PEARSON NEW INTERNATIONAL EDITION

# Modern Physics
Randy Harris
Second Edition

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# Table of Contents

# The Dawn of a New Age

**Chapter Outline**



Introductory classical physics covers a body of knowledge that can explain a vast amount of what goes on around us, from the everyday to the abstract. Why does a bicycle go forward when the rider pushes downward on a pedal? What forms can energy take, and if it is conserved, why can't we reuse indefinitely? How do waves propagate energy and information without any matter moving from source to receiver? What are electric and magnetic fields?

However, early in the 20th century, revolutionary ideas arose that shook classical physics to its foundation. Even the most basic and "obvious" truths about space and time and the nature of the matter around us came under assault. In this chapter, we discuss several telling signs that something was wrong with the classical picture, then we survey the path ahead. (Certain topics from classical physics are particularly important to modern physics. A review of these topics is available on the companion website.)

## 1 Troubling Questions

The core topics of classical physics are mechanics, electromagnetism, and thermodynamics. Let us take a look at some of the problems that loomed in these areas at the dawn of the modern age.

Classical mechanics attained a cohesive form in the late 1600s with the work of Sir Isaac Newton. Triumphant in explaining the behaviors of macroscopic objects at ordinary speeds, Newton's work reigned unchallenged for centuries. As convincing as anything were its successes in celestial applications. When the orbit of the planet Uranus was found to deviate slightly from what Newton's laws predicted it should be, the instinctive response was to attribute the deviation not to any failure of those laws but rather to some unseen heavenly body. Newton's laws predicted its location, and Neptune was later found right where it should be. After such satisfying confirmation, it was natural to expect further examples. Mercury's orbit is somewhat elliptical, but the ellipse is not retraced again and again. It precesses—that is, its points of maximum radius advance slightly with each orbit about the Sun. Newton's laws predicted one rate of precession; actual observation differed. The discrepancy was again thought to be due to an unseen planet, but in this case, none could be found.

Maturing in the mid-19th century with James Clerk Maxwell's completion and integration of the laws of Gauss, Ampere, and Faraday, electromagnetism has been extremely successful. It superbly explains the physics behind telephonic communication and electrical power supply, which had become commonplace by the end of the 19th century, and it continues to prove its validity in countless applications to this day. Early on, however, doubts were raised about the theory's prediction of waves of electromagnetic radiation. For one thing, the lack of explicit reference to a medium of propagation seemed to put light in a special category among wave phenomena. Another problem had to do with the energy expected in electromagnetic radiation exchanged with matter. Charges in the matter should jiggle around at rates dependent on the temperature, producing and absorbing electromagnetic energy as they do. A standard wave calculation predicted that the electromagnetic intensity nearby should be infinite! Yet another perplexing question concerned the ability of light to eject electrons from a metal, known as the photoelectric effect. In the classical view, a light wave is simply a pair of self-propagating electric and magnetic fields spread diffusely through some region of space. If light encounters an electron in a metal, these fields should be able to transfer energy to the electron and knock it out of the metal. Light of low intensity might require considerable time to deposit enough energy, but a high intensity should knock electrons out at a high rate and—owing to its stronger electric and magnetic fields—should eject them with greater kinetic energy. In fact, even very low-intensity light can eject electrons immediately, and the kinetic energy of an ejected electron is completely independent of the light's intensity. The frequency of the light seemed to be the deciding factor, and classical electromagnetism could not explain why.

By the end of the 19th century, statistical thermodynamics had become one of the cornerstones of physics. Its laws had been established, and correct predictions were being made. A major step forward was the formulation of the equipartition theorem. This says that each independent degree of freedom possessed by a particle in a thermodynamic system should manifest $\frac{1}{2}k_BT$ of energy on average, where $k_B$ is the Boltzmann constant and $T$ is the temperature. There are three dimensions of translational freedom, leading to the famous formula for the average translational kinetic energy of a particle: $\frac{3}{2}k_BT$. In a solid, each atom has three additional degrees of freedom due to elastic potential energy in each dimension. With six total degrees of freedom per atom, the equipartition theorem predicts that a solid should have a heat capacity—energy per degree per mole—of $3k_BN_{Av}$, where $N_{Av}$ is Avogadro's number. This prediction supported an early empirical observation that many solids seem to have a heat capacity very near this value. However, even solids that adhered to the prediction at ordinary temperatures deviated noticeably at low temperatures, where the heat capacity seemed to drop off toward zero. To classical statistical physics, this was thoroughly baffling.

Perhaps attracting more scrutiny than any other classically inexplicable phenomenon at the turn of the 20th century was the subject of atomic spectra. Atoms emit only certain wavelengths of light. Why? With the discovery of the electron by J. J. Thomson in 1898, hopes of explaining spectra rose—for a small charged particle somehow jiggling around in an atom should emit electromagnetic radiation. Neither the proton nor the atomic nucleus was yet known, and Thomson's model of the atom assumed that its electrons were embedded in a uniform sea of positive charge. No plausible oscillations of the electrons in this model could explain the observations. About a dozen years later, the work of Ernest Rutherford and his students produced the now-familiar model of electrons orbiting a positive nucleus. Although the quantum age had by then begun, Rutherford's nuclear model was still classical, and it actually compounded the mystery. An orbiting electron would be accelerating continuously, and any time a charged particle accelerates, it radiates electromagnetic energy. The nuclear atom should be unstable, with the electron spiralling into the nucleus! Perhaps not surprisingly, this model also failed to explain the spectral evidence. Some of its basic elements survived, but it was to be profoundly altered by the new paradigm.

