PEARSON NEW INTERNATIONAL EDITION

Analysis with an Introduction to Proof

Steven R. Lay

Fifth Edition

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Logic and Proof

To understand mathematics and mathematical arguments, it is necessary to have a solid understanding of logic and the way in which known facts can be combined to prove new facts. Although many people consider themselves to be logical thinkers, the thought patterns developed in everyday living are only suggestive of and not totally adequate for the precision required in mathematics. In this chapter we take a careful look at the rules of logic and the way in which mathematical arguments are constructed. Section 1 presents the logical connectives that enable us to build compound statements from simpler ones. Section 2 discusses the role of quantifiers. Sections 3 and 4 analyze the structure of mathematical proofs and illustrate the various proof techniques by means of examples.

Section 1 LOGICAL CONNECTIVES

The language of mathematics consists primarily of declarative sentences. If a sentence can be classified as true or false, it is called a **statement**. The truth or falsity of a statement is known as its truth value. For a sentence to be a statement, it is not necessary that we actually know whether it is true or false, but it must clearly be the case that it is one or the other.

1.1 EXAMPLE Consider the following sentences.

(a) Two plus two equals four.
(b) Every continuous function is differentiable.
(c) $x^2 - 5x + 6 = 0$.
(d) A circle is the only convex set in the plane that has the same width in each direction.
(e) Every even number greater than 2 is the sum of two primes.

Sentences (a) and (b) are statements since (a) is true and (b) is false. Sentence (c), on the other hand, is true for some x and false for others. If we have a particular context in mind, then (c) will be a statement. In Section 2 we shall see how to remove this ambiguity. Sentences (d) and (e) are more difficult. You may or may not know whether they are true or false, but it is certain that each sentence must be one or the other. Thus (d) and (e) are both statements. [It turns out that (d) can be shown to be false, and the truth value of (e) has not yet been established.†]

1.2 PRACTICE Which of the sentences are statements?

(a) If x is a real number, then $x^2 \geq 0$.
(b) Seven is a prime number.
(c) Seven is an even number.
(d) This sentence is false.

In studying mathematical logic we shall not be concerned with the truth value of any particular simple statement. To be a statement, it must be either true or false (and not both), but it is immaterial which condition applies. What will be important is how the truth value of a compound statement is determined by the truth values of its simpler parts.

In everyday English conversation we have a variety of ways to change or combine statements. A simple statement‡ like

It is windy.

can be negated to form the statement

It is *not* windy.

† Sentence (e) is known as the Goldbach conjecture after the Prussian mathematician Christian Goldbach, who made this conjecture in a letter to Leonhard Euler in 1742. Using computers it has been verified for all even numbers up to 10^{14} but has not yet been proved for *every* even number. For a good discussion of the history of this problem, see Hofstadter (1979). Recent results are reported in Deshouillers et al. (1998).

‡ It may be questioned whether or not the sentence "It is windy" is a statement, since the term "windy" is so vague. If we *assume* that "windy" is given a precise definition, then in a particular place at a particular time, "It is windy" will be a statement. It is customary to assume precise definitions when we use descriptive language in an example. This problem does not arise in a mathematical context because the definitions *are* precise.

The compound statement

It is windy *and* the waves are high.

is made up of two parts: "It is windy" and "The waves are high." These two parts can also be combined in other ways. For example,

It is windy *or* the waves are high.
If it is windy, *then* the waves are high.
It is windy *if and only if* the waves are high.

The italicized words above (*not*, *and*, *or*, *if* … *then*, *if and only if*) are called **sentential connectives**. Their use in mathematical writing is similar to (but not identical with) their everyday usage. To remove any possible ambiguity, we shall look carefully at each and specify its precise mathematical meaning.

Let p stand for a given statement. Then $\sim p$ (read *not* p) represents the logical opposite (**negation**) of p. When p is true, $\sim p$ is false; when p is false, $\sim p$ is true. This can be summarized in a truth table:

p	$\sim p$
T	F
F	T

where T stands for true and F stands for false.

1.3 EXAMPLE Let p, q, and r be given as follows:

p: Today is Monday.
q: Five is an even number.
r: The set of integers is countable.

Then their negations can be written as

$\sim p$: Today is not Monday.
$\sim q$: Five is not an even number.
or
Five is an odd number.
$\sim r$: The set of integers is not countable.
or
The set of integers is uncountable.

The connective *and* is used in logic in the same way as it is in ordinary language. If p and q are statements, then the statement p *and* q (called the **conjunction** of p and q and denoted by $p \wedge q$) is true only when both p and q are true, and it is false otherwise.

1.4 PRACTICE Complete the truth table for $p \wedge q$. Note that we have to use four lines in this table to include all possible combinations of truth values of p and q.

p	q	$p \wedge q$
T	T	
T	F	
F	T	
F	F	

The connective *or* is used to form a compound statement known as a **disjunction**. In common English the word *or* can have two meanings. In the sentence

We are going to paint our house yellow or green.

the intended meaning is *yellow or green, but not both*. This is known as the exclusive meaning of *or*. On the other hand, in the sentence

Do you want cake or ice cream for dessert?

the intended meaning may include the possibility of having both. This inclusive meaning is the *only* way the word *or* is used in logic. Thus, if we denote the disjunction *p or q* by $p \vee q$, we have the following truth table:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

A statement of the form

If p, then q.

is called an **implication** or a **conditional** statement. The if-statement p in the implication is called the **antecedent** and the then-statement q is called the **consequent**. To decide on an appropriate truth table for implication, let us consider the following sentence:

If it stops raining by Saturday, then I will go to the football game.

If a friend made a statement like this, under what circumstances could you call him a liar? Certainly, if the rain stops and he doesn't go, then he did not tell the truth. But what if the rain doesn't stop? He hasn't said what he will do then, so whether he goes or not, either is all right.

Although it might be argued that other interpretations make equally good sense, mathematicians have agreed that an implication will be called false only when the antecedent is true and the consequent is false. If we denote the implication "if *p*, then *q*" by $p \Rightarrow q$, we obtain the following table:

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

It is important to recognize that in mathematical writing the conditional statement can be disguised in several equivalent forms. Thus the following expressions all mean exactly the same thing:

if p, then q	q provided that p
p implies q	q whenever p
p only if q	p is a sufficient condition for q
q if p	q is a necessary condition for p

1.5 PRACTICE Identify the antecedent and the consequent in each of the following statements.

(a) If n is an integer, then $2n$ is an even number.
(b) You can work here only if you have a college degree.
(c) The car will not run whenever you are out of gas.
(d) Continuity is a necessary condition for differentiability.

One way to visualize an implication $R \Rightarrow S$ is to picture two sets R and S, with R inside S. Figure 1 shows several objects of different shapes. Some are round and some are not round. Some are solid and some are not solid. Objects that are round are in set R and objects that are solid are in set S.

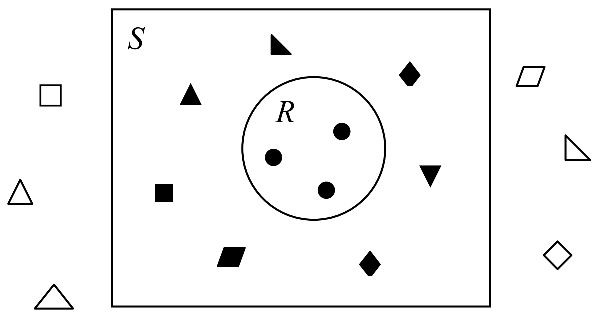

Figure 1 $R \Rightarrow S$

We see that the relationship between R and S in Figure 1 can be stated in several equivalent ways:

- If an object is round (R), then it is solid (S).
- An object is solid (S) whenever it is round (R).
- An object is solid (S) provided that it is round (R).
- Being round (R) is a sufficient condition for an object to be solid (S). (It is sufficient to know that an object is round to conclude that it is solid.)
- Being solid (S) is a necessary condition for an object to be round (R). (It is necessary for an item to be solid in order for it to be round.)

