PEARSON NEW INTERNATIONAL EDITION

Classical Mechanics

Goldstein Safko Poole

Third Edition

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Preface to the Third Edition

The first edition of this text appeared in 1950, and it was so well received that it went through a second printing the very next year. Throughout the next three decades it maintained its position as the acknowledged standard text for the introductory Classical Mechanics course in graduate level physics curricula throughout the United States, and in many other countries around the world. Some major institutions also used it for senior level undergraduate Mechanics. Thirty years later, in 1980, a second edition appeared which was "a through-going revision of the first edition." The preface to the second edition contains the following statement: "I have tried to retain, as much as possible, the advantages of the first edition while taking into account the developments of the subject itself, its position in the curriculum, and its applications to other fields." This is the philosophy which has guided the preparation of this third edition twenty more years later.

The second edition introduced one additional chapter on Perturbation Theory, and changed the ordering of the chapter on Small Oscillations. In addition it added a significant amount of new material which increased the number of pages by about 68%. This third edition adds still one more new chapter on Nonlinear Dynamics or Chaos, but counterbalances this by reducing the amount of material in several of the other chapters, by shortening the space allocated to appendices, by considerably reducing the bibliography, and by omitting the long lists of symbols. Thus the third edition is comparable in size to the second.

In the chapter on relativity we have abandoned the complex Minkowski space in favor of the now standard real metric. Two of the authors prefer the complex metric because of its pedagogical advantages (HG) and because it fits in well with Clifford Algebra formulations of Physics (CPP), but the desire to prepare students who can easily move forward into other areas of theory such as field theory and general relativity dominated over personal preferences. Some modern notation such as 1-forms, mapping and the wedge product is introduced in this chapter.

The chapter on Chaos is a necessary addition because of the current interest in nonlinear dynamics which has begun to play a significant role in applications of classical dynamics. The majority of classical mechanics problems and applications in the real world include nonlinearities, and it is important for the student to have a grasp of the complexities involved, and of the new properties that can emerge. It is also important to realize the role of fractal dimensionality in chaos.

New sections have been added and others combined or eliminated here and there throughout the book, with the omissions to a great extent motivated by the desire not to extend the overall length beyond that of the second edition. A section

was added on the Euler and Lagrange exact solutions to the three body problem. In several places phase space plots and Lissajous figures were appended to illustrate solutions. The damped-driven pendulum was discussed as an example that explains the workings of Josephson junctions. The symplectic approach was clarified by writing out some of the matrices. The harmonic oscillator was treated with anisotropy, and also in polar coordinates. The last chapter on continua and fields was formulated in the modern notation introduced in the relativity chapter. The significances of the special unitary group in two dimensions SU(2) and the special orthogonal group in three dimensions SO(3) were presented in more up-to-date notation, and an appendix was added on groups and algebras. Special tables were introduced to clarify properties of ellipses, vectors, vector fields and 1-forms, canonical transformations, and the relationships between the spacetime and symplectic approaches.

Several of the new features and approaches in this third edition had been mentioned as possibilities in the preface to the second edition, such as properties of group theory, tensors in non-Euclidean spaces, and "new mathematics" of theoretical physics such as manifolds. The reference to "One area omitted that deserves special attention—nonlinear oscillation and associated stability questions" now constitutes the subject matter of our new Chapter 11 "Classical Chaos." We debated whether to place this new chapter after Perturbation theory where it fits more logically, or before Perturbation theory where it is more likely to be covered in class, and we chose the latter. The referees who reviewed our manuscript were evenly divided on this question.

The mathematical level of the present edition is about the same as that of the first two editions. Some of the mathematical physics, such as the discussions of hermitean and unitary matrices, was omitted because it pertains much more to quantum mechanics than it does to classical mechanics, and little used notations like dyadics were curtailed. Space devoted to power law potentials, Cayley-Klein parameters, Routh's procedure, time independent perturbation theory, and the stress-energy tensor was reduced. In some cases reference was made to the second edition for more details. The problems at the end of the chapters were divided into "derivations" and "exercises," and some new ones were added.

The authors are especially indebted to Michael A. Unseren and Forrest M. Hoffman of the Oak Ridge National laboratory for their 1993 compilation of errata in the second edition that they made available on the Internet. It is hoped that not too many new errors have slipped into this present revision. We wish to thank the students who used this text in courses with us, and made a number of useful suggestions that were incorporated into the manuscript. Professors Thomas Sayetta and the late Mike Schuette made helpful comments on the Chaos chapter, and Professors Joseph Johnson and James Knight helped to clarify our ideas on Lie Algebras. The following professors reviewed the manuscript and made many helpful suggestions for improvements: Yoram Alhassid, Yale University; Dave Ellis, University of Toledo; John Gruber, San Jose State; Thomas Handler, University of Tennessee; Daniel Hong, Lehigh University; Kara Keeter, Idaho State University; Carolyn Lee; Yannick Meurice, University of Iowa; Daniel

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A list of corrections for all printings are on the Web at ⟨http://astro.physics.sc.edu/goldstein/goldstein.html⟩. Additions to this listing may be emailed to the address given on that page.

It has indeed been an honor for two of us (CPP and JLS) to collaborate as co-authors of this third edition of such a classic book fifty years after its first appearance. We have admired this text since we first studied Classical Mechanics from the first edition in our graduate student days (CPP in 1953 and JLS in 1960), and each of us used the first and second editions in our teaching throughout the years. Professor Goldstein is to be commended for having written and later enhanced such an outstanding contribution to the classic Physics literature.

Above all we register our appreciation and acknolwedgement in the words of Psalm 19,1:

$$Οἱ\ οὐρανοι\ διηγοῦνται\ δοξαν\ Θεοῦ$$

Flushing, New York HERBERT GOLDSTEIN
Columbia, South Carolina CHARLES P. POOLE, JR.
Columbia, South Carolina JOHN L. SAFKO
July, 2002

Preface to the Second Edition

The prospect of a second edition of *Classical Mechanics*, almost thirty years after initial publication, has given rise to two nearly contradictory sets of reactions. On the one hand it is claimed that the adjective "classical" implies the field is complete, closed, far outside the mainstream of physics research. Further, the first edition has been paid the compliment of continuous use as a text since it first appeared. Why then the need for a second edition? The contrary reaction has been that a second edition is long overdue. More important than changes in the subject matter (which have been considerable) has been the revolution in the attitude towards classical mechanics in relation to other areas of science and technology. When it appeared, the first edition was part of a movement breaking with older ways of teaching physics. But what were bold new ventures in 1950 are the commonplaces of today, exhibiting to the present generation a slightly musty and old-fashioned air. Radical changes need to be made in the presentation of classical mechanics.

In preparing this second edition, I have attempted to steer a course somewhere between these two attitudes. I have tried to retain, as much as possible, the advantages of the first edition (as I perceive them) while taking some account of the developments in the subject itself, its position in the curriculum, and its applications to other fields. What has emerged is a thorough-going revision of the first edition. Hardly a page of the text has been left untouched. The changes have been of various kinds:

Errors (some egregious) that I have caught, or which have been pointed out to me, have of course been corrected. It is hoped that not too many new ones have been introduced in the revised material.

The chapter on small oscillations has been moved from its former position as the penultimate chapter and placed immediately after Chapter 5 on rigid body motion. This location seems more appropriate to the usual way mechanics courses are now being given. Some material relating to the Hamiltonian formulation has therefore had to be removed and inserted later in (the present) Chapter 8.

A new chapter on perturbation theory has been added (Chapter 11). The last chapter, on continuous systems and fields, has been greatly expanded, in keeping with the implicit promise made in the Preface to the first edition.

New sections have been added throughout the book, ranging from one in Chapter 3 on Bertrand's theorem for the central-force potentials giving rise to closed orbits, to the final section of Chapter 12 on Noether's theorem. For the most part these sections contain completely new material.

In various sections arguments and proofs have been replaced by new ones that seem simpler and more understandable, e.g., the proof of Euler's theorem in Chapter 4. Occasionally, a line of reasoning presented in the first edition has been supplemented by a different way of looking at the problem. The most important example is the introduction of the symplectic approach to canonical transformations, in parallel with the older technique of generating functions. Again, while the original convention for the Euler angles has been retained, alternate conventions, including the one common in quantum mechanics, are mentioned and detailed formulas are given in an appendix.

As part of the fruits of long experience in teaching courses based on the book, the body of exercises at the end of each chapter has been expanded by more than a factor of two and a half. The bibliography has undergone similar expansion, reflecting the appearance of many valuable texts and monographs in the years since the first edition. In deference to—but not in agreement with—the present neglect of foreign languages in graduate education in the United States, references to foreign-language books have been kept down to a minimum.

The choices of topics retained and of the new material added reflect to some degree my personal opinions and interests, and the reader might prefer a different selection. While it would require too much space (and be too boring) to discuss the motivating reasons relative to each topic, comment should be made on some general principles governing my decisions. The question of the choice of mathematical techniques to be employed is a vexing one. The first edition attempted to act as a vehicle for introducing mathematical tools of wide usefulness that might be unfamiliar to the student. In the present edition the attitude is more one of caution. It is much more likely now than it was 30 years ago that the student will come to mechanics with a thorough background in matrix manipulation. The section on matrix properties in Chapter 4 has nonetheless been retained, and even expanded, so as to provide a convenient reference of needed formulas and techniques. The cognoscenti can, if they wish, simply skip the section. On the other hand, very little in the way of newer mathematical tools has been introduced. Elementary properties of group theory are given scattered mention throughout the book. Brief attention is paid in Chapters 6 and 7 to the manipulation of tensors in non-Euclidean spaces. Otherwise, the mathematical level in this edition is pretty much the same as in the first. It is more than adequate for the physics content of the book, and alternate means exist in the curriculum for acquiring the mathematics needed in other branches of physics. In particular the "new mathematics" of theoretical physics has been deliberately excluded. No mention is made of manifolds or diffeomorphisms, of tangent fibre bundles or invariant tori. There are certain highly specialized areas of classical mechanics where the powerful tools of global analysis and differential topology are useful, probably essential. However, it is not clear to me that they contribute to the understanding of the physics of classical mechanics at the level sought in this edition. To introduce these mathematical concepts, and their applications, would swell the book beyond bursting, and serve, probably, only to obscure the physics. Theoretical physics, current trends to the contrary, is not merely mathematics.

In line with this attitude, the complex Minkowski space has been retained for most of the discussion of special relativity in order to simplify the mathematics. The bases for this decision (which it is realized goes against the present fashion) are given in detail on pages 292–293.

It is certainly true that classical mechanics today is far from being a closed subject. The last three decades have seen an efflorescence of new developments in classical mechanics, the tackling of new problems, and the application of the techniques of classical mechanics to far-flung reaches of physics and chemistry. It would clearly not be possible to include discussions of all of these developments here. The reasons are varied. Space limitations are obviously important. Also, popular fads of current research often prove ephemeral and have a short lifetime. And some applications require too extensive a background in other fields, such as solid-state physics or physical chemistry. The selection made here represents something of a personal compromise. Applications that allow simple descriptions and provide new insights are included in some detail. Others are only briefly mentioned, with enough references to enable the student to follow up his awakened curiosity. In some instances I have tried to describe the current state of research in a field almost entirely in words, without mathematics, to provide the student with an overall view to guide further exploration. One area omitted deserves special mention—nonlinear oscillation and associated stability questions. The importance of the field is unquestioned, but it was felt that an adequate treatment deserves a book to itself.

With all the restrictions and careful selection, the book has grown to a size probably too large to be covered in a single course. A number of sections have been written so that they may be omitted without affecting later developments and have been so marked. It was felt however that there was little need to mark special "tracks" through the book. Individual instructors, familiar with their own special needs, are better equipped to pick and choose what they feel should be included in the courses they give.

I am grateful to many individuals who have contributed to my education in classical mechanics over the past thirty years. To my colleagues Professors Frank L. DiMaggio, Richard W. Longman, and Dean Peter W. Likins I am indebted for many valuable comments and discussions. My thanks go to Sir Edward Bullard for correcting a serious error in the first edition, especially for the gentle and gracious way he did so. Professor Boris Garfinkel of Yale University very kindly read and commented on several of the chapters and did his best to initiate me into the mysteries of celestial mechanics. Over the years I have been the grateful recipient of valuable corrections and suggestions from many friends and strangers, among whom particular mention should be made of Drs. Eric Ericsen (of Oslo University), K. Kalikstein, J. Neuberger, A. Radkowsky, and Mr. W. S. Pajes. Their contributions have certainly enriched the book, but of course I alone am responsible for errors and misinterpretations. I should like to add a collective acknowledgment and thanks to the authors of papers on classical mechanics that have appeared during the last three decades in the *American Journal of Physics*, whose pages I hope I have perused with profit.

The staff at Addison-Wesley have been uniformly helpful and encouraging. I want especially to thank Mrs. Laura R. Finney for her patience with what must have seemed a never-ending process, and Mrs. Marion Howe for her gentle but persistent cooperation in the fight to achieve an acceptable printed page.

To my father, Harry Goldstein ז"ל, I owe more than words can describe for his lifelong devotion and guidance. But I wish at least now to do what he would not permit in his lifetime—to acknowledge the assistance of his incisive criticism and careful editing in the preparation of the first edition. I can only hope that the present edition still reflects something of his insistence on lucid and concise writing.

I wish to dedicate this edition to those I treasure above all else on this earth, and who have given meaning to my life—to my wife, Channa, and our children, Penina Perl, Aaron Meir, and Shoshanna.

And above all I want to register the thanks and acknowledgment of my heart, in the words of Daniel (2:23):

לך אלה אבהתי מהודא ומשבח אנה
די חכמתא וגבורתא יהבת לי

Kew Gardens Hills, New York
January 1980

HERBERT GOLDSTEIN

Preface to the First Edition

An advanced course in classical mechanics has long been a time-honored part of the graduate physics curriculum. The present-day function of such a course, however, might well be questioned. It introduces no new physical concepts to the graduate student. It does not lead him directly into current physics research. Nor does it aid him, to any appreciable extent, in solving the practical mechanics problems he encounters in the laboratory.

Despite this arraignment, classical mechanics remains an indispensable part of the physicist's education. It has a twofold role in preparing the student for the study of modern physics. First, classical mechanics, in one or another of its advanced formulations, serves as the springboard for the various branches of modern physics. Thus, the technique of action-angle variables is needed for the older quantum mechanics, the Hamilton-Jacobi equation and the principle of least action provide the transition to wave mechanics, while Poisson brackets and canonical transformations are invaluable in formulating the newer quantum mechanics. Secondly, classical mechanics affords the student an opportunity to master many of the mathematical techniques necessary for quantum mechanics while still working in terms of the familiar concepts of classical physics.

Of course, with these objectives in mind, the traditional treatment of the subject, which was in large measure fixed some fifty years ago, is no longer adequate. The present book is an attempt at an exposition of classical mechanics which does fulfill the new requirements. Those formulations which are of importance for modern physics have received emphasis, and mathematical techniques usually associated with quantum mechanics have been introduced wherever they result in increased elegance and compactness. For example, the discussion of central force motion has been broadened to include the kinematics of scattering and the classical solution of scattering problems. Considerable space has been devoted to canonical transformations, Poisson bracket formulations, Hamilton-Jacobi theory, and action-angle variables. An introduction has been provided to the variational principle formulation of continuous systems and fields. As an illustration of the application of new mathematical techniques, rigid body rotations are treated from the standpoint of matrix transformations. The familiar Euler's theorem on the motion of a rigid body can then be presented in terms of the eigenvalue problem for an orthogonal matrix. As a consequence, such diverse topics as the inertia tensor, Lorentz transformations in Minkowski space, and resonant frequencies of small oscillations become capable of a unified mathematical treatment. Also, by this

technique it becomes possible to include at an early stage the difficult concepts of reflection operations and pseudotensor quantities, so important in modern quantum mechanics. A further advantage of matrix methods is that "spinors" can be introduced in connection with the properties of Cayley-Klein parameters.