## 2 A Glimpse of the New World

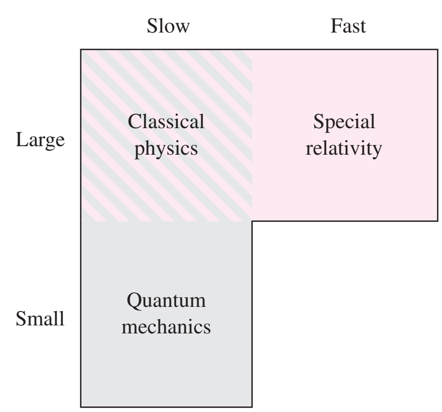
Much of modern physics rests on two basic ideas: First, space and time are not the absolutes they might seem to be. Second, things we might think of as particles may behave as waves, and vice versa. Although these ideas are now accepted as fundamental to physics, they initially met with considerable opposition.

The main reason they remained hidden for so long is that they involve behaviors not easily observed. And the reason they were not universally welcomed is that they often seem counterintuitive. These two reasons are related. A behavior certainly would not be counterintuitive if it were subject to simple observation. By the same token, we cannot justifiably claim to possess intuition about a phenomenon we cannot observe. Loosely speaking, modern physics is the study of the small and the fast, but no one has ever actually seen an individual atom nor has anyone ever traveled at a significant fraction of the speed of light relative to Earth. We must be careful not to apply preexisting notions based on experience to situations in which we have no experience.

The branch of modern physics dealing with space and time is special relativity. Although it is often said to be the physics that applies when objects or frames of reference are moving at very high speed, this is rather misleading, for it makes relativity sound like a special case. Relativity agrees with classical physics at low speeds, but it also succeeds at speeds comparable to that of light, where classical physics fails. Thus, classical mechanics is the special case. Of the many startling claims of special relativity, one is particularly helpful as an introduction and preparation for the challenges ahead: If passengers on a (very) high-speed train confirm that clocks at the front and back of their train strike noon simultaneously, observers on the ground will confirm that these clocks do not strike noon simultaneously. This discrepancy is very small at ordinary speeds, but it is *not* an optical illusion.

The other main branch of modern physics is quantum mechanics, which has its own challenging notions. In classical mechanics and electromagnetism, we treat the electron as a particle, but in small confines, it behaves as a diffuse wave. It does not have a specific location! All we can know are probabilities of finding the electron *if* an attempt were made to do so, and the probabilities are related to the amplitude of the wave. These claims are met with amusement and disbelief by some people, but upon them rest major areas of science, such as chemistry, modern electronics, materials science, modern optics (lasers), nuclear physics, and a host of others. We say that quantum mechanics applies in the realm of very small confines, not usually open to casual scrutiny, but as in the case of relativity, this is somewhat inaccurate. Quantum mechanics is correct for small and large and converges to the special case of classical mechanics in the limit of large things.

It is natural to ask how classical mechanics can be a special case of two different things. The figure schematically depicts the realms of applicability of the different theories. The special relativity we study in this text is valid only for large things. It is not correct quantum mechanically. Similarly, the quantum mechanics we study is, with a few noted exceptions, not relativistically correct; rather, it is valid only for slow-moving things. Classical physics is the region where these two overlap. The region conspicuously missed by both regions is the realm of the small and fast. This is the focus of high-energy physics—the search for the fundamental structure of the universe.




Classical physics is tremendously successful in its realm, but modern physics is truly an eye-opener. On to the new world!

# Waves and Particles I: Electromagnetic Radiation Behaving as Particles

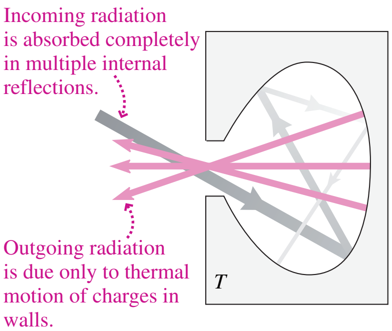
**Chapter Outline**



We now begin our investigation of quantum mechanics. In some sense, quantum mechanics is the study of small things—so small that it is essentially impossible to observe them without affecting their very behavior. For example, the simplest way of observing an object is to look at it. But to do that, light must be bounced off it, and light carries energy, some of which will unavoidably be transferred to the object. Ordinarily, the effect is inconsequential, but if the object is very small, such as a single electron, it might be significant. Thus, we shouldn't be too surprised that the behavior might vary, depending on how the observation is made.

A cornerstone of quantum mechanics is **wave-particle duality**: Things may behave as waves or as discrete particles, depending on the situation. The "situation" might be imposed by a deliberate experiment or governed simply by the dimensions of the region where the thing is confined. Two of the most important things we study are massive objects and electromagnetic radiation. In classical situations, our observations reveal electromagnetic radiation behaving as waves and massive objects as discrete particles. We now look at the nonclassical side of the coin. In this chapter, we study the complementary topic—electromagnetic radiation behaving as a collection of discrete particles. We begin with a brief look at the discovery that sparked the quantum revolution.

