

$$\text{Slope} = m = \frac{-2 - 4}{-10 - 5} = \frac{-6}{-15} = \frac{2}{5}$$

Either $(5, 4)$ or $(-10, -2)$ can be used in the point-slope form with $m = 2/5$. If $(x_1, y_1) = (5, 4)$, then

$$y - y_1 = m(x - x_1)$$

$$y - 4 = \frac{2}{5}(x - 5) \qquad y_1 = 4, m = \tfrac{2}{5}, x_1 = 5$$

$$y - 4 = \frac{2}{5}x - 2 \qquad \text{Distributive property}$$

$$y = \frac{2}{5}x + 2. \qquad \text{Add 4 to both sides.}$$

Check that the same result is found if $(x_1, y_1) = (-10, -2)$. TRY YOUR TURN 4

YOUR TURN 4 Find the equation of the line through $(2, 9)$ and $(5, 3)$. Put your answer in slope-intercept form.

EXAMPLE 6 Horizontal Line

Find an equation of the line through $(8, -4)$ and $(-2, -4)$.

SOLUTION Find the slope.

$$m = \frac{-4 - (-4)}{-2 - 8} = \frac{0}{-10} = 0$$

Choose, say, $(8, -4)$ as (x_1, y_1).

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (-4) &= 0(x - 8) && y_1 = -4, m = 0, x_1 = 8 \\ y + 4 &= 0 && 0(x - 8) = 0 \\ y &= -4 \end{aligned}$$

Plotting the given ordered pairs and drawing a line through the points show that the equation $y = -4$ represents a horizontal line. See Figure 5(a). Every horizontal line has a slope of zero and an equation of the form $y = k$, where k is the y-value of all ordered pairs on the line.

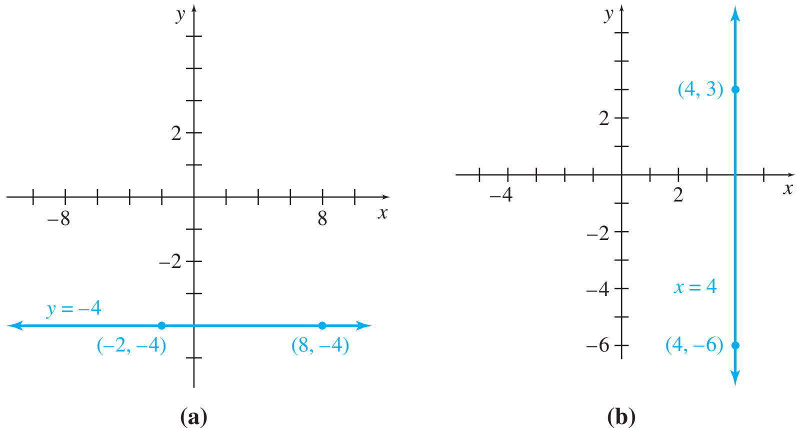


FIGURE 5

EXAMPLE 7 Vertical Line

Find an equation of the line through $(4, 3)$ and $(4, -6)$.

SOLUTION The slope of the line is

$$m = \frac{-6 - 3}{4 - 4} = \frac{-9}{0},$$

which is undefined. Since both ordered pairs have x-coordinate 4, the equation is $x = 4$. Because the slope is undefined, the equation of this line cannot be written in the slope-intercept form.

Again, plotting the given ordered pairs and drawing a line through them show that the graph of $x = 4$ is a vertical line. See Figure 5(b).

Slope of Horizontal and Vertical Lines

The slope of a horizontal line is 0.

The slope of a vertical line is undefined.

The different forms of linear equations discussed in this section are summarized below. The slope-intercept and point-slope forms are equivalent ways to express the equation of a nonvertical line. The slope-intercept form is simpler for a final answer, but you may find the point-slope form easier to use when you know the slope of a line and a point through which the line passes. The slope-intercept form is often considered the standard form. Any line that is not vertical has a unique slope-intercept form but can have many point-slope forms for its equation.

Equations of Lines

Equation	Description
$y = mx + b$	**Slope-intercept form:** slope m, y-intercept b
$y - y_1 = m(x - x_1)$	**Point-slope form:** slope m, line passes through (x_1, y_1)
$x = k$	**Vertical line:** x-intercept k, no y-intercept (except when $k = 0$), undefined slope
$y = k$	**Horizontal line:** y-intercept k, no x-intercept (except when $k = 0$), slope 0

Parallel and Perpendicular Lines

One application of slope involves deciding whether two lines are parallel, which means that they never intersect. Since two parallel lines are equally "steep," they should have the same slope. Also, two lines with the same "steepness" are parallel.

Parallel Lines

Two lines are **parallel** if and only if they have the same slope, or if they are both vertical.

EXAMPLE 8 Parallel Line

Find the equation of the line that passes through the point $(3, 5)$ and is parallel to the line $2x + 5y = 4$.

SOLUTION The slope of $2x + 5y = 4$ can be found by writing the equation in slope-intercept form. To put the equation in this form, solve for y.

$$2x + 5y = 4$$

$$y = -\frac{2}{5}x + \frac{4}{5} \qquad \text{Subtract } 2x \text{ from both sides and divide both sides by 5.}$$

This result shows that the slope is $-2/5$, the coefficient of x. Since the lines are parallel, $-2/5$ is also the slope of the line whose equation we want. This line passes through $(3, 5)$. Substituting $m = -2/5$, $x_1 = 3$, and $y_1 = 5$ into the point-slope form gives

$$y - y_1 = m(x - x_1)$$

$$y - 5 = -\frac{2}{5}(x - 3)$$

$$y - 5 = -\frac{2}{5}x + \frac{6}{5} \qquad \text{Distributive property}$$

$$y = -\frac{2}{5}x + \frac{6}{5} + 5 \cdot \frac{5}{5} \qquad \text{Add 5 to both sides and get a common denominator.}$$

$$y = -\frac{2}{5}x + \frac{31}{5}.$$

TRY YOUR TURN 5

YOUR TURN 5 Find the equation of the line that passes through the point $(4, 5)$ and is parallel to the line $3x - 6y = 7$. Put your answer in slope-intercept form.

As already mentioned, two nonvertical lines are parallel if and only if they have the same slope. Two lines having slopes with a product of -1 are perpendicular. A proof of this fact, which depends on similar triangles from geometry, is given as Exercise 43 in this section.

Perpendicular Lines

Two lines are **perpendicular** if and only if the product of their slopes is -1, or if one is vertical and the other horizontal.

EXAMPLE 9 Perpendicular Line

Find the equation of the line L passing through the point $(3, 7)$ and perpendicular to the line having the equation $5x - y = 4$.

SOLUTION To find the slope, write $5x - y = 4$ in slope-intercept form:

$$y = 5x - 4.$$

The slope is 5. Since the lines are perpendicular, if line L has slope m, then

$$5m = -1$$
$$m = -\frac{1}{5}.$$

Now substitute $m = -1/5$, $x_1 = 3$, and $y_1 = 7$ into the point-slope form.

$$y - 7 = -\frac{1}{5}(x - 3)$$

$$y - 7 = -\frac{1}{5}x + \frac{3}{5} \qquad \text{Distribute.}$$

$$y = -\frac{1}{5}x + \frac{3}{5} + 7 \cdot \frac{5}{5} \qquad \text{Add 7 to both sides and get a common denominator.}$$

$$y = -\frac{1}{5}x + \frac{38}{5}$$

TRY YOUR TURN 6

YOUR TURN 6 Find the equation of the line passing through the point $(3, 2)$ and perpendicular to the line having the equation $2x + 3y = 4$.

The next example uses the equation of a line to analyze real-world data. In this example, we are looking at how one variable changes over time. To simplify the arithmetic, we will *rescale* the variable representing time, although computers and calculators have made rescaling less important than in the past. Here it allows us to work with smaller numbers, and, as you will see, find the y-intercept of the line more easily. We will use rescaling on many examples throughout this book. When we do, it is important to be consistent.

EXAMPLE 10 Prevalence of Cigarette Smoking

In recent years, the percentage of the U.S. population age 18 and older who smoke has decreased at a roughly constant rate, from 20.9% in 2005 to 18.1% in 2012. ***Source: Centers for Disease Control and Prevention***.

(a) Find the equation describing this linear relationship.

SOLUTION Let t represent time in years, with $t = 0$ representing 2000. With this rescaling, the year 2005 corresponds to $t = 5$ and the year 2012 corresponds to $t = 12$. Let y represent the percentage of the population who smoke. The two ordered pairs representing the given information are then $(5, 20.9)$ and $(12, 18.1)$. The slope of the line through these points is

$$m = \frac{18.1 - 20.9}{12 - 5} = \frac{-2.8}{7} = -0.4.$$

This means that, on average, the percentage of the adult population who smoke is decreasing by about 0.4% per year.

Using $m = -0.4$ in the point-slope form, and choosing $(t_1, y_1) = (5, 20.9)$, gives the required equation.

$$y - 20.9 = -0.4(t - 5)$$
$$y - 20.9 = -0.4t + 2 \qquad \text{Distributive property}$$
$$y = -0.4t + 22.9 \qquad \text{Add 20.9 to both sides.}$$

We could have used the other point $(12, 18.1)$ and found the same answer. Instead, we'll use this to check our answer by observing that $-0.4(12) + 22.9 = 18.1$, which agrees with the y-value at $t = 12$.

(b) One objective of Healthy People 2020 (a campaign of the U.S. Department of Health and Human Services) is to reduce the percentage of U.S. adults who smoke to 12% or less by the year 2020. If this decline in smoking continued at the same rate, will they meet this objective?

SOLUTION Using the same rescaling, $t = 20$ corresponds to the year 2020. Substituting this value into the above equation gives

$$y = -0.4(20) + 22.9 = 14.9.$$

Continuing at this rate, an estimated 14.9% of the adult population will still smoke in 2020, and the objective of Healthy People 2020 will not be met.

Notice that if the formula from part (a) of Example 10 is valid for all nonnegative t, then eventually y becomes 0:

$$-0.4t + 22.9 = 0$$
$$-0.4t = -22.9 \qquad \text{Subtract 22.9 from both sides.}$$
$$t = \frac{-22.9}{-0.4} = 57.25 \approx 57^*, \qquad \text{Divide both sides by } -0.4.$$

which indicates that 57 years from 2000 (in the year 2057), 0% of the U.S. adult population will smoke. Of course, it is still possible that in 2057 there will be adults who smoke; the trend of recent years may not continue. Most equations are valid for some specific set of numbers. It is highly speculative to extrapolate beyond those values.

On the other hand, people in business and government often need to make some prediction about what will happen in the future, so a tentative conclusion based on past trends may be better than no conclusion at all. There are also circumstances, particularly in the physical sciences, in which theoretical reasons imply that the trend will continue.

Graph of a Line

It can be shown that every equation of the form $ax + by = c$ has a straight line as its graph, assuming a and b are not both 0. Although just two points are needed to determine a line, it is a good idea to plot a third point as a check.

EXAMPLE 11 Graph of a Line

Graph $y = -\frac{3}{2}x + 6$.

SOLUTION There are several ways to graph a line.

Method I Plot Points

Note that the y-intercept is 6, so the point $(0, 6)$ is on the line. Next, by substituting $x = 2$ and $x = 4$ into the equation, we find the points $(2, 3)$ and $(4, 0)$. (We could use any values for x, but we chose even numbers so the value of y would be an integer.) These three points are

*The symbol $\approx$ means "is approximately equal to."

plotted in Figure 6(a). A line is drawn through them in Figure 6(b). (We need to find only two points on the line. The third point gives a confirmation of our work.)