1.6 PRACTICE In Figure 1, which of the following is correct?

(a) An object is solid (S) only if it is round (R).
(b) An object is round (R) only if it is solid (S).

The statement "p if and only if q" is the conjunction of the two conditional statements $p \Rightarrow q$ and $q \Rightarrow p$. A statement in this form is called a **biconditional** and is denoted by $p \Leftrightarrow q$. In written form the abbreviation "iff" is sometimes used instead of "if and only if." The truth table for the biconditional can be obtained by analyzing the compound statement $(p \Rightarrow q) \wedge (q \Rightarrow p)$ a step at a time.

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$(p \Rightarrow q) \wedge (q \Rightarrow p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Thus we see that $p \Leftrightarrow q$ is true precisely when p and q have the same truth values.

1.7 PRACTICE Construct a truth table for each of the following compound statements.

(a) $\sim(p \wedge q) \Leftrightarrow [(\sim p) \vee (\sim q)]$
(b) $\sim(p \vee q) \Leftrightarrow [(\sim p) \wedge (\sim q)]$
(c) $\sim(p \Rightarrow q) \Leftrightarrow [p \wedge (\sim q)]$

In Practice 1.7 we find that each of the compound statements is true in all cases. Such a statement is called a **tautology**. When a biconditional statement is a tautology, it shows that the two parts of the biconditional are logically equivalent. That is, the two component statements have the same truth tables.

We shall encounter many more tautologies in the next few sections. They are very useful in changing a statement from one form into an equivalent statement in a different (one hopes simpler) form. In 1.7(a) we see that the negation of a conjunction is logically equivalent to the disjunction of the negations. Similarly, in 1.7(b) we learn that the negation of a disjunction is the conjunction of the negations. In 1.7(c) we find that the negation of an implication is *not* another implication, but rather it is the conjunction of the antecedent and the negation of the consequent.

1.8 EXAMPLE Using Practice 1.7(a), we see that the negation of

The set S is compact and convex.

can be written as

The set S is not compact or it is not convex.

This example also illustrates that using equivalent forms in logic does not depend on knowing the meaning of the terms involved. It is the form of the statement that is important. Whether or not we happen to know the definition of "compact" and "convex" is of no consequence in forming the negation above.

1.9 PRACTICE Use the tautologies in Practice 1.7 to write out a negation of each of the following statements.

(a) Seven is prime or 2 + 2 = 4.
(b) If M is bounded, then M is compact.
(c) If roses are red and violets are blue, then I love you.

Review of Key Terms in Section 1

Statement	Implication	Biconditional
Negation	Conditional	Tautology
Conjunction	Antecedent	
Disjunction	Consequent	

ANSWERS TO PRACTICE PROBLEMS

1.2 (a), (b), and (c)

1.4

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

1.5 (a) Antecedent: n is an integer
Consequent: $2n$ is an even number
(b) Antecedent: you can work here
Consequent: you have a college degree
(c) Antecedent: you are out of gas
Consequent: the car will not run
(d) Antecedent: differentiability
Consequent: continuity

1.6 Statement (b) is correct. If one of the objects is not solid, then it cannot possibly be round.

1.7 Sometimes we condense a truth table by writing the truth values under the part of a compound expression to which they apply.

(a)

p	q	$\sim(p \wedge q)$	$\Leftrightarrow$	$[(\sim p) \vee (\sim q)]$
T	T	F T	T	F F F
T	F	T F	T	F T T
F	T	T F	T	T T F
F	F	T F	T	T T T

(b)

p	q	$\sim(p \vee q)$	$\Leftrightarrow$	$[(\sim p) \wedge (\sim q)]$
T	T	F T	T	F F F
T	F	F T	T	F F T
F	T	F T	T	T F F
F	F	T F	T	T T T

(c)

p	q	$\sim(p \Rightarrow q)$	$\Leftrightarrow$	$[p \wedge (\sim q)]$
T	T	F T	T	T F F
T	F	T F	T	T T T
F	T	F T	T	F F F
F	F	F T	T	F F T

1.9 (a) Seven is not prime and $2 + 2 \neq 4$.
(b) M is bounded and M is not compact.
(c) Roses are red and violets are blue, but I do not love you.

1 *EXERCISES*

*Exercises marked with * are used in later sections, and exercises marked with ☆ have hints or solutions in the back of the chapter.*

1. Mark each statement True or False. Justify each answer.

(a) In order to be classified as a statement, a sentence must be true.
(b) Some statements are both true and false.
(c) When statement p is true, its negation $\sim p$ is false.
(d) A statement and its negation may both be false.
(e) In mathematical logic, the word "or" has an inclusive meaning.

2. Mark each statement True or False. Justify each answer.

(a) In an implication $p \Rightarrow q$, statement p is referred to as the proposition.
(b) The only case where $p \Rightarrow q$ is false is when p is true and q is false.
(c) "If p, then q" is equivalent to "p whenever q."
(d) The negation of a conjunction is the disjunction of the negations of the individual parts.
(e) The negation of $p \Rightarrow q$ is $q \Rightarrow p$.

3. Write the negation of each statement. ☆

(a) The 3×3 identity matrix is singular.
(b) The function $f(x) = \sin x$ is bounded on $\mathbb{R}$.
(c) The functions f and g are linear.
(d) Six is prime or seven is odd.
(e) If x is in D, then $f(x) < 5$.
(f) If (a_n) is monotone and bounded, then (a_n) is convergent.
(g) If f is injective, then S is finite or denumerable.

4. Write the negation of each statement.

(a) The function $f(x) = x^2 - 9$ is continuous at $x = 3$.
(b) The relation R is reflexive or symmetric.
(c) Four and nine are relatively prime.
(d) x is in A or x is not in B.
(e) If $x < 7$, then $f(x)$ is not in C.
(f) If (a_n) is convergent, then (a_n) is monotone and bounded.
(g) If f is continuous and A is open, then $f^{-1}(A)$ is open.

5. Identify the antecedent and the consequent in each statement. ☆

(a) M has a zero eigenvalue whenever M is singular.
(b) Linearity is a sufficient condition for continuity.
(c) A sequence is Cauchy only if it is bounded.
(d) $x < 3$ provided that $y > 5$.

6. Identify the antecedent and the consequent in each statement.

(a) A sequence is convergent if it is Cauchy.
(b) Convergence is a necessary condition for boundedness.
(c) Orthogonality implies invertability.
(d) K is closed and bounded only if K is compact.

7. Construct a truth table for each statement.

(a) $p \Rightarrow \sim q$ ☆
(b) $[p \wedge (p \Rightarrow q)] \Rightarrow q$
(c) $[p \Rightarrow (q \wedge \sim q)] \Leftrightarrow \sim p$ ☆

8. Construct a truth table for each statement.

(a) $p \vee \sim q$
(b) $p \wedge \sim p$
(c) $[(\sim q) \wedge (p \Rightarrow q)] \Rightarrow \sim p$

9. Indicate whether each statement is True or False. ☆

(a) $3 \leq 5$ and 11 is odd.
(b) $3^2 = 8$ or $2 + 3 = 5$.
(c) $5 > 8$ or 3 is even.
(d) If 6 is even, then 9 is odd.
(e) If $8 < 3$, then $2^2 = 5$.
(f) If 7 is odd, then 10 is prime.
(g) If 8 is even and 5 is not prime, then $4 < 7$.
(h) If 3 is odd or $4 > 6$, then $9 \leq 5$.
(i) If both $5 - 3 = 2$ and $5 + 3 = 2$, then $9 = 4$.
(j) It is not the case that 5 is even or 7 is prime.