Several additional departures have been unhesitatingly made. All too often, special relativity receives no connected development except as part of a highly specialized course which also covers general relativity. However, its vital importance in modern physics requires that the student be exposed to special relativity at an early stage in his education. Accordingly, Chapter 6 has been devoted to the subject. Another innovation has been the inclusion of velocity-dependent forces. Historically, classical mechanics developed with the emphasis on static forces dependent on position only, such as gravitational forces. On the other hand, the velocity-dependent electromagnetic force is constantly encountered in modern physics. To enable the student to handle such forces as early as possible, velocity-dependent potentials have been included in the structure of mechanics from the outset, and have been consistently developed throughout the text.

Still another new element has been the treatment of the mechanics of continuous systems and fields in Chapter 11, and some comment on the choice of material is in order. Strictly interpreted, the subject could include all of elasticity, hydrodynamics, and acoustics, but these topics lie outside the prescribed scope of the book, and adequate treatises have been written for most of them. In contrast, no connected account is available on the classical foundations of the variational principle formulation of continuous systems, despite its growing importance in the field theory of elementary particles. The theory of fields can be carried to considerable length and complexity before it is necessary to introduce quantization. For example, it is perfectly feasible to discuss the stress-energy tensor, microscopic equations of continuity, momentum space representations, etc., entirely within the domain of classical physics. It was felt, however, that an adequate discussion of these subjects would require a sophistication beyond what could naturally be expected of the student. Hence it was decided, for this edition at least, to limit Chapter 11 to an elementary description of the Lagrangian and Hamiltonian formulation of fields.

The course for which this text is designed normally carries with it a prerequisite of an intermediate course in mechanics. For both the inadequately prepared graduate student (an all too frequent occurrence) and the ambitious senior who desires to omit the intermediate step, an effort was made to keep the book self-contained. Much of Chapters 1 and 3 is therefore devoted to material usually covered in the preliminary courses.

With few exceptions, no more mathematical background is required of the student than the customary undergraduate courses in advanced calculus and vector analysis. Hence considerable space is given to developing the more complicated mathematical tools as they are needed. An elementary acquaintance with Maxwell's equations and their simpler consequences is necessary for understanding the sections on electromagnetic forces. Most entering graduate students have had at least one term's exposure to modern physics, and frequent advantage has

been taken of this circumstance to indicate briefly the relation between a classical development and its quantum continuation.

A large store of exercises is available in the literature on mechanics, easily accessible to all, and there consequently seemed little point to reproducing an extensive collection of such problems. The exercises appended to each chapter therefore have been limited, in the main, to those which serve as extensions of the text, illustrating some particular point or proving variant theorems. Pedantic museum pieces have been studiously avoided.

The question of notation is always a vexing one. It is impossible to achieve a completely consistent and unambiguous system of notation that is not at the same time impracticable and cumbersome. The customary convention has been followed by indicating vectors by bold face Roman letters. In addition, matrix quantities of whatever rank, and tensors other than vectors, are designated by bold face sans serif characters, thus: **A**. An index of symbols is appended at the end of the book, listing the initial appearance of each meaning of the important symbols. Minor characters, appearing only once, are not included.

References have been listed at the end of each chapter, for elaboration of the material discussed or for treatment of points not touched on. The evaluations accompanying these references are purely personal, of course, but it was felt necessary to provide the student with some guide to the bewildering maze of literature on mechanics. These references, along with many more, are also listed at the end of the book. The list is not intended to be in any way complete, many of the older books being deliberately omitted. By and large, the list contains the references used in writing this book, and must therefore serve also as an acknowledgement of my debt to these sources.

The present text has evolved from a course of lectures on classical mechanics that I gave at Harvard University, and I am grateful to Professor J. H. Van Vleck, then Chairman of the Physics Department, for many personal and official encouragements. To Professor J. Schwinger, and other colleagues I am indebted for many valuable suggestions. I also wish to record my deep gratitude to the students in my courses, whose favorable reaction and active interest provided the continuing impetus for this work.

תושלב"ע

Cambridge, Mass.
March 1950

HERBERT GOLDSTEIN

CHAPTER 1

Survey of the Elementary Principles

The motion of material bodies formed the subject of some of the earliest research pursued by the pioneers of physics. From their efforts there has evolved a vast field known as analytical mechanics or dynamics, or simply, mechanics. In the present century the term "classical mechanics" has come into wide use to denote this branch of physics in contradistinction to the newer physical theories, especially quantum mechanics. We shall follow this usage, interpreting the name to include the type of mechanics arising out of the special theory of relativity. It is the purpose of this book to develop the structure of classical mechanics and to outline some of its applications of present-day interest in pure physics. Basic to any presentation of mechanics are a number of fundamental physical concepts, such as space, time, simultaneity, mass, and force. For the most part, however, these concepts will not be analyzed critically here; rather, they will be assumed as undefined terms whose meanings are familiar to the reader.

1.1 ■ MECHANICS OF A PARTICLE

Let $\mathbf{r}$ be the radius vector of a particle from some given origin and $\mathbf{v}$ its vector velocity:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}. \tag{1.1}$$

The *linear momentum* $\mathbf{p}$ of the particle is defined as the product of the particle mass and its velocity:

$$\mathbf{p} = m\mathbf{v}. \tag{1.2}$$

In consequence of interactions with external objects and fields, the particle may experience forces of various types, e.g., gravitational or electrodynamic; the vector sum of these forces exerted on the particle is the total force $\mathbf{F}$. The mechanics of the particle is contained in *Newton's second law of motion*, which states that there exist frames of reference in which the motion of the particle is described by the differential equation

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \equiv \dot{\mathbf{p}}, \tag{1.3}$$

or

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}). \tag{1.4}$$

In most instances, the mass of the particle is constant and Eq. (1.4) reduces to

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} = m\mathbf{a}, \tag{1.5}$$

where $\mathbf{a}$ is the vector acceleration of the particle defined by

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2}. \tag{1.6}$$

The equation of motion is thus a differential equation of second order, assuming $\mathbf{F}$ does not depend on higher-order derivatives.

A reference frame in which Eq. (1.3) is valid is called an *inertial* or *Galilean system*. Even within classical mechanics the notion of an inertial system is something of an idealization. In practice, however, it is usually feasible to set up a coordinate system that comes as close to the desired properties as may be required. For many purposes, a reference frame fixed in Earth (the "laboratory system") is a sufficient approximation to an inertial system, while for some astronomical purposes it may be necessary to construct an inertial system (or inertial frame) by reference to distant galaxies.

Many of the important conclusions of mechanics can be expressed in the form of conservation theorems, which indicate under what conditions various mechanical quantities are constant in time. Equation (1.3) directly furnishes the first of these, the

Conservation Theorem for the Linear Momentum of a Particle: If the total force, $\mathbf{F}$*, is zero, then* $\dot{\mathbf{p}} = 0$ *and the linear momentum,* $\mathbf{p}$*, is conserved.*

The angular momentum of the particle about point O, denoted by $\mathbf{L}$, is defined as

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}, \tag{1.7}$$

where $\mathbf{r}$ is the radius vector from O to the particle. Notice that the order of the factors is important. We now define the *moment of force* or *torque* about O as

$$\mathbf{N} = \mathbf{r} \times \mathbf{F}. \tag{1.8}$$

The equation analogous to (1.3) for $\mathbf{N}$ is obtained by forming the cross product of $\mathbf{r}$ with Eq. (1.4):

$$\mathbf{r} \times \mathbf{F} = \mathbf{N} = \mathbf{r} \times \frac{d}{dt}(m\mathbf{v}). \tag{1.9}$$

Equation (1.9) can be written in a different form by using the vector identity:

$$\frac{d}{dt}(\mathbf{r} \times m\mathbf{v}) = \mathbf{v} \times m\mathbf{v} + \mathbf{r} \times \frac{d}{dt}(m\mathbf{v}), \tag{1.10}$$

where the first term on the right obviously vanishes. In consequence of this identity, Eq. (1.9) takes the form

$$\mathbf{N} = \frac{d}{dt}(\mathbf{r} \times m\mathbf{v}) = \frac{d\mathbf{L}}{dt} \equiv \dot{\mathbf{L}}. \tag{1.11}$$

Note that both $\mathbf{N}$ and $\mathbf{L}$ depend on the point O, about which the moments are taken.

As was the case for Eq. (1.3), the torque equation, (1.11), also yields an immediate conservation theorem, this time the

Conservation Theorem for the Angular Momentum of a Particle: If the total torque, $\mathbf{N}$, *is zero then* $\dot{\mathbf{L}} = 0$, *and the angular momentum* $\mathbf{L}$ *is conserved.*

Next consider the work done by the external force $\mathbf{F}$ upon the particle in going from point 1 to point 2. By definition, this work is

$$W_{12} = \int_1^2 \mathbf{F} \cdot d\mathbf{s}. \tag{1.12}$$

For constant mass (as will be assumed from now on unless otherwise specified), the integral in Eq. (1.12) reduces to

$$\int \mathbf{F} \cdot d\mathbf{s} = m \int \frac{d\mathbf{v}}{dt} \cdot \mathbf{v}\, dt = \frac{m}{2} \int \frac{d}{dt}(v^2)\, dt,$$

and therefore

$$W_{12} = \frac{m}{2}(v_2^2 - v_1^2). \tag{1.13}$$

The scalar quantity $mv^2/2$ is called the kinetic energy of the particle and is denoted by T, so that the work done is equal to the change in the kinetic energy:

$$W_{12} = T_2 - T_1. \tag{1.14}$$

If the force field is such that the work W_{12} is the same for any physically possible path between points 1 and 2, then the force (and the system) is said to be *conservative*. An alternative description of a conservative system is obtained by imagining the particle being taken from point 1 to point 2 by one possible path and then being returned to point 1 by another path. The independence of W_{12} on the particular path implies that the work done around such a closed circuit is zero, i.e.:

$$\oint \mathbf{F} \cdot d\mathbf{s} = 0. \tag{1.15}$$

Physically it is clear that a system cannot be conservative if friction or other dissipation forces are present, because $F \cdot ds$ due to friction is always positive and the integral cannot vanish.

By a well-known theorem of vector analysis, a necessary and sufficient condition that the work, W_{12}, be independent of the physical path taken by the particle is that $\mathbf{F}$ be the gradient of some scalar function of position:

$$\mathbf{F} = -\nabla V(\mathbf{r}), \tag{1.16}$$

where V is called the *potential*, or *potential energy*. The existence of V can be inferred intuitively by a simple argument. If W_{12} is independent of the path of integration between the end points 1 and 2, it should be possible to express W_{12} as the change in a quantity that depends only upon the positions of the end points. This quantity may be designated by $-V$, so that for a differential path length we have the relation

$$\mathbf{F} \cdot d\mathbf{s} = -dV$$

or

$$F_s = -\frac{\partial V}{\partial s},$$

which is equivalent to Eq. (1.16). Note that in Eq. (1.16) we can add to V any quantity constant in space, without affecting the results. Hence *the zero level of V is arbitrary.*

For a conservative system, the work done by the forces is

$$W_{12} = V_1 - V_2. \tag{1.17}$$

Combining Eq. (1.17) with Eq. (1.14), we have the result

$$T_1 + V_1 = T_2 + V_2, \tag{1.18}$$

which states in symbols the

Energy Conservation Theorem for a Particle: If the forces acting on a particle are conservative, then the total energy of the particle, $T + V$, is conserved.

The force applied to a particle may in some circumstances be given by the gradient of a scalar function that depends explicitly on both the position of the particle and the time. However, the work done on the particle when it travels a distance ds,

$$\mathbf{F} \cdot d\mathbf{s} = -\frac{\partial V}{\partial s}\, ds,$$

is then no longer the total change in $-V$ during the displacement, since V also changes explicitly with time as the particle moves. Hence, the work done as the

particle goes from point 1 to point 2 is no longer the difference in the function V between those points. While a total energy $T + V$ may still be defined, it is not conserved during the course of the particle's motion.

1.2 ■ MECHANICS OF A SYSTEM OF PARTICLES

In generalizing the ideas of the previous section to systems of many particles, we must distinguish between the *external forces* acting on the particles due to sources outside the system, and *internal forces* on, say, some particle i due to all other particles in the system. Thus, the equation of motion (Newton's second law) for the ith particle is written as

$$\sum_j \mathbf{F}_{ji} + \mathbf{F}_i^{(e)} = \dot{\mathbf{p}}_i, \tag{1.19}$$

where $\mathbf{F}_i^{(e)}$ stands for an external force, and $\mathbf{F}_{ji}$ is the internal force on the ith particle due to the jth particle ($\mathbf{F}_{ii}$, naturally, is zero). We shall assume that the $\mathbf{F}_{ij}$ (like the $\mathbf{F}_i^{(e)}$) obey Newton's third law of motion in its original form: that the forces two particles exert on each other are equal and opposite. This assumption (which does not hold for all types of forces) is sometimes referred to as the *weak law of action and reaction.*

Summed over all particles, Eq. (1.19) takes the form

$$\frac{d^2}{dt^2}\sum_i m_i \mathbf{r}_i = \sum_i \mathbf{F}_i^{(e)} + \sum_{\substack{i,j \\ i \neq j}} \mathbf{F}_{ji}. \tag{1.20}$$

The first sum on the right is simply the total external force $\mathbf{F}^{(e)}$, while the second term vanishes, since the law of action and reaction states that each pair $\mathbf{F}_{ij} + \mathbf{F}_{ji}$ is zero. To reduce the left-hand side, we define a vector $\mathbf{R}$ as the average of the radii vectors of the particles, weighted in proportion to their mass:

$$\mathbf{R} = \frac{\sum m_i \mathbf{r}_i}{\sum m_i} = \frac{\sum m_i \mathbf{r}_i}{M}. \tag{1.21}$$

The vector $\mathbf{R}$ defines a point known as the *center of mass*, or more loosely as the center of gravity, of the system (cf. Fig. 1.1). With this definition, (1.20) reduces to

$$M\frac{d^2\mathbf{R}}{dt^2} = \sum_i \mathbf{F}_i^{(e)} \equiv \mathbf{F}^{(e)}, \tag{1.22}$$

which states that the center of mass moves as if the total external force were acting on the entire mass of the system concentrated at the center of mass. Purely internal forces, if the obey Newton's third law, therefore have no effect on the

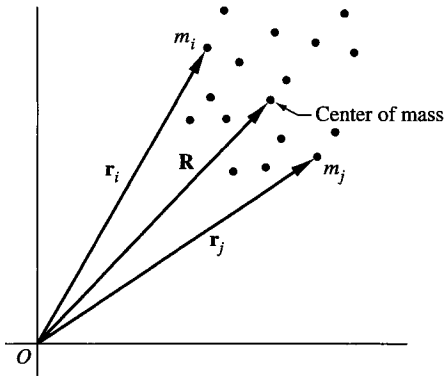

FIGURE 1.1 The center of mass of a system of particles.

motion of the center of mass. An oft-quoted example is the motion of an exploding shell—the center of mass of the fragments traveling as if the shell were still in a single piece (neglecting air resistance). The same principle is involved in jet and rocket propulsion. In order that the motion of the center of mass be unaffected, the ejection of the exhaust gases at high velocity must be counterbalanced by the forward motion of the vehicle at a slower velocity.