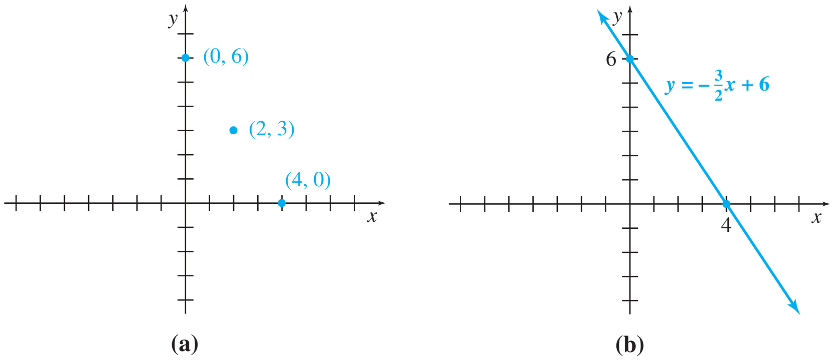


FIGURE 6

Method 2
Find Intercepts

We already found the y-intercept $(0, 6)$ by setting $x = 0$. To find the x-intercept, let $y = 0$.

$$0 = -\frac{3}{2}x + 6$$

$$\frac{3}{2}x = 6 \qquad \text{Add } \tfrac{3}{2}x \text{ to both sides.}$$

$$x = \frac{6}{3/2} = 6\left(\frac{2}{3}\right) = 4 \qquad \text{Divide both sides by } \tfrac{3}{2}.$$

This gives the point $(4, 0)$ that we found earlier. Once the x- and y-intercepts are found, we can draw a line through them.

Method 3
Use the Slope and y-Intercept

Observe that the slope of $-3/2$ means that every time x increases by 2, y decreases by 3. So start at the y-intercept $(0, 6)$ and go 2 across and 3 down to the point $(2, 3)$. Again, go 2 across and 3 down to the point $(4, 0)$. Draw the line through these points.

Not every line has two distinct intercepts; the graph in the next example does not cross the x-axis, and so it has no x-intercept.

EXAMPLE 12 Graph of a Horizontal Line

Graph $y = -3$.

SOLUTION The equation $y = -3$, or equivalently, $y = 0x - 3$, always gives the same y-value, -3, for any value of x. Therefore, no value of x will make $y = 0$, so the graph has no x-intercept. As we saw in Example 6, the graph of such an equation is a horizontal line parallel to the x-axis. In this case the y-intercept is -3, as shown in Figure 7.

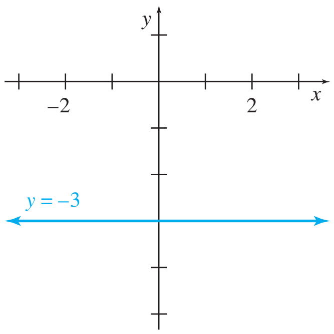


FIGURE 7

Linear equations allow us to set up simple mathematical models for real-life situations. In almost every case, linear (or any other reasonably simple) equations provide only approximations to real-world situations. Nevertheless, these are often remarkably useful approximations.

EXAMPLE 13 Tuition

APPLY IT

The table on the left lists the average annual cost (in dollars) of tuition and fees at public four-year colleges for selected years. ***Source: The College Board***.

(a) Plot the cost of public colleges by letting $t = 0$ correspond to 2000. Are the data *exactly* linear? Could the data be *approximated* by a linear equation?

SOLUTION The data are plotted in Figure 8(a) in a figure known as a **scatterplot**. Although it is not exactly linear, it is approximately linear and could be approximated by a linear equation.

Cost of Public College

Year	Tuition and Fees
2000	3508
2002	4098
2004	5126
2006	5804
2008	6599
2010	7629
2012	8646
2013	8893

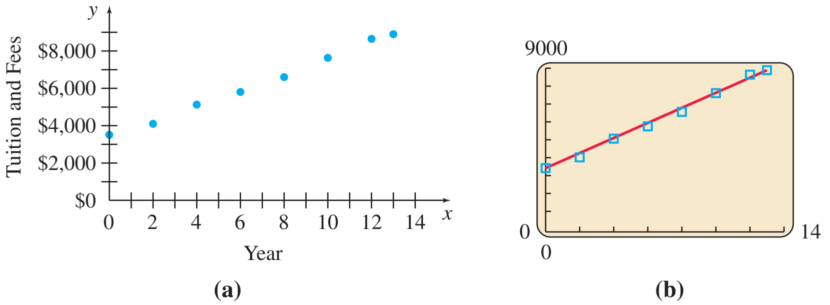


FIGURE 8

(b) Use the points $(0, 3508)$ and $(13, 8893)$ to determine an equation that models the data.

SOLUTION We first find the slope of the line as follows:

$$m = \frac{8893 - 3508}{13 - 0} = \frac{5385}{13} \approx 414.2.$$

We have rounded to four digits, noting that we cannot expect more accuracy in our answer than in the data, which are accurate to four digits. The slope indicates that the average annual cost of tuition and fees at public four-year colleges is increasing by about $414 per year.

Using the slope-intercept form of the line, $y = mt + b$, with $m = 414.2$ and $b = 3508$, gives

$$y = 414.2t + 3508.$$

TECHNOLOGY NOTE

A graphing calculator plot of this line and the data points are shown in Figure 8(b). Notice that the points closely fit the line. More details on how to construct this graphing calculator plot are given at the end of this example.

(c) Discuss the accuracy of using this equation to estimate the cost of public colleges in the year 2030.

SOLUTION The year 2030 corresponds to the year $t = 30$, for which the equation predicts a cost of

$$y = 414.2(30) + 3508 = 15{,}934, \quad \text{or} \quad \$15{,}934.$$

The year 2030 is many years in the future, however. Many factors could affect the tuition, and the actual figure for 2030 could turn out to be very different from our prediction.

TECHNOLOGY NOTE

You can plot data with a TI-84 Plus C graphing calculator using the following steps.

1. Store the data in lists. Select the `EDIT` command in the `STAT` menu. Enter the x-values in `L1` and y-values in `L2`.
2. Define the stat plot. Go to `STAT PLOT` and select `Plot1`. Select `On`. Be sure that `xlist` is `L1` and `ylist` is `L2`.
3. Define the viewing window.
4. To graph the line, select `Y=` and enter the equation in `Y1`.
5. Display the graph by selecting `GRAPH`.

Consult the calculator's instruction booklet or the *Graphing Calculator and Excel Spreadsheet Manual*, available with this book, for specific instructions. See the calculator–generated graph in Figure 8(b), which includes the points and line from Example 13. Notice how the line closely approximates the data.

1.1 WARM-UP EXERCISES

W1. Evaluate $\dfrac{15-(-3)}{-2-4}$. *(Sec. R.1)*

Solve each equation for *y*. *(Sec. R.4)*

W2. $y-(-3)=-2(x+5)$

W3. $y-\dfrac{1}{2}=\dfrac{2}{5}\left(x+\dfrac{1}{3}\right)$

W4. $2x-3y=7$

1.1 EXERCISES

Find the slope of each line.

1. Through $(6,-10)$ and $(0,11)$

2. Through $(5,-4)$ and $(1,3)$

3. Through $(3,10)$ and $(3,3)$

4. Through $(1,5)$ and $(-2,5)$

5. $y=2.7x$

6. $y=3x-2$

7. $4x-5y=10$

8. $4x+7y=1$

9. $x=-7$

10. The x-axis

11. $y=18$

12. $y=-6$

13. A line parallel to $30x-5y=-20$

14. A line perpendicular to $8x=2y-5$

In Exercises 15–24, find an equation in slope-intercept form for each line.

15. Through $(4,8)$, $m=-3$

16. Through $(2,4)$, $m=-1$

17. Through $(5,-13)$ and $m=0$

18. Through $(-8,1)$, with undefined slope

19. Through $(3,4)$ and $(1,7)$

20. Through $(4,4)$ and $(2,5)$

21. Through $(2/3,1/2)$ and $(1/4,-2)$

22. Through $(1/6,-1/3)$ and $(5/6,5)$

23. Through $(2,4)$ and $(2,1)$

24. Through $(-1,3)$ and $(0,3)$

In Exercises 25–34, find an equation for each line in the form $ax+by=c$, where a, b, and c are integers with no factor common to all three and $a \geq 0$.

25. x-intercept -8, y-intercept -2

26. x-intercept -2, y-intercept 4

27. Vertical, through $(7,-8)$

28. Vertical, through $(-8,-5)$

29. Through $(-10,45)$, parallel to $9x+2y=17$

30. Through $(2,-5)$, parallel to $2x-y=-4$

31. Through $(4,9)$, perpendicular to $3x+y=2$

32. Through $(-2,6)$, perpendicular to $2x-3y=5$

33. The line with y-intercept 9, perpendicular to $x+4y=11$

34. The line with x-intercept $-2/3$ and perpendicular to $2x-y=4$

35. Do the points $(4,3)$, $(2,0)$, and $(-18,-12)$ lie on the same line? Explain why or why not. (*Hint:* Find the slopes between the points.)

36. Find k so that the line through $(4,-1)$ and $(k,2)$ is

(a) parallel to $2x+3y=6$,

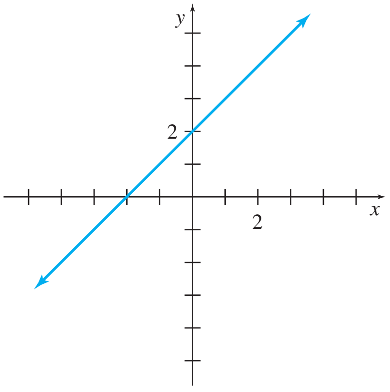
(b) perpendicular to $5x-2y=-1$.

37. Use slopes to show that the quadrilateral with vertices at $(1,3)$, $(-5/2,2)$, $(-7/2,4)$, and $(2,1)$ is a parallelogram.

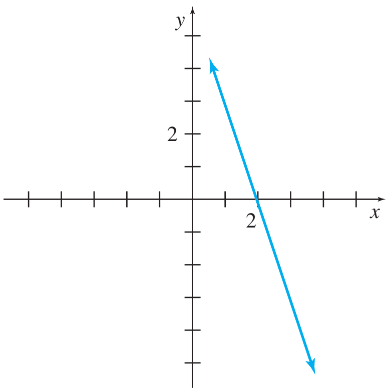
38. Use slopes to show that the square with vertices at $(-2,5)$, $(4,5)$, $(4,-1)$, and $(-2,-1)$ has diagonals that are perpendicular.

For the lines in Exercises 39 and 40, which of the following is closest to the slope of the line? (a) 1 (b) 2 (c) 3 (d) 21 (e) 22 (f) −3

39.

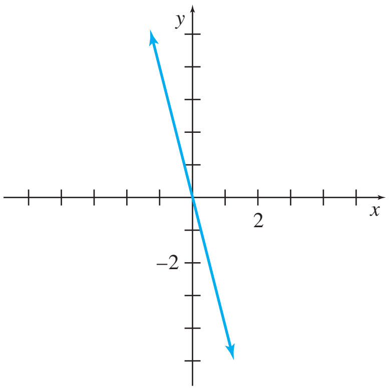


40.

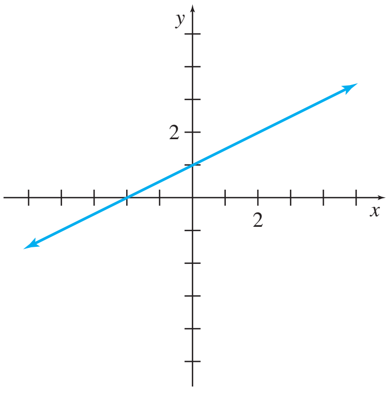


In Exercises 41 and 42, estimate the slope of the lines.

41.



42.



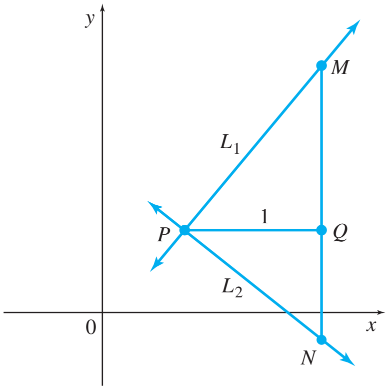
43. To show that two perpendicular lines, neither of which is vertical, have slopes with a product of −1, go through the following steps. Let line L_1 have equation $y = m_1x + b_1$, and let L_2 have equation $y = m_2x + b_2$, with $m_1 > 0$ and $m_2 < 0$. Assume that L_1 and L_2 are perpendicular, and use right triangle MPN shown in the figure. Prove each of the following statements.

(a) MQ has length m_1.

(b) QN has length $-m_2$.

(c) Triangles MPQ and PNQ are similar.

(d) $m_1/1 = 1/(-m_2)$ and $m_1m_2 = -1$



44. Consider the equation $\dfrac{x}{a} + \dfrac{y}{b} = 1$.

(a) Show that this equation represents a line by writing it in the form $y = mx + b$.

(b) Find the x- and y-intercepts of this line.

(c) Explain in your own words why the equation in this exercise is known as the intercept form of a line.