10. Indicate whether each statement is True or False.

(a) $2 + 3 = 5$ and 5 is even.
(b) $3 + 4 = 5$ or $4 + 5 = 6$.
(c) 7 is even or 6 is not prime.
(d) If $4 + 4 = 8$, then 9 is prime.
(e) If 6 is prime, then $8 < 6$.
(f) If $6 < 2$, then $4 + 4 = 8$.
(g) If 8 is prime or 7 is odd, then 9 is even.
(h) If $2 + 5 = 7$ only if $3 + 4 = 8$, then $3^2 = 9$.
(i) If both $5 - 3 = 2$ and $5 + 3 = 8$, then $8 - 3 = 4$.
(j) It is not the case that 5 is not prime and 3 is odd.

11. Let p be the statement "The figure is a polygon," and let q be the statement "The figure is a circle." Express each of the following statements in symbols. ☆

(a) The figure is a polygon, but it is not a circle.
(b) The figure is a polygon or a circle, but not both.
(c) If the figure is not a circle, then it is a polygon.
(d) The figure is a circle whenever it is not a polygon.
(e) The figure is a polygon iff it is not a circle.

12. Let m be the statement "x is perpendicular to M," and let n be the statement "x is perpendicular to N." Express each of the following statements in symbols.

(a) x is perpendicular to N but not perpendicular to M.
(b) x is not perpendicular to M, nor is it perpendicular to N.
(c) x is perpendicular to N only if x is perpendicular to M.
(d) x is not perpendicular to N provided it is perpendicular to M.
(e) It is not the case that x is perpendicular to M and perpendicular to N.

13. Define a new sentential connective ∇, called *nor*, by the following truth table.

p	q	$p \nabla q$
T	T	F
T	F	F
F	T	F
F	F	T

(a) Use a truth table to show that $p \nabla p$ is logically equivalent to $\sim p$.
(b) Complete a truth table for $(p \nabla p) \nabla (q \nabla q)$.
(c) Which of our basic connectives ($p \wedge q$, $p \vee q$, $p \Rightarrow q$, $p \Leftrightarrow q$) is logically equivalent to $(p \nabla p) \nabla (q \nabla q)$?

14. Use truth tables to verify that each of the following is a tautology. Parts (a) and (b) are called *commutative laws*, parts (c) and (d) are *associative laws*, and parts (e) and (f) are *distributive laws*.

(a) $(p \wedge q) \Leftrightarrow (q \wedge p)$
(b) $(p \vee q) \Leftrightarrow (q \vee p)$
(c) $[p \wedge (q \wedge r)] \Leftrightarrow [(p \wedge q) \wedge r]$
(d) $[p \vee (q \vee r)] \Leftrightarrow [(p \vee q) \vee r]$
(e) $[p \wedge (q \vee r)] \Leftrightarrow [(p \wedge q) \vee (p \wedge r)]$
(f) $[p \vee (q \wedge r)] \Leftrightarrow [(p \vee q) \wedge (p \vee r)]$

Section 2 QUANTIFIERS

In Section 1 we found that the sentence

$$x^2 - 5x + 6 = 0$$

needed to be considered within a particular context in order to become a statement. When a sentence involves a variable such as x, it is customary to use functional notation when referring to it. Thus we write

$$p(x): x^2 - 5x + 6 = 0$$

to indicate that $p(x)$ is the sentence "$x^2 - 5x + 6 = 0$." For a specific value of x, $p(x)$ becomes a statement that is either true or false. For example, $p(2)$ is true and $p(4)$ is false.

When a variable is used in an equation or an inequality, we assume that the general context for the variable is the set of real numbers, unless we are told otherwise. Within this context we may remove the ambiguity of $p(x)$ by using a quantifier. The sentence

$$\text{For every } x,\ x^2 - 5x + 6 = 0.$$

is a statement since it is false. In symbols we write

$$\forall\, x,\ p(x),$$

where the **universal quantifier** $\forall$ is read, "For every…," "For all…," "For each…," or a similar equivalent phrase. The sentence

$$\text{There exists an } x \text{ such that } x^2 - 5x + 6 = 0.$$

is also a statement, and it is true. In symbols we write

$$\exists\, x \ni p(x),$$

where the **existential quantifier** $\exists$ is read, "There exists…," "There is at least one...," or something equivalent. The symbol $\ni$ is just a shorthand notation for the phrase "such that."

2.1 EXAMPLE The statement

There exists a number less than 7.

can be written

$$\exists\, x \ni x < 7$$

or in the abbreviated form

$$\exists\, x < 7,$$

where it is understood that x is to represent a real number. Sometimes the quantifier is not explicitly written down, as in the statement

If x is greater than 1, then x^2 is greater than 1.

The intended meaning is

$$\forall\, x,\ \text{if } x > 1, \text{ then } x^2 > 1.$$

In general, if a variable is used in the antecedent of an implication without being quantified, then the universal quantifier is assumed to apply.

2.2 PRACTICE Rewrite each statement using $\exists$, $\forall$, and $\ni$, as appropriate.

(a) There exists a positive number x such that $x^2 = 5$.
(b) For every positive number M, there is a positive number N such that $N < 1/M$.
(c) If $n \geq N$, then $|f_n(x) - f(x)| \leq 3$ for all x in A.
(d) No positive number x satisfies the equation $f(x) = 5$.

Having seen several examples of how existential and universal quantifiers are used, let us now consider how quantified statements are negated. Consider the statement

Everyone in the room is awake.

What condition must apply to the people in the room in order for the statement to be false? Must everyone be asleep? No, it is sufficient that at least one person be asleep. On the other hand, in order for the statement

Someone in the room is asleep.

to be false, it must be the case that everyone is awake. Symbolically, if

$$p(x)\text{: } x \text{ is awake,}$$

then

$$\sim[\forall\, x,\ p(x)] \Leftrightarrow [\exists\, x \ni \sim p(x)],$$

where the symbol "~" represents negation. Similarly,

$$\sim[\exists\, x \ni p(x)] \Leftrightarrow [\forall\, x,\ \sim p(x)].$$

2.3 EXAMPLE Let us look at several more quantified statements and derive their negations. Notice in part (b) that the inequality "$0 < g(y) \le 1$" is a conjunction of two inequalities "$0 < g(y)$" and "$g(y) \le 1$." Thus its negation is a disjunction. In part (c), note that the "and" between x and y is not a logical connective that needs to be negated. That is, the negation of "for all x and y in A" is "there exist x and y in A." In a complicated statement like (d), it is helpful to work through the negation one step at a time. Fortunately, (d) is about as messy as it will get.

(a) Statement: For every x in A, $f(x) > 5$.

$$\forall\, x \text{ in } A,\ f(x) > 5.$$

Negation: $\exists\, x$ in $A \ni f(x) \le 5$.

$$\text{There is an } x \text{ in } A \text{ such that } f(x) \le 5.$$

(b) Statement: There exists a positive number y such that $0 < g(y) \le 1$.

$$\exists\, y > 0 \ni 0 < g(y) \le 1.$$

Negation: $\forall\, y > 0,\ g(y) \le 0$ or $g(y) > 1$.

$$\text{For every positive number } y,\ g(y) \le 0 \text{ or } g(y) > 1.$$

(c) Statement: For all x and y in A, there exists z in B such that $x + y = z$.

$$\forall\, x \text{ and } y \text{ in } A,\ \exists\, z \text{ in } B \ni x + y = z.$$

Negation: $\exists\, x$ and y in $A \ni \forall\, z$ in B, $x + y \ne z$.

$$\text{There exist } x \text{ and } y \text{ in } A \text{ such that for all } z \text{ in } B,\ x + y \ne z.$$

(d) Statement:

$$\forall\, \varepsilon > 0\ \exists\, N \ni \forall\, n, \text{ if } n \ge N, \text{ then } \forall\, x \text{ in } S,\ |f_n(x) - f(x)| < \varepsilon.$$