**Figure 1** Radiation exits a cavity through a hole, which behaves as a blackbody.




## 1 Blackbody Radiation: A New Fundamental Constant

The quantum age dawned with the work of Max Planck in the year 1900. Planck was trying to find a theory that would explain **blackbody radiation**. All materials emit electromagnetic radiation, because they contain charged particles that jiggle around, and an accelerating charge radiates electromagnetic energy. The amount of energy radiated depends on the average energy of the motion, which, in turn, depends on the temperature. For example, coals radiate invisible infrared energy even when cold, but when heated, they emit more radiation, much of it in the red end of the spectrum. They visibly glow "red hot." Most materials, however, also *reflect* electromagnetic energy. A **blackbody** is defined to be any object from which electromagnetic radiation emanates solely due to the thermal motion of its charges. Any radiation that *strikes* it must be absorbed rather than reflected, hence the name. (The term must not be taken too literally. The Sun's surface, from which reflection is insignificant, is a blackbody.)

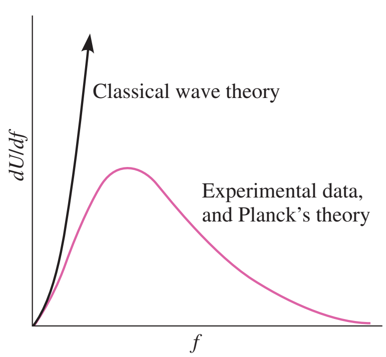
While coal is a good approximation, fabricating a true blackbody might seem problematic. Imagine, however, an object with an interior cavity and a small hole connecting it to the exterior, depicted in Figure 1. Any radiation entering the hole would reflect from the cavity's inner surface many times, losing energy to the object at each reflection. Essentially none would reflect back directly through the hole. On the other hand, all areas of the inner surface contain charges in thermal motion, constantly absorbing electromagnetic energy and reradiating it as they jiggle around. They will furthermore be in equilibrium with the electromagnetic energy in the cavity—the charges and the radiation will have the same temperature, $T$. The portion of the radiation leaking out of the small hole will be characteristic of this temperature, so the *hole* behaves as a blackbody of temperature $T$.

Experiment demonstrates that the energy emitted by a blackbody, or equivalently a cavity, is small at low frequency, reaches a maximum, then falls again toward zero thereafter. This is illustrated by the experimental curve in Figure 2, which plots electromagnetic energy $dU$ per frequency range $df$, known as **spectral energy density**. Classical *theory*, on the other hand, differed. If the electromagnetic radiation in a cavity behaves strictly as sinusoidally oscillating *waves* of arbitrary amplitude, the average energy of a wave of any given frequency should be $k_B T$. Multiplying this by a factor that accounts for the *number* of different waves per frequency range $df$ in volume $V$, the classical prediction for spectral energy density is

$$\frac{dU}{df} = k_B T \times \frac{8\pi V}{c^3} f^2 \qquad \text{Spectral energy density via classical wave theory}$$

Something is certainly wrong here, for as Figure 2 shows, this parabolic function diverges as $f$ increases without bound. If true, all materials would radiate infinite power.

**Figure 2** Experiment shows that as frequency increases, the blackbody spectral energy density reaches a maximum, then falls off. Classical wave theory predicts a divergence.



Planck found that he could match the experimental data with a curious assumption: The energy at frequency $f$ is somehow restricted to $E = nhf$, where $n$ is an integer and $h$ is a constant. The specific error in classical wave theory is in the average energy of a given wave, which is obtained by integrating over an assumed continuum of possible energy values. Under Planck's assumption, these values are discrete, so *the integral becomes a sum*, with a notably different result. Replacing the $k_B T$ of wave theory with Planck's result, the spectral energy density becomes

$$\frac{dU}{df} = \frac{hf}{e^{hf/k_B T} - 1} \times \frac{8\pi V}{c^3} f^2 \tag{1}$$

Planck's spectral energy density

which fits the experimental curve perfectly.

Planck's constant
$h = 6.63 \times 10^{-34}\ \text{J}\cdot\text{s}$

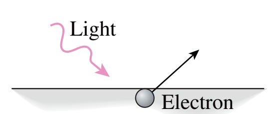
The value Planck quoted for $h$, the now famous **Planck's constant**, was $6.63 \times 10^{-34}\ \text{J}\cdot\text{s}$. Naturally, it was the value that matched his formula to the experimental curve. While it is tempting to question why he couldn't derive it, the fact that it was even *possible* to match the curve was significant. Although varying $h$ would change the energy density's magnitude at all frequencies, it would not alter its shape. Had there been no merit in Planck's assumption, it would have been the most bizarre coincidence for that shape to match the experimental curve. With hindsight, we now realize that Planck's constant *cannot* be derived, for it is one of nature's fundamental constants (e.g., the universal gravitational constant $G$), all of which are a matter of experimental observation. For the discovery, Planck was awarded the 1918 Nobel Prize.

Planck's spectral energy density is the crucial link between temperature and electromagnetic radiation. Interestingly, although the assumption $E = nhf$, on which Planck based his formula, might suggest electromagnetic radiation behaving as an integral number of particles of energy $hf$, Planck hesitated at the new frontier—others carried the revolution forward. Let us take a look at the next major step.

## 2 The Photoelectric Effect

Classically, electromagnetic radiation is a wave. The energy it carries is diffuse, distributed continuously along a broad wave front, and its intensity—energy per unit time per unit area—is proportional to $E_0^2$, where $E_0$ is the amplitude of the electric field oscillations. In the 1880s, Heinrich Hertz demonstrated that a light beam directed at the surface of a metal could liberate electrons. This is called the **photoelectric effect**—light producing a flow of electricity—and is depicted in Figure 3. It was also known that a certain amount of energy is required simply to free an electron. The electron is bound to the metal; pulling it loose takes energy; and any surplus becomes the freed electron's kinetic energy. The minimum energy required to free an electron, the **work function** $\phi$, is a characteristic of the particular metal. Table 1 lists some values (subject to variation, depending on impurities and other factors).