Graph each equation.

45. $y = x - 1$

46. $y = x - 11$

47. $y = -4x + 9$

48. $y = -6x + 12$

49. $2x - 3y = 12$

50. $2x - y = 8$

51. $3y - 7x = -21$

52. $8x - 6y = -24$

53. $y = -2$

54. $y = -6$

55. $x + 5 = 0$

56. $y + 8 = 0$

57. $y = 2x$

58. $y = -4x$

59. $x + 4y = 0$

60. $3x - 5y = 0$

APPLICATIONS

Business and Economics

61. Sales The sales of a small company were $13,000 in its third year of operation and $37,000 in its seventh year. Let y represent sales in the xth year of operation. Assume that the data can be approximated by a straight line.

(a) Find the slope of the sales line, and give an equation for the line in the form $y = mx + b$.

(b) Use your answer from part (a) to find out how many years must pass before the sales surpass $50,000.

62. Cost The total cost for a bakery to produce 150 gourmet cupcakes is \$225, while the total cost to produce 175 gourmet cupcakes is \$247. Let y represent the total cost for x gourmet cupcakes. Assume that a straight line can approximate the data.

(a) Find and interpret the slope of the cost line for the gourmet cupcakes.

(b) Determine an equation that models the data. Write the equation in the slope-intercept form.

(c) Use your answer from part (b) to determine an approximation of the cost of 200 gourmet cupcakes.

63. Tuition The table lists the annual cost (in dollars) of tuition and fees at private four-year colleges for selected years. (See Example 13.) ***Source: The College Board***.

Year	Tuition and Fees
2000	16,072
2002	18,060
2004	20,045
2006	22,308
2008	24,818
2010	26,766
2012	28,989
2013	30,094

(a) Sketch a graph of the data. Do the data appear to lie roughly along a straight line?

(b) Let $t = 0$ correspond to the year 2000. Use the points $(0, 16{,}072)$ and $(13, 30{,}094)$ to determine a linear equation that models the data. What does the slope of the graph of the equation indicate?

(c) Discuss the accuracy of using this equation to estimate the cost of private college in 2025.

64. Use of Cellular Telephones The following table shows the subscribership of cellular telephones in the United States (in millions) for selected years between 2000 and 2012. ***Source: The World Almanac and Book of Facts 2014***.

Year	2000	2004	2008	2012
Subscribers (in millions)	109.48	182.14	270.33	326.48

(a) Plot the data by letting $t = 0$ correspond to 2000. Discuss how well the data fit a straight line.

(b) Determine a linear equation that approximates the number of subscribers using the points $(0, 109.48)$ and $(12, 326.48)$.

(c) Repeat part (b) using the points $(4, 182.14)$ and $(12, 326.48)$.

(d) Discuss why your answers to parts (b) and (c) are similar but not identical.

(e) Using your equations from parts (b) and (c), approximate the number of cellular phone subscribers in the year 2010. Compare your result with the actual value of 296.29 million.

65. Consumer Price Index The Consumer Price Index (CPI) is a measure of the change in the cost of goods over time. The index was 100 for the three-year period centered on 1983. For simplicity, we will assume that the CPI was exactly 100 in 1983. Then the CPI of 229.6 in 2012 indicates that an item that cost \$1.00 in 1983 would cost \$2.30 in 2012. The CPI has been increasing approximately linearly over the last few decades. ***Source: The World Almanac and Book of Facts 2014***.

(a) Use this information to determine an equation for the CPI in terms of t, which represents the years since 1980.

(b) Based on the answer to part (a), what was the predicted value of the CPI in 2000? Compare this estimate with the actual CPI of 172.2.

(c) Describe the rate at which the annual CPI is changing.

Life Sciences

66. HIV Infection The time interval between a person's initial infection with HIV and that person's eventual development of AIDS symptoms is an important issue. The method of infection with HIV affects the time interval before AIDS develops. One study of HIV patients who were infected by intravenous drug use found that 17% of the patients had AIDS after 4 years, and 33% had developed the disease after 7 years. The relationship between the time interval and the percentage of patients with AIDS can be modeled accurately with a linear equation. ***Source: Epidemiologic Review***.

(a) Write a linear equation $y = mt + b$ that models these data, using the ordered pairs $(4, 0.17)$ and $(7, 0.33)$.

(b) Use your equation from part (a) to predict the number of years before half of these patients will have AIDS.

67. Exercise Heart Rate To achieve the maximum benefit for the heart when exercising, your heart rate (in beats per minute) should be in the target heart rate zone. The lower limit of this zone is found by taking 70% of the difference between 220 and your age. The upper limit is found by using 85%. ***Source: Physical Fitness***.

(a) Find formulas for the upper and lower limits (u and l) as linear equations involving the age x.

(b) What is the target heart rate zone for a 20-year-old?

(c) What is the target heart rate zone for a 40-year-old?

(d) Two women in an aerobics class stop to take their pulse and are surprised to find that they have the same pulse. One woman is 36 years older than the other and is working at the upper limit of her target heart rate zone. The younger woman is working at the lower limit of her target heart rate zone. What are the ages of the two women, and what is their pulse?

(e) Run for 10 minutes, take your pulse, and see if it is in your target heart rate zone. (After all, this is listed as an exercise!)

68. Ponies Trotting A study found that the peak vertical force on a trotting Shetland pony increased linearly with the pony's speed, and that when the force reached a critical level, the pony switched from a trot to a gallop. For one pony, the critical force was 1.16 times its body weight. It experienced a force of 0.75 times its body weight at a speed of 2 meters per second and a force of 0.93 times its body weight at 3 meters per second. At what speed did the pony switch from a trot to a gallop? ***Source: Science***.

69. Life Expectancy Some scientists believe there is a limit to how long humans can live. One supporting argument is that during the past century, life expectancy from age 65 has increased more slowly than life expectancy from birth, so eventually these two will be equal, at which point, according to these scientists, life expectancy should increase no further. In 1900, life expectancy at birth was 46 yr, and life expectancy at age 65 was 76 yr. In 2010, these figures had risen to 78.7 and 84.1, respectively. In both cases, the increase in life expectancy has been linear. Using these assumptions and the data given, find the maximum life expectancy for humans. ***Source: Science***.

Social Sciences

70. Child Mortality Rate The mortality rate for children under 5 years of age around the world has been declining in a roughly linear fashion in recent years. The rate per 1000 live births was 90 in 1990 and 48 in 2012. ***Source: World Health Organization***.

(a) Determine a linear equation that approximates the mortality rate in terms of time t, where t represents the number of years since 1900.

(b) One of the Millennium Development Goals (MDG) of the World Health Organization is to reduce the mortality rate for children under 5 years of age to 30 by 2015. If this trend were to continue, in what year would this goal be reached?

71. Immigration In 1950, there were 249,187 immigrants admitted to the United States. In 2012, the number was 1,031,631. ***Source: 2012 Yearbook of Immigration Statistics***.

(a) Assuming that the change in immigration is linear, write an equation expressing the number of immigrants, y, in terms of t, the number of years after 1900.

(b) Use your result in part (a) to predict the number of immigrants admitted to the United States in 2015.

(c) Considering the value of the y-intercept in your answer to part (a), discuss the validity of using this equation to model the number of immigrants throughout the entire 20th century.

72. Marriage The following table lists the U.S. median age at first marriage for men and women. The age at which both groups marry for the first time seems to be increasing at a roughly linear rate in recent decades. Let t correspond to the number of years since 1980. ***Source: U.S. Census Bureau***.

Age at First Marriage				
Year	1980	1990	2000	2010
Men	24.7	26.1	26.8	28.2
Women	22.0	23.9	25.1	26.1

(a) Find a linear equation that approximates the data for men, using the data for the years 1980 and 2010.

(b) Repeat part (a) using the data for women.

(c) Which group seems to have the faster increase in median age at first marriage?

(d) According to the equation from part (a), in what year will the men's median age at first marriage reach 30?

(e) When the men's median age at first marriage is 30, what will the median age be for women?

Physical Sciences

73. Galactic Distance The table lists the distances (in megaparsecs where 1 megaparsec $\approx 3.1 \times 10^{19}$ km) and velocities (in kilometers per second) of four galaxies moving rapidly away from Earth. ***Source: Astronomical Methods and Calculations, and Fundamental Astronomy***.

Galaxy	Distance	Velocity
Virga	15	1600
Ursa Minor	200	15,000
Corona Borealis	290	24,000
Bootes	520	40,000

(a) Plot the data points, letting x represent distance and y represent velocity. Do the points lie in an approximately linear pattern?

(b) Write a linear equation $y = mx$ to model these data, using the ordered pair $(520, 40{,}000)$.

(c) The galaxy Hydra has a velocity of 60,000 km per sec. Use your equation to approximate how far away it is from Earth.

(d) The value of m in the equation is called the *Hubble constant*. The Hubble constant can be used to estimate the age of the universe A (in years) using the formula

$$A = \frac{9.5 \times 10^{11}}{m}.$$

Approximate A using your value of m.

74. Global Warming In 1990, the Intergovernmental Panel on Climate Change predicted that the average temperature on Earth would rise 0.3°C per decade in the absence of international controls on greenhouse emissions. Let t measure the time in years since 1970, when the average global temperature was 15°C. ***Source: Science News***.

(a) Find a linear equation giving the average global temperature in degrees Celsius in terms of t, the number of years since 1970.

(b) Scientists have estimated that the sea level will rise by 65 cm if the average global temperature rises to 19°C. According to your answer to part (a), when would this occur?

General Interest

75. News Sources A survey asked respondents where they got news "yesterday" (the day before they participated in the survey). In 2006, about 40% of respondents got their news from the newspaper, while 23% got their news online. In 2012, about 29% got their news from the newspaper, while 39% got their news online. Both news sources changed at a roughly linear rate. ***Source: State of the Media***.

(a) Find a linear equation expressing the percent of respondents, y_n, who got their news from the newspaper in terms of t, the years since 2000.

(b) Find a linear equation expressing the percent of respondents, y_o, who got their news online in terms of t, the years since 2000.

(c) Find the rate of change over time for the percentage of respondents for each source of news.

YOUR TURN ANSWERS

1. $1/3$
2. $y = (3/2)x + 6$
3. $-8/3$
4. $y = -2x + 13$
5. $y = (1/2)x + 3$
6. $y = (3/2)x - 5/2$

1.2 Linear Functions and Applications

APPLY IT **How many units must be sold for a firm to break even?**

In Example 7 in this section, this question will be answered using a linear function.

As we saw in the previous section, many situations involve two variables related by a linear equation. For such a relationship, when we express the variable y in terms of x, we say that y is a **linear function** of x. This means that for any allowed value of x (the **independent variable**), we can use the equation to find the corresponding value of y (the **dependent variable**). Examples of equations defining linear functions include $y = 2x + 3$, $y = -5$, and $2x - 3y = 7$, which can be written as $y = (2/3)x - (7/3)$. Equations in the form $x = k$, where k is a constant, do not define linear functions. All other linear equations define linear functions.

$f(x)$ Notation

Letters such as f, g, or h are often used to name functions. For example, f might be used to name the function defined by

$$y = 5 - 3x.$$

To show that this function is named f, it is common to replace y with $f(x)$ (read "f of x") to get

$$f(x) = 5 - 3x.$$

By choosing **2** as a value of x, $f(x)$ becomes

$$f(2) = 5 - 3 \cdot 2 = 5 - 6 = -1.$$

The corresponding ordered pair is $(2, -1)$. In a similar manner,

$$f(-4) = 5 - 3(-4) = 17, \qquad f(0) = 5, \qquad f(-6) = 23,$$

and so on.

EXAMPLE 1 Function Notation

Let $g(x) = -4x + 5$. Find $g(3)$, $g(0)$, $g(-2)$, and $g(b)$.

SOLUTION To find $g(3)$, substitute 3 for x.

$$g(3) = -4(3) + 5 = -12 + 5 = -7$$

Similarly,

$$g(0) = -4(0) + 5 = 0 + 5 = 5,$$
$$g(-2) = -4(-2) + 5 = 8 + 5 = 13,$$

and

$$g(b) = -4b + 5.$$

TRY YOUR TURN 1

YOUR TURN 1
Calculate $g(-5)$.

We summarize the discussion below.

Linear Function

A relationship f defined by

$$y = f(x) = mx + b,$$

for real numbers m and b, is a **linear function**.