Negation:

$$\exists\, \varepsilon > 0 \ni \sim[\exists\, N \ni \forall\, n, \text{ if } n \ge N, \text{ then } \forall\, x \text{ in } S,\ |f_n(x) - f(x)| < \varepsilon],$$

or

$$\exists\, \varepsilon > 0 \ni \forall\, N, \sim[\forall\, n, \text{ if } n \ge N, \text{ then } \forall\, x \text{ in } S,\ |f_n(x) - f(x)| < \varepsilon],$$

or

$$\exists\, \varepsilon > 0 \ni \forall\, N\ \exists\, n \ni \sim[\text{if } n \ge N, \text{ then } \forall\, x \text{ in } S,\ |f_n(x) - f(x)| < \varepsilon],$$

or

$$\exists\, \varepsilon > 0 \ni \forall\, N\ \exists\, n \ni n \ge N \text{ and } \sim[\forall\, x \text{ in } S,\ |f_n(x) - f(x)| < \varepsilon],$$

or

$$\exists\, \varepsilon > 0 \ni \forall\, N\ \exists\, n \ni n \ge N \text{ and } \exists\, x \text{ in } S \ni |f_n(x) - f(x)| \ge \varepsilon.$$

2.4 PRACTICE Write the negation of each statement in Practice 2.2.

It is important to realize that the order in which quantifiers are used affects the truth value. For example, when talking about real numbers, the statement

$$\forall\, x\, \exists\, y \ni y > x$$

is true. That is, given any real number x there is always a real number y that is greater than that x. But the statement

$$\exists\, y \ni \forall\, x,\ y > x$$

is false, since there is no fixed real number y that is greater than every real number. Thus care must be taken when reading (and writing) quantified statements so that the order of the quantifiers is not inadvertently changed.

2.5 PRACTICE Determine the truth value of each statement. Assume that x and y are real numbers. Justify your answers.

(a) $\forall\, x\, \exists\, y \ni\ x + y = 3$.

(b) $\exists\, x \ni \forall\, y,\ x + y \neq 3$.

Review of Key Terms in Section 2

Universal quantifier "$\forall$" Existential quantifier "$\exists$" Such that "$\ni$"

ANSWERS TO PRACTICE PROBLEMS

2.2 (a) $\exists\, x > 0 \ni x^2 = 5$.

(b) $\forall\, M > 0\, \exists\, N > 0 \ni N < 1/M$.

(c) $\forall\, n$, if $n \geq N$, then $\forall\, x$ in A, $|f_n(x) - f(x)| \leq 3$.

(d) The words "no" and "none" are universal quantifiers in a negative sense. In general, the statement "None of them are $P(x)$" is equivalent to "All of them are not $P(x)$." Thus the statement can be written as "$\forall\, x > 0,\ f(x) \neq 5$."

2.4 (a) $\forall\, x > 0, x^2 \neq 5$.

(b) $\exists\, M > 0 \ni \forall\, N > 0, N \geq 1/M$.

(c) $\exists\, n \ni n \geq N$ and $\exists\, x$ in $A \ni |f_n(x) - f(x)| > 3$.

(d) $\exists\, x > 0 \ni f(x) = 5$.

2.5 (a) The statement says, "For every x, there exists a y such that $x + y = 3$." We want to know if given any x, can we find a y that makes $x + y$ equal to 3. The answer is "yes," because the equation can be solved for y in terms of x. So the statement is true, and we can justify it by saying, “Given any x, let $y = 3 - x$.” This not only states that there is such a y, but it identifies what y is.

(b) The statement says, “There exists an x such that for all y, $x + y \neq 3$.” To show this is true, we might try to find an x that had the desired property. But this is hard to do since given any x, there is always a y that makes $x + y = 3$, namely $y = 3 - x$. So statement (b) is false. And the justification for this is the sentence “Given any x, let $y = 3 - x$.” Note that statement (b) is the negation of statement (a). So it should come as no surprise that (a) being true means (b) is false, and the justification for both conclusions is the same.

2 *EXERCISES*

*Exercises marked with * are used in later sections, and exercises marked with ☆ have hints or solutions in the back of the chapter.*

1. Mark each statement True or False. Justify each answer.

(a) The symbol “$\forall$” means “for every.”
(b) The negation of a universal statement is another universal statement.
(c) The symbol “$\ni$” is read “such that.”

2. Mark each statement True or False. Justify each answer.

(a) The symbol “$\exists$” means “there exist several.”
(b) If a variable is used in the antecedent of an implication without being quantified, then the universal quantifier is assumed to apply.
(c) The order in which quantifiers are used affects the truth value.

3. Write the negation of each statement. ☆

(a) Some pencils are red.
(b) All chairs have four legs.
(c) No one on the basketball team is over 6 feet 4 inches tall.
(d) $\exists\, x > 2 \ni f(x) = 7$.
(e) $\forall\, x$ in A, $\exists\, y > 2 \ni 0 < f(y) < f(x)$.
(f) If $x > 3$, then $\exists\, \varepsilon > 0 \ni x^2 > 9 + \varepsilon$.

4. Write the negation of each statement.

(a) Everyone likes Robert.
(b) Some students work part-time.
(c) No square matrices are triangular.
(d) $\exists\, x$ in $B \ni f(x) > k$.
(e) If $x > 5$, then $f(x) < 3$ or $f(x) > 7$.
(f) If x is in A, then $\exists\, y$ in $B \ni f(x) < f(y)$.

5. Determine the truth value of each statement, assuming x is a real number. Justify your answer. ☆

(a) $\exists\, x$ in the interval $[2,4] \ni x < 7$.
(b) $\forall\, x$ in the interval $[2,4]$, $x < 7$.
(c) $\exists\, x \ni x^2 = 5$.
(d) $\forall\, x, x^2 = 5$.
(e) $\exists\, x \ni x^2 \neq -3$.
(f) $\forall\, x, x^2 \neq -3$.
(g) $\exists\, x \ni x \div x = 1$
(h) $\forall\, x, x \div x = 1$.

6. Determine the truth value of each statement, assuming x is a real number. Justify your answer.

(a) $\exists\, x$ in the interval $[3,5] \ni x \geq 4$.
(b) $\forall\, x$ in the interval $[3,5]$, $x \geq 4$
(c) $\exists\, x \ni x^2 \neq 3$.
(d) $\forall\, x, x^2 \neq 3$.
(e) $\exists\, x \ni x^2 = -5$.
(f) $\forall\, x, x^2 = -5$.
(g) $\exists\, x \ni x - x = 0$.
(h) $\forall\, x, x - x = 0$.

7. Below are two strategies for determining the truth value of a statement involving a positive number x and another statement $P(x)$.
(i) Find some $x > 0$ such that $P(x)$ is true.
(ii) Let x be the name for any number greater than 0 and show $P(x)$ is true.

For each statement below, indicate which strategy is more appropriate.

(a) $\forall\, x > 0, P(x)$. ☆
(b) $\exists\, x > 0 \ni P(x)$. ☆
(c) $\exists\, x > 0 \ni \sim P(x)$.
(d) $\forall\, x > 0, \sim P(x)$.

8. Which of the following best identifies f as a constant function, where x and y are real numbers.

(a) $\exists\, x \ni \forall\, y,\ f(x) = y$.
(b) $\forall\, x\, \exists\, y \ni\ f(x) = y$.
(c) $\exists\, y \ni \forall\, x,\ f(x) = y$.
(d) $\forall\, y\, \exists\, x \ni\ f(x) = y$.

9. Determine the truth value of each statement, assuming that x and y are real numbers. Justify your answer. ☆

(a) $\forall\, x$ and $y, x \leq y$.
(b) $\exists\, x$ and $y \ni x \leq y$.
(c) $\forall\, x, \exists\, y \ni x \leq y$.
(d) $\exists\, x \ni \forall\, y, x \leq y$.