**Figure 3** The photoelectric effect: Light liberating an electron from a metal surface.




If light were strictly a wave, this effect should have several specific traits. First, if light of one wavelength is able to eject electrons, then light of any wavelength should be able to do it. Independent of the wavelength, the rate at which energy arrives (the intensity)—and therefore the rate at which electrons

**TABLE 1**

| Metal | Work Function $\phi$ (in eV) |
|---|---|
| Cesium | 1.9 |
| Potassium | 2.2 |
| Sodium | 2.3 |
| Magnesium | 3.7 |
| Zinc | 4.3 |
| Chromium | 4.4 |
| Tungsten | 4.5 |

are ejected—could be made arbitrarily large simply by increasing $E_0$. Second, if the intensity is low, then even though electrons might still be ejected, a measurable time lag should arise. Because a wave is diffuse, considerable time might be needed for enough energy to accumulate in the electron's vicinity. (See Exercise 16.) Finally, at any given frequency, if the intensity is increased, the departing electrons should be more energetic. A stronger electric field should produce a larger acceleration.

Imagine the experimenter's surprise when weak light of 500 nm wavelength ejects electrons from sodium, with no time lag, while light of 600 nm wavelength cannot, even at *many times* the intensity. Moreover, the energy of the electrons liberated by the 500 nm light is completely independent of the intensity. Classically, this cannot be explained!

In 1905, Albert Einstein proposed the following explanation: The light is behaving as a collection of particles, called **photons**, each with energy given by

Energy of a photon

$$E = hf \tag{2}$$

where $h$ is Planck's constant. A given electron is ejected by a *single* photon, with the photon transferring all its energy to the electron and then disappearing—multiple photons very rarely gang up on one electron. If the light's frequency is too low, such that the photon energy $hf$ is less than the work function $\phi$, then there is simply insufficient energy in any given photon to free an electron. *So none are freed, no matter how high the intensity;* no matter how abundant the photons. (The photon energy becomes internal energy or reflected light.) However, if the frequency is high enough, such that $hf > \phi$, then electrons can be ejected. The kinetic energy given to the electron would then be the difference between the photon's energy and the energy $\phi$ required to free the electron from the metal.

Photoelectric effect

$$\mathrm{KE}_{\mathrm{max}} = hf - \phi \tag{3}$$

The subscript "max" arises because $\phi$ is the energy needed to free the *least* strongly bound electrons. Others may also be freed, but less of the photon's energy would then be left for kinetic energy.

Einstein's interpretation of the photoelectric effect explains not only the observation that a certain minimum frequency is required but also the other classically unexpected results. If a single photon—a *particle of concentrated energy* rather than a diffuse wave—does have enough energy, ejection should be immediate, with no time lag. Also, the electron's kinetic energy should depend only on the energy of the single photon—the frequency—not on how many strike the metal per unit time (the intensity). In all respects, Einstein's explanation agrees with the experimental evidence, and the achievement earned him the 1921 Nobel Prize in physics.

## EXAMPLE 1

Light of 380 nm wavelength is directed at a metal electrode. To determine the energy of electrons ejected, an opposing electrostatic potential difference is established between it and another electrode, as shown in Figure 4. The current of photoelectrons from one to the other is stopped completely when the potential difference is

1.10 V. Determine (a) the work function of the metal and (b) the maximum-wavelength light that can eject electrons from this metal.

**SOLUTION**

(a) In the region between the electrodes, the electrons lose kinetic energy as they gain potential energy. If a potential energy difference of $qV = (1.6 \times 10^{-19}\,\text{C})(1.10\,\text{V}) = 1.76 \times 10^{-19}\,\text{J} = 1.10\,\text{eV}$ is the most they can surmount, their kinetic energy leaving the first electrode must be no larger than 1.10 eV. The potential difference that barely stops the flow is known as the **stopping potential**. Using equation (3),

$$1.76 \times 10^{-19}\,\text{J} = (6.63 \times 10^{-34}\,\text{J}\cdot\text{s})\left(\frac{3 \times 10^8\,\text{m/s}}{380 \times 10^{-9}\,\text{m}}\right) - \phi$$

$$\Rightarrow \phi = 3.47 \times 10^{-19}\,\text{J} = 2.17\,\text{eV}$$

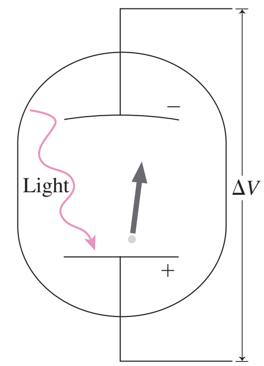
(b) If the wavelength of the light were increased to $\lambda'$, the frequency—and thus the photon energy—would decrease. The limit for ejecting electrons is when an incoming photon has only enough energy to free an electron from the metal, with none left for kinetic energy. Again using equation (3),

$$0 = hf' - \phi = (6.63 \times 10^{-34}\,\text{J}\cdot\text{s})\left(\frac{3 \times 10^8\,\text{m/s}}{\lambda'}\right) - 3.47 \times 10^{-19}\,\text{J}$$

$$\Rightarrow \lambda' = 573\,\text{nm}$$

Wavelengths longer than 573 nm have insufficient energy per photon, so no photoelectrons are produced. The maximum wavelength for which electrons are freed is called the **threshold wavelength**, and the corresponding minimum frequency is the **threshold frequency**.