Linear functions are used to model many applications in business and science, as shown in the next examples.

Supply and Demand

The supply and demand relationship is one of the fundamental concepts of economics. The **supply** is the quantity of a certain good that producers are willing to provide. The **demand** is the quantity of the good that consumers are willing to buy.

The **Law of Demand** states that, if all other factors remain equal, as the price of an item increases, consumers are less likely to buy an increasingly expensive item, and so the demand for the item decreases. On the other hand, the **Law of Supply** states that as the price of an item increases, producers are more likely to see a profit in selling the item, and so the supply of the item increases. It is important to note that suppliers cannot always react quickly to a change in demand or a change in price.

Economists study the interaction between price, supply, and demand. If the price increases, the increase in the quantity supplied and decrease in the quantity demanded can eventually result in a surplus, which causes the price to fall. These countervailing trends tend to move the price, as well as the quantity supplied and demanded, toward an equilibrium value.

For example, during the late 1980s and early 1990s, the consumer demand for cranberries (and all of their healthy benefits) soared. The quantity demanded surpassed the quantity supplied, causing a shortage, and cranberry prices rose dramatically. As prices increased, growers wanted to increase their profits, so they planted more acres of cranberries. Unfortunately, cranberries take 3 to 5 years from planting until they can first be harvested. As growers waited and prices increased, consumer demand decreased. When the cranberries were finally harvested, the supply overwhelmed the demand and a huge surplus occurred, causing the price of cranberries to drop in the late 1990s. Other factors were involved in this situation, but the relationship between price, supply, and demand was nonetheless typical. ***Source: Agricultural Marketing Resource Center***.

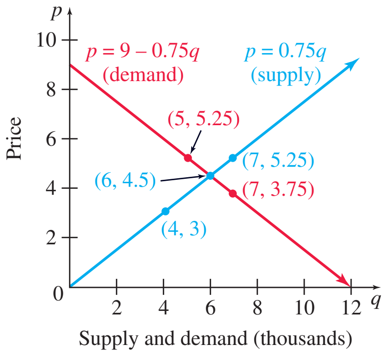
Although economists consider price to be the independent variable, they have the unfortunate habit of plotting price, usually denoted by p, on the vertical axis, while everyone else graphs the independent variable on the horizontal axis. This custom was started by the English economist Alfred Marshall (1842–1924). To abide by this custom, we will write p, the price, as a function of q, the quantity produced, and plot p on the vertical axis. But remember, it is really *price* that determines how much consumers demand and producers supply, not the other way around.

Supply and demand functions are not necessarily linear, the simplest kind of function. Yet most functions are approximately linear if a small enough piece of the graph is taken, allowing applied mathematicians to often use linear functions for simplicity. That approach will be taken in this chapter.

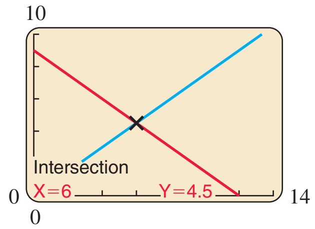
EXAMPLE 2 Supply and Demand

Suppose that Greg Tobin, manager of a giant supermarket chain, has studied the supply and demand for watermelons. He has noticed that the demand increases as the price decreases. He has determined that the quantity (in thousands) demanded weekly, q, and the price (in dollars) per watermelon, p, are related by the linear function

$$p = D(q) = 9 - 0.75q. \quad \text{Demand function}$$



(a)



(b)

FIGURE 9

(a) Find the quantity demanded at a price of \$5.25 per watermelon and at a price of \$3.75 per watermelon.

SOLUTION To find the quantity demanded at a price of \$5.25 per watermelon, replace p in the demand function with 5.25 and solve for q.

$$\begin{aligned} 5.25 &= 9 - 0.75q \\ -3.75 &= -0.75q && \text{Subtract 9 from both sides.} \\ 5 &= q && \text{Divide both sides by } -0.75. \end{aligned}$$

Thus, at a price of \$5.25, the quantity demanded is 5000 watermelons.

Similarly, replace p with 3.75 to find the demand when the price is \$3.75. Verify that this leads to $q = 7$. When the price is lowered from \$5.25 to \$3.75 per watermelon, the quantity demanded increases from 5000 to 7000 watermelons.

(b) Greg also noticed that the quantity of watermelons supplied decreased as the price decreased. Price p and supply q are related by the linear function

$$p = S(q) = 0.75q. \qquad \text{Supply function}$$

Find the quantity supplied at a price of \$5.25 per watermelon and at a price of \$3.00 per watermelon.

SOLUTION Substitute 5.25 for p in the supply function, $p = 0.75q$, to find that $q = 7$, so the quantity supplied is 7000 watermelons. Similarly, replacing p with 3 in the supply equation gives a quantity supplied of 4000 watermelons. If the price decreases from \$5.25 to \$3.00 per watermelon, the quantity supplied also decreases, from 7000 to 4000 watermelons.

(c) Graph both functions on the same axes.

SOLUTION The results of part (a) are written as the ordered pairs $(5, 5.25)$ and $(7, 3.75)$. The line through those points is the graph of the demand function, $p = 9 - 0.75q$, shown in red in Figure 9(a). We used the ordered pairs $(7, 5.25)$ and $(4, 3)$ from the work in part (b) to graph the supply function, $p = 0.75q$, shown in blue in Figure 9(a).

TRY YOUR TURN 2

YOUR TURN 2 Find the quantity of watermelon demanded and supplied at a price of \$3.30 per watermelon.

TECHNOLOGY NOTE

A calculator-generated graph of the lines representing the supply and demand functions in Example 2 is shown in Figure 9(b). To get this graph, the equation of each line, using x and y instead of q and p, was entered, along with an appropriate window. The coordinates of the intersection point, as shown at the bottom of the graph, are found by selecting `intersect` from the `CALC` menu.

NOTE Not all supply and demand problems will have the same scale on both axes. It helps to consider the intercepts of both the supply graph and the demand graph to decide what scale to use. For example, in Figure 9, the y-intercept of the demand function is 9, so the scale should allow values from 0 to at least 9 on the vertical axis. The x-intercept of the demand function is 12, so the values on the x-axis must go from 0 to 12.

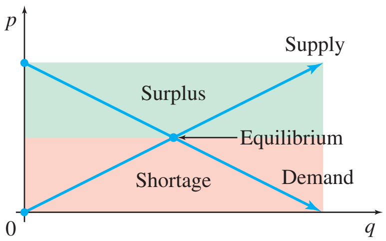


FIGURE 10

As shown in the graphs of Figure 9, both the supply graph and the demand graph pass through the point $(6, 4.5)$. If the price of a watermelon is more than \$4.50, the quantity supplied will exceed the quantity demanded and there will be a **surplus** of watermelons. At a price less than \$4.50, the quantity demanded will exceed the quantity supplied and there will be a **shortage** of watermelons. Only at a price of \$4.50 will quantity demanded and supplied be equal. For this reason, \$4.50 is called the *equilibrium price*. When the price is \$4.50, quantity demanded and supplied both equal 6000 watermelons, the *equilibrium quantity*. Figure 10 illustrates a general supply and demand situation.

Equilibrium

When supply and demand are equal, the economy is said to be at **equilibrium**. The **equilibrium price** of the commodity is the price found at the point where the supply and demand graphs for that commodity intersect. The **equilibrium quantity** is the quantity demanded and supplied at that same point.

Note that in the real marketplace, prices are constantly changing in reaction to fluctuations in the supply and demand, and so therefore, equilibrium can rarely be achieved except in theory.

EXAMPLE 3 Equilibrium Quantity

Use algebra to find the equilibrium quantity and price for the watermelons in Example 2.

SOLUTION The equilibrium quantity is found when the prices from both supply and demand are equal. Set the two expressions for p equal to each other and solve.

$$9 - 0.75q = 0.75q$$

$$9 = 1.5q \qquad \text{Add } 0.75q \text{ to both sides.}$$

$$6 = q \qquad \text{Divide both sides by } 1.5.$$

The equilibrium quantity is 6000 watermelons, the same answer found earlier.

The equilibrium price can be found by plugging the value of $q = 6$ into either the demand or the supply function. Using the demand function,

$$p = D(6) = 9 - 0.75(6) = 4.5.$$

The equilibrium price is \$4.50, as we found earlier. Check your work by also plugging $q = 6$ into the supply function.

TRY YOUR TURN 3

YOUR TURN 3 Repeat Example 3 using the demand equation $D(q) = 10 - 0.85q$ and the supply equation $S(q) = 0.4q$.

TECHNOLOGY NOTE

You may prefer to find the equilibrium quantity by solving the equation with your calculator. Or, if your calculator has a TABLE feature, you can use it to find the value of q that makes the two expressions equal.

Another important issue is how, in practice, the equations of the supply and demand functions can be found. Data need to be collected, and if they lie perfectly along a line, then the equation can easily be found with any two points. What usually happens, however, is that the data are scattered, and there is no line that goes through all the points. In this case we must find a line that approximates the linear trend of the data as closely as possible (assuming the points lie approximately along a line) as in Example 13 in the previous section. This is usually done by the *method of least squares*, also referred to as *linear regression.* We will discuss this method in Section 1.3.

Cost Analysis

The cost of manufacturing an item commonly consists of two parts. The first is a **fixed cost** for designing the product, setting up a factory, training workers, and so on. Within broad limits, the fixed cost is constant for a particular product and does not change as more items are made. The second part is a *cost per item* for labor, materials, packing, shipping, and so on. The total value of this second cost *does* depend on the number of items made.

EXAMPLE 4 Cost Analysis

A small company decides to produce video games. The owners find that the fixed cost for creating the game is \$5000, after which they must spend \$12 to produce each individual copy of the game. Find a formula $C(x)$ for the cost as a linear function of x, the number of games produced.

SOLUTION Notice that $C(0) = 5000$, since \$5000 must be spent even if no games are produced. Also, $C(1) = 5000 + 12 = 5012$, and $C(2) = 5000 + 2 \cdot 12 = 5024$. In general,

$$C(x) = 5000 + 12x,$$

because every time x increases by 1, the cost should increase by \$12. The number 12 is also the slope of the graph of the cost function; the slope gives us the cost to produce one additional item.

In economics, **marginal cost** is the rate of change of cost $C(x)$ at a level of production x and is equal to the slope of the cost function at x. It approximates the cost of producing one additional item. In fact, some books define the marginal cost to be the cost of producing one additional item. With *linear functions*, these two definitions are equivalent, and the marginal cost, which is equal to the slope of the cost function, is *constant*. For instance, in the video game example, the marginal cost of each game is \$12. For other types of functions, these two definitions are only approximately equal. Marginal cost is important to management in making decisions in areas such as cost control, pricing, and production planning.

The work in Example 4 can be generalized. Suppose the total cost to make x items is given by the linear cost function $C(x) = mx + b$. The fixed cost is found by letting $x = 0$:

$$C(0) = m \cdot 0 + b = b;$$

thus, the fixed cost is b dollars. The additional cost of each additional item, the marginal cost, is m, the slope of the line $C(x) = mx + b$.

Linear Cost Function

In a cost function of the form $C(x) = mx + b$, the m represents the marginal cost and b the fixed cost. Conversely, if the fixed cost of producing an item is b and the marginal cost is m, then the **linear cost function** $C(x)$ for producing x items is $C(x) = mx + b$.

EXAMPLE 5 Cost Function

The marginal cost to make x batches of a prescription medication is \$10 per batch, while the cost to produce 100 batches is \$1500. Find the cost function $C(x)$, given that it is linear.

SOLUTION Since the cost function is linear, it can be expressed in the form $C(x) = mx + b$. The marginal cost is \$10 per batch, which gives the value for m. Using $x = 100$ and $C(x) = 1500$ in the point-slope form of the line gives

$$\begin{aligned} C(x) - 1500 &= 10(x - 100) \\ C(x) - 1500 &= 10x - 1000 && \text{Distributive property} \\ C(x) &= 10x + 500. && \text{Add 1500 to both sides.} \end{aligned}$$

The cost function is given by $C(x) = 10x + 500$, where the fixed cost is \$500.

TRY YOUR TURN 4

YOUR TURN 4 Repeat Example 5, using a marginal cost of \$15 per batch and a cost of \$1930 to produce 80 batches.