By Eq. (1.21) the total linear momentum of the system,

$$\mathbf{P} = \sum m_i \frac{d\mathbf{r}_i}{dt} = M\frac{d\mathbf{R}}{dt}, \tag{1.23}$$

is the total mass of the system times the velocity of the center of mass. Consequently, the equation of motion for the center of mass, (1.23), can be restated as the

Conservation Theorem for the Linear Momentum of a System of Particles: If the total external force is zero, the total linear momentum is conserved.

We obtain the total angular momentum of the system by forming the cross product $\mathbf{r}_i \times \mathbf{p}_i$ and summing over i. If this operation is performed in Eq. (1.19), there results, with the aid of the identity, Eq. (1.10),

$$\sum_i (\mathbf{r}_i \times \dot{\mathbf{p}}_i) = \sum_i \frac{d}{dt}(\mathbf{r}_i \times \mathbf{p}_i) = \dot{\mathbf{L}} = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{(e)} + \sum_{\substack{i,j \\ i \neq j}} \mathbf{r}_i \times \mathbf{F}_{ji}. \tag{1.24}$$

The last term on the right in (1.24) can be considered a sum of the pairs of the form

$$\mathbf{r}_i \times \mathbf{F}_{ji} + \mathbf{r}_j \times \mathbf{F}_{ij} = (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ji}, \tag{1.25}$$

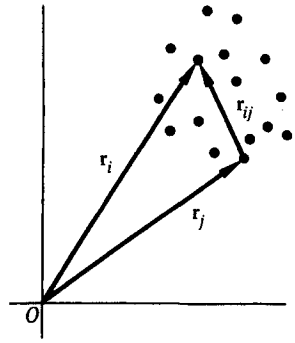

FIGURE 1.2 The vector $\mathbf{r}_{ij}$ between the ith and jth particles.

using the equality of action and reaction. But $\mathbf{r}_i - \mathbf{r}_j$ is identical with the vector $\mathbf{r}_{ij}$ from j to i (cf. Fig. 1.2), so that the right-hand side of Eq. (1.25) can be written as

$$\mathbf{r}_{ij} \times \mathbf{F}_{ji}.$$

If the internal forces between two particles, in addition to being equal and opposite, also lie along the line joining the particles—a condition known as the *strong law of action and reaction*—then all of these cross products vanish. The sum over pairs is zero under this assumption and Eq. (1.24) may be written in the form

$$\frac{d\mathbf{L}}{dt} = \mathbf{N}^{(e)}. \tag{1.26}$$

The time derivative of the total angular momentum is thus equal to the moment of the external force about the given point. Corresponding to Eq. (1.26) is the

Conservation Theorem for Total Angular Momentum: $\mathbf{L}$ *is constant in time if the applied (external) torque is zero.*

(It is perhaps worthwhile to emphasize that this is a *vector* theorem; i.e., L_z will be conserved if $N_z^{(e)}$ is zero, even if $N_x^{(e)}$ and $N_y^{(e)}$ are not zero.)

Note that the conservation of linear momentum in the absence of applied forces assumes that the weak law of action and reaction is valid for the internal forces. The conservation of the total angular momentum of the system in the absence of applied torques requires the validity of the strong law of action and reaction—that the internal forces in addition be *central*. Many of the familiar physical forces, such as that of gravity, satisfy the strong form of the law. But it is possible to find forces for which action and reaction are equal even though the forces are not central (see below). In a system involving moving charges, the forces between the charges predicted by the Biot-Savart law may indeed violate both forms of

the action and reaction law.* Equations (1.23) and (1.26), and their corresponding conservation theorems, are not applicable in such cases, at least in the form given here. Usually it is then possible to find some generalization of **P** or **L** that is conserved. Thus, in an isolated system of moving charges it is the sum of the mechanical angular momentum and the electromagnetic "angular momentum" of the field that is conserved.

Equation (1.23) states that the total linear momentum of the system is the same as if the entire mass were concentrated at the center of mass and moving with it. The analogous theorem for angular momentum is more complicated. With the origin O as reference point, the total angular momentum of the system is

$$\mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{p}_i .$$

Let $\mathbf{R}$ be the radius vector from O to the center of mass, and let $\mathbf{r}'_i$ be the radius vector from the center of mass to the ith particle. Then we have (cf. Fig. 1.3)

$$\mathbf{r}_i = \mathbf{r}'_i + \mathbf{R} \tag{1.27}$$

and

$$\mathbf{v}_i = \mathbf{v}'_i + \mathbf{v}$$

where

$$\mathbf{v} = \frac{d\mathbf{R}}{dt}$$

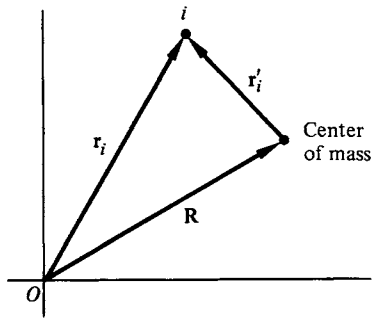

FIGURE 1.3 The vectors involved in the shift of reference point for the angular momentum.

*If two charges are moving uniformly with parallel velocity vectors that are not perpendicular to the line joining the charges, then the net mutual forces are equal and opposite but do not lie along the vector between the charges. Consider, further, two charges moving (instantaneously) so as to "cross the T," i.e., one charge moving directly at the other, which in turn is moving at right angles to the first. Then the second charge exerts a nonvanishing magnetic force on the first, without experiencing any magnetic reaction force at that instant.

is the velocity of the center of mass relative to O, and

$$\mathbf{v}_i' = \frac{d\mathbf{r}_i'}{dt}$$

is the velocity of the ith particle relative to the center of mass of the system. Using Eq. (1.27), the total angular momentum takes on the form

$$\mathbf{L} = \sum_i \mathbf{R} \times m_i \mathbf{v} + \sum_i \mathbf{r}_i' \times m_i \mathbf{v}_i' + \left(\sum_i m_i \mathbf{r}_i'\right) \times \mathbf{v} + \mathbf{R} \times \frac{d}{dt} \sum_i m_i \mathbf{r}_i'.$$

The last two terms in this expression vanish, for both contain the factor $\sum m_i \mathbf{r}_i'$, which, it will be recognized, defines the radius vector of the center of mass in the very coordinate system whose origin is the center of mass and is therefore a null vector. Rewriting the remaining terms, the total angular momentum about O is

$$\mathbf{L} = \mathbf{R} \times M\mathbf{v} + \sum_i \mathbf{r}_i' \times \mathbf{p}_i'. \tag{1.28}$$

In words, Eq. (1.28) says that the total angular momentum about a point O is the angular momentum of motion concentrated at the center of mass, plus the angular momentum of motion about the center of mass. The form of Eq. (1.28) emphasizes that in general $\mathbf{L}$ depends on the origin O, through the vector $\mathbf{R}$. Only if the center of mass is at rest with respect to O will the angular momentum be independent of the point of reference. In this case, the first term in (1.28) vanishes, and $\mathbf{L}$ always reduces to the angular momentum taken about the center of mass.

Finally, let us consider the energy equation. As in the case of a single particle, we calculate the work done by all forces in moving the system from an initial configuration 1, to a final configuration 2:

$$W_{12} = \sum_i \int_1^2 \mathbf{F}_i \cdot d\mathbf{s}_i = \sum_i \int_1^2 \mathbf{F}_i^{(e)} \cdot d\mathbf{s}_i + \sum_{\substack{i,j \\ i \neq j}} \int_1^2 \mathbf{F}_{ji} \cdot d\mathbf{s}_i. \tag{1.29}$$

Again, the equations of motion can be used to reduce the integrals to

$$\sum_i \int_1^2 \mathbf{F}_i \cdot d\mathbf{s}_i = \sum_i \int_1^2 m_i \dot{\mathbf{v}}_i \cdot \mathbf{v}_i dt = \sum_i \int_1^2 d\left(\frac{1}{2} m_i v_i^2\right).$$

Hence, the work done can still be written as the difference of the final and initial kinetic energies:

$$W_{12} = T_2 - T_1,$$

where T, the total kinetic energy of the system, is

$$T = \frac{1}{2} \sum_i m_i v_i^2. \tag{1.30}$$

Making use of the transformations to center-of-mass coordinates, given in Eq. (1.27), we may also write T as

$$T = \frac{1}{2}\sum_i m_i(\mathbf{v} + \mathbf{v}_i') \cdot (\mathbf{v} + \mathbf{v}_i')$$

$$= \frac{1}{2}\sum_i m_i v^2 + \frac{1}{2}\sum_i m_i v_i'^2 + \mathbf{v} \cdot \frac{d}{dt}\left(\sum_i m_i \mathbf{r}_i'\right),$$

and by the reasoning already employed in calculating the angular momentum, the last term vanishes, leaving

$$T = \frac{1}{2}Mv^2 + \frac{1}{2}\sum_i m_i v_i'^2 \tag{1.31}$$

The kinetic energy, like the angular momentum, thus also consists of two parts: the kinetic energy obtained if all the mass were concentrated at the center of mass, plus the kinetic energy of motion about the center of mass.

Consider now the right-hand side of Eq. (1.29). In the special case that the external forces are derivable in terms of the gradient of a potential, the first term can be written as

$$\sum_i \int_1^2 \mathbf{F}_i^{(e)} \cdot d\mathbf{s}_i = -\sum_i \int_1^2 \nabla_i V_i \cdot d\mathbf{s}_i = -\sum_i V_i \Big|_1^2,$$

where the subscript i on the del operator indicates that the derivatives are with respect to the components of $\mathbf{r}_i$. If the internal forces are also conservative, then the mutual forces between the ith and jth particles, $\mathbf{F}_{ij}$ and $\mathbf{F}_{ji}$, can be obtained from a potential function V_{ij}. To satisfy the strong law of action and reaction, V_{ij} can be a function only of the distance between the particles:

$$V_{ij} = V_{ij}(|\,\mathbf{r}_i - \mathbf{r}_j\,|). \tag{1.32}$$

The two forces are then automatically equal and opposite,

$$\mathbf{F}_{ji} = -\nabla_i V_{ij} = +\nabla_j V_{ij} = -\mathbf{F}_{ij}, \tag{1.33}$$

and lie along the line joining the two particles,

$$\nabla V_{ij}(|\,\mathbf{r}_i - \mathbf{r}_j\,|) = (\mathbf{r}_i - \mathbf{r}_j) f, \tag{1.34}$$

where f is some scalar function. If V_{ij} were also a function of the difference of some other pair of vectors associated with the particles, such as their velocities or (to step into the domain of modern physics) their intrinsic "spin" angular momenta, then the forces would still be equal and opposite, but would not necessarily lie along the direction between the particles.

When the forces are all conservative, the second term in Eq. (1.29) can be rewritten as a sum over *pairs* of particles, the terms for each pair being of the form

$$-\int_1^2 (\nabla_i V_{ij} \cdot d\mathbf{s}_i + \nabla_j V_{ij} \cdot d\mathbf{s}_j).$$

If the difference vector $\mathbf{r}_i - \mathbf{r}_j$ is denoted by $\mathbf{r}_{ij}$, and if ∇_{ij} stands for the gradient with respect to $\mathbf{r}_{ij}$, then

$$\nabla_i V_{ij} = \nabla_{ij} V_{ij} = -\nabla_j V_{ij},$$

and

$$d\mathbf{s}_i - d\mathbf{s}_j = d\mathbf{r}_i - d\mathbf{r}_j = d\mathbf{r}_{ij},$$

so that the term for the ij pair has the form

$$-\int \nabla_{ij} V_{ij} \cdot d\mathbf{r}_{ij}.$$

The total work arising from internal forces then reduces to

$$-\frac{1}{2}\sum_{\substack{i,j \\ i\neq j}} \int_1^2 \nabla_{ij} V_{ij} \cdot d\mathbf{r}_{ij} = -\frac{1}{2}\sum_{\substack{i,j \\ i\neq j}} V_{ij}\Big|_1^2. \tag{1.35}$$

The factor $\frac{1}{2}$ appears in Eq. (1.35) because in summing over *both* i and j each member of a given pair is included twice, first in the i summation and then in the j summation.

From these considerations, it is clear that if the external and internal forces are both derivable from potentials it is possible to define a *total potential energy*, V, of the system,

$$V = \sum_i V_i + \frac{1}{2}\sum_{\substack{i,j \\ i\neq j}} V_{ij}, \tag{1.36}$$

such that the total energy $T + V$ is conserved, the analog of the conservation theorem (1.18) for a single particle.

The second term on the right in Eq. (1.36) will be called the internal potential energy of the system. In general, it need not be zero and, more important, it may vary as the system changes with time. Only for the particular class of systems known as *rigid bodies* will the internal potential always be constant. Formally, a rigid body can be defined as a system of particles in which the distances r_{ij} are fixed and cannot vary with time. In such case, the vectors $d\mathbf{r}_{ij}$ can only be perpendicular to the corresponding $\mathbf{r}_{ij}$, and therefore to the $\mathbf{F}_{ij}$. Therefore, in a rigid body the *internal forces do no work*, and the internal potential must remain

constant. Since the total potential is in any case uncertain to within an additive constant, an unvarying internal potential can be completely disregarded in discussing the motion of the system.

1.3 ■ CONSTRAINTS

From the previous sections one might obtain the impression that all problems in mechanics have been reduced to solving the set of differential equations (1.19):

$$m_i\ddot{\mathbf{r}}_i = \mathbf{F}_i^{(e)} + \sum_j \mathbf{F}_{ji}.$$

One merely substitutes the various forces acting upon the particles of the system, turns the mathematical crank, and grinds out the answers! Even from a purely physical standpoint, however, this view is oversimplified. For example, it may be necessary to take into account the *constraints* that limit the motion of the system. We have already met one type of system involving constraints, namely rigid bodies, where the constraints on the motions of the particles keep the distances r_{ij} unchanged. Other examples of constrained systems can easily be furnished. The beads of an abacus are constrained to one-dimensional motion by the supporting wires. Gas molecules within a container are constrained by the walls of the vessel to move only *inside* the container. A particle placed on the surface of a solid sphere is subject to the constraint that it can move only on the surface or in the region exterior to the sphere.

Constraints may be classified in various ways, and we shall use the following system. If the conditions of constraint can be expressed as equations connecting the coordinates of the particles (and possibly the time) having the form

$$f(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \ldots, t) = 0, \tag{1.37}$$

then the constraints are said to be *holonomic*. Perhaps the simplest example of holonomic constraints is the rigid body, where the constraints are expressed by equations of the form

$$(\mathbf{r}_i - \mathbf{r}_j)^2 - c_{ij}^2 = 0.$$

A particle constrained to move along any curve or on a given surface is another obvious example of a holonomic constraint, with the equations defining the curve or surface acting as the equations of a constraint.

Constraints not expressible in this fashion are called nonholonomic. The walls of a gas container constitute a nonholonomic constraint. The constraint involved in the example of a particle placed on the surface of a sphere is also nonholonomic, for it can be expressed as an inequality

$$r^2 - a^2 \geq 0$$

(where a is the radius of the sphere), which is not in the form of (1.37). Thus, in a gravitational field a particle placed on the top of the sphere will slide down the surface part of the way but will eventually fall off.