**Figure 4** Current between the electrodes stops when the opposing potential energy difference equals the maximum kinetic energy of the photoelectrons.




The central point in Einstein's explanation of the photoelectric effect is that electromagnetic radiation appears to be behaving as a collection of *particles*, each with a discrete energy. Something that is discrete, as opposed to continuous, is said to be **quantized**. In the photoelectric effect, the energy in light is quantized.

## EXAMPLE 2

How many photons per second emanate from a 10 mW 633 nm laser?

**SOLUTION**

For each photon,

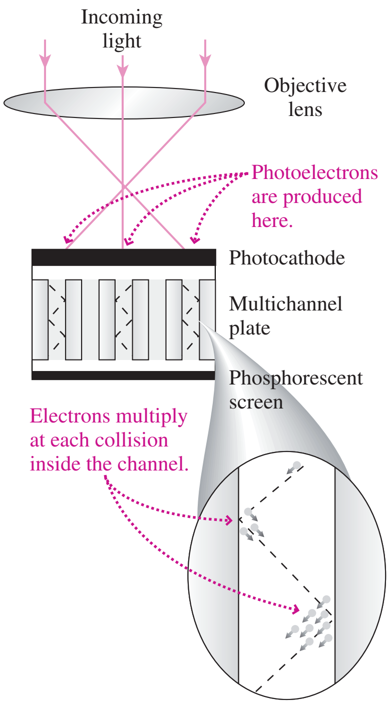
$$E = hf = h\frac{c}{\lambda} = (6.63 \times 10^{-34}\,\text{J}\cdot\text{s})\left(\frac{3 \times 10^8\,\text{m/s}}{633 \times 10^{-9}\,\text{m}}\right) = 3.14 \times 10^{-19}\,\text{J}$$

To find number of particles per unit time, we divide energy per unit time by energy per particle:

$$\frac{\text{number of particles}}{\text{time}} = \frac{10 \times 10^{-3}\,\text{J/s}}{3.14 \times 10^{-19}\,\text{J/particle}} = 3.18 \times 10^{16}\,\text{particles/s}$$

Clearly, photons are rather "small," and it is easy to see how a light beam could appear continuous.

**Figure 5** In a night vision device, a light image becomes an image of free electrons, amplified in a multichannel plate and then revealed on a screen.




The photoelectric effect has long been used in simple light sensors, where light intensity registers as a photocurrent, but it is also used in more sophisticated ways. Let us take a look at one.

### REAL-WORLD EXAMPLE NIGHT VISION

By "replacing" a photon with an electron, whose charge makes it easier to "amplify," the photoelectric effect is a common front end on optical imaging systems. One example is the night vision device (NVD). A typical NVD is shown schematically in Figure 5. An objective lens focuses an optical image onto a thin piece of material, called a photocathode, where the photoelectric effect transforms it into an image of freed electrons. Naturally, the dimmer the light, the fewer the photoelectrons. Amplification is accomplished via a microchannel plate. This element has hundreds of thousands of channels per square centimeter. An electron entering a channel at one end, driven through by a potential difference, knocks off other electrons at each collision with the channel walls, emerging at the other end with about 10,000 fellow electrons. (Macroscopic objects that work this way are known as photomultiplier tubes and are widely used in astronomy and particle physics, as well as in medical imaging.) This greatly amplified signal then strikes a phosphorescent screen to produce the final visible image in the phosphor's characteristic color.

Optimizing photocathode materials is front-line research, but we can easily grasp one of the basic constraints.

#### Applying the Physics

(a) If we wish a light-sensing device relying on the photoelectric effect to be sensitive to the entire visible spectrum (400–700 nm), explain why zinc would be a poor choice of photocathode. See Table 1. (b) A common photocathode containing the alkali metals cesium, potassium, and sodium has a very low effective work function of about 1.4 eV. What wavelengths can it "see"?

**SOLUTION**

(a) Zinc has a work function of 4.3 eV. Example 1 showed that a material with a *lower* work function has a threshold wavelength of 573 nm. Thus, wavelengths in the orange and red end of the spectrum would not liberate any photoelectrons. Zinc would "see" even less of the visible spectrum.

(b) The work function equals the energy of the longest-wavelength photon that can free an electron. Before finding this wavelength, we note that it is quite common to express wavelengths in nanometers and energies in electronvolts, so a value for the product $hc$ in these units is very convenient. Exercise 27 shows that it is 1240 eV · nm. Thus,

$hc = 1240\ \text{eV} \cdot \text{nm}$

$$E = \frac{hc}{\lambda} \rightarrow 1.4\ \text{eV} = \frac{1240\ \text{eV}\cdot\text{nm}}{\lambda}$$

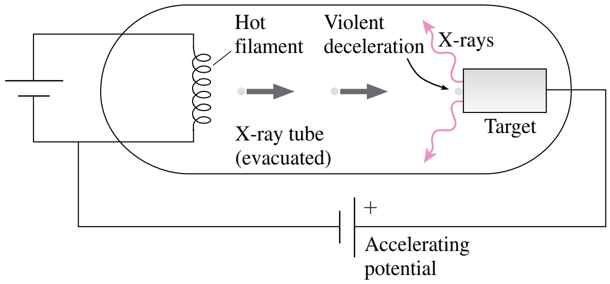
$$\Rightarrow \quad \lambda = 886\ \text{nm}$$

This material can see all wavelengths in the visible spectrum and well into the infrared.