EXAMPLE 6 Cost Function

A company has fixed costs of \$12,500. The total cost to produce 1000 widgets is \$13,260. (A widget is a term often used by economists to refer to an abstract unit of production.) Find the cost function $C(x)$, given that it is linear.

SOLUTION The linear cost function is $C(x) = mx + b$. Here the fixed cost is given: $b = 12{,}500$. We need to find the marginal cost, m. One way to find the marginal cost is to use the value of b, along with $x = 1000$ and $C(x) = 13{,}260$, in the linear cost function form and solve for m.

$$C(x) = mx + b$$

$$13{,}260 = m \cdot 1000 + 12{,}500 \qquad C(x) = 13{,}260,\ x = 1000, \text{ and } b = 12{,}500$$

$$760 = 1000m \qquad \text{Subtract 12,500 from both sides.}$$

$$0.76 = m \qquad \text{Divide both sides by 1000.}$$

YOUR TURN 5 Repeat Example 6, using a fixed cost of \$7145 and a cost of \$7965 to produce 100 items.

The cost function is given by $C(x) = 0.76x + 12{,}500$, where the marginal cost is \$0.76.

Note that the marginal cost, m, could have also been found by using the slope formula, $m = (y_2 - y_1)/(x_2 - x_1)$, with the points $(0, 12{,}500)$ and $(1000, 13{,}260)$.

TRY YOUR TURN 5

Break-Even Analysis

The goal of every company is to make a profit. A company can make a profit only if the revenue received from its customers exceeds the cost of producing and selling its goods and services. Companies use break-even analysis to analyze the revenue and cost of sales, and to determine the amount that revenues can fall while still making a profit.

Break-Even Analysis

Let p be the price per unit and let x be the number of units sold (demand).

The **revenue,** $R(x)$, is the amount of money that a company receives; it is the product of the price per unit p and the number of units sold x. That is,

$$R(x) = px.$$

The **profit**, $P(x)$, is the money a company makes after paying its costs; it is the difference between revenue $R(x)$ and cost $C(x)$. That is,

$$P(x) = R(x) - C(x).$$

The number of units, x, at which revenue just equals cost is the **break-even quantity**. It is found by setting revenue equal to cost. That is,

$$R(x) = C(x).$$

(The break-even quantity can also be found by setting profit equal to zero; $P(x) = 0$.)

The **break-even point** is the corresponding ordered pair, and indicates the point of zero loss or profit.

EXAMPLE 7 Break-Even Analysis

A firm producing poultry feed finds that the total cost $C(x)$ in dollars of producing and selling x units is given by

$$C(x) = 20x + 100.$$

Management plans to charge \$24 per unit for the feed.

(a) Determine the revenue function.

SOLUTION Revenue is the product of the price per unit, $p = 24$, and the number of units sold, x. That is,

$$R(x) = 24x.$$

(b) How many units must be sold for the firm to break even?

APPLY IT

SOLUTION The firm will break even (no profit and no loss) as long as revenue just equals cost. Set the revenue function equal to the cost function and solve.

$$R(x) = C(x)$$
$$24x = 20x + 100$$
$$4x = 100$$
$$x = 25$$

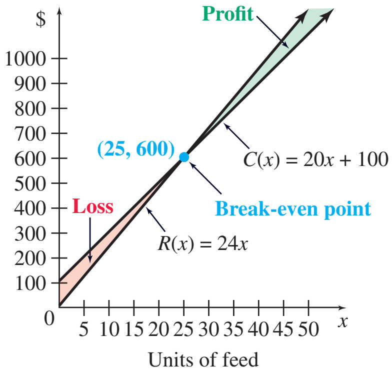


FIGURE 11

The firm breaks even by selling 25 units, which is the break-even quantity. The graphs of $R(x) = 24x$ and $C(x) = 20x + 100$ are shown in Figure 11. The break-even point (where $x = 25$) is shown on the graph. If the company sells more than 25 units (if $x > 25$), it makes a profit. If it sells fewer than 25 units, it loses money.

(c) What is the profit if 100 units of feed are sold?

SOLUTION Use the formula for profit $P(x)$.

$$P(x) = R(x) - C(x)$$
$$= 24x - (20x + 100)$$
$$= 4x - 100$$

Then $P(100) = 4(100) - 100 = 300$. The firm will make a profit of $300 from the sale of 100 units of feed.

(d) How many units must be sold to produce a profit of $900?

SOLUTION Let $P(x) = 900$ in the equation $P(x) = 4x - 100$ and solve for x.

$$900 = 4x - 100$$
$$1000 = 4x$$
$$x = 250$$

Sales of 250 units will produce $900 profit. **TRY YOUR TURN 6**

YOUR TURN 6 Repeat Example 7(d), using a cost function $C(x) = 35x + 250$, a charge of $58 per unit, and a profit of $8030.

Temperature

One of the most common linear relationships found in everyday situations deals with temperature. Recall that water freezes at 32° Fahrenheit and 0° Celsius, while it boils at 212° Fahrenheit and 100° Celsius.* The ordered pairs $(0, 32)$ and $(100, 212)$ are graphed in Figure 12 on axes showing Fahrenheit (F) as a function of Celsius (C). The line joining them is the graph of the function.

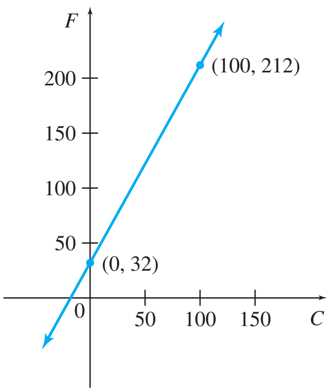


FIGURE 12

EXAMPLE 8 Temperature

Derive an equation relating F and C.

SOLUTION To derive the required linear equation, first find the slope using the given ordered pairs, $(0, 32)$ and $(100, 212)$.

$$m = \frac{212 - 32}{100 - 0} = \frac{180}{100} = \frac{9}{5}$$

*Gabriel Fahrenheit (1686–1736), a German physicist, invented his scale with 0° representing the temperature of an equal mixture of ice and ammonium chloride (a type of salt), and 96° as the temperature of the human body. (It is often said, erroneously, that Fahrenheit set 100° as the temperature of the human body. Fahrenheit's own words are quoted in *A History of the Thermometer and Its Use in Meteorology* by W. E. Knowles, Middleton: The Johns Hopkins Press, 1966, p. 75.) The Swedish astronomer Anders Celsius (1701–1744) set 0° and 100° as the freezing and boiling points of water.

The F-intercept of the graph is 32, so by the slope-intercept form, the equation of the line is

$$F = \frac{9}{5}C + 32.$$

With simple algebra this equation can be rewritten to give C in terms of F:

$$C = \frac{5}{9}(F - 32).$$

1.2 WARM-UP EXERCISES

W1. Evaluate $3(x-2)^2 + 6(x+4) - 5x + 4$ for $x = 5$. *(Sec. R.1)*

W2. Graph the line $y = 7 - 2.5x$. *(Sec. 1.1)*

1.2 EXERCISES

For Exercises 1–10, let $f(x) = 7 - 5x$ and $g(x) = 2x - 3$. Find the following.

1. $f(3)$

2. $f(4)$

3. $f(-2)$

4. $f(-1)$

5. $g(1.5)$

6. $g(2.5)$

7. $g(-1/2)$

8. $g(-3/4)$

9. $f(t)$

10. $g(k^2)$

In Exercises 11–14, decide whether the statement is true or false.

11. To find the x-intercept of the graph of a linear function, we solve $y = f(x) = 0$, and to find the y-intercept, we evaluate $f(0)$.

12. The graph of $f(x) = -5$ is a vertical line.

13. The slope of the graph of a linear function cannot be undefined.

14. The graph of $f(x) = ax$ is a straight line that passes through the origin.

15. Describe what fixed costs and marginal costs mean to a company.

16. In a few sentences, explain why the price of a commodity not already at its equilibrium price should move in that direction.

17. Explain why a linear function may not be adequate for describing the supply and demand functions.

18. In your own words, describe the break-even quantity, how to find it, and what it indicates.

Write a linear cost function for each situation. Identify all variables used.

19. A ski resort charges a snowboard rental fee of \$40 plus \$8.5 per hour.

20. An Internet site for downloading music charges a \$10 registration fee plus 99 cents per downloaded song.

21. A parking garage charges \$9 plus 55 cents per half-hour.

22. For a one-day rental, a car rental firm charges \$44 plus 28 cents per mile.

Assume that each situation can be expressed as a linear cost function. Find the cost function in each case.

23. Fixed cost: \$200; 60 items cost \$2600 to produce.

24. Fixed cost: \$35; 8 items cost \$395 to produce.

25. Marginal cost: \$225; 140 parts cost \$43,000 to produce.

26. Marginal cost: \$120; 700 items cost \$96,500 to produce.

APPLICATIONS

Business and Economics

27. Supply and Demand Suppose that the demand and price for strawberries are related by

$$p = D(q) = 5 - 0.25q,$$

where p is the price (in dollars) and q is the quantity demanded (in hundreds of quarts). Find the price at each level of demand.

(a) 0 quarts **(b)** 400 quarts **(c)** 840 quarts

Find the quantity demanded for the strawberries at each price.

(d) \$4.50 **(e)** \$3.25

(f) \$2.40 **(g)** Graph $p = 5 - 0.25q$.

Suppose the price and supply of strawberries is related by

$$p = S(q) = 0.25q,$$

where p is the price (in dollars) and q is the quantity supplied (in hundreds of quarts) of strawberries. Find the quantity supplied at each price.

(h) \$0 **(i)** \$2 **(j)** \$4.50

(k) Graph $p = 0.75q$ on the same axis used for part (g).

(l) Find the equilibrium quantity and the equilibrium price.

28. Supply and Demand Suppose that the demand and price for a certain model of a youth wristwatch are related by

$$p = D(q) = 20 - 1.5q,$$

where p is the price (in dollars) and q is the quantity demanded (in hundreds) Find the price at each level of demand.

(a) 0 watches **(b)** 300 watches **(c)** 600 watches

Find the quantity demanded for the watch at each price.

(d) \$8 **(e)** \$5

(f) \$2 **(g)** Graph $p = 20 - 1.5q$.

Suppose the price and supply of the watch are related by

$$p = (q) = 1.25q,$$

where p is the price (in dollars) and q is the quantity supplied (in hundreds) of watches. Find the quantity supplied at each price.

(h) \$0 **(i)** \$9 **(j)** \$20

(k) Graph $p = 1.25q$ on the same axis used for part (g).

(l) Find the equilibrium quantity and the equilibrium price.

29. Supply and Demand Let the supply and demand functions for sugar be given by

$$p = S(q) = 1.4q - 0.6 \quad \text{and} \quad p = D(q) = -2q + 3.2,$$

where p is the price per pound and q is the quantity in thousands of pounds.

(a) Graph these on the same axes.

(b) Find the equilibrium quantity and the equilibrium price.

30. Supply and Demand Let the supply and demand functions for butter scotch ice cream be given by

$$p = S(q) = \frac{5}{7}q \text{ and}$$

$$p = D(q) = 200 - \frac{3}{7}q,$$

where p is the price in dollars and q is the number of 10-gallon tubs.

(a) Graph these on the same axes.

(b) Find the equilibrium quantity and the equilibrium price.

31. Supply and Demand Suppose that the supply function for honey is $p = S(q) = 0.5q + 2.5$, where p is the price in dollars for an 8-oz container and q is the quantity in barrels. Suppose also that the equilibrium price is \$5 and the demand is 4 barrels when the price is \$6.5. Find an equation for the demand function, assuming it is linear.

32. Supply and Demand Suppose that the supply function for walnuts is $p = S(q) = 0.25q + 3.6$, where p is the price in dollars per pound and q is the quantity in bushels. Suppose also that the equilibrium price is \$5.85, and the demand is 4 bushels when the price is \$7.60. Find an equation for the demand function, assuming it is linear.

33. Break-Even Analysis Producing x units of tacos costs $C(x) = 3x + 30$; revenue is $R(x) = 18x$, where $C(x)$ and $R(x)$ are in dollars.

(a) What is the break-even quantity?

(b) What is the profit from 70 units?

(c) How many units will produce a profit of \$600?

34. Break-Even Analysis To produce x units of a religious medal costs $C(x) = 12x + 39$. The revenue is $R(x) = 25x$. Both $C(x)$ and $R(x)$ are in dollars.

(a) Find the break-even quantity.

(b) Find the profit from 250 units.