Constraints are further classified according to whether the equations of constraint contain the time as an explicit variable (rheonomous) or are not explicitly dependent on time (scleronomous). A bead sliding on a rigid curved wire fixed in space is obviously subject to a scleronomous constraint; if the wire is moving in some prescribed fashion, the constraint is rheonomous. Note that if the wire moves, say, as a reaction to the bead's motion, then the time dependence of the constraint enters in the equation of the constraint only through the coordinates of the curved wire (which are now part of the system coordinates). The overall constraint is then scleronomous.

Constraints introduce two types of difficulties in the solution of mechanical problems. First, the coordinates r_i are no longer all independent, since they are connected by the equations of constraint; hence the equations of motion (1.19) are not all independent. Second, the forces of constraint, e.g., the force that the wire exerts on the bead (or the wall on the gas particle), is not furnished a priori. They are among the unknowns of the problem and must be obtained from the solution we seek. Indeed, imposing constraints on the system is simply another method of stating that there are forces present in the problem that cannot be specified directly but are known rather in terms of their effect on the motion of the system.

In the case of holonomic constraints, the first difficulty is solved by the introduction of *generalized coordinates*. So far we have been thinking implicitly in terms of Cartesian coordinates. A system of N particles, free from constraints, has $3N$ independent coordinates or *degrees of freedom*. If there exist holonomic constraints, expressed in k equations in the form (1.37), then we may use these equations to eliminate k of the $3N$ coordinates, and we are left with $3N - k$ independent coordinates, and the system is said to have $3N - k$ degrees of freedom. This elimination of the dependent coordinates can be expressed in another way, by the introduction of new, $3N - k$, independent variables $q_1, q_2, \ldots, q_{3N-k}$ in terms of which the old coordinates $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N$ are expressed by equations of the form

$$\begin{aligned} \mathbf{r}_1 &= \mathbf{r}_1(q_1, q_2, \ldots, q_{3N-k}, t) \\ &\vdots \\ \mathbf{r}_N &= \mathbf{r}_N(q_1, q_2, \ldots, q_{3N-k}, t) \end{aligned} \tag{1.38}$$

containing the constraints in them implicitly. These are *transformation* equations from the set of $(\mathbf{r}_l)$ variables to the (q_l) set, or alternatively Eqs. (1.38) can be considered as parametric representations of the $(\mathbf{r}_l)$ variables. It is always assumed that we can also transform back from the (q_l) to the $(\mathbf{r}_l)$ set, i.e., that Eqs. (1.38) combined with the k equations of constraint can be inverted to obtain any q_i as a function of the $(\mathbf{r}_l)$ variable and time.

Usually the generalized coordinates, q_l, unlike the Cartesian coordinates, will not divide into convenient groups of three that can be associated together to form vectors. Thus, in the case of a particle constrained to move *on* the surface of a sphere, the two angles expressing position on the sphere, say latitude and longitude, are obvious possible generalized coordinates. Or, in the example of a double pendulum moving in a plane (two particles connected by an inextensible light rod and suspended by a similar rod fastened to one of the particles), satisfactory generalized coordinates are the two angles θ_1, θ_2. (Cf. Fig. 1.4.) Generalized coordinates, in the sense of coordinates other than Cartesian, are often useful in systems without constraints. Thus, in the problem of a particle moving in an external central force field ($V = V(r)$), there is no constraint involved, but it is clearly more convenient to use spherical polar coordinates than Cartesian coordinates. Do not, however, think of generalized coordinates in terms of conventional orthogonal position coordinates. All sorts of quantities may be invoked to serve as generalized coordinates. Thus, the amplitudes in a Fourier expansion of $\mathbf{r}_j$ may be used as generalized coordinates, or we may find it convenient to employ quantities with the dimensions of energy or angular momentum.

If the constraint is nonholonomic, the equations expressing the constraint cannot be used to eliminate the dependent coordinates. An oft-quoted example of a nonholonomic constraint is that of an object rolling on a rough surface without slipping. The coordinates used to describe the system will generally involve angular coordinates to specify the orientation of the body, plus a set of coordinates describing the location of the point of contact on the surface. The constraint of "rolling" connects these two sets of coordinates; they are not independent. A change in the position of the point of contact inevitably means a change in its orientation. Yet we cannot reduce the number of coordinates, for the "rolling" condition is not expressible as a equation between the coordinates, in the manner of (1.37). Rather, it is a condition on the *velocities* (i.e., the point of contact is stationary), a differential condition that can be given in an integrated form only *after* the problem is solved.

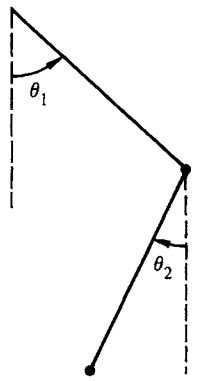

FIGURE 1.4 Double pendulum.

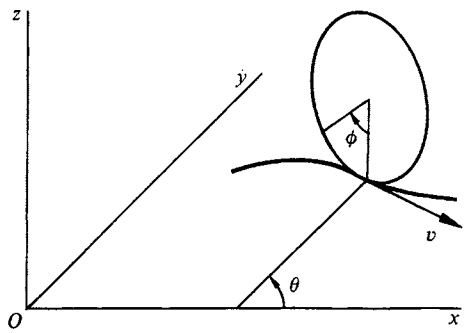

FIGURE 1.5 Vertical disk rolling on a horizontal plane.

A simple case will illustrate the point. Consider a disk rolling on the horizontal xy plane constrained to move so that the plane of the disk is always vertical. The coordinates used to describe the motion might be the x, y coordinates of the center of the disk, an angle of rotation ϕ about the axis of the disk, and an angle θ between the axis of the disk and say, the x axis (cf. Fig 1.5). As a result of the constraint the velocity of the center of the disk, **v**, has a magnitude proportional to $\dot{\phi}$,

$$v = a\dot{\phi},$$

where a is the radius of the disk, and its direction is perpendicular to the axis of the disk:

$$\dot{x} = v\sin\theta,$$
$$\dot{y} = -v\cos\theta.$$

Combining these conditions, we have two *differential* equations of constraint:

$$\begin{aligned} dx - a\sin\theta d\phi &= 0, \\ dy + a\cos\theta d\phi &= 0. \end{aligned} \tag{1.39}$$

Neither of Eqs. (1.39) can be integrated without in fact solving the problem; i.e., we cannot find an integrating factor $f(x, y, \theta, \phi)$ that will turn either of the equations into exact differentials (cf. Derivation 4).* Hence, the constraints cannot be reduced to the form of Eq. (1.37) and are therefore nonholonomic. Physically we can see that there can be no direct functional relation between ϕ and the other coordinates x, y, and θ by noting that at any point on its path the disk can be

*In principle, an integrating factor can always be found for a first-order differential equation of constraint in systems involving only two coordinates and such constraints are therefore holonomic. A familiar example is the two-dimensional motion of a circle rolling on an inclined plane.

made to roll around in a circle tangent to the path and of arbitrary radius. At the end of the process, x, y, and θ have been returned to their original values, but ϕ has changed by an amount depending on the radius of the circle.

Nonintegrable *differential* constraints of the form of Eqs. (1.39) are of course not the only type of nonholonomic constraints. The constraint conditions may involve higher-order derivatives, or may appear in the form of inequalities, as we have seen.

Partly because the dependent coordinates can be eliminated, problems involving holonomic constraints are always amenable to a formal solution. But there is no general way to attack nonholonomic examples. True, if the constraint is nonintegrable, the differential equations of constraint can be introduced into the problem along with the differential equations of motion, and the dependent equations eliminated, in effect, by the method of Lagrange multipliers.

We shall return to this method at a later point. However, the more vicious cases of nonholonomic constraint must be tackled individually, and consequently in the development of the more formal aspects of classical mechanics, it is almost invariably assumed that any constraint, if present, is holonomic. This restriction does not greatly limit the applicability of the theory, despite the fact that many of the constraints encountered in everyday life are nonholonomic. The reason is that the entire concept of constraints imposed in the system through the medium of wires or surfaces or walls is particularly appropriate only in macroscopic or large-scale problems. But today physicists are more interested in atomic and nuclear problems. On this scale all objects, both in and out of the system, consist alike of molecules, atoms, or smaller particles, exerting definite forces, and the notion of constraint becomes artificial and rarely appears. Constraints are then used only as mathematical idealizations to the actual physical case or as classical approximations to a quantum-mechanical property, e.g., rigid body rotations for "spin." Such constraints are always holonomic and fit smoothly into the framework of the theory.

To surmount the second difficulty, namely, that the forces of constraint are unknown a priori, we should like to so formulate the mechanics that the forces of constraint disappear. We need then deal only with the known applied forces. A hint as to the procedure to be followed is provided by the fact that in a particular system with constraints, i.e., a rigid body, the work done by internal forces (which are here the forces of constraint) vanishes. We shall follow up this clue in the ensuing sections and generalize the ideas contained in it.

1.4 ■ D'ALEMBERT'S PRINCIPLE AND LAGRANGE'S EQUATIONS

A virtual (infinitesimal) displacement of a system refers to a change in the configuration of the system as the result of any arbitrary infinitesimal change of the coordinates $\delta\mathbf{r}_i$, *consistent with the forces and constraints imposed on the system at the given instant* t. The displacement is called virtual to distinguish it from an actual displacement of the system occurring in a time interval dt, during which

the forces and constraints may be changing. Suppose the system is in equilibrium; i.e., the total force on each particle vanishes, $\mathbf{F}_i = 0$. Then clearly the dot product $\mathbf{F}_i \cdot \delta\mathbf{r}_i$, which is the virtual work of the force $\mathbf{F}_i$ in the displacement $\delta\mathbf{r}_i$, also vanishes. The sum of these vanishing products over all particles must likewise be zero:

$$\sum_i \mathbf{F}_i \cdot \delta\mathbf{r}_i = 0. \tag{1.40}$$

As yet nothing has been said that has any new physical content. Decompose $\mathbf{F}_i$ into the applied force, $\mathbf{F}_i^{(a)}$, and the force of constraint, $\mathbf{f}_i$,

$$\mathbf{F}_i = \mathbf{F}_i^{(a)} + \mathbf{f}_i, \tag{1.41}$$

so that Eq. (1.40) becomes

$$\sum_i \mathbf{F}_i^{(a)} \cdot \delta\mathbf{r}_i + \sum_i \mathbf{f}_i \cdot \delta\mathbf{r}_i = 0. \tag{1.42}$$

We now restrict ourselves to systems for which *the net virtual work of the forces of constraint is zero*. We have seen that this condition holds true for rigid bodies and it is valid for a large number of other constraints. Thus, if a particle is constrained to move on a surface, the force of constraint is perpendicular to the surface, while the virtual displacement must be tangent to it, and hence the virtual work vanishes. This is no longer true if sliding friction forces are present, and we must exclude such systems from our formulation. The restriction is not unduly hampering, since the friction is essentially a macroscopic phenomenon. On the other hand, the forces of rolling friction do not violate this condition, since the forces act on a point that is momentarily at rest and can do no work in an infinitesimal displacement consistent with the rolling constraint. Note that if a particle is constrained to a surface that is itself moving in time, the force of constraint is instantaneously perpendicular to the surface and the work during a virtual displacement is still zero even though the work during an actual displacement in the time dt does not necessarily vanish.

We therefore have as the condition for equilibrium of a system that the virtual work of the *applied forces* vanishes:

$$\sum_i \mathbf{F}_i^{(a)} \cdot \delta\mathbf{r}_i = 0. \tag{1.43}$$

Equation (1.43) is often called the *principle of virtual work*. Note that the coefficients of $\delta\mathbf{r}_i$ can no longer be set equal to zero; i.e., in general $\mathbf{F}_i^{(a)} \neq 0$, since the $\delta\mathbf{r}_i$ are not completely independent but are connected by the constraints. In order to equate the coefficients to zero, we must transform the principle into a form involving the virtual displacements of the q_i, which are independent. Equation (1.43) satisfies our needs in that it does not contain the $\mathbf{f}_i$, but it deals only with statics; we want a condition involving the general motion of the system.

To obtain such a principle, we use a device first thought of by James Bernoulli and developed by D'Alembert. The equation of motion,

$$\mathbf{F}_i = \dot{\mathbf{p}}_i,$$

can be written as

$$\mathbf{F}_i - \dot{\mathbf{p}}_i = 0,$$

which states that the particles in the system will be in equilibrium under a force equal to the actual force plus a "reversed effective force" $-\dot{\mathbf{p}}_i$. Instead of (1.40), we can immediately write

$$\sum_i (\mathbf{F}_i - \dot{\mathbf{p}}_i) \cdot \delta\mathbf{r}_i = 0, \tag{1.44}$$

and, making the same resolution into applied forces and forces of constraint, there results

$$\sum_i (\mathbf{F}_i^{(a)} - \dot{\mathbf{p}}_i) \cdot \delta\mathbf{r}_i + \sum_i \mathbf{f}_i \cdot \delta\mathbf{r}_i = 0.$$

We again restrict ourselves to systems for which the virtual work of the forces of constraint vanishes and therefore obtain

$$\sum_i (\mathbf{F}_i^{(a)} - \dot{\mathbf{p}}_i) \cdot \delta\mathbf{r}_i = 0, \tag{1.45}$$

which is often called *D'Alembert's principle*. We have achieved our aim, in that the forces of constraint no longer appear, and the superscript $^{(a)}$ can now be dropped without ambiguity. It is still not in a useful form to furnish equations of motion for the system. We must now transform the principle into an expression involving virtual displacements of the generalized coordinates, which are then independent of each other (for holonomic constraints), so that the coefficients of the δq_i can be set separately equal to zero.

The translation from $\mathbf{r}_i$ to q_j language starts from the transformation equations (1.38),

$$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, \ldots, q_n, t) \tag{1.45'}$$

(assuming n independent coordinates), and is carried out by means of the usual "chain rules" of the calculus of partial differentiation. Thus, $\mathbf{v}_i$ is expressed in terms of the $\dot{q}_k$ by the formula

$$\mathbf{v}_i \equiv \frac{d\mathbf{r}_i}{dt} = \sum_k \frac{\partial \mathbf{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \mathbf{r}_i}{\partial t}. \tag{1.46}$$

Similarly, the arbitrary virtual displacement $\delta \mathbf{r}_i$ can be connected with the virtual displacements δq_i by

$$\delta \mathbf{r}_i = \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j. \tag{1.47}$$

Note that no variation of time, δt, is involved here, since a virtual displacement by definition considers only displacements of the coordinates. (Only then is the virtual displacement perpendicular to the force of constraint if the constraint itself is changing in time.)

In terms of the generalized coordinates, the virtual work of the $\mathbf{F}_i$ becomes

$$\begin{aligned} \sum_i \mathbf{F}_i \cdot \delta \mathbf{r}_i &= \sum_{i,j} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j \\ &= \sum_j Q_j \delta q_j, \end{aligned} \tag{1.48}$$

where the Q_j are called the components of the *generalized force*, defined as

$$Q_j = \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}. \tag{1.49}$$

Note that just as the q's need not have the dimensions of length, so the Q's do not necessarily have the dimensions of force, but $Q_j \delta q_j$ must always have the dimensions of work. For example, Q_j might be a torque N_j and dq_j a differential angle $d\theta_j$, which makes $N_j \, d\theta_j$ a differential of work.