## 3 The Production of X-Rays

We use the name **X-rays** for electromagnetic radiation whose wavelengths are in the $10^{-2}$ nm to 10 nm region of the spectrum. The name was coined by Wilhelm Röentgen, who first studied the radiation. Among other things, he

**Figure 6** X-rays are produced when electrons "boiled" off a hot filament are accelerated into a metal target.

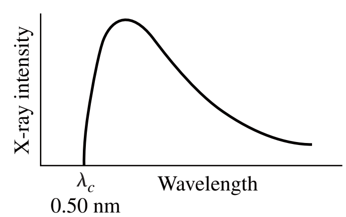



found that X-rays could expose photographic film after passing through a solid object, such as a human being. For his work, he received the first Nobel Prize in physics in 1901. Nowadays, X-rays are an important tool in many areas of research, but their production also gives key evidence of electromagnetic radiation's particle nature.

As shown in Figure 6, X-rays can be produced by smashing high-speed electrons into a metal target. When they hit, these violently decelerating charges produce much radiation, called **bremsstrahlung**, a German word meaning "braking radiation." The fact that electromagnetic radiation can be produced this way is not surprising from the classical perspective, but if it is strictly a wave, we might expect it to cover the entire spectrum. Although the total energy is limited by the number of electrons arriving per unit time, there is no reason waves shouldn't emerge with some amplitude at all wavelengths. But this isn't what happens.

Figure 7 depicts the spectrum produced when electrons of kinetic energy 25 keV strike a molybdenum target. A broad range is apparent,[1] but none of wavelength less than $\lambda_c = 0.050$ nm. This is called the **cutoff wavelength**, and there is no classical explanation for so sharp a termination of the spectrum.

**Figure 7** The X-ray spectrum produced when 25 keV electrons strike a molybdenum target.




The nonclassical explanation is that electromagnetic radiation of a given frequency simply cannot be of arbitrarily small amplitude. If the radiation is quantized, the minimum energy allowed at frequency $f$ is $hf$, a *single photon*. We cannot produce half a photon, so if multiple electrons do not combine their energies into a single photon, no photon could ever be produced of energy greater than the kinetic energy of a single electron. Is this the case? Setting the kinetic energy of an incoming electron equal to the energy of one photon,

$$25 \times 10^3 \text{ eV} = \frac{hc}{\lambda} = \frac{1240 \text{ eV} \cdot \text{nm}}{\lambda}$$
$$\Rightarrow \lambda = 0.050 \text{ nm}$$

This is fairly convincing evidence. Electrons apparently do not combine their energies.

[1]Not shown are characteristic X-ray spikes due to interactions specific to the elements in the target material rather than simple deceleration.

## 4 The Compton Effect

We now consider yet another phenomenon that couldn't be explained by the classical view of electromagnetic radiation as strictly waves, one that uncovers another important property of photons. The situation is the scattering of electromagnetic radiation from free stationary electrons. According to classical electromagnetic wave theory, the electrons would oscillate and therefore reradiate, or "scatter," electromagnetic energy in all directions *initially at the same frequency as the incoming radiation*.

Arthur Holly Compton, investigating this phenomenon with X-rays and using carbon atoms as the source of electrons, found that some radiation scattered backward immediately with a wavelength significantly longer than that of the incoming X-rays. (The electrons involved are initially bound to carbon atoms, but so weakly as to be effectively free. See Exercise 30.) Compton's explanation treated the X-rays as a collection of photons, each with a discrete energy but also with another property we usually associate with a particle—momentum. Signs point this way from two directions:

1. According to special relativity, an object with zero mass should have momentum related to its energy by

$$E = pc$$

2. According to classical electromagnetic wave theory, electromagnetic waves do carry momentum. However, for a diffuse wave, we speak of momentum *density*, which is related to the energy density by

$$\frac{\text{energy}}{\text{volume}} = \frac{\text{momentum}}{\text{volume}} \times c$$

With two such compatible clues, it seems reasonable that the momentum of a photon might be given by

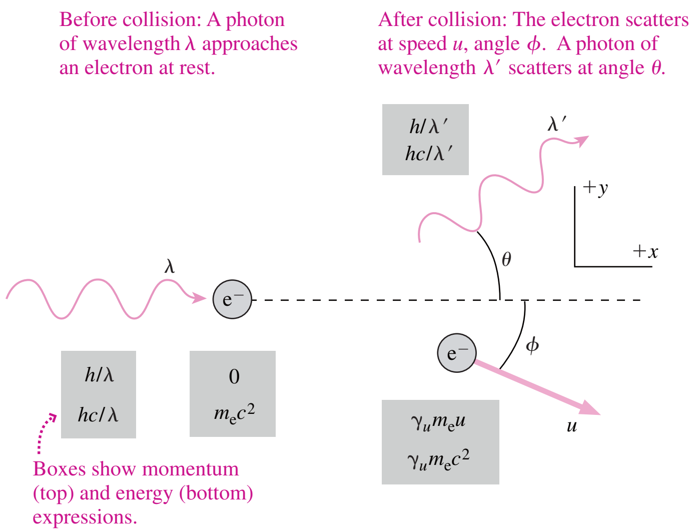
$$p = \frac{E}{c} = \frac{hf}{c} = \frac{h}{\lambda}$$

Is this true? Compton provided the first experimental evidence. Let us set the stage.