(c) Find the number of units that must be produced for a profit of \$130.

35. T-Shirt Cost Joanne Wendelken sells silk-screened T-shirts at community festivals and crafts fairs. Her marginal cost to produce one T-shirt is \$3.50. Her total cost to produce 60 T-shirts is \$300, and she sells them for \$9 each.

(a) Find the linear cost function for Joanne's T-shirt production.

(b) How many T-shirts must she produce and sell in order to break even?

(c) How many T-shirts must she produce and sell to make a profit of \$500?

36. Publishing Costs Alfred Juarez owns a small publishing house specializing in Latin American poetry. His fixed cost to produce a typical poetry volume is \$525, and his total cost to produce 1000 copies of the book is \$2675. His books sell for \$4.95 each.

(a) Find the linear cost function for Alfred's book production.

(b) How many poetry books must he produce and sell in order to break even?

(c) How many books must he produce and sell to make a profit of \$1000?

37. Marginal Cost of Coffee The manager of a restaurant found that the cost to produce 100 cups of coffee is \$11.02, while the cost to produce 400 cups is \$40.12. Assume the cost $C(x)$ is a linear function of x, the number of cups produced.

(a) Find a formula for $C(x)$.

(b) What is the fixed cost?

(c) Find the total cost of producing 1000 cups.

(d) Find the total cost of producing 1001 cups.

(e) Find the marginal cost of the 1001st cup.

(f) What is the marginal cost of *any* cup and what does this mean to the manager?

38. Marginal Cost of a New Plant In deciding whether to set up a new manufacturing plant, company analysts have decided that a linear function is a reasonable estimation for the total cost $C(x)$ in dollars to produce x items. They estimate the cost to produce

5000 items as \$400,000 and the cost to produce 12,000 items as \$750,000.

(a) Find a formula for $C(x)$.

(b) Find the fixed cost.

(c) Find the total cost to produce 30,000 items.

(d) What is the marginal cost of the items to be produced in this plant and what does this mean to the manager?

Break-Even Analysis **You are the manager of a firm. You are considering the manufacture of a new product, so you ask the accounting department for cost estimates and the sales department for sales estimates. After you receive the data, you must decide whether to go ahead with production of the new product. Analyze the data in Exercises 39–42 (find a break-even quantity) and then decide what you would do in each case. Also write the profit function.**

39. $C(x) = 85x + 900$; $R(x) = 105x$; no more than 38 units can be sold.

40. $C(x) = 105x + 6000$; $R(x) = 250x$; no more than 400 units can be sold.

41. $C(x) = 70x + 500$; $R(x) = 60x$ (*Hint*: What does a negative break-even quantity mean?)

42. $C(x) = 1000x + 5000$; $R(x) = 900x$

43. **Break-Even Analysis** Suppose that the fixed cost for a product is \$400 and the break-even quantity is 80. Find the marginal profit (the slope of the linear profit function).

44. **Break-Even Analysis** Suppose that the fixed cost for a product is \$650 and the break-even quantity is 25. Find the marginal profit (the slope of the linear profit function).

Life Sciences

45. **Adélie Penguin Chicks** The energy requirements for walking ($f(x)$ in W/kg) were determined for 8 Adélie penguin chicks with respect to walking speed (x in m/s). The relationship can be approximated by the linear function

$$f(x) = 41.3x + 8.9,$$

for speeds between 0 and 0.3 m/s. ***Source: Journal of Comparative Physiology B***.

(a) Approximately what is the energy requirement for walking at a speed of 0.2 m/s?

(b) Find and interpret the slope of the line.

46. **Tobacco Deaths** The U.S. Centers for Disease Control and Prevention projects that tobacco could soon be the leading cause of death in the world. In 1990, 35 million years of healthy life were lost globally due to tobacco. This quantity was rising linearly at a rate of about 28 million years each decade. In contrast, 100 million years of healthy life were lost due to diarrhea, with the rate falling linearly 22 million years each decade. ***Source: Science***.

(a) Write the years of healthy life in millions lost globally to tobacco as a linear function $f_t(t)$ of the years, t, since 1990.

(b) Write the years of healthy life in millions lost globally to diarrhea as a linear function $f_d(t)$ of the years, t, since 1990.

(c) Using your answers to parts (a) and (b), find in what year the amount of healthy life lost to tobacco was expected to first equal that lost to diarrhea.

Physical Sciences

47. **Temperature** Use the formula for conversion between Fahrenheit and Celsius derived in Example 8 to convert each temperature.

(a) 58°F to Celsius **(b)** −20°F to Celsius

(c) 50°C to Fahrenheit

48. **Body Temperature** You may have heard that the average temperature of the human body is 98.6°. Recent experiments show that the actual figure is closer to 98.2°. The figure of 98.6 comes from experiments done by Carl Wunderlich in 1868. But Wunderlich measured the temperatures in degrees Celsius and rounded the average to the nearest degree, giving 37°C as the average temperature. ***Source: Science News***.

(a) What is the Fahrenheit equivalent of 37°C?

(b) Given that Wunderlich rounded to the nearest degree Celsius, his experiments tell us that the actual average human body temperature is somewhere between 36.5°C and 37.5°C. Find what this range corresponds to in degrees Fahrenheit.

49. **Temperature** Find the temperature at which the Celsius and Fahrenheit temperatures are numerically equal.

General Interest

50. **Education Cost** A recent budget for the California State University system projected a fixed cost of \$486,000 at each of five off-campus centers, plus a marginal cost of \$1140 per student. ***Source: California State University***.

(a) Find a formula for the cost at each center, $C(x)$, as a linear function of x, the number of students.

(b) The budget projected 500 students at each center. Calculate the total cost at each center.

(c) Suppose, due to budget cuts, that each center is limited to \$1 million. What is the maximum number of students that each center can then support?

YOUR TURN ANSWERS

1. 25

2. 7600 and 4400

3. 8000 watermelons and \$3.20 per watermelon

4. $C(x) = 15x + 730$ **5.** $C(x) = 8.2x + 7145$ **6.** 360

1.3 The Least Squares Line

APPLY IT **How has the accidental death rate in the United States changed over time?**
In Example 1 in this section, we show how to answer such questions using the method of least squares.

Accidental Death Rate	
Year	Death Rate
1920	71.2
1930	80.5
1940	73.4
1950	60.3
1960	52.1
1970	56.2
1980	46.5
1990	36.9
2000	34.0
2010	39.1

We use past data to find trends and to make tentative predictions about the future. The only assumption we make is that the data are related linearly—that is, if we plot pairs of data, the resulting points will lie close to some line. This method cannot give exact answers. The best we can expect is that, if we are careful, we will get a reasonable approximation.

The table lists the number of accidental deaths per 100,000 people in the United States through the past century. ***Source: National Center for Health Statistics***. If you were a manager at an insurance company, these data could be very important. You might need to make some predictions about how much you will pay out next year in accidental death benefits, and even a very tentative prediction based on past trends is better than no prediction at all.

The first step is to draw a **scatterplot**, as we have done in Figure 13. Notice that the points lie approximately along a line, which means that a linear function may give a good approximation of the data. If we select two points and find the line that passes through them, as we did in Section 1.1, we will get a different line for each pair of points, and in some cases the lines will be very different. We want to draw one line that is simultaneously close to all the points on the graph, but many such lines are possible, depending upon how we define the phrase "simultaneously close to all the points." How do we decide on the best possible line? Before going on, you might want to try drawing the line you think is best on Figure 13 below.

The line used most often in applications is that in which the sum of the squares of the vertical distances from the data points to the line is as small as possible. Such a line is called the **least squares line**. The least squares line for the data in Figure 13 is drawn in Figure 14. How does the line compare with the one you drew on Figure 13? It may not be exactly the same, but should appear similar.

In Figure 14, the vertical distances from the points to the line are indicated by d_1, d_2, and so on, up through d_{10} (read "d-sub-one, d-sub-two, d-sub-three," and so on). For n points, corresponding to the n pairs of data, the least squares line is found by minimizing the sum $(d_1)^2 + (d_2)^2 + (d_3)^2 + \cdots + (d_n)^2$.

We often use **summation notation** to write the sum of a list of numbers. The Greek letter sigma, Σ, is used to indicate "the sum of." For example, we write the sum $x_1 + x_2 + \cdots + x_n$, where n is the number of data points, as

$$x_1 + x_2 + \cdots + x_n = \Sigma x.$$

Similarly, Σxy means $x_1y_1 + x_2y_2 + \cdots + x_ny_n$, and so on.

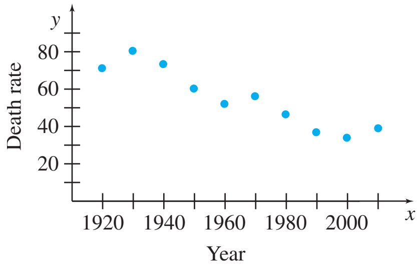


FIGURE 13

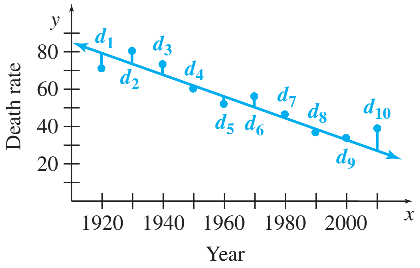


FIGURE 14

CAUTION Note that Σx^2 means $x_1^2 + x_2^2 + \cdots + x_n^2$, which is *not* the same as squaring Σx. When we square Σx, we write it as $(\Sigma x)^2$.

For the least squares line, the sum of the distances we are to minimize, $d_1^2 + d_2^2 + \cdots + d_n^2$, is written as

$$d_1^2 + d_2^2 + \cdots + d_n^2 = \Sigma d^2.$$

To calculate the distances, we let $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$ be the actual data points and we let the least squares line be $Y = mx + b$. We use Y in the equation instead of y to distinguish the predicted values (Y) from the y-value of the given data points. The predicted value of Y at x_1 is $Y_1 = mx_1 + b$, and the distance, d_1, between the actual y-value y_1 and the predicted value Y_1 is

$$d_1 = |Y_1 - y_1| = |mx_1 + b - y_1|.$$

Likewise,

$$d_2 = |Y_2 - y_2| = |mx_2 + b - y_2|,$$

and

$$d_n = |Y_n - y_n| = |mx_n + b - y_n|.$$

The sum to be minimized becomes

$$\begin{aligned}\Sigma d^2 &= (mx_1 + b - y_1)^2 + (mx_2 + b - y_2)^2 + \cdots + (mx_n + b - y_n)^2 \\ &= \Sigma(mx + b - y)^2,\end{aligned}$$

where $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$ are known and m and b are to be found.

The method of minimizing this sum requires advanced techniques in multivariable calculus and is not given here. To obtain the equation for the least squares line, a system of equations must be solved, producing the following formulas for determining the slope m and y-intercept b.*

Least Squares Line

The **least squares line** $Y = mx + b$ that gives the best fit to the data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ has slope m and y-intercept b given by

$$m = \frac{n(\Sigma xy) - (\Sigma x)(\Sigma y)}{n(\Sigma x^2) - (\Sigma x)^2} \quad \text{and} \quad b = \frac{\Sigma y - m(\Sigma x)}{n}.$$

EXAMPLE 1 Least Squares Line

APPLY IT Calculate the least squares line for the accidental death rate data.

SOLUTION

Method 1 Calculating by Hand

To find the least squares line for the given data, we first find the required sums. To reduce the size of the numbers, we rescale the year data. Let x represent the years since 1900, so that, for example, $x = 20$ corresponds to the year 1920. Let y represent the death rate. We then calculate the values in the xy, x^2, and y^2 columns and find their totals. (The column headed y^2 will be used later.) Note that the number of data points is $n = 10$.

*See Exercise 9 at the end of this section.

Least Squares Calculations				
x	y	xy	x^2	y^2
20	71.2	1424	400	5069.44
30	80.5	2415	900	6480.25
40	73.4	2936	1600	5387.56
50	60.3	3015	2500	3636.09
60	52.1	3126	3600	2714.41
70	56.2	3934	4900	3158.44
80	46.5	3720	6400	2162.25
90	36.9	3321	8100	1361.61
100	34.0	3400	10,000	1156.00
110	39.1	4301	12,100	1528.81
$\Sigma x = 650$	$\Sigma y = 550.2$	$\Sigma xy = 31{,}592$	$\Sigma x^2 = 50{,}500$	$\Sigma y^2 = 32{,}654.86$

Putting the column totals into the formula for the slope m, we get

$$m = \frac{n(\Sigma xy) - (\Sigma x)(\Sigma y)}{n(\Sigma x^2) - (\Sigma x)^2} \qquad \text{Formula for } m$$

$$= \frac{10(31{,}592) - (650)(550.2)}{10(50{,}500) - (650)^2} \qquad \text{Substitute from the table.}$$

$$= \frac{315{,}920 - 357{,}630}{505{,}000 - 422{,}500} \qquad \text{Multiply.}$$

$$= \frac{-41{,}710}{82{,}500} \qquad \text{Subtract.}$$

$$= -0.5055757576 \approx -0.506.$$

The significance of m is that the death rate per 100,000 people is tending to drop (because of the negative) at a rate of 0.506 per year.

Now substitute the value of m and the column totals in the formula for b:

$$b = \frac{\Sigma y - m(\Sigma x)}{n} \qquad \text{Formula for } b$$

$$= \frac{550.2 - (-0.5055757576)(650)}{10} \qquad \text{Substitute.}$$

$$= \frac{550.2 - (-328.6242424)}{10} \qquad \text{Multiply.}$$

$$= \frac{878.8242424}{10} = 87.88242424 \approx 87.9.$$

Substitute m and b into the least squares line, $Y = mx + b$; the least squares line that best fits the 10 data points has equation

$$Y = -0.506x + 87.9.$$

This gives a mathematical description of the relationship between the year and the number of accidental deaths per 100,000 people. The equation can be used to predict y from a given value of x, as we will show in Example 2. As we mentioned before, however, caution must be exercised when using the least squares equation to predict data points that are far from the range of points on which the equation was modeled.

CAUTION In computing m and b, we rounded the final answer to three digits because the original data were known only to three digits. It is important, however, *not* to round any of the intermediate results (such as Σx^2) because round-off error may have a detrimental effect on the accuracy of the answer. Similarly, it is important not to use a rounded-off value of m when computing b.

Method 2 Graphing Calculator

The calculations for finding the least squares line are often tedious, even with the aid of a calculator. Fortunately, many calculators can calculate the least squares line with just a few keystrokes. For purposes of illustration, we will show how the least squares line in this example is found with a TI-84 Plus C graphing calculator.

We begin by entering the data into the calculator. We will be using the first two lists, called L_1 and L_2. Choosing the `STAT` menu, then choosing the fourth entry `ClrList`, we enter L_1, L_2, to indicate the lists to be cleared. Now we press `STAT` again and choose the first entry `EDIT`, which brings up the blank lists. As before, rescale the years (letting 20 correspond to 1920, and so on) and then enter the numbers in L_1. We put the death rate in L_2, giving the screen shown in Figure 15.

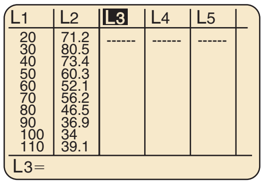


FIGURE 15

Quit the editor, press `STAT` again, and choose `CALC` instead of `EDIT`. Then choose item 4 `LinReg` $(ax + b)$ to get the values of a (the slope) and b (the y-intercept) for the least squares line, as shown in Figure 16. With a and b rounded to three decimal places, the least squares line is $Y = -0.506x + 87.9$. A graph of the data points and the line is shown in Figure 17.

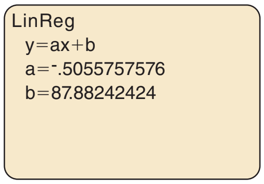


FIGURE 16

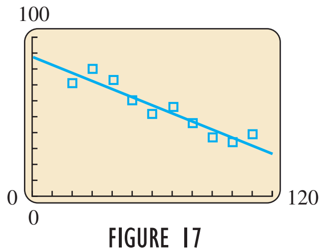


FIGURE 17

For more details on finding the least squares line with a graphing calculator, see the *Graphing Calculator and Excel Spreadsheet Manual* available with this book.

Method 3 Spreadsheet

Many computer spreadsheet programs can also find the least squares line. Figure 18 shows the scatterplot and least squares line for the accidental death rate data using an Excel spreadsheet. The scatterplot was found using the Marked Scatter chart from the Gallery and the line was found using the Add Trendline command under the Chart menu. For details, see the *Graphing Calculator and Excel Spreadsheet Manual* available with this book.

YOUR TURN 1 Calculate the least squares line for the following data.

x	1	2	3	4	5	6
y	3	4	6	5	7	8

TRY YOUR TURN 1

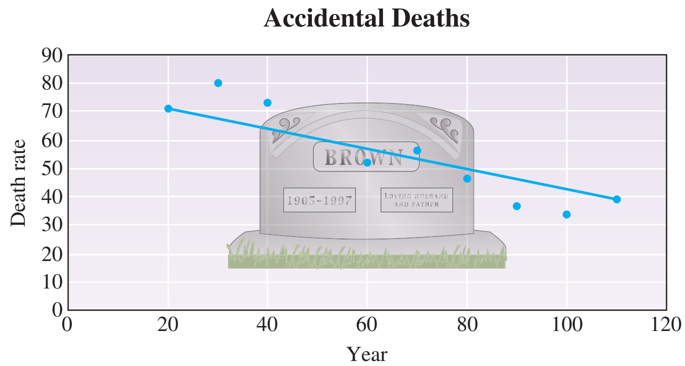


FIGURE 18

EXAMPLE 2 Least Squares Line

What do we predict the accidental death rate to be in 2020?

SOLUTION Use the least squares line equation given above with $x = 120$.

$$\begin{aligned} Y &= -0.506x + 87.9 \\ &= -0.506(120) + 87.9 \\ &\approx 27.2 \end{aligned}$$

The accidental death rate in 2020 is predicted to be about 27.2 per 100,000 population. In this case, we will have to wait until the 2020 data become available to see how accurate our prediction is. We have observed, however, that the accidental death rate began to go up after 2000 and was 39.1 per 100,000 population in 2010. This illustrates the danger of extrapolating beyond the data.

EXAMPLE 3 Least Squares Line

In what year is the death rate predicted to drop below 26 per 100,000 population?

SOLUTION Let $Y = 26$ in the equation above and solve for x.

$$\begin{aligned} 26 &= -0.506x + 87.9 \\ -61.9 &= -0.506x && \text{Subtract 87.9 from both sides.} \\ x &\approx 122.3 && \text{Divide both sides by } -0.506. \end{aligned}$$

This implies that sometime in the year 2022 (122 years after 1900) the death rate drops below 26 per 100,000 population.

Correlation

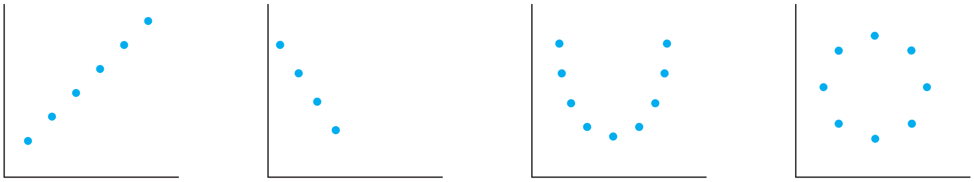
Although the least squares line can always be found, it may not be a good model. For example, if the data points are widely scattered, no straight line will model the data accurately. One measure of how well the original data fits a straight line is the **correlation coefficient**, denoted by r, which can be calculated by the following formula.

Correlation Coefficient

$$r = \frac{n(\Sigma xy) - (\Sigma x)(\Sigma y)}{\sqrt{n(\Sigma x^2) - (\Sigma x)^2} \cdot \sqrt{n(\Sigma y^2) - (\Sigma y)^2}}$$

Although the expression for r looks daunting, remember that each of the summations, Σx, Σy, Σxy, and so on, are just the totals from a table like the one we prepared for the data on accidental deaths. Also, with a calculator, the arithmetic is no problem! Furthermore, statistics software and many calculators can calculate the correlation coefficient for you.

The correlation coefficient measures the strength of the linear relationship between two variables. It was developed by statistics pioneer Karl Pearson (1857–1936). The correlation coefficient r is between 1 and -1 or is equal to 1 or -1. Values of exactly 1 or -1 indicate that the data points lie *exactly* on the least squares line. If $r = 1$, the least squares line has a positive slope; $r = -1$ gives a negative slope. If $r = 0$, there is no linear correlation between the data points (but some *nonlinear* function might provide an excellent fit for the data). A correlation coefficient of zero may also indicate that the data fit a horizontal line. To investigate what is happening, it is always helpful to sketch a scatterplot of the data. Some scatterplots that correspond to these values of r are shown in Figure 19 on the next page.



r close to 1 | r close to -1 | r close to 0 | r close to 0

FIGURE 19

A value of r close to 1 or -1 indicates the presence of a linear relationship. The exact value of r necessary to conclude that there is a linear relationship depends upon n, the number of data points, as well as how confident we want to be of our conclusion. For details, consult a text on statistics.*

EXAMPLE 4 Correlation Coefficient

Find r for the data on accidental death rates in Example 1.

SOLUTION

Method 1 Calculating by Hand

From the table in Example 1,

$$\Sigma x = 650, \ \Sigma y = 550.2, \ \Sigma xy = 31{,}592, \ \Sigma x^2 = 50{,}500,$$
$$\Sigma y^2 = 32{,}654.86, \quad \text{and} \quad n = 10.$$

Substituting these values into the formula for r gives

$$r = \frac{n(\Sigma xy) - (\Sigma x)(\Sigma y)}{\sqrt{n(\Sigma x^2) - (\Sigma x)^2} \cdot \sqrt{n(\Sigma y^2) - (\Sigma y)^2}} \qquad \text{Formula for } r$$

$$= \frac{10(31{,}592) - (650)(550.2)}{\sqrt{10(50{,}500) - (650)^2} \cdot \sqrt{10(32{,}654.86) - (550.2)^2}} \qquad \text{Substitute.}$$

$$= \frac{315{,}920 - 357{,}630}{\sqrt{505{,}000 - 422{,}500} \cdot \sqrt{326{,}548.6 - 302{,}720.04}} \qquad \text{Multiply.}$$

$$= \frac{-41{,}710}{\sqrt{82{,}500} \cdot \sqrt{23{,}828.56}} \qquad \text{Subtract.}$$

$$= \frac{-41{,}710}{44{,}337.97695} \qquad \text{Take square roots and multiply.}$$

$$= -0.9407285327 \approx -0.941.$$

This is a high correlation, which agrees with our observation that the data fit a line quite well.

Method 2 Graphing Calculator

Most calculators that give the least squares line will also give the correlation coefficient. To do this on the TI-84 Plus C, press the second function `CATALOG` and go down the list to the entry `DiagnosticOn`. Press `ENTER` at that point, then press `STAT`, `CALC`, and choose item 4 to get the display in Figure 20. The result is the same as we got by hand. The command `DiagnosticOn` need be entered only once, and the correlation coefficient will always appear in the future.

```
LinReg
 y=ax+b
 a=-.5055757576
 b=87.88242424
 r²=.8849701723
 r=-.9407285327
```

FIGURE 20

*For example, see *Introductory Statistics*, 9th edition, by Neil A. Weiss, Boston, Mass.: Pearson, 2012.

Method 3
Spreadsheet

Many computer spreadsheet programs have a built-in command to find the correlation coefficient. For example, in Excel, use the command "= `CORREL(A1:A10,B1:B10)`" to find the correlation of the 10 data points stored in columns A and B. For more details, see the *Graphing Calculator and Excel Spreadsheet Manual* available with this text.

YOUR TURN 2 Find r for the following data.

x	1	2	3	4	5	6
y	3	4	6	5	7	8

TRY YOUR TURN 2

The square of the correlation coefficient gives the fraction of the variation in y that is explained by the linear relationship between x and y. Consider Example 4, where $r^2 = (-0.941)^2 = 0.885$. This means that 88.5% of the variation in y is explained by the linear relationship found earlier in Example 1. The remaining 11.5% comes from the scattering of the points about the line.

EXAMPLE 5 Average Expenditure per Pupil Versus Test Scores

Many states and school districts debate whether or not increasing the amount of money spent per student will guarantee academic success. The following scatterplot shows the average eighth grade reading score on the National Assessment of Education Progress (NAEP) for the 50 states and the District of Columbia in 2011 plotted against the average expenditure per pupil in 2011. Explore how the correlation coefficient is affected by the inclusion of the District of Columbia, which spent \$18,475 per pupil and had a NAEP score of 242. ***Source: U.S. Census Bureau and National Center for Education Statistics***.