We turn now to the other other term involved in Eq. (1.45), which may be written as

$$\sum_i \dot{\mathbf{p}}_i \cdot \delta \mathbf{r}_i = \sum_i m_i \ddot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i.$$

Expressing $\delta \mathbf{r}_i$ by (1.47), this becomes

$$\sum_{i,j} m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j.$$

Consider now the relation

$$\sum_i m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = \sum_i \left[\frac{d}{dt} \left(m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \right) - m_i \dot{\mathbf{r}}_i \cdot \frac{d}{dt} \left(\frac{\partial \mathbf{r}_i}{\partial q_j} \right) \right]. \tag{1.50}$$

In the last term of Eq. (1.50) we can interchange the differentiation with respect to t and q_j, for, in analogy to (1.46),

$$\frac{d}{dt}\left(\frac{\partial \mathbf{r}_i}{\partial q_j}\right) = \frac{\partial \dot{\mathbf{r}}_i}{\partial q_j} = \sum_k \frac{\partial^2 \mathbf{r}_i}{\partial q_j \partial q_k}\dot{q}_k + \frac{\partial^2 \mathbf{r}_i}{\partial q_j \partial t},$$
$$= \frac{\partial \mathbf{v}_i}{\partial q_j},$$

by Eq. (1.46). Further, we also see from Eq. (1.46) that

$$\frac{\partial \mathbf{v}_i}{\partial \dot{q}_j} = \frac{\partial \mathbf{r}_i}{\partial q_j}. \tag{1.51}$$

Substitution of these changes in (1.50) leads to the result that

$$\sum_i m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = \sum_i \left[\frac{d}{dt}\left(m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial \dot{q}_j}\right) - m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial q_j}\right],$$

and the second term on the left-hand side of Eq. (1.45) can be expanded into

$$\sum_j \left\{\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}\left(\sum_i \frac{1}{2} m_i v_i^2\right)\right] - \frac{\partial}{\partial q_j}\left(\sum_i \frac{1}{2} m_i v_i^2\right) - Q_j\right\} \delta q_j.$$

Identifying $\sum_i \frac{1}{2} m_i v_i^2$ with the system kinetic energy T, D'Alembert's principle (cf. Eq. (1.45)) becomes

$$\sum_j \left\{\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right] - Q_j\right\} \delta q_j = 0. \tag{1.52}$$

Note that in a system of Cartesian coordinates the partial derivative of T with respect to q_j vanishes. Thus, speaking in the language of differential geometry, this term arises from the curvature of the coordinates q_j. In polar coordinates, e.g., it is in the partial derivative of T with respect to the angle coordinate that the centripetal acceleration term appears.

Thus far, no restriction has been made on the nature of the constraints other than that they be workless in a virtual displacement. The variables q_j can be any set of coordinates used to describe the motion of the system. If, however, the constraints are holonomic, then it is possible to find sets of independent coordinates q_j that contain the constraint conditions implicitly in the transformation equations (1.38). Any virtual displacement δq_j is then independent of δq_k, and therefore the only way for (1.52) to hold is for the individual coefficients to vanish:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j. \tag{1.53}$$

There are n such equations in all.

When the forces are derivable from a scalar potential function V,

$$\mathbf{F}_i = -\nabla_i V.$$

Then the generalized forces can be written as

$$Q_j = \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = -\sum_i \nabla_i V \cdot \frac{\partial \mathbf{r}_i}{\partial q_j},$$

which is exactly the same expression for the partial derivative of a function $-V(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N, t)$ with respect to q_j:

$$Q_j \equiv -\frac{\partial V}{\partial q_j}. \tag{1.54}$$

See Eq. (1.47). Equations (1.53) can then be rewritten as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial (T - V)}{\partial q_j} = 0. \tag{1.55}$$

The equations of motion in the form (1.55) are not necessarily restricted to conservative systems; only if V is not an explicit function of time is the system conservative (cf. p. 4). As here defined, the potential V does not depend on the generalized velocities. Hence, we can include a term in V in the partial derivative with respect to $\dot{q}_j$:

$$\frac{d}{dt}\left(\frac{\partial (T - V)}{\partial \dot{q}_j}\right) - \frac{\partial (T - V)}{\partial q_j} = 0.$$

Or, defining a new function, the *Lagrangian* L, as

$$L = T - V, \tag{1.56}$$

the Eqs. (1.53) become

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0, \tag{1.57}$$

expressions referred to as "Lagrange's equations."

Note that for a particular set of equations of motion there is no unique choice of Lagrangian such that Eqs. (1.57) lead to the equations of motion in the given generalized coordinates. Thus, in Derivations 8 and 10 it is shown that if $L(q, \dot{q}, t)$ is an approximate Lagrangian and $F(q, t)$ is *any* differentiable function of the generalized coordinates and time, then

$$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{dF}{dt} \tag{1.57'}$$

is a Lagrangian also resulting in the same equations of motion. It is also often possible to find alternative Lagrangians beside those constructed by this prescription (see Exercise 20). While Eq. (1.56) is always a suitable way to construct a Lagrangian for a conservative system, it does not provide the *only* Lagrangian suitable for the given system.

1.5 ■ VELOCITY-DEPENDENT POTENTIALS AND THE DISSIPATION FUNCTION

Lagrange's equations can be put in the form (1.57) even if there is no potential function, V, in the usual sense, providing the generalized forces are obtained from a function $U(q_j, \dot{q}_j)$ by the prescription

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right). \tag{1.58}$$

In such case, Eqs. (1.57) still follow from Eqs. (1.53) with the Lagrangian given by

$$L = T - U. \tag{1.59}$$

Here U may be called a "generalized potential," or "velocity-dependent potential." The possibility of using such a "potential" is not academic; it applies to one very important type of force field, namely, the electromagnetic forces on moving charges. Considering its importance, a digression on this subject is well worthwhile.

Consider an electric charge, q, of mass m moving at a velocity, $\mathbf{v}$, in an otherwise charge-free region containing both an electric field, $\mathbf{E}$, and a magnetic field, $\mathbf{B}$, which may depend upon time and position. The charge experiences a force, called the Lorentz force, given by

$$\mathbf{F} = q[\mathbf{E} + (\mathbf{v} \times \mathbf{B})]. \tag{1.60}$$

Both $\mathbf{E}(x, y, z, t)$ and $\mathbf{B}(x, y, z, t)$ are continuous functions of time and position derivable from a scalar potential $\phi(x, y, z, t)$ and a vector potential $\mathbf{A}(x, y, z, t)$ by

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \tag{1.61a}$$

and

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{1.61b}$$

The force on the charge can be derived from the following velocity-dependent potential energy

$$U = q\phi - q\mathbf{A} \cdot \mathbf{v}, \tag{1.62}$$

so the Lagrangian, $L = T - U$, is

$$L = \tfrac{1}{2}mv^2 - q\phi + q\mathbf{A} \cdot \mathbf{v}. \tag{1.63}$$

Considering just the x-component of Lagrange's equations gives

$$m\ddot{x} = q\left(v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_y}{\partial x} + v_z\frac{\partial A_z}{\partial x}\right) - q\left(\frac{\partial \phi}{\partial x} + \frac{dA_x}{dt}\right). \tag{1.64}$$

The total time derivative of A_x is related to the partial time derivative through

$$\begin{aligned}\frac{dA_x}{dt} &= \frac{\partial A_x}{\partial t} + \mathbf{v}\cdot\nabla A_x \\ &= \frac{\partial A_x}{\partial t} + v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_x}{\partial y} + v_z\frac{\partial A_x}{\partial z}.\end{aligned} \tag{1.65}$$

Equation (1.61b) gives

$$(\mathbf{v}\times\mathbf{B})_x = v_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) + v_z\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right).$$

Combining these expressions gives the equation of motion in the x-direction

$$m\ddot{x} = q\left[\mathrm{E}_x + (\mathbf{v}\times\mathbf{B})_x\right]. \tag{1.66}$$

On a component-by-component comparison, Eqs. (1.66) and (1.60) are identical, showing that the Lorentz force equation is derivable from Eqs. (1.61) and (1.62).

Note that if not all the forces acting on the system are derivable from a potential, then Lagrange's equations can always be written in the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j,$$

where L contains the potential of the conservative forces as before, and Q_j represents the forces *not* arising from a potential. Such a situation often occurs when frictional forces are present. It frequently happens that the frictional force is proportional to the velocity of the particle, so that its x-component has the form

$$F_{fx} = -k_x v_x.$$

Frictional forces of this type may be derived in terms of a function $\mathcal{F}$, known as *Rayleigh's dissipation function*, and defined as

$$\mathcal{F} = \frac{1}{2}\sum_i\left(k_x v_{ix}^2 + k_y v_{iy}^2 + k_z v_{iz}^2\right), \tag{1.67}$$

where the summation is over the particles of the system. From this definition it is clear that

$$F_{f_{x_i}} = -\frac{\partial \mathcal{F}}{\partial v_{x_i}},$$

or, symbolically,

$$\mathbf{F}_f = -\nabla_v \mathcal{F}. \tag{1.68}$$

We can also give a physical interpretation to the dissipation function. The work done *by* the system *against* friction is

$$dW_f = -\mathbf{F}_f \cdot d\mathbf{r} = -\mathbf{F}_f \cdot \mathbf{v}\, dt = \left(k_x v_x^2 + k_y v_y^2 + k_z v_z^2\right) dt.$$

Hence, $2\mathcal{F}$ is the rate of energy dissipation due to friction. The component of the generalized force resulting from the force of friction is then given by

$$\begin{aligned} Q_j = \sum_i \mathbf{F}_{fi} \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} &= -\sum \nabla_v \mathcal{F} \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \\ &= -\sum \nabla_v \mathcal{F} \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_j}, \qquad \text{by (1.51)}, \\ &= -\frac{\partial \mathcal{F}}{\partial \dot{q}_j}. \end{aligned} \tag{1.69}$$

An example is Stokes' law, whereby a sphere of radius a moving at a speed v, in a medium of viscosity η experiences the frictional drag force $\mathbf{F}_f = -6\pi\eta a\mathbf{v}$. The Lagrange equations with dissipation become

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} + \frac{\partial \mathcal{F}}{\partial \dot{q}_j} = 0, \tag{1.70}$$

so that two scalar functions, L and $\mathcal{F}$, must be specified to obtain the equations of motion.

1.6 ■ SIMPLE APPLICATIONS OF THE LAGRANGIAN FORMULATION

The previous sections show that for systems where we can define a Lagrangian, i.e., holonomic systems with applied forces derivable from an ordinary or generalized potential and workless constraints, we have a very convenient way of setting up the equations of motion. We were led to the Lagrangian formulation by the desire to eliminate the forces of constraint from the equations of motion, and in achieving this goal we have obtained many other benefits. In setting up the original form of the equations of motion, Eqs. (1.19), it is necessary to work with many *vector* forces and accelerations. With the Lagrangian method we only deal with two *scalar* functions, T and V, which greatly simplifies the problem.

A straightforward routine procedure can now be established for all problems of mechanics to which the Lagrangian formulation is applicable. We have only to write T and V in generalized coordinates, form L from them, and substitute in (1.57) to obtain the equations of motion. The needed transformation of T and V from Cartesian coordinates to generalized coordinates is obtained by applying the

transformation equations (1.38) and (1.45′). Thus, T is given in general by

$$T = \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i \left(\sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \mathbf{r}_i}{\partial t} \right)^2 .$$

It is clear that on carrying out the expansion, the expression for T in generalized coordinates will have the form

$$T = M_0 + \sum_j M_j \dot{q}_j + \frac{1}{2} \sum_{j,k} M_{jk} \dot{q}_j \dot{q}_k, \tag{1.71}$$

where M_0, M_j, M_{jk} are definite functions of the $\mathbf{r}$'s and t and hence of the q's and t. In fact, a comparison shows that

$$M_0 = \sum_i \frac{1}{2} m_i \left(\frac{\partial \mathbf{r}_i}{\partial t} \right)^2 ,$$

$$M_j = \sum_i m_i \frac{\partial \mathbf{r}_i}{\partial t} \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}, \tag{1.72}$$

and

$$M_{jk} = \sum_i m_i \frac{\partial \mathbf{r}_i}{\partial q_j} \cdot \frac{\partial \mathbf{r}_i}{\partial q_k}.$$

Thus, the kinetic energy of a system can always be written as the sum of three homogeneous functions of the generalized velocities,

$$T = T_0 + T_1 + T_2, \tag{1.73}$$

where T_0 is independent of the generalized velocities, T_1 is linear in the velocities, and T_2 is quadratic in the velocities. If the transformation equations do not contain the time explicitly, as may occur when the constraints are independent of time (scleronomous), then only the last term in Eq. (1.71) is nonvanishing, and T is always a homogeneous quadratic form in the generalized velocities.

Let us now consider simple examples of this procedure:

1. Single particle in space
   (a) Cartesian coordinates
   (b) Plane polar coordinates
2. Atwood's machine
3. Time-dependent constraint—bead sliding on rotating wire

1. (a) *Motion of one particle: using Cartesian coordinates.* The generalized forces needed in Eq. (1.53) are obviously F_x, F_y, and F_z. Then

$$T = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right),$$

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial y} = \frac{\partial T}{\partial z} = 0,$$

$$\frac{\partial T}{\partial \dot{x}} = m\dot{x}, \qquad \frac{\partial T}{\partial \dot{y}} = m\dot{y}, \qquad \frac{\partial T}{\partial \dot{z}} = m\dot{z},$$

and the equations of motion are

$$\frac{d}{dt}(m\dot{x}) = F_x, \quad \frac{d}{dt}(m\dot{y}) = F_y, \quad \frac{d}{dt}(m\dot{z}) = F_z. \tag{1.74}$$

We are thus led back to the original Newton's equations of motion.

(b) *Motion of one particle: using plane polar coordinates.* Here we must express T in terms of $\dot{r}$ and $\dot{\theta}$. The transformation equations, Eqs. (1.38), are

$$x = r\cos\theta$$

$$y = r\sin\theta.$$

By analogy to (1.46), the velocities are given by

$$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta,$$

$$\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta.$$

The kinetic energy $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$ then reduces formally to

$$T = \tfrac{1}{2}m\left[\dot{r}^2 + \left(r\dot{\theta}\right)^2\right]. \tag{1.75}$$

An alternative derivation of Eq. (1.75) is obtained by recognizing that the plane polar components of the velocity are $\dot{r}$ along $\mathbf{r}$, and $r\dot{\theta}$ along the direction perpendicular to r, denoted by the unit vector $\hat{\boldsymbol{\theta}}$. Hence, the square of the velocity expressed in polar coordinates is simply $\dot{r}^2 + (r\dot{\theta})^2$. With the aid of the expression

$$d\mathbf{r} = \hat{\mathbf{r}}\,dr + r\hat{\boldsymbol{\theta}}\,d\theta + \hat{\mathbf{k}}\,dz$$

for the differential position vector, $d\mathbf{r}$, in cylindrical coordinates, restricted to the plane $\mathbf{z} = 0$ where $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are unit vectors in the $\mathbf{r}$ and $\boldsymbol{\theta}$-directions, respectively, the components of the generalized force can be obtained from the definition, Eq. (1.49),

$$Q_r = \mathbf{F}\cdot\frac{\partial \mathbf{r}}{\partial r} = \mathbf{F}\cdot\hat{\mathbf{r}} = F_r,$$

$$Q_\theta = \mathbf{F}\cdot\frac{\partial \mathbf{r}}{\partial \theta} = \mathbf{F}\cdot r\hat{\boldsymbol{\theta}} = rF_\theta,$$

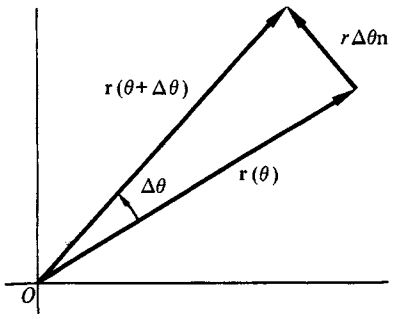

FIGURE 1.6 Derivative of r with respect to θ.

since the derivative of $\mathbf{r}$ with respect to θ is, by the definition of a derivative, a vector in the direction of $\hat{\theta}$ (cf. Fig. 1.6). There are two generalized coordinates, and therefore two Lagrange equations. The derivatives occurring in the r equation are

$$\frac{\partial T}{\partial r} = mr\dot{\theta}^2, \qquad \frac{\partial T}{\partial \dot{r}} = m\dot{r}, \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) = m\ddot{r},$$

and the equation itself is

$$m\ddot{r} - mr\dot{\theta}^2 = F_r,$$

the second term being the centripetal acceleration term. For the θ equation, we have the derivatives

$$\frac{\partial T}{\partial \theta} = 0, \qquad \frac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta}, \qquad \frac{d}{dt}\left(mr^2\dot{\theta}\right) = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta},$$

so that the equation becomes

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = rF_\theta.$$

Note that the left side of the equation is just the time derivative of the angular momentum, and the right side is exactly the applied torque, so that we have simply rederived the torque equation (1.26), where $L = mr^2\dot{\theta}$ and $N^{(e)} = rF_\theta$.