### A Two-Particle Collision

In the particle view of electromagnetic radiation, the interaction of X-rays and electrons is simply a collection of separate two-particle collisions between photon and electron, for which we may now express momentum and energy conservation. As shown in Figure 8, we assume that an X-ray photon of wavelength $\lambda$ strikes a stationary electron, and afterward, the electron scatters

**Figure 8** Momentum and energy when a photon strikes a free electron.




at speed $u$ and angle $\phi$, while a scattered photon of wavelength $\lambda'$ departs at $\theta$. Using $hc/\lambda$ for the photon energy (more convenient than the equivalent $hf$), we have

Momentum conserved:

$$x\text{-component:}\quad \frac{h}{\lambda} = \frac{h}{\lambda'}\cos\theta + \gamma_u m_e u \cos\phi \tag{4}$$

$$y\text{-component:}\quad 0 = \frac{h}{\lambda'}\sin\theta - \gamma_u m_e u \sin\phi \tag{5}$$

Energy conserved:

$$h\frac{c}{\lambda} + m_e c^2 = h\frac{c}{\lambda'} + \gamma_u m_e c^2 \tag{6}$$

As it happens, the electron often moves very fast after the collision, so we must use the relativistically correct expressions for its momentum and energy. The photon expressions are already correct; *non*relativistic ones do not exist for things that always move at $c$.

Equations (4) to (6) have been found to agree completely with experimental observations when a photon collides with a free electron. All tests since Compton's original work have reaffirmed the conclusion: The momentum of a photon is given by

$$p = \frac{h}{\lambda} \tag{7}$$

Momentum of a photon

The Compton effect's most striking departure from classical expectation is the large and immediate wavelength shift in the scattered radiation. We clarify this by eliminating the electron speed $u$ and scattering angle $\phi$ from among equations (4) to (6). (The somewhat lengthy algebra is left to Exercise 38.) What remains is

Compton effect

$$\lambda' - \lambda = \frac{h}{m_e c}(1 - \cos\theta) \tag{8}$$

The difference in wavelength between the incident and scattered photons depends only on the angle of scatter, a discovery that won for Compton the 1927 Nobel Prize. We see also that the scattered photon is always of longer wavelength than the incident. Its energy is always less, which makes sense, for kinetic energy is given to the electron. In particular, the maximum increase in wavelength is for backward scatter of the photon, $\theta = 180°$, because a "head-on" collision imparts the maximum possible energy to the electron. (*Note:* In Compton's experiment, some radiation of the incident wavelength *was* scattered at all angles, because some X-rays effectively interact with a much heavier mass—the whole atom—giving a negligible wavelength shift. See Exercise 33.)

## EXAMPLE 3

An X-ray photon of 0.0500 nm wavelength strikes a free, stationary electron, and the scattered photon departs at 90° from the initial photon direction. Determine the momenta of the incident photon, the scattered photon, and the electron.

**SOLUTION**

For the incident photon,

$$p_{\text{incident}} = \frac{h}{\lambda} = \frac{6.63 \times 10^{-34}\,\text{J}\cdot\text{s}}{0.05 \times 10^{-9}\,\text{m}} = 1.33 \times 10^{-23}\,\text{kg}\cdot\text{m/s}$$

We may solve (8) for the scattered photon's wavelength.

$$\lambda' - 0.0500 \times 10^{-9}\,\text{m} = \frac{6.63 \times 10^{-34}\,\text{J}\cdot\text{s}}{(9.11 \times 10^{-31}\,\text{kg})(3 \times 10^{8}\,\text{m/s})}(1 - \cos 90°)$$

$$\Rightarrow \lambda' = 0.0524\,\text{nm}$$

Thus,

$$p' = \frac{h}{\lambda'} = \frac{6.63 \times 10^{-34}\,\text{J}\cdot\text{s}}{0.0524 \times 10^{-9}\,\text{m}} = 1.26 \times 10^{-23}\,\text{kg}\cdot\text{m/s}$$

For the electron's direction, we use equations (4) and (5).

$$\frac{h}{\lambda} = \frac{h}{\lambda'}\cos 90° + \gamma_u m_e u \cos\phi \quad\rightarrow\quad \frac{h}{0.0500\,\text{nm}} = \gamma_u m_e u \cos\phi$$

$$0 = \frac{h}{\lambda'}\sin 90° - \gamma_u m_e u \sin\phi \quad\rightarrow\quad \frac{h}{0.0524\,\text{nm}} = \gamma_u m_e u \sin\phi$$

Dividing the bottom equation ($y$) by the top ($x$), $\gamma_u m_e u$ cancels.

$$\frac{0.0500}{0.0524} = \tan\phi \quad \Rightarrow \quad \phi = 43.6°$$

Reinserting in either gives the magnitude of the electron's momentum.

$$0 = \frac{6.63 \times 10^{-34}\ \text{J}\cdot\text{s}}{0.0524 \times 10^{-9}\ \text{m}}\sin 90° - \gamma_u m_e u \sin 43.6°$$

$$\Rightarrow \quad \gamma_u m_e u = 1.83 \times 10^{-23}\ \text{kg}\cdot\text{m/s}$$

Thus,

$p_e = 1.83 \times 10^{-23}$ kg·m/s at 43.6° from direction of incident photon.

The vector momentum addition is shown in Figure 9. Even though massless, X-rays are very energetic particles. The electron is "bumped" to $2 \times 10^7$ m/s, nearly one-tenth the speed of light.