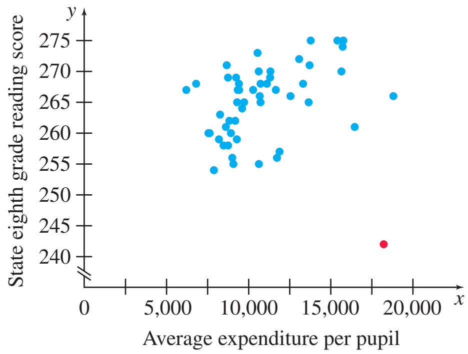


FIGURE 21

SOLUTION A spreadsheet was used to create a plot of the points shown in Figure 21. Washington D.C. corresponds to the red point in the lower right, which is noticeably separate from all the other points. Using the original data, the correlation coefficient when Washington D.C. is included is 0.1798, indicating that there is not a strong linear correlation. Excluding Washington D.C. raises the correlation coefficient to 0.4427, which is a somewhat stronger indication of a linear correlation. This illustrates that one extreme data point that is separate from the others, known as an **outlier**, can have a strong effect on the correlation coefficient.

Even if the correlation between average expenditure per pupil and reading score in Example 5 was high, this would not prove that spending more per pupil causes high reading scores. To prove this would require further research. It is a common statistical fallacy to assume that correlation implies causation. Perhaps the correlation is due to a third underlying variable. In Example 5, perhaps states with wealthier families spend more per pupil, and the students read better because wealthier families have greater access to reading material. Determining the truth requires careful research methods that are beyond the scope of this textbook.

1.3 EXERCISES

1. Suppose a positive linear correlation is found between two quantities. Does this mean that one of the quantities increasing causes the other to increase? If not, what does it mean?

2. Given a set of points, the least squares line formed by letting x be the independent variable will not necessarily be the same as the least squares line formed by letting y be the independent variable. Give an example to show why this is true.

3. For the following table of data,

x	1	2	3	4	5	6	7	8	9	10
y	0	0.5	1	2	2.5	3	3	4	4.5	5

(a) draw a scatterplot.

(b) calculate the correlation coefficient.

(c) calculate the least squares line and graph it on the scatterplot.

(d) predict the y-value when x is 11.

The following problem is reprinted from the November 1989 *Actuarial Examination on Applied Statistical Methods. Source: Society of Actuaries.*

4. You are given

X	6.8	7.0	7.1	7.2	7.4
Y	0.8	1.2	0.9	0.9	1.5

Determine r^2, the coefficient of determination for the regression of Y on X. Choose one of the following. (*Note:* The coefficient of determination is defined as the square of the correlation coefficient.)

(a) 0.3 **(b)** 0.4 **(c)** 0.5 **(d)** 0.6 **(e)** 0.7

5. Consider the following table of data.

x	1	1	2	2	9
y	1	2	1	2	9

(a) Calculate the least squares line and the correlation coefficient.

(b) Repeat part (a), but this time delete the last point.

(c) Draw a graph of the data, and use it to explain the dramatic difference between the answers to parts (a) and (b).

6. Consider the following table of data.

x	1	2	3	4	9
y	1	2	3	4	−20

(a) Calculate the least squares line and the correlation coefficient.

(b) Repeat part (a), but this time delete the last point.

(c) Draw a graph of the data, and use it to explain the dramatic difference between the answers to parts (a) and (b).

7. Consider the following table of data.

x	1	2	3	4
y	1	1	1	1.1

(a) Calculate the correlation coefficient.

(b) Sketch a graph of the data.

(c) Based on how closely the data fits a straight line, is your answer to part (a) surprising? Discuss the extent to which the correlation coefficient describes how well the data fit a horizontal line.

8. Consider the following table of data.

x	0	1	2	3	4
y	4	1	0	1	4

(a) Calculate the least squares line and the correlation coefficient.

(b) Sketch a graph of the data.

(c) Comparing your answers to parts (a) and (b), does a correlation coefficient of 0 mean that there is no relationship between the x- and y- values? Would some curve other than a line fit the data better? Explain.

9. The formulas for the least squares line were found by solving the system of equations

$$nb + (\Sigma x)m = \Sigma y$$
$$(\Sigma x)b + (\Sigma x^2)m = \Sigma xy.$$

Solve the above system for b and m to show that

$$m = \frac{n(\Sigma xy) - (\Sigma x)(\Sigma y)}{n(\Sigma x^2) - (\Sigma x)^2} \quad \text{and}$$

$$b = \frac{\Sigma y - m(\Sigma x)}{n}.$$

APPLICATIONS

Business and Economics

10. **Consumer Durable Goods** Suppose the total value of consumer durable goods (goods purchased by households for their nonbusiness use with a life expectancy of at least three years) has grown at an approximately linear rate in recent years. The annual data for ten years can be summarized as follows, where x represents the years since the base year and y the total value of consumer durable goods in trillions of dollars.

$n = 10$ $\quad \Sigma x = 75$ $\quad \Sigma x^2 = 645$

$\Sigma y = 60$ $\quad \Sigma y^2 = 400$ $\quad \Sigma xy = 480$

(a) Find an equation for the least squares line.

(b) Use your result from part (a) to predict the total value of consumer durable goods in the year 2025.

(c) If this growth continues linearly, in what year will the total value of consumer durable goods first reach at least 9 trillion dollars?

(d) Find and interpret the correlation coefficient.

11. Decrease in Banks Suppose the number of banks in a country has been dropping steadily since 1984, and the trend in recent years has been roughly linear. The annual data for ten years can be summarized as follows, where x represents the years since the base year and y the number of banks in thousands.

$$n = 10 \qquad \Sigma x = 75 \qquad \Sigma x^2 = 645$$
$$\Sigma y = 45 \qquad \Sigma y^2 = 205.177 \qquad \Sigma xy = 323$$

(a) Find an equation for the least squares line.

(b) Use your result from part (a) to predict the number of banks in the year 2025.

(c) If this trend continues linearly, in what year will the number of banks in the country drop below 1000?

(d) Find and interpret the correlation coefficient.

12. Internet The percent of households with Internet use at home has been growing steadily, as shown by the following table. ***Source: U.S. Census Bureau***.

Year	2000	2003	2007	2009	2012
Percent of Households	41.5	54.7	61.7	68.7	74.8

(a) Find an equation for the least squares line, letting x equal the number of years since 2000.

(b) Based on your answer to part (a), at approximately what rate is the percent of households with Internet use at home growing per year?

(c) Use your result from part (a) to predict the percent of households with Internet use at home in the year 2015.

(d) If this trend continues linearly, in what year will the percent of households with Internet use at home first exceed 90%?

(e) Find and interpret the correlation coefficient.

13. Landlines The percent of U.S. households with telephone landlines has decreased at a roughly linear rate, as shown by the following table. ***Source: Centers for Disease Control and Prevention***.

Year	Percent of Households
2005	89.7
2006	84.1
2007	81.9
2008	77.9
2009	73.5
2010	68.2
2011	63.8
2012	59.6

(a) Find the equation of the least squares line, letting x equal the number of years since 2000.

(b) Based on your answer to part (a), at approximately what rate is the percent of households with landlines decreasing per year?

(c) Use your result from part (a) to predict the percent of households with landlines in the year 2015.

(d) If this trend continues linearly, in what year will the percent of households with landlines first dip below 40%?

(e) Find and interpret the correlation coefficient.

14. Consumer Credit The total amount of consumer credit has been increasing steadily in recent years. The following table gives the total U.S. outstanding consumer credit (in billions of dollars). ***Source: Federal Reserve***.

Year	Consumer Credit	Year	Consumer Credit
2004	2219.5	2009	2553.5
2005	2319.8	2010	2648.1
2006	2415.0	2011	2757.0
2007	2551.9	2012	2924.3
2008	2592.1	2013	3099.2

(a) Find an equation for the least squares line, letting x equal the number of years since 2000.

(b) Based on your answer to part (a), at approximately what rate is consumer credit growing per year?

(c) Use your result from part (a) to predict the amount of consumer credit in 2015.

(d) If this trend continues linearly, in what year will the total debt first exceed \$4000 billion?

(e) Find and interpret the correlation coefficient.

15. Mean Earnings The mean earnings (in dollars) of workers 18 years and over, by educational attainment, have increased steadily over the years and are given in the following table. Let x equal the number of years since 1900. ***Source: U.S. Census***.

Year	High School Graduate	Bachelor's Degree	Year	High School Graduate	Bachelor's Degree
1976	8393	13,033	1996	22,154	38,112
1978	9834	15,291	1998	23,594	43,782
1980	11,314	18,075	2000	25,692	49,595
1982	12,560	20,272	2002	27,280	51,194
1984	13,893	23,072	2004	28,631	51,568
1986	15,120	26,511	2006	31,071	56,788
1988	16,750	28,344	2008	31,283	58,613
1990	17,820	31,112	2010	31,003	57,621
1992	18,737	32,629	2012	32,630	60,159
1994	20,248	37,224			

(a) Plot the data for high school graduates and for workers with a bachelor's degree. Do the data points lie in a linear pattern?

(b) Find an equation for the least squares line for the mean earnings of high school graduates. Calculate and interpret the correlation coefficient.

(c) Based on your answer from part (b), at approximately what rate are the mean earnings of high school graduates growing per year?

(d) Find an equation for the least squares line for the mean earnings of workers with a bachelor's degree. Calculate and interpret the correlation coefficient.

(e) Based on your answer from part (d), at approximately what rate are the mean earnings of workers with a bachelor's degree growing per year?

(f) If these trends continue linearly, in what year will the mean earnings for high school graduates exceed \$75,000? In what year will workers with a bachelor's degree exceed \$75,000?

16. Air Fares Using Expedia, a discount travel website, the American Airline prices for a one-way flight from New York City to various cities were recorded. The following table gives the distances from New York City to 14 selected cities, along with the airfare to each of these cities.

City	Distance, x (miles)	Price, y (dollars)
Chicago	802	191
Denver	1771	183
Kansas City	1198	193
Little Rock	1238	247
Los Angeles	2786	316
Minneapolis	1207	206
Nashville	892	151
Philadelphia	95	290
Phoenix	2411	223
Portland	2885	321
Reno	2705	288
St. Louis	948	152
San Diego	2762	267
Seattle	2815	341

(a) Plot the data. Do the data points lie in a linear pattern?

(b) Find the correlation coefficient. Combining this with your answer to part (a), does the cost of a ticket tend to go up with the distance flown?

(c) Find the equation for the least squares line, and use it to find the approximate marginal cost per mile to fly.

(d) Identify the outlier in the scatterplot. Discuss the reason why there would be a difference in price to this city.

Life Sciences

17. Bird Eggs The average length and width of various bird eggs are given in the following table. ***Source: National Council of Teachers of Mathematics***.

Bird Name	Width (cm)	Length (cm)
Canada goose	5.8	8.6
Robin	1.5	1.9
Turtledove	2.3	3.1
Hummingbird	1.0	1.0
Raven	3.3	5.0

(a) Plot the points, putting the length on the y-axis and the width on the x-axis. Do the data appear to be linear?

(b) Find the least squares line, and plot it on the same graph as the data.

(c) Suppose there are birds with eggs even smaller than those of hummingbirds. Would the equation found in part (b) continue to make sense for all positive widths, no matter how small? Explain.

(d) Find the correlation coefficient.

18. Size of Hunting Parties In the 1960s, the famous researcher Jane Goodall observed that chimpanzees hunt and eat meat as part of their regular diet. Sometimes chimpanzees hunt alone, while other times they form hunting parties. The following table summarizes research on chimpanzee hunting parties, giving the size of the hunting party and the percentage of successful hunts. ***Source: American Scientist and Mathematics Teacher***.

Number of Chimps in Hunting Party	Percentage of Successful Hunts
1	20
2	30
3	28
4	42
5	40
6	58
7	45
8	62
9	65
10	63
12	75
13	75
14	78
15	75
16	82

(a) Plot the data. Do the data points lie in a linear pattern?

(b) Find the correlation coefficient. Combining this with your answer to part (a), does the percentage of successful hunts tend to increase with the size of the hunting party?

(c) Find the equation of the least squares line, and graph it on your scatterplot.