10. Determine the truth value of each statement, assuming that x and y are real numbers. Justify your answer.

(a) $\forall x, \exists\, y \ni xy = 0.$
(b) $\forall x, \exists\, y \ni xy = 1.$
(c) $\exists\, y \ni \forall\, x, xy = 1.$
(d) $\forall x, \exists\, y \ni xy = x.$

11. Determine the truth value of each statement, assuming that x, y, and z are real numbers. Justify your answer. ☆

(a) $\exists\, x \ni \forall\, y\, \exists\, z \ni x + y = z.$
(b) $\exists\, x$ and $y \ni \forall\, z, x + y = z.$
(c) $\forall\, x$ and $y, \exists\, z \ni y - z = x.$
(d) $\forall\, x$ and $y, \exists\, z \ni xz = y.$
(e) $\exists\, x \ni \forall\, y$ and z, $z > y$ implies that $z > x + y.$
(f) $\forall\, x, \exists\, y$ and $z \ni z > y$ implies that $z > x + y.$

12. Determine the truth value of each statement, assuming that x, y, and z are real numbers. Justify your answer.

(a) $\forall\, x$ and $y, \exists\, z \ni x + y = z.$
(b) $\forall\, x\, \exists\, y \ni \forall\, z, x + y = z.$
(c) $\exists\, x \ni \forall\, y, \exists\, z \ni xz = y.$
(d) $\forall\, x$ and $y, \exists\, z \ni yz = x.$
(e) $\forall\, x\, \exists\, y \ni \forall\, z$, $z > y$ implies that $z > x + y.$
(f) $\forall\, x$ and $y, \exists\, z \ni z > y$ implies that $z > x + y.$

Exercises 13 to 21 give certain properties of functions. You are to do two things: (a) rewrite the defining conditions in logical symbolism using $\forall$, $\exists$, $\ni$, and $\Rightarrow$, as appropriate; and (b) write the negation of part (a) using the same symbolism. It is not necessary that you understand precisely what each term means.

Example: A function f is odd if for every x, $f(-x) = -f(x)$.

(a) defining condition: $\forall\, x,\ f(-x) = -f(x)$.

(b) negation: $\exists\, x \ni\ f(-x) \neq -f(x)$.

13. A function f is *even* if for every x, $f(-x) = f(x)$. ☆

14. A function f is *periodic* if there exists a $k > 0$ such that for every x, $f(x+k) = f(x)$.

15. A function f is *increasing* if for every x and y, if $x \le y$, then $f(x) \le f(y)$. ☆

16. A function f is *strictly decreasing* if for every x and y, if $x < y$, then $f(x) > f(y)$.

17. A function $f: A \to B$ is *injective* if for every x and y in A, if $f(x) = f(y)$, then $x = y$. ☆

18. A function $f: A \to B$ is *surjective* if for every y in B there exists an x in A such that $f(x) = y$.

19. A function $f: D \to R$ is *continuous* at $c \in D$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(c)| < \varepsilon$ whenever $|x - c| < \delta$ and $x \in D$. ☆

20. A function f is *uniformly continuous on a set S* if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever x and y are in S and $|x - y| < \delta$.

21. The real number L is the *limit* of the function $f: D \to R$ at the point c if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x \in D$ and $0 < |x - c| < \delta$. ☆

22. Consider the following sentences:

(a) The nucleus of a carbon atom consists of protons and neutrons.
(b) Jesus Christ rose from the dead and is alive today.
(c) Every differentiable function is continuous.

Each of these sentences has been affirmed by some people at some time as being "true." Write an essay on the nature of truth, comparing and contrasting its meaning in these (and possibly other) contexts. You might also want to consider some of the following questions: To what extent is truth absolute? To what extent can truth change with time? To what extent is truth based on opinion? To what extent are people free to accept as true anything they wish?

Section 3 TECHNIQUES OF PROOF: I

In the first two sections we introduced some of the vocabulary of logic and mathematics. Our aim is to be able to read and write mathematics, and this requires more than just vocabulary. It also requires syntax. That is, we need to understand how statements are combined to form the mysterious mathematical entity known as a proof. Since this topic tends to be intimidating to many students, let us ease into it gently by first considering the two main types of logical reasoning: inductive reasoning and deductive reasoning.

3.1 EXAMPLE Consider the function $f(n) = n^2 + n + 17$. If we evaluate this function for various positive integers, we observe that we always seem to obtain a prime number. (Recall that a positive integer n is prime if $n > 1$ and its only positive divisors are 1 and n.) For example,

$$f(1) = 19$$
$$f(2) = 23$$
$$f(3) = 29$$
$$f(4) = 37$$
$$\vdots$$
$$f(8) = 89$$
$$\vdots$$
$$f(12) = 173$$
$$\vdots$$
$$f(15) = 257$$

and all these numbers (as well as the ones skipped over) are prime. On the basis of this experience we might conjecture that the function $f(n) = n^2 + n + 17$ will always produce a prime number when n is a positive integer. Drawing a conclusion of this sort is an example of **inductive reasoning**. On the basis of looking at individual cases we make a general conclusion.

If we let $p(n)$ be the sentence "$n^2 + n + 17$ is a prime number" and we understand that n refers to a positive integer, then we can ask, is

$$\forall\, n,\ p(n)$$

a true statement? Have we proved it is true?

It is important to realize that indeed we have *not* proved that it is true. We have shown that

$$\exists\, n \ni p(n)$$

is true. In fact, we know that $p(n)$ is true for many n. But we have not proved that it is true for *all n*. How can we come up with a proof? It turns out that we cannot, since the statement "$\forall\, n,\ p(n)$" happens to be false.

How do we know that it is false? We know that it is false because we can think of an example where $n^2 + n + 17$ is not prime. (Such an example is called a **counterexample**.) One such counterexample is $n = 17$:

$$17^2 + 17 + 17 \;=\; 17(17 + 1 + 1) \;=\; 17 \cdot 19.$$

There are others as well. For example, when $n = 16$,

$$\begin{aligned} 16^2 + 16 + 17 \;&=\; 16(16 + 1) + 17 \\ &=\; 16(17) + 17 \;=\; (16 + 1)(17) \;=\; 17^2, \end{aligned}$$

but it only takes one counterexample to prove that "$\forall\, n,\ p(n)$" is false.

On the basis of Example 3.1 we might infer that inductive reasoning is of little value. Although it is true that the conclusions drawn from inductive reasoning have not been proved logically, they can be very useful. Indeed, this type of reasoning is the basis for most if not all scientific experimentation. It is also often the source of the conjectures that when proved become the theorems of mathematics.

3.2 PRACTICE Provide counterexamples to the following statements.

(a) All birds can fly.
(b) Every continuous function is differentiable.

3.3 EXAMPLE Consider the function $g(n, m) = n^2 + n + m$, where n and m are understood to be positive integers. In Example 3.1 we saw that $g(16, 17) = 16^2 + 16 + 17 = 17^2$. We might also observe that

$$\begin{aligned} g(1,2) &= 1^2 + 1 + 2 = 4 = 2^2 \\ g(2,3) &= 2^2 + 2 + 3 = 9 = 3^2 \\ &\vdots \\ g(5,6) &= 5^2 + 5 + 6 = 36 = 6^2 \\ &\vdots \\ g(12,13) &= 12^2 + 12 + 13 = 169 = 13^2. \end{aligned}$$

On the basis of these examples (using inductive reasoning) we can form the conjecture "$\forall\, n,\, q(n)$," where $q(n)$ is the statement

$$g(n, n+1) = (n+1)^2.$$

It turns out that our conjecture this time is true, and we can prove it. Using the familiar laws of algebra, we have

$$\begin{aligned} g(n, n+1) &= n^2 + n + (n+1) && \text{[definition of } g(n, n+1)\text{]} \\ &= n^2 + 2n + 1 && \text{[since } n + n = 2n\text{]} \\ &= (n+1)(n+1) && \text{[by factoring]} \\ &= (n+1)^2 && \text{[definition of } (n+1)^2\text{].} \end{aligned}$$

Since our reasoning at each step does not depend on n being any specific integer, we conclude that "$\forall\, n,\, q(n)$" is true.