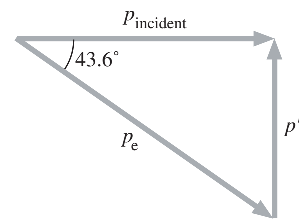



**Figure 9** Momentum conservation in photon-electron collision.

If treating X-rays as a collection of photons predicts so well what we experimentally observe, why is the *wave* theory of electromagnetic radiation still around? The answer is deceptively simple: The behavior of electromagnetic radiation is just as wave theory predicts, *if* the wavelength is long. Particle and wave behaviors converge at long wavelengths. Of course, this "answer" raises a thornier question: Can a line be drawn between wave and particle? This issue we take up in Section 6. For now, we simply note that if instead of a 0.0500 nm wavelength X-ray, we "hit" the electron in Example 3 with visible light of 500 nm wavelength, the wavelength *change* predicted by equation (8) would be the same as before. The *percent* change, rather than the approximate 5% in the example, would be only 0.0005%. Without great precision, the scattered radiation would seem to be the same wavelength as the incident—the prediction of classical wave theory.

Here we have one example of the **correspondence principle**. This principle, which should be viewed as a guideline rather than a quantitative "law" of physics, states that a nonclassical theory should agree with the previous classical one in the appropriate limit. For example, special relativity is a nonclassical theory that agrees with classical mechanics in the limit of small velocities. To this we add that the nonclassical particle theory of electromagnetic radiation agrees with classical wave theory in the limit of long wavelengths. As the wavelength of a beam of electromagnetic radiation is increased, the energy per photon decreases. A given intensity would then comprise a larger number of less-energetic photons and would begin to exhibit the behavior we expect of a continuous wave. (Occurring only for *short* wavelengths, the photoelectric effect might appear to violate the principle, but no "classical expectation" is really valid, because electron binding in solids is fundamentally quantum mechanical.)

In connection with the correspondence principle, it is often said that classical behavior follows in the limit that $h$ goes to zero. It is true that photon energies and momenta tend to be small because $h$ is so small. If $h$ were indeed zero, $E$ and $p$ (i.e., $hf$ and $h/\lambda$) would be zero, and light would never behave as a granular collection of particles. However, Planck's constant is a fundamental *constant* of nature—the constant associated with quantum phenomena—and as such, never "goes" anywhere. Thus, it is better to say that classical behavior follows in the limit of long wavelengths.

## An Inelastic Collision

The photon-electron collision in the Compton effect is necessarily elastic. Because the mass/internal energy of fundamental particles like the electron cannot change, neither can the system's kinetic energy. In fact, we may rearrange the energy-conservation equation (6) as $E = E' + (\gamma_u - 1)m_e c^2$. Photons have only kinetic energy (no mass/internal energy), so this is a statement of *kinetic* energy conservation. (*Note:* The electron kinetic energy is the relativistically correct form.) Let us reinforce our grasp of the conservation laws by considering a system in which mass *is* subject to change.

### EXAMPLE 4

A neutron and proton bound together by the "strong force" is called a **deuteron**. It is the nucleus of the hydrogen isotope **deuterium**, also known as "heavy hydrogen." A helium nucleus is two protons and two neutrons bound together by the same force. Suppose a very energetic photon strikes a helium nucleus and breaks it into two deuterons, each departing at $0.6c$, as depicted in Figure 10. The photon vanishes in the process. (a) What was the photon's energy? (b) In what directions do the deuterons depart? (The mass of a deuteron is 2.01355 u, and of a helium nucleus, 4.00151 u, where 1 u $= 1.66 \times 10^{-27}$ kg.)

**SOLUTION**

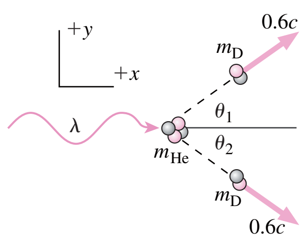
(a) Let us use a subscript He for helium and D for a deuteron. Before the collision, we have a photon and a stationary helium nucleus. Afterward, we have two deuterons moving at $0.6c$. Energy is conserved:

$$h\frac{c}{\lambda} + m_{He}c^2 = 2 \times \gamma_{0.6c}\, m_D c^2$$

Noting that $\gamma_{0.6c} = 1.25$, we have

$$\begin{aligned} h\frac{c}{\lambda} &= c^2(2 \times 1.25\, m_D - m_{He}) \\ &= c^2[2 \times 1.25(2.01355\text{ u}) - 4.00151\text{ u}] \\ &= c^2(1.0324\text{ u}) \\ &= (3 \times 10^8\text{ m/s})^2(1.0324\text{ u} \times 1.66 \times 10^{-27}\text{ kg/u}) \\ &= 1.54 \times 10^{-10}\text{ J} \cong 1\text{ GeV} \end{aligned}$$

**Figure 10** A photon disintegrates a nucleus.



An energetic photon, indeed! This exceeds the entire mass-energy of a proton, $(1.67 \times 10^{27}\ \text{kg})\,(3 \times 10^{8}\ \text{m/s})^2 = 1.5 \times 10^{-10}$ J. In the collision we are analyzing, almost all the photon's energy goes into kinetic energy of the deuterons; the system's final mass (4.0271) is only slightly higher than the initial (4.00151).

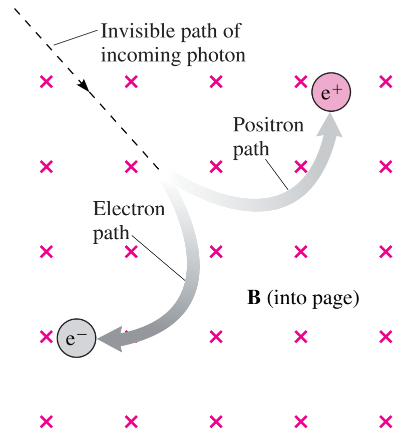