2. *Atwood's machine*—(See Fig. 1.7) an example of a conservative system with holonomic, scleronomous constraint (the pulley is assumed frictionless and massless). Clearly there is only one independent coordinate x, the position of the other weight being determined by the constraint that the length of the rope between them is l. The potential energy is

$$V = -M_1gx - M_2g(l - x),$$

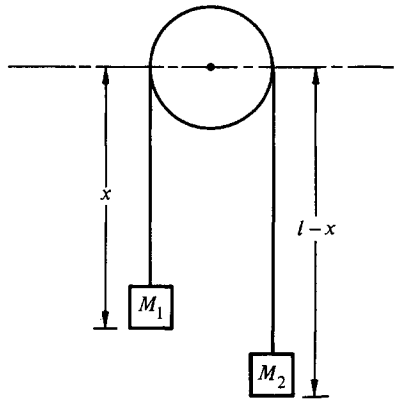

FIGURE 1.7 Atwood's machine.

while the kinetic energy is

$$T = \tfrac{1}{2}(M_1 + M_2)\dot{x}^2.$$

Combining the two, the Lagrangian has the form

$$L = T - V = \tfrac{1}{2}(M_1 + M_2)\dot{x}^2 + M_1 g x + M_2 g(l - x).$$

There is only one equation of motion, involving the derivatives

$$\frac{\partial L}{\partial x} = (M_1 - M_2)\, g,$$
$$\frac{\partial L}{\partial \dot{x}} = (M_1 + M_2)\,\dot{x},$$

so that we have

$$(M_1 + M_2)\,\ddot{x} = (M_1 - M_2)\, g,$$

or

$$\ddot{x} = \frac{M_1 - M_2}{M_1 + M_2} g,$$

which is the familiar result obtained by more elementary means. This trivial problem emphasizes that the forces of constraint—here the tension in the rope—appear nowhere in the Lagrangian formulation. By the same token, neither can the tension in the rope be found directly by the Lagrangian method.

3. *A bead (or ring) sliding on a uniformly rotating wire in a force-free space.* The wire is straight, and is rotated uniformly about some fixed axis perpendicular to the wire. This example has been chosen as a simple illustration of a constraint

being time dependent, with the rotation axis along z and the wire in the xy plane. The transformation equations explicitly contain the time.

$$x = r\cos\omega t, \qquad (\omega = \text{angular velocity of rotation})$$

$$y = r\sin\omega t. \qquad (r = \text{distance along wire from rotation axis})$$

While we could then find T (here the same as L) by the same procedure used to obtain (1.71), it is simpler to take over (1.75) directly, expressing the constraint by the relation $\dot{\theta} = \omega$:

$$T = \tfrac{1}{2}m\left(\dot{r}^2 + r^2\omega^2\right).$$

Note that T is not a homogeneous quadratic function of the generalized velocities, since there is now a term not involving $\dot{r}$. The equation of motion is then

$$m\ddot{r} - mr\omega^2 = 0$$

or

$$\ddot{r} = r\omega^2,$$

which is the familiar simple harmonic oscillator equation with a change of sign. The solution $r = e^{\omega t}$ for a bead initially at rest on the wire shows that the bead moves exponentially outwards. Again, the method cannot furnish the force of constraint that keeps the bead on the wire. Equation (1.26) with the angular momentum, $L = mr^2\omega = m\omega r_0^2 e^{2\omega t}$, provides the force $F = N/r$, which produces the constraint force, $F = 2mr_0\omega^2 e^{\omega t}$, acting perpendicular to the wire and the axis of rotation.

DERIVATIONS

1. Show that for a single particle with constant mass the equation of motion implies the following differential equation for the kinetic energy:

$$\frac{dT}{dt} = \mathbf{F}\cdot\mathbf{v},$$

while if the mass varies with time the corresponding equation is

$$\frac{d(mT)}{dt} = \mathbf{F}\cdot\mathbf{p}.$$

2. Prove that the magnitude R of the position vector for the center of mass from an arbitrary origin is given by the equation

$$M^2R^2 = M\sum_i m_i r_i^2 - \frac{1}{2}\sum_{i\neq j} m_i m_j r_{ij}^2.$$

3. Suppose a system of two particles is known to obey the equations of motion, Eqs. (1.22) and (1.26). From the equations of the motion of the individual particles show that the internal forces between particles satisfy both the weak and the strong laws of action and reaction. The argument may be generalized to a system with arbitrary number of particles, thus proving the converse of the arguments leading to Eqs. (1.22) and (1.26).

4. The equations of constraint for the rolling disk, Eqs. (1.39), are special cases of general linear differential equations of constraint of the form

$$\sum_{i=1}^{n} g_i(x_1, \ldots, x_n) dx_i = 0.$$

A constraint condition of this type is holonomic only if an integrating function $f(x_1, \ldots, x_n)$ can be found that turns it into an exact differential. Clearly the function must be such that

$$\frac{\partial (f g_i)}{\partial x_j} = \frac{\partial (f g_j)}{\partial x_i}$$

for all $i \neq j$. Show that no such integrating factor can be found for either of Eqs. (1.39).

5. Two wheels of radius a are mounted on the ends of a common axle of length b such that the wheels rotate independently. The whole combination rolls without slipping on a plane. Show that there are two nonholonomic equations of constraint,

$$\cos\theta dx + \sin\theta dy = 0$$
$$\sin\theta dx - \cos\theta dy = \tfrac{1}{2}a\left(d\phi + d\phi'\right),$$

(where θ, ϕ, and ϕ' have meanings similar to those in the problem of a single vertical disk, and (x, y) are the coordinates of a point on the axle midway between the two wheels) and one holonomic equation of constraint,

$$\theta = C - \frac{a}{b}(\phi - \phi'),$$

where C is a constant.

6. A particle moves in the xy plane under the constraint that its velocity vector is always directed towards a point on the x axis whose abscissa is some given function of time $f(t)$. Show that for $f(t)$ differentiable, but otherwise arbitrary, the constraint is nonholonomic.

7. Show that Lagrange's equations in the form of Eqs. (1.53) can also be written as

$$\frac{\partial \dot{T}}{\partial \dot{q}_j} - 2\frac{\partial T}{\partial q_j} = Q_j.$$

These are sometimes known as the *Nielsen* form of the Lagrange equations.

8. If L is a Lagrangian for a system of n degrees of freedom satisfying Lagrange's equations, show by direct substitution that

$$L' = L + \frac{dF(q_1, \ldots, q_n, t)}{dt}$$

also satisfies Lagrange's equations where F is any arbitrary, but differentiable, function of its arguments.

9. The electromagnetic field is invariant under a gauge transformation of the scalar and vector potential given by

$$\mathbf{A} \to \mathbf{A} + \nabla\psi(\mathbf{r}, \mathbf{t}),$$
$$\phi \to \phi - \frac{1}{c}\frac{\partial \psi}{\partial t},$$

where ψ is arbitrary (but differentiable). What effect does this gauge transformation have on the Lagrangian of a particle moving in the electromagnetic field? Is the motion affected?

10. Let $q_1, \ldots, q_n$ be a set of independent generalized coordinates for a system of n degrees of freedom, with a Lagrangian $L(q, \dot{q}, t)$. Suppose we transform to another set of independent coordinates $s_1, \ldots, s_n$ by means of transformation equations

$$q_i = q_i(s_1, \ldots, s_n, t), \qquad i = 1, \ldots, n.$$

(Such a transformation is called a *point transformation.*) Show that if the Lagrangian function is expressed as a function of s_j, $\dot{s}_j$, and t through the equations of transformation, then L satisfies Lagrange's equations with respect to the s coordinates:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}_j}\right) - \frac{\partial L}{\partial s_j} = 0.$$

In other words, the form of Lagrange's equations is invariant under a point transformation.

EXERCISES

11. Consider a uniform thin disk that rolls without slipping on a horizontal plane. A horizontal force is applied to the center of the disk and in a direction parallel to the plane of the disk.

(a) Derive Lagrange's equations and find the generalized force.

(b) Discuss the motion if the force is not applied parallel to the plane of the disk.

12. The *escape velocity* of a particle on Earth is the minimum velocity required at Earth's surface in order that the particle can escape from Earth's gravitational field. Neglecting the resistance of the atmosphere, the system is conservative. From the conservation theorem for potential plus kinetic energy show that the escape velocity for Earth, ignoring the presence of the Moon, is 11.2 km/s.

13. Rockets are propelled by the momentum reaction of the exhaust gases expelled from the tail. Since these gases arise from the reaction of the fuels carried in the rocket, the mass of the rocket is not constant, but decreases as the fuel is expended. Show that the equation of motion for a rocket projected vertically upward in a uniform gravitational

field, neglecting atmospheric friction, is

$$m\frac{dv}{dt} = -v'\frac{dm}{dt} - mg,$$

where m is the mass of the rocket and v' is the velocity of the escaping gases relative to the rocket. Integrate this equation to obtain v as a function of m, assuming a constant time rate of loss of mass. Show, for a rocket starting initially from rest, with v' equal to 2.1 km/s and a mass loss per second equal to 1/60th of the initial mass, that in order to reach the escape velocity the ratio of the weight of the fuel to the weight of the empty rocket must be almost 300!

14. Two points of mass m are joined by a rigid weightless rod of length l, the center of which is constrained to move on a circle of radius a. Express the kinetic energy in generalized coordinates.

15. A point particle moves in space under the influence of a force derivable from a generalized potential of the form

$$U(\mathbf{r}, \mathbf{v}) = V(r) + \boldsymbol{\sigma} \cdot \mathbf{L},$$

where $\mathbf{r}$ is the radius vector from a fixed point, $\mathbf{L}$ is the angular momentum about that point, and $\boldsymbol{\sigma}$ is a fixed vector in space.

(a) Find the components of the force on the particle in both Cartesian and spherical polar coordinates, on the basis of Eq. (1.58).

(b) Show that the components in the two coordinate systems are related to each other as in Eq. (1.49).

(c) Obtain the equations of motion in spherical polar coordinates.

16. A particle moves in a plane under the influence of a force, acting toward a center of force, whose magnitude is

$$F = \frac{1}{r^2}\left(1 - \frac{\dot{r}^2 - 2\ddot{r}r}{c^2}\right),$$

where r is the distance of the particle to the center of force. Find the generalized potential that will result in such a force, and from that the Lagrangian for the motion in a plane. (The expression for F represents the force between two charges in Weber's electrodynamics.)

17. A nucleus, originally at rest, decays radioactively by emitting an electron of momentum 1.73 MeV/c, and at right angles to the direction of the electron a neutrino with momentum 1.00 MeV/c. (The MeV, million electron volt, is a unit of energy used in modern physics, equal to 1.60×10^{-13} J. Correspondingly, MeV/c is a unit of linear momentum equal to 5.34×10^{-22} kg·m/s.) In what direction does the nucleus recoil? What is its momentum in MeV/c? If the mass of the residual nucleus is 3.90×10^{-25} kg what is its kinetic energy, in electron volts?

18. A Lagrangian for a particular physical system can be written as

$$L' = \frac{m}{2}\left(a\dot{x}^2 + 2b\dot{x}\dot{y} + c\dot{y}^2\right) - \frac{K}{2}\left(ax^2 + 2bxy + cy^2\right),$$

where a, b, and c are arbitrary constants but subject to the condition that $b^2 - ac \neq 0$.

What are the equations of motion? Examine particularly the two cases $a = 0 = c$ and $b = 0, c = -a$. What is the physical system described by the above Lagrangian? Show that the usual Lagrangian for this system as defined by Eq. (1.56) is related to L' by a point transformation (cf. Derivation 10). What is the significance of the condition on the value of $b^2 - ac$?

19. Obtain the Lagrange equations of motion for a spherical pendulum, i.e., a mass point suspended by a rigid weightless rod.

20. A particle of mass m moves in one dimension such that it has the Lagrangian

$$L = \frac{m^2\dot{x}^4}{12} + m\dot{x}^2V(x) - V^2(x),$$

where V is some differentiable function of x. Find the equation of motion for $x(t)$ and describe the physical nature of the system on the basis of this equation.

21. Two mass points of mass m_1 and m_2 are connected by a string passing through a hole in a smooth table so that m_1 rests on the table surface and m_2 hangs suspended. Assuming m_2 moves only in a vertical line, what are the generalized coordinates for the system? Write the Lagrange equations for the system and, if possible, discuss the physical significance any of them might have. Reduce the problem to a single second-order differential equation and obtain a first integral of the equation. What is its physical significance? (Consider the motion only until m_1 reaches the hole.)

22. Obtain the Lagrangian and equations of motion for the double pendulum illustrated in Fig. 1.4, where the lengths of the pendula are l_1 and l_2 with corresponding masses m_1 and m_2.

23. Obtain the equation of motion for a particle falling vertically under the influence of gravity when frictional forces obtainable from a dissipation function $\frac{1}{2}kv^2$ are present. Integrate the equation to obtain the velocity as a function of time and show that the maximum possible velocity for a fall from rest is $v = mg/k$.

24. A spring of rest length L_a (no tension) is connected to a support at one end and has a mass M attached at the other. Neglect the mass of the spring, the dimension of the mass M, and assume that the motion is confined to a vertical plane. Also, assume that the spring only stretches without bending but it can swing in the plane.

(a) Using the angular displacement of the mass from the vertical and the length that the string has stretched from its rest length (hanging with the mass m), find Lagrange's equations.

(b) Solve these equations for small stretching and angular displacements.

(c) Solve the equations in part (a) to the next order in both stretching and angular displacement. This part is amenable to hand calculations. Using some reasonable assumptions about the spring constant, the mass, and the rest length, discuss the motion. Is a resonance likely under the assumptions stated in the problem?

(d) (For analytic computer programs.) Consider the spring to have a total mass $m \ll M$. Neglecting the bending of the spring, set up Lagrange's equations correctly to first order in m and the angular and linear displacements.

(e) (For numerical computer analysis.) Make sets of reasonable assumptions of the constants in part (a) and make a single plot of the two coordinates as functions of time.

CHAPTER 2

Variational Principles and Lagrange's Equations

2.1 ■ HAMILTON'S PRINCIPLE

The derivation of Lagrange's equations presented in Chapter 1 started from a consideration of the instantaneous state of the system and small virtual displacements about the instantaneous state, i.e., from a "differential principle" such as D'Alembert's principle. It is also possible to obtain Lagrange's equations from a principle that considers the entire motion of the system between times t_1 and t_2, and small virtual variations of this motion from the actual motion. A principle of this nature is known as an "integral principle."