Now that we have proved the general statement "$\forall\, n,\, q(n)$," we can apply it to any particular case. Thus we know that

$$g(124, 125) = 125^2$$

without having to do any computation. This is an example of **deductive reasoning**: applying a general principle to a particular case. Most of the proofs encountered in mathematics are based on this type of reasoning.

3.4 PRACTICE In what way was deductive reasoning used in Example 3.3 to prove $\forall\, n,\, q(n)$?

The most common type of mathematical theorem can be symbolized as $p \Rightarrow q$, where p and q may be compound statements. To assert that $p \Rightarrow q$ is a theorem is to claim that $p \Rightarrow q$ is a tautology; that is, that it is always true. From Section 1 we know that $p \Rightarrow q$ is true unless p is true and q is false. Thus, to prove that p implies q, we have to show that whenever p is true it follows that q must be true. When an implication $p \Rightarrow q$ is identified as a theorem, it is customary to refer to p as the **hypothesis** and q as the **conclusion**.

The construction of a proof of the implication $p \Rightarrow q$ can be thought of as building a bridge of logical statements to connect the hypothesis p with the conclusion q. The building blocks that go into the bridge consist of four kinds of statements: (1) definitions, (2) assumptions or axioms that are accepted as true, (3) theorems that have previously been established as true, and (4) statements that are logically implied by the earlier statements in the proof. The logical equivalences discussed in Section 1 provide alternate ways to join the blocks together. When actually building the bridge, it may not be at all obvious which blocks to use or in what order to use them. This is where experience is helpful, together with perseverance, intuition, and sometimes a good bit of luck.

In building a bridge from the hypothesis p to the conclusion q, it is often useful to start at both ends and work toward the middle. That is, we might begin by asking, "What must I know in order to conclude that q is true?" Call this q_1. Then ask, "What must I know to conclude that q_1 is true?" Call this q_2. Continue this process as long as it is productive, thus obtaining a sequence of implications:

$$\cdots \Rightarrow q_2 \Rightarrow q_1 \Rightarrow q.$$

Then look at the hypothesis p and ask, "What can I conclude from p that will lead me toward q?" Call this p_1. Then ask, "What can I conclude from p_1?" Continue this process as long as it is productive, thus obtaining

$$p \Rightarrow p_1 \Rightarrow p_2 \cdots.$$

We hope that at some point the part of the bridge leaving p will join with the part that arrives at q, forming a complete span:

$$p \Rightarrow p_1 \Rightarrow p_2 \Rightarrow \cdots \Rightarrow q_2 \Rightarrow q_1 \Rightarrow q.$$

3.5 EXAMPLE Let us return to the result proved in Example 3.3 to illustrate the process just described. We begin by writing the theorem in the form $p \Rightarrow q$. One way of doing this is as follows: "If $g(n, m) = n^2 + n + m$, then $g(n, n+1) = (n+1)^2$." Symbolically, we identify the hypothesis

$$p: g(n, m) = n^2 + n + m$$

and the conclusion

$$q: g(n, n+1) = (n+1)^2.$$

In asking what statement will imply q, there are many possible answers. One simple answer is to use the definition of a square and let

$$q_1: g(n, n+1) = (n+1)(n+1).$$

By multiplying out the product $(n+1)(n+1)$, we obtain

$$q_2: g(n, n+1) = n^2 + 2n + 1.$$

Now certainly $q_2 \Rightarrow q_1 \Rightarrow q$, but it is not clear how we might back up further. Thus we turn to the hypothesis p and ask what we can conclude. Since we wish to know something about $g(n, n+1)$, the first step is to use the definition of g. That is, let

$$p_1: g(n, n+1) = n^2 + n + (n+1).$$

It is clear that $p_1 \Rightarrow q_2$, so the complete bridge is now formed:

$$p \Rightarrow p_1 \Rightarrow q_2 \Rightarrow q_1 \Rightarrow q.$$

This is essentially what was written in Example 3.3.

Associated with an implication $p \Rightarrow q$ there is a related implication $\sim q \Rightarrow \sim p$, called the **contrapositive**. Using a truth table, it is easy to see that an implication and its contrapositive are logically equivalent. Thus one way of proving an implication is to prove its contrapositive.

3.6 PRACTICE

(a) Use a truth table to verify that $p \Rightarrow q$ and $\sim q \Rightarrow \sim p$ are logically equivalent.
(b) Is $p \Rightarrow q$ logically equivalent to $q \Rightarrow p$?

3.7 EXAMPLE

The contrapositive of the theorem "If $7m$ is an odd number, then m is an odd number" is "If m is not an odd number, then $7m$ is not an odd number" or, equivalently, "If m is an even number, then $7m$ is an even number." (Recall that a number m is even if it can be written as $2k$ for some integer k. If a number is not even, then it is odd. It is to be understood here that we are talking about integers.) Using the contrapositive, we can construct a proof of the theorem as follows:

Hypothesis: m is an even number.

$m = 2k$ for some integer k	[definition]
$7m = 7(2k)$	[known property of equality]
$7m = 2(7k)$	[known property of multiplication]
$7k$ is an integer	[since k is an integer]

Conclusion: $7m$ is an even number.

[since $7m$ is 2 times the integer $7k$]

This is much easier than trying to show directly that $7m$ being odd implies that m is odd.

3.8 PRACTICE Write the contrapositive of each implication in Practice 1.5.

In Practice 3.6(b) we saw that $p \Rightarrow q$ is not logically equivalent to $q \Rightarrow p$. The implication $q \Rightarrow p$ is called the **converse** of $p \Rightarrow q$. It is possible for an implication to be false, while its converse is true. Thus we cannot prove $p \Rightarrow q$ by showing $q \Rightarrow p$.

3.9 EXAMPLE The implication "If $m^2 > 0$, then $m > 0$" is false, but its converse "If $m > 0$, then $m^2 > 0$" is true.

3.10 PRACTICE Write the converse of each implication in Practice 1.5.

Another implication that is closely related to $p \Rightarrow q$ is the **inverse** $\sim p \Rightarrow \sim q$. The inverse implication is not logically equivalent to $p \Rightarrow q$, but it is logically equivalent to the converse. In fact, the inverse is the contrapositive of the converse.

3.11 PRACTICE Use a truth table to show that the inverse and the converse of $p \Rightarrow q$ are logically equivalent.

Looking at the contrapositive form of an implication is a useful tool in proving theorems. Since it is a property of the logical structure and does not depend on the subject matter, it can be used in any setting involving an implication. There are many more tautologies that can be used in the same way. Some of the more common ones are listed in the next example.

3.12 EXAMPLE The following tautologies are useful in constructing proofs. The first two indicate, for example, that an "if and only if" theorem $p \Leftrightarrow q$ can be proved by establishing $p \Rightarrow q$ and its converse $q \Rightarrow p$ or by showing $p \Rightarrow q$ and its inverse $\sim p \Rightarrow \sim q$. The letter c is used to represent a statement that is always false. Such a statement is called a **contradiction**. While this list of tautologies need not be memorized, it will be helpful if each one is studied carefully to see just what it is saying. In the next section we illustrate the use of several of these tautologies.

(a) $(p \Leftrightarrow q) \Leftrightarrow [(p \Rightarrow q) \wedge (q \Rightarrow p)]$
(b) $(p \Leftrightarrow q) \Leftrightarrow [(p \Rightarrow q) \wedge (\sim p \Rightarrow \sim q)]$
(c) $(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$
(d) $p \vee \sim p$
(e) $(p \wedge \sim p) \Leftrightarrow c$
(f) $(\sim p \Rightarrow c) \Leftrightarrow p$
(g) $(p \Rightarrow q) \Leftrightarrow [(p \wedge \sim q) \Rightarrow c]$
(h) $[p \wedge (p \Rightarrow q)] \Rightarrow q$
(i) $[(p \Rightarrow q) \wedge \sim q] \Rightarrow \sim p$
(j) $[(p \vee q) \wedge \sim p] \Rightarrow q$
(k) $(p \wedge q) \Rightarrow p$
(l) $[(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow (p \Rightarrow r)$
(m) $[(p_1 \Rightarrow p_2) \wedge (p_2 \Rightarrow p_3) \wedge \cdots \wedge (p_{n-1} \Rightarrow p_n)] \Rightarrow (p_1 \Rightarrow p_n)$
(n) $[(p \wedge q) \Rightarrow r] \Leftrightarrow [p \Rightarrow (q \Rightarrow r)]$
(o) $[(p \Rightarrow q) \wedge (r \Rightarrow s) \wedge (p \vee r)] \Rightarrow (q \vee s)$
(p) $[p \Rightarrow (q \vee r)] \Leftrightarrow [(p \wedge \sim q) \Rightarrow r]$
(q) $[(p \vee q) \Rightarrow r] \Leftrightarrow [(p \Rightarrow r) \wedge (q \Rightarrow r)]$

Review of Key Terms in Section 3

Inductive reasoning	Hypothesis	Converse
Counterexample	Conclusion	Inverse
Deductive reasoning	Contrapositive	Contradiction

ANSWERS TO PRACTICE PROBLEMS

3.2 (a) Any flightless bird, such as an ostrich. (b) The absolute value function is continuous for all real numbers, but it is not differentiable at the origin.

3.4 The general rules about factoring polynomials were applied to the specific polynomial $n^2 + n + (n+1)$.

3.6 (a)

p	q	$(p \Rightarrow q)$	$\Leftrightarrow$	$[(\sim q)$	$\Rightarrow$	$(\sim p)]$
T	T	T	T	F	T	F
T	F	F	T	T	F	F
F	T	T	T	F	T	T
F	F	T	T	T	T	T

(b) No, $p \Rightarrow q$ is not logically equivalent to $q \Rightarrow p$. The biconditional between them is not a tautology.

p	q	$(p \Rightarrow q)$	$\Leftrightarrow$	$(q \Rightarrow p)$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	F
F	F	T	T	T

3.8 (a) If $2n$ is an odd number, then n is not an integer.
(b) If you do not have a college degree, then you cannot work here.
(c) If the car runs, then you are not out of gas.
(d) If a function is not continuous, then it is not differentiable.

3.10 (a) If $2n$ is an even number, then n is an integer.
(b) If you have a college degree, then you can work here.
(c) If the car does not run, then you are out of gas.
(d) If a function is continuous, then it is differentiable.

3.11

p	q	$(q \Rightarrow p)$	$\Leftrightarrow$	$[(\sim p)$	$\Rightarrow$	$(\sim q)]$
T	T	T	T	F	T	F
T	F	T	T	F	T	T
F	T	F	T	T	F	F
F	F	T	T	T	T	T

3 *EXERCISES*

*Exercises marked with * are used in later sections, and exercises marked with ☆ have hints or solutions in the back of the chapter.*

1. Mark each statement True or False. Justify each answer.

(a) When an implication $p \Rightarrow q$ is used as a theorem, we refer to p as the antecedent.

(b) The contrapositive of $p \Rightarrow q$ is $\sim p \Rightarrow \sim q$.

(c) The inverse of $p \Rightarrow q$ is $\sim q \Rightarrow \sim p$.
(d) To prove "$\forall\, n,\ p(n)$" is true, it takes only one example.
(e) To prove "$\exists\, n \ni p(n)$" is true, it takes only one example.

2. Mark each statement True or False. Justify each answer.

(a) When an implication $p \Rightarrow q$ is used as a theorem, we refer to q as the conclusion.
(b) A statement that is always false is called a lie.
(c) The converse of $p \Rightarrow q$ is $q \Rightarrow p$.
(d) To prove "$\forall\, n,\ p(n)$" is false, it takes only one counterexample.
(e) To prove "$\exists\, n \ni p(n)$" is false, it takes only one counterexample.

3. Write the contrapositive of each implication. ☆

(a) If all roses are red, then some violets are blue.
(b) A is not invertible if there exists a nontrivial solution to $A\mathbf{x} = \mathbf{0}$.
(c) If f is continuous and C is connected, then $f(C)$ is connected.

4. Write the converse of each implication in Exercise 3.

5. Write the inverse of each implication in Exercise 3.

6. Provide a counterexample for each statement.

(a) For every real number x, if $x^2 > 9$ then $x > 3$.
(b) For every integer n, we have $n^3 \geq n$.
(c) For all real numbers $x \geq 0$, we have $x^2 \leq x^3$.
(d) Every triangle is a right triangle.
(e) For every positive integer n, $n^2 + n + 41$ is prime.
(f) Every prime is an odd number.
(g) No integer greater than 100 is prime.
(h) $3^n + 2$ is prime for all positive integers n.
(i) For every integer $n > 3$, $3n$ is divisible by 6.
(j) If x and y are unequal positive integers and xy is a perfect square, then x and y are perfect squares.
(k) For every real number x, there exists a real number y such that $xy = 2$.
(l) The reciprocal of a real number $x \geq 1$ is a real number y such that $0 < y < 1$.
(m) No rational number satisfies the equation $x^3 + (x - 1)^2 = x^2 + 1$.
(n) No rational number satisfies the equation $x^4 + (1/x) - \sqrt{x+1} = 0$.

***7.** Suppose p and q are integers. Recall that an integer m is even iff $m = 2k$ for some integer k and m is odd iff $m = 2k + 1$ for some integer k. Prove the following. [You may use the fact that the sum of integers and the product of integers are again integers.]

(a) If p is odd and q is odd, then $p + q$ is even.
(b) If p is odd and q is odd, then pq is odd.
(c) If p is odd and q is odd, then $p + 3q$ is even.
(d) If p is odd and q is even, then $p + q$ is odd.
(e) If p is even and q is even, then $p + q$ is even.

(f) If p is even or q is even, then pq is even.
(g) If pq is odd, then p is odd and q is odd.
(h) If p^2 is even, then p is even. ☆
(i) If p^2 is odd, then p is odd.

8. Let f be the function given by $f(x) = 4x + 7$. Use the contrapositive implication to prove the statement: If $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

9. In each part, a list of hypotheses is given. These hypotheses are assumed to be true. Using tautologies from Example 3.12, you are to establish the desired conclusion. Indicate which tautology you are using to justify each step. ☆

(a) Hypotheses: $r \Rightarrow \sim s$, $t \Rightarrow s$
Conclusion: $r \Rightarrow \sim t$
(b) Hypotheses: r, $\sim t$, $(r \wedge s) \Rightarrow t$
Conclusion: $\sim s$
(c) Hypotheses: $r \Rightarrow \sim s$, $\sim r \Rightarrow \sim t$, $\sim t \Rightarrow u$, $v \Rightarrow s$
Conclusion: $\sim v \vee u$

10. Repeat Exercise 9 for the following hypotheses and conclusions.

(a) Hypotheses: $\sim r$, $(\sim r \wedge s) \Rightarrow r$
Conclusion: $\sim s$
(b) Hypotheses: $\sim t$, $(r \vee s) \Rightarrow t$
Conclusion: $\sim s$
(c) Hypotheses: $r \Rightarrow \sim s$, $t \Rightarrow u$, $s \vee t$
Conclusion: $\sim r \vee u$

11. Assume that the following two hypotheses are true: (1) If the basketball center is healthy or the point guard is hot, then the team will win and the fans will be happy; and (2) if the fans are happy or the coach is a millionaire, then the college will balance the budget. Derive the following conclusion: If the basketball center is healthy, then the college will balance the budget. Using letters to represent the simple statements, write out a formal proof in the format of Exercise 9.

Section 4 TECHNIQUES OF PROOF: II

Mathematical theorems and proofs do not occur in isolation, but always in the context of some mathematical system. For example, in Section 3 when we discussed a conjecture related to prime numbers, the natural context of that discussion was the positive integers. In Example 3.7 when talking about odd and even numbers, the context was the set of all integers. Very often a theorem will make no explicit reference to the mathematical system in which it is being proved; it must be inferred from the context. Usually, this causes no difficulty, but if there is a possibility of ambiguity, the careful writer will explicitly name the system being considered.