Before presenting the integral principle, the meaning attached to the phrase "motion of the system between times t_1 and t_2" must first be stated in more precise language. The instantaneous configuration of a system is described by the values of the n generalized coordinates $q_1, \ldots, q_n$, and corresponds to a particular point in a Cartesian hyperspace where the q's form the n coordinate axes. This n-dimensional space is therefore known as configuration space. As time goes on, the state of the system changes and the system point moves in configuration space tracing out a curve, described as "the path of motion of the system." The "motion of the system," as used above, then refers to the motion of the system point along this path in *configuration space*. Time can be considered formally as a parameter of the curve; to each point on the path there is associated one or more values of the time. Note that configuration space has no necessary connection with the physical three-dimensional space, just as the generalized coordinates are not necessarily position coordinates. The path of motion in configuration space has no resemblance to the path in space of any actual particle; each point on the path represents the *entire* system configuration at some given instant of time.

The integral *Hamilton's principle* describes the motion of those mechanical systems for which all forces (except the forces of constraint) are derivable from a generalized scalar potential that may be a function of the coordinates, velocities, and time. Such systems will be denoted as *monogenic*. Where the potential is an explicit function of position coordinates only, then a monogenic system is also conservative (cf. Section 1.2).

For monogenic systems, Hamilton's principle can be stated as

> *The motion of the system from time t_1 to time t_2 is such that the line integral (called the* action *or the* action integral*),*

$$I = \int_{t_1}^{t_2} L\,dt, \tag{2.1}$$

where $L = T - V$, has a stationary value for the actual path of the motion.

That is, out of all possible paths by which the system point could travel from its position at time t_1 to its position at time t_2, it will actually travel along that path for which the value of the integral (2.1) is stationary. By the term "stationary value" for a line integral, we mean that the integral along the given path has the same value to within first-order infinitesimals as that along all neighboring paths (i.e., those that differ from it by infinitesimal displacements). (Cf. Fig. 2.1.) The notion of a stationary value for a line integral thus corresponds in ordinary function theory to the vanishing of the first derivative.

We can summarize Hamilton's principle by saying that the motion is such that the *variation* of the line integral I for fixed t_1 and t_2 is zero:

$$\delta I = \delta \int_{t_1}^{t_2} L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)\,dt = 0. \tag{2.2}$$

Where the system constraints are holonomic, Hamilton's principle, Eq. (2.2), is both a necessary and sufficient condition for Lagrange's equations, Eqs. (1.57). Thus, it can be shown that Hamilton's principle follows directly from Lagrange's equations. Instead, however, we shall prove the converse, namely, that Lagrange's equations follow from Hamilton's principle, as being the more important theorem. That Hamilton's principle is a sufficient condition for deriving the equations of motion enables us to construct the mechanics of monogenic systems from Hamilton's principle as the basic postulate rather than Newton's laws of motion. Such a formulation has advantages; e.g., since the integral I is obviously invariant to the system of generalized coordinates used to express L, the equations of motion must always have the Lagrangian form no matter how the generalized coordinates

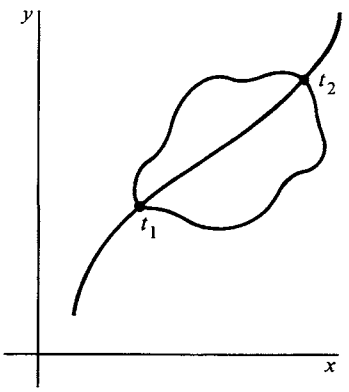

FIGURE 2.1 Path of the system point in configuration space.

are transformed. More important, the formulation in terms of a variational principle is the route that is generally followed when we try to describe apparently nonmechanical systems in the mathematical clothes of classical mechanics, as in the theory of fields.

2.2 ■ SOME TECHNIQUES OF THE CALCULUS OF VARIATIONS

Before demonstrating that Lagrange's equations do follow from (2.2), we must first examine the methods of the calculus of variations, for a chief problem of this calculus is to find the curve for which some given line integral has a stationary value. (See website for necessary comments.)

Consider first the problem in an essentially one-dimensional form: We have a function $f(y, \dot{y}, x)$ defined on a path $y = y(x)$ between two values x_1 and x_2, where $\dot{y}$ is the derivative of y with respect to x. We wish to find a particular path $y(x)$ such that the line integral J of the function f between x_1 and x_2,

$$\dot{y} \equiv \frac{dy}{dx},$$

$$J = \int_{x_1}^{x_2} f(y, \dot{y}, x)\, dx, \tag{2.3}$$

has a stationary value relative to paths differing infinitesimally from the correct function $y(x)$. The variable x here plays the role of the parameter t, and we consider only such varied paths for which $y(x_1) = y_1$, $y(x_2) = y_2$. (Cf. Fig. 2.2.) Note that Fig. 2.2 does *not* represent configuration space. In the one-dimensional configuration space, both the correct and varied paths are the segment of the straight line connecting y_1 and y_2; the paths differ only in the functional relation between y and x. The problem is one-dimensional, y is not a coordinate, it is a function of x.

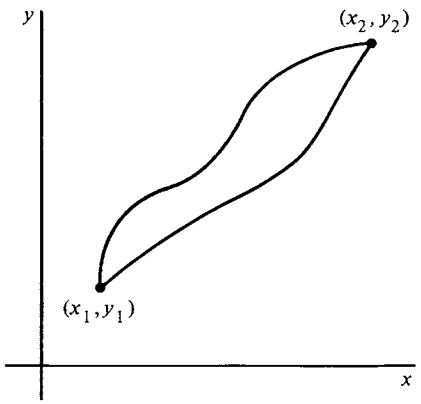

FIGURE 2.2 Varied paths of the function of $y(x)$ in the one-dimensional extremum problem.

We put the problem in a form that enables us to use the familiar apparatus of the differential calculus for finding the stationary points of a function. Since J must have a stationary value for the correct path relative to *any* neighboring path, the variation must be zero relative to *some* particular set of neighboring paths labeled by an infinitesimal parameter α. Such a set of paths might be denoted by $y(x, \alpha)$, with $y(x, 0)$ representing the correct path. For example, if we select any function $\eta(x)$ that vanishes at $x = x_1$ and $x = x_2$, then a possible set of varied paths is given by

$$y(x, \alpha) = y(x, 0) + \alpha\eta(x). \tag{2.4}$$

For simplicity, it is assumed that both the correct path $y(x)$ and the auxiliary function $\eta(x)$ are well-behaved functions—continuous and nonsingular between x_1 and x_2, with continuous first and second derivatives in the same interval. For any such parametric family of curves, J in Eq. (2.3) is also a function of α:

$$J(\alpha) = \int_{x_1}^{x_2} f\,(y(x, \alpha), \dot{y}(x, \alpha), x)\,dx, \tag{2.5}$$

and the condition for obtaining a stationary point is the familiar one that

$$\left(\frac{dJ}{d\alpha}\right)_{\alpha=0} = 0. \tag{2.6}$$

By the usual methods of differentiating under the integral sign, we find that

$$\frac{dJ}{d\alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \alpha}\right) dx. \tag{2.7}$$

Consider the second of these integrals:

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \alpha}\,dx = \int_{x_1}^{x_2} \frac{\partial f}{\partial \dot{y}}\frac{\partial^2 y}{\partial x\,\partial \alpha}\,dx.$$

Integrating by parts, the integral becomes

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial \dot{y}}\frac{\partial^2 y}{\partial x\,\partial \alpha}\,dx = \frac{\partial f}{\partial \dot{y}}\frac{\partial y}{\partial \alpha}\bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right)\frac{\partial y}{\partial \alpha}\,dx. \tag{2.8}$$

The conditions on all the varied curves are that they pass through the points (x_1, y_1), (x_2, y_2), and hence the partial derivative of y with respect to α at x_1 and x_2 must vanish. Therefore, the first term of (2.8) vanishes and Eq. (2.7) reduces to

$$\frac{dJ}{d\alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}}\right)\frac{\partial y}{\partial \alpha}\,dx.$$

The condition for a stationary value, Eq. (2.6), is therefore equivalent to the equation

$$\int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial \dot{y}} \right) \left(\frac{\partial y}{\partial \alpha} \right)_0 dx = 0. \tag{2.9}$$

Now, the partial derivative of y with respect to α occurring in Eq. (2.9) is a function of x that is arbitrary except for continuity and end point conditions. For example, for the particular parametric family of varied paths given by Eq. (2.4), it is the arbitrary function $\eta(x)$. We can therefore apply to Eq (2.9) the so-called "fundamental lemma" of the calculus of variations, which says if

$$\int_{x_1}^{x_2} M(x)\eta(x)\, dx = 0 \tag{2.10}$$

for all arbitrary functions $\eta(x)$ continuous through the second derivative, then $M(x)$ must identically vanish in the interval (x_1, x_2). While a formal mathematical proof of the lemma can be found in texts on the calculus of variations, the validity of the lemma is easily seen intuitively. We can imagine constructing a function η that is positive in the immediate vicinity of any chosen point in the interval and zero everywhere else. Equation (2.10) can then hold only if $M(x)$ vanishes at that (arbitrarily) chosen point, which shows M must be zero throughout the interval. From Eq. (2.9) and the fundamental lemma, it therefore follows that J can have a stationary value only if

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial \dot{y}} \right) = 0. \tag{2.11}$$

The differential quantity,

$$\left(\frac{\partial y}{\partial \alpha} \right)_0 d\alpha \equiv \delta y, \tag{2.12}$$

represents the infinitesimal departure of the varied path from the correct path $y(x)$ at the point x and thus corresponds to the virtual displacement introduced in Chapter 1 (hence the notation δy). Similarly, the infinitesimal variation of J about the correct path can be designated

$$\left(\frac{dJ}{d\alpha} \right)_0 d\alpha \equiv \delta J. \tag{2.13}$$

The assertion that J is stationary for the correct path can thus be written

$$\delta J = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial \dot{y}} \right) \delta y\, dx = 0,$$

requiring that $y(x)$ satisfy the differential equation (2.11). The δ-notation, introduced through Eqs. (2.12) and (2.13), may be used as a convenient shorthand for treating the variation of integrals, remembering always that it stands for the manipulation of parametric families of varied paths such as Eq. (2.4).

Some simple examples of the application of Eq. (2.11) (which clearly resembles a Lagrange equation) may now be considered:

1. *Shortest distance between two points in a plane.* An element of length in a plane is

$$ds = \sqrt{dx^2 + dy^2}$$

and the total length of any curve going between points 1 and 2 is

$$I = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

The condition that the curve be the shortest path is that I be a minimum. This is an example of the extremum problem as expressed by Eq. (2.3), with

$$f = \sqrt{1 + \dot{y}^2}.$$

Substituting in (2.11) with

$$\frac{\partial f}{\partial y} = 0, \qquad \frac{\partial f}{\partial \dot{y}} = \frac{\dot{y}}{\sqrt{1 + \dot{y}^2}},$$

we have

$$\frac{d}{dx}\left(\frac{\dot{y}}{\sqrt{1 + \dot{y}^2}}\right) = 0$$

or

$$\frac{\dot{y}}{\sqrt{1 + \dot{y}^2}} = c,$$

where c is constant. This solution can be valid only if

$$\dot{y} = a,$$

where a is a constant related to c by

$$a = \frac{c}{\sqrt{1 - c^2}}.$$

But this is clearly the equation of a straight line,

$$y = ax + b,$$

where b is another constant of integration. Strictly speaking, the straight line has only been proved to be an extremum path, but for this problem it is obviously also a minimum. The constants of integration, a and b, are determined by the condition that the curve pass through the two end points, (x_1, y_1), (x_2, y_2).

In a similar fashion we can obtain the shortest distance between two points on a sphere, by setting up the arc length on the surface of the sphere in terms of the angle coordinates of position on the sphere. In general, curves that give the shortest distance between two points on a given surface are called the *geodesics* of the surface. (See website about part 2 below.)

2. *Minimum surface of revolution.* Suppose we form a surface of revolution by taking some curve passing between two fixed end points (x_1, y_1) and (x_2, y_2) defining the xy plane, and revolving it about the y axis (cf. Fig. 2.3a). The problem then is to find that curve for which the surface area is a minimum. The area of a strip of the surface is $2\pi x\, ds = 2\pi x\sqrt{1+\dot{y}^2}\, dx$, and the total area is

$$2\pi \int_1^2 x\sqrt{1+\dot{y}^2}\, dx.$$

The extremum of this integral is again given by (2.11) where

$$f = x\sqrt{1+\dot{y}^2}$$

and

$$\frac{\partial f}{\partial y} = 0, \qquad \frac{\partial f}{\partial \dot{y}} = \frac{x\dot{y}}{\sqrt{1+\dot{y}^2}}.$$

Equation (2.11) becomes in this case

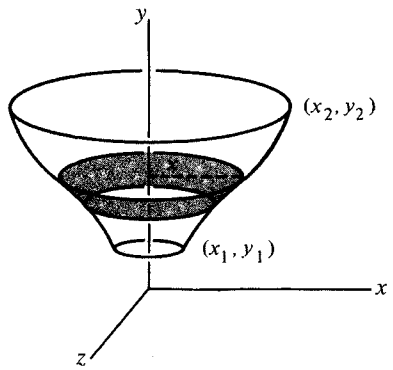

FIGURE 2.3a Minimum surface of revolution. Note that this figure is drawn for y_1 and y_2 having the same sign relative to the rotation axis. This is not assumed in the general solution.

$$\frac{d}{dx}\left(\frac{x\dot{y}}{\sqrt{1+\dot{y}^2}}\right) = 0$$

or

$$\frac{x\dot{y}}{\sqrt{1+\dot{y}^2}} = a,$$

where a is some constant of integration clearly smaller than the minimum value of x. Squaring the above equation and factoring terms, we have

$$\dot{y}^2(x^2 - a^2) = a^2,$$

or solving,

$$\frac{dy}{dx} = \frac{a}{\sqrt{x^2 - a^2}}.$$

The general solution of this differential equation, in light of the nature of a, is

$$y = a\int \frac{dx}{\sqrt{x^2 - a^2}} + b = a \operatorname{arc\,cosh}\frac{x}{a} + b$$

or

$$x = a\cosh\frac{y-b}{a},$$

which is the equation of a catenary. Again the two constants of integration, a and b, are determined in principle by the requirements that the curve pass through the two given end points, as shown in Fig. 2.3b.

Curves satisfying the preceding equation all scale as x/a and y/a with one independent parameter b/a. This suggests that when the solutions are examined in detail they turn out to be a great deal more complicated than these considera-

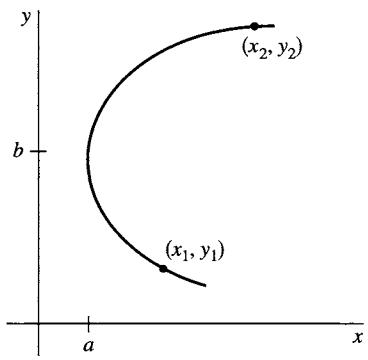

FIGURE 2.3b General catenary solution for minimum surface of revolution.

tions suggest. For some pairs of end points, unique constants of integration a and b can be found. But for other end points, it is possible to draw two different catenary curves through the end points, while for additional cases no possible values can be found for a and b. Further, recall that Eq. (2.11) represents a condition for finding curves $y(x)$ continuous through the second derivative that render the integral stationary. The catenary solutions therefore do not always represent minimum values, but may represent "inflection points" where the length of the curve is stationary but not minimum.

For certain combinations of end points (an example is x_1 and x_2 both positive and both much smaller than $y_2 - y_1$), the absolute minimum in the surface of revolution is provided (cf. Exercise 8) by a curve composed of straight line segments—from the first end point parallel to the x axis until the y axis is reached, then along the y axis until the point $(0, y_2)$ and then out in a straight line to the second end point corresponding to the area $\pi(x_1^2 + x_2^2)$. This curve results when $a = 0$, forcing either $x = 0$ or $y =$ constant. Since this curve has discontinuous first derivatives, we should not expect to find it as a solution to Eq. (2.11).

This example is valuable in emphasizing the restrictions that surround the derivation and the meaning of the stationary condition. Exercises 7 and 8 examine the conditions for the pathological behavior for a symmetric example. More information can be found in many texts on the calculus of variations.

3. *The brachistochrone problem.* (See Fig. 2.4a.) This well-known problem is to find the curve joining two points, along which a particle falling from rest under the influence of gravity travels from the higher to the lower point in the least time.

If v is the speed along the curve, then the time required to fall an arc length ds is ds/v, and the problem is to find a minimum of the integral

$$t_{12} = \int_1^2 \frac{ds}{v}.$$

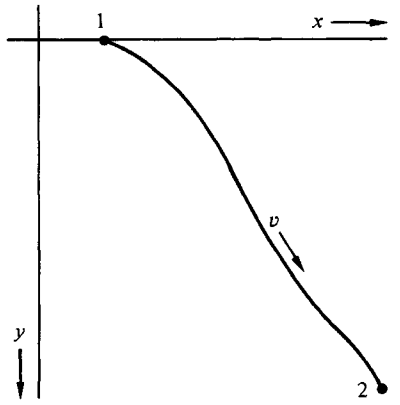

FIGURE 2.4a The brachistochrone problem.

If y is measured down from the initial point of release, the conservation theorem for the energy of the particle can be written as

$$\tfrac{1}{2}mv^2 = mgy$$

or

$$v = \sqrt{2gy}.$$

Then the expression for t_{12} becomes

$$t_{12} = \int_1^2 \frac{\sqrt{1+\dot{y}^2}}{\sqrt{2gy}}\,dx,$$

and f is identified as

$$f = \sqrt{\frac{1+\dot{y}^2}{2gy}}.$$

The integration of Eq. (2.11) with this form for f is straightforward and is left as an exercise.

The parametric solution in terms of its one parameter, a, given by

$$x = a(\phi - \sin\phi),\quad y = a(1 - \cos\phi),$$

is sketched in Fig. 2.4b for the first cycle ($0 \le x \le 2\pi a$) and the beginning of the second cycle. Three cases of solutions are indicated. A power-series expansion of the solution for the limit $y \ll a$ gives

$$y = a\sqrt[3]{\frac{9}{2}(x/a)^2}.$$

The brachistochrone problem is famous in the history of mathematics, for it was the analysis of this problem by John Bernoulli that led to the formal foundation of the calculus of variations.

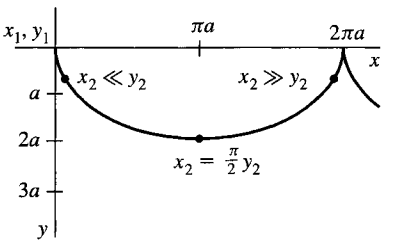

FIGURE 2.4b Cycloid solution to the brachistochrone problem showing positions on the curve for the three cases $x_2 \ll y_2$, $x_2 = \frac{\pi}{2}y_2$, and $x_2 \gg y_2$.

2.3 ■ DERIVATION OF LAGRANGE'S EQUATIONS FROM HAMILTON'S PRINCIPLE

The fundamental problem of the calculus of variations is easily generalized to the case where f is a function of many independent variables y_i, and their derivatives $\dot{y}_i$. (Of course, all these quantities are considered as functions of the parametric variable x.) Then a variation of the integral J,

$$\delta J = \delta \int_1^2 f(y_1(x); y_2(x), \ldots, \dot{y}_1(x); \dot{y}_2(x), \ldots, x)\, dx, \tag{2.14}$$

is obtained, as before, by considering J as a function of parameter α that labels a possible set of curves $y_1(x, \alpha)$. Thus, we may introduce α by setting

$$\begin{aligned} y_1(x, \alpha) &= y_1(x, 0) + \alpha\eta_1(x), \\ y_2(x, \alpha) &= y_2(x, 0) + \alpha\eta_2(x), \\ &\vdots \qquad\quad \vdots \qquad\quad \vdots \end{aligned} \tag{2.15}$$

where $y_1(x, 0)$, $y_2(x, 0)$, etc., are the solutions of the extremum problem (to be obtained) and η_1, η_2, etc., are independent functions of x that vanish at the end points and that are continuous through the second derivative, but otherwise are completely arbitrary.

The calculation proceeds as before. The variation of J is given in terms of

$$\frac{\partial J}{\partial \alpha}\, d\alpha = \int_1^2 \sum_i \left(\frac{\partial f}{\partial y_i} \frac{\partial y_i}{\partial \alpha}\, d\alpha + \frac{\partial f}{\partial \dot{y}_i} \frac{\partial \dot{y}_i}{\partial \alpha}\, d\alpha \right) dx. \tag{2.16}$$

Again we integrate by parts the integral involved in the second sum of Eq. (2.16):

$$\int_1^2 \frac{\delta f}{\partial \dot{y}_i} \frac{\partial^2 y_i}{\partial \alpha\, \partial x}\, dx = \frac{\partial f}{\partial \dot{y}_i} \frac{\partial y_i}{\partial \alpha} \bigg|_1^2 - \int_1^2 \frac{\partial y_i}{\partial \alpha} \frac{d}{dx} \left(\frac{\partial f}{\partial \dot{y}_i} \right) dx,$$

where the first term vanishes because all curves pass through the fixed end points. Substituting in (2.16), δJ becomes

$$\delta J = \int_1^2 \sum_i \left(\frac{\partial f}{\partial y_i} - \frac{d}{dx} \frac{\partial f}{\partial \dot{y}_i} \right) \delta y_i\, dx, \tag{2.17}$$

where, in analogy with (2.12), the variation δy_i is

$$\delta y_i = \left(\frac{\partial y_i}{\partial \alpha} \right)_0 d\alpha.$$

Since the y variables are independent, the variations δy_i are independent (e.g., the functions $\eta_i(x)$ will be independent of each other). Hence, by an obvious extension of the fundamental lemma (cf. Eq. (2.10)), the condition that δJ is zero

requires that the coefficients of the δy_i separately vanish:

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}_i} = 0, \qquad i = 1, 2, \ldots, n. \tag{2.18}$$

Equations (2.18) represent the appropriate generalization of (2.11) to several variables and are known as the *Euler-Lagrange differential equations*. Their solutions represent curves for which the variation of an integral of the form given in (2.14) vanishes. Further generalizations of the fundamental variational problem are easily possible. Thus, we can take f as a function of higher derivatives $\ddot{y}$, $\dddot{y}$, etc., leading to equations different from (2.18). Or we can extend it to cases where there are several parameters x_j and the integral is then multiple, with f also involving as variables derivatives of y_i with respect to each of the parameters x_j. Finally, it is possible to consider variations in which the end points are *not* held fixed.

For present purposes, what we have derived here suffices, for the integral in Hamilton's principle,

$$I = \int_1^2 L(q_i, \dot{q}_i, t)\, dt, \tag{2.19}$$

has just the form stipulated in (2.14) with the transformation

$$x \to t$$

$$y_i \to q_i$$

$$f(y_i, \dot{y}_i, x) \to L(q_i, \dot{q}_i, t).$$

In deriving Eqs. (2.18), we assumed that the y_i variables are independent. The corresponding condition in connection with Hamilton's principle is that the generalized coordinates q_i be independent, which requires that the constraints be holonomic. The Euler-Lagrange equations corresponding to the integral I then become the Lagrange equations of motion,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \qquad i = 1, 2, \ldots, n,$$

and we have accomplished our original aim, to show that Lagrange's equations follow from Hamilton's principle—for monogenic systems with holonomic constraints.

2.4 ■ EXTENDING HAMILTON'S PRINCIPLE TO SYSTEMS WITH CONSTRAINTS

In Section 1.3 we solved problems with holonomic constraints by choosing coordinates such that the constraint equations (1.37) become a trivial $0 = 0$ set of

equations. In this section we show that Hamilton's principle can be used to solve systems with holonomic constraints as well as certain types of non-holonomic systems.

First consider holonomic constraints. When we derive Lagrange's equations from either Hamilton's or D'Alembert's principle, the holonomic constraints appear in the last step when the variations in the q_i were considered independent of each other. However, the virtual displacements in the δq_I's may not be consistent with the constraints. If there are n variables and m constraint equations f_α of the form of Eq. (1.37), the extra virtual displacements are eliminated by the method of *Lagrange undetermined multipliers*.

We modify the integral in Eq. (2.19) to be

$$I = \int_1^2 \left(L + \sum_{\alpha=1}^{m} \lambda_\alpha f_a \right) dt, \tag{2.20}$$

and allow the q_α and the λ_α to vary independently to obtain $n + m$ equations. The variations of the λ_α's give the m constraint equations. The variations of the q_i's give

$$\delta I = \int_1^2 dt \left(\sum_{i=1}^{n} \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} + \sum_{\alpha=1}^{m} \lambda_a \frac{\partial f_a}{\partial q_i} \right) \delta q_i \right) = 0. \tag{2.21}$$

However, the δq_i's are not independent. We choose the λ_α's so that m of the equations are satisfied for arbitrary δq_i, and then choose the variations of the δq_i in the remaining $n - m$ equations independently. Thus we obtain m equations of the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} + \sum_{\alpha=1}^{m} \lambda_a \frac{\partial f_a}{\partial q_k} = 0, \tag{2.22}$$

for $k = 1, \ldots, m$. The equality follows from the choice of the λ_α's. We also have the same expressions as Eq. (2.22) for $k = m + 1, \ldots, n$, where the equality follows from the virtual variations of the δq_i's.

This solves the system at the expense of introducing m functions λ_α. We can understand this by considering that Eqs. (2.22), for $k = 1, \ldots, n$, can be written as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = -\sum_{\alpha=1}^{m} \lambda_a \frac{\partial f_a}{\partial q_k} = Q_k, \tag{2.23}$$

where the Q_k are generalized forces. The functions, Q_κ, have the magnitudes of the forces needed to produce the individual constraints; however, since the choice of the "+" in the third term of Eq. (2.22) is arbitrary, we can mathematically determine only the magnitudes of these generalized forces. You need to understand the physics to determine their directions.

As an example, consider a smooth solid hemisphere of radius a placed with its flat side down and fastened to the Earth whose gravitational acceleration is g. Place a small mass M at the top of the hemisphere with an infinitesimal displacement off center so the mass slides down without friction. Choose coordinates x, y, z centered on the base of the hemisphere with z vertical and the x-z plane containing the initial motion of the mass.

Let θ be the angle from the top of the sphere to the mass. The Lagrangian is $L = \frac{1}{2}M(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$. The initial conditions allow us to ignore the y coordinate, so the constraint equation is $a - \sqrt{x^2 + z^2} = 0$. Expressing the problem in terms of $r^2 = x^2 + z^2$ and $x/z = \cos\theta$, Lagrange's equations are $Ma\dot{\theta}^2 - Mg\cos\theta + \lambda = 0$, and $Ma^2\ddot{\theta} + Mga\sin\theta = 0$. Solve the second equation and then the first to obtain

$$\dot{\theta}^2 = -\frac{2g}{a}\cos\theta + \frac{2g}{a} \quad \text{and} \quad \lambda = Mg(3\cos\theta - 2).$$

So λ is the magnitude of the force keeping the particle on the sphere and since $\lambda = 0$ when $\theta = \cos^{-1}(\frac{2}{3})$, the mass leaves the sphere at that angle.

In general, nonholonomic constraints cannot be expressed by a variational principle. One of the exceptions is semi-holonomic constraints where the constraints can be written as a set of functions of the form

$$f_\alpha(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n; t) = 0, \tag{2.24}$$

where $\alpha = 1, 2, \ldots, m$. Equation (2.24) commonly appears in the restricted form

$$f_\alpha = \sum_{k=1}^{n} a_{ak}\dot{q}_k + a_0 = 0, \tag{2.25}$$

where the f_α are a set of nonintegrable differential expressions and the $a_{\alpha k}$ and a_0 are functions of the q_i and t. In these cases, since we cannot integrate the constraints, there are more variables than equations. However, we can treat the variations in the same fashion as before by writing*

$$\delta \int_{t_1}^{t_2} \left(L + \sum_{\alpha=1}^{m} \mu_a f_a \right) dt = 0, \tag{2.26}$$

where the symbol μ is used to distinguish these multipliers from the holonomic Lagrange multipliers. If we assume that $\mu_\alpha = \mu_\alpha(t)$, the equations resulting from the virtual displacements are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = Q_k = -\sum_{\alpha=1}^{m} \mu_a \frac{\partial f_a}{\partial \dot{q}_k}, \tag{2.27}$$

*J. Ray, *Amer. J. Phys.* **34** (1202), 1969; E. J. Saletan & A. H. Comer, *Amer. J. Phys.* **38** (892–897), 1970.

and the $\delta\mu_\alpha$ give the equations of constraint (2.23). These two sets (Eq. (2.26) and (2.27)) together constitute $n + m$ equations for the $n + m$ unknowns. Hence they can be interpreted as equivalent to an $n + m$ holonomic system with generalized forces Q_k. The generalization to $\mu_\alpha = \mu_\alpha(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n; t)$ is straightforward.

As an example, consider a particle with the Lagrangian

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z) \tag{2.28}$$

subject to the nonholonomic constraint

$$f(x, \dot{x}, y, \dot{y}, z) = \dot{x}y^2 + x\dot{y} + kz = 0, \tag{2.29}$$

with k a constant. The resulting equations of motion are

$$m\ddot{x} + \mu y^2 - \frac{\partial V}{\partial x} = 0, \tag{2.30a}$$

$$m\ddot{y} + \mu x - \frac{\partial V}{\partial y} = 0, \tag{2.30b}$$

and

$$m\ddot{z} - \frac{\partial V}{\partial z} = 0. \tag{2.30c}$$

We now solve the four equations ((2.29) and (3.30)) to find $x(t)$, $y(t)$, $z(t)$, and the multiplier $\mu(t)$.

In this process we have obtained more information than was originally sought. Not only do we get the q_k's we set out to find, but we also get $m\lambda_l$'s. What is the physical significance of the λ_l's? Suppose we remove the constraints on the system, but instead apply external forces Q'_k in such a manner as to keep the motion of the system unchanged. The equations of motion likewise remain the same. Clearly these extra applied forces must be equal to the forces of constraint, for they are the forces applied to the system so as to satisfy the condition of constraint. Under the influence of these forces Q'_k, the equations of motion are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = Q'_k. \tag{2.31}$$

But these must be identical with Eqs. (2.24). Hence, we can identify (2.25) with Q'_k, the generalized forces of constraint. In this type of problem we really do not eliminate the forces of constraint from the formulation. They are supplied as part of the answer.

Although it is not obvious, the version of Hamilton's principle adopted here for semiholonomic systems also requires that the constraints do no work in virtual displacements. This can be most easily seen by rewriting Hamilton's principle in