When dealing with quantified statements, it is particularly important to know exactly what system is being considered. For example, the statement

$$\forall\, x,\ \sqrt{x^2} = x$$

is true in the context of the positive numbers but is false when considering all real numbers. Similarly,

$$\exists\, x \ni x^2 = 25 \text{ and } x < 3$$

is false for positive numbers and true for real numbers. When you learn about set notation, it will become easier to be precise in indicating the context of a particular quantified statement. For now, we have to write it out with words.

To prove a universal statement

$$\forall\, x,\ p(x),$$

we let x represent an arbitrary member from the system under consideration and then show that statement $p(x)$ is true. The only properties that we can use about x are those that apply to all the members of the system. For example, if the system consists of the integers, we cannot use the property that x is even, since this does not apply to all the integers.

To prove an existential statement

$$\exists\, x \ni p(x),$$

we have to prove that there is at least one member x in the system for which $p(x)$ is true. The most direct way of doing this is to construct (produce, guess, etc.) a specific x that has the required property. Unfortunately, there is no surefire way to always find a particular x that will work. If the hypothesis in the theorem contains a quantified statement, this can sometimes be helpful, but often it is just a matter of working on both ends of the logical bridge until you can get them to meet in the middle.

4.1 EXAMPLE To illustrate the process of writing a proof with quantifiers, consider the following:

THEOREM: For every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$1 - \delta < x < 1 + \delta \quad \text{implies that} \quad 5 - \varepsilon < 2x + 3 < 5 + \varepsilon.$$

We are asked to prove that something is true for each positive number ε. Thus we begin by letting ε be an arbitrary positive number. We need to use this ε to find a positive δ with the property that

$$1 - \delta < x < 1 + \delta \quad \text{implies that} \quad 5 - \varepsilon < 2x + 3 < 5 + \varepsilon.$$

Let us begin with the consequent of the implication. We want to have

$$5 - \varepsilon < 2x + 3 < 5 + \varepsilon .$$

This will be true if

$$2 - \varepsilon < 2x < 2 + \varepsilon ,$$

and this in turn will follow from

$$1 - \frac{\varepsilon}{2} < x < 1 + \frac{\varepsilon}{2}.$$

Thus we see that choosing δ to be $\varepsilon/2$ will meet the required condition. In writing down the proof in a formal manner we would simply set δ equal to $\varepsilon/2$ and then show that this particular δ will work.

Proof: Given any $\varepsilon > 0$, let $\delta = \varepsilon/2$. Then δ is also positive and whenever

$$1 - \delta < x < 1 + \delta$$

we have

$$1 - \frac{\varepsilon}{2} < x < 1 + \frac{\varepsilon}{2},$$

so that

$$2 - \varepsilon < 2x < 2 + \varepsilon,$$

and

$$5 - \varepsilon < 2x + 3 < 5 + \varepsilon,$$

as required. ♦[†]

In some situations it is possible to prove an existential statement in an indirect way without actually producing any specific member of the system. One indirect method is to use the contrapositive form of the implication and another is to use a proof by contradiction.

The two basic forms of a proof by contradiction are based on tautologies (f) and (g) in Example 3.12. Tautology (f) has the form

$$(\sim p \Rightarrow c) \Leftrightarrow p.$$

If we wish to conclude a statement p, we can do so by showing that the negation of p leads to a contradiction. Tautology (g) has the form

$$(p \Rightarrow q) \Leftrightarrow [(p \wedge \sim q) \Rightarrow c].$$

If we wish to conclude that p implies q, we can do so by showing that p and not q leads to a contradiction. In either case the contradiction can involve part of the hypothesis or some other statement that is known to be true.

[†] The symbol ♦ is used here to denote the end of a formal proof.

4.2 PRACTICE Use truth tables to verify that $(\sim p \Rightarrow c) \Leftrightarrow p$ and $[(p \wedge \sim q) \Rightarrow c] \Leftrightarrow (p \Rightarrow q)$ are tautologies.

4.3 EXAMPLE To illustrate an indirect proof of an existential statement, consider the following:

THEOREM: Let f be an integrable function. If $\int_0^1 f(x)\,dx \neq 0$, then there exists a point x in the interval $[0,1]$ such that $f(x) \neq 0$.

Symbolically, we have $p \Rightarrow q$, where

$$p: \int_0^1 f(x)\,dx \neq 0$$

$$q: \exists\, x \text{ in } [0,1] \ni f(x) \neq 0.$$

The contrapositive implication, $\sim q \Rightarrow \sim p$, can be written as

$$\text{If for every } x \text{ in } [0,1],\ f(x) = 0, \text{ then } \int_0^1 f(x) = 0.$$

This is much easier to prove. Instead of having to conclude the existence of an x in $[0,1]$ with a particular property, we are given that every x in $[0,1]$ has a different property. Indeed, the proof now follows directly from the definition of the integral, since each of the terms in any upper or lower Riemann sum will be zero.

4.4 EXAMPLE To illustrate a proof by contradiction, consider the following:

THEOREM: Let x be a real number. If $x > 0$, then $1/x > 0$.

Symbolically, we have $p \Rightarrow q$, where

$$p: x > 0$$

$$q: \frac{1}{x} > 0.$$

Tautology (g) in Example 3.12 says that $p \Rightarrow q$ is equivalent to $(p \wedge \sim q) \Rightarrow c$, where c represents a contradiction. Thus we begin by supposing $x > 0$ and $1/x \leq 0$. Since $x > 0$, we can multiply both sides of the inequality $1/x \leq 0$ by x to obtain

$$(x)\left(\frac{1}{x}\right) \leq (x)(0).$$

But $(x)(1/x) = 1$ and $(x)(0) = 0$, so we have $1 \leq 0$, a contradiction to the (presumably known) fact that $1 > 0$. Having shown that $p \wedge \sim q$ leads to a contradiction, we conclude that $p \Rightarrow q$.

Another tautology in Example 3.12 that deserves special attention is statement (q):

$$[(p \vee q) \Rightarrow r] \Leftrightarrow [(p \Rightarrow r) \wedge (q \Rightarrow r)].$$

Some proofs naturally divide themselves into the consideration of two (or more) cases. For example, integers are either odd or even. Real numbers are positive, negative, or zero. It may be that different arguments are required for each case. It is tautology (q) that shows us how to combine the cases.

4.5 EXAMPLE Suppose we wish to prove that if x is a real number, then $x \le |x|$. Symbolically, we have $s \Rightarrow r$, where

$$s: \ x \text{ is a real number}$$

$$r: \ x \le |x|.$$

First, we recall the definition of absolute value:

$$|x| = \begin{cases} x, & \text{if } x \ge 0, \\ -x, & \text{if } x < 0. \end{cases}$$

Since this definition is divided into two parts, it is natural to divide our proof into two cases. Thus statement s is replaced by the equivalent disjunction $p \vee q$, where

$$p: x \ge 0 \quad \text{and} \quad q: x < 0.$$

Our theorem now is to prove $(p \vee q) \Rightarrow r$, and we do this by showing that $(p \Rightarrow r) \wedge (q \Rightarrow r)$. The actual proof could be written as follows:

> Let x be an arbitrary real number. Then $x \ge 0$ or $x < 0$. If $x \ge 0$, then by definition, $x = |x|$. On the other hand, if $x < 0$, then $-x > 0$, so that $x < 0 < -x = |x|$. Thus, in either case, we have $x \le |x|$. ♦

4.6 PRACTICE In proving a theorem that relates to factoring positive integers greater than 1, what two cases might reasonably be considered?

An alternative form of proof by cases arises when the consequent of an implication involves a disjunction. In this situation, tautology (p) of Example 3.12 is often helpful:

$$[p \Rightarrow (q \vee r)] \Leftrightarrow [(p \wedge \sim q) \Rightarrow r].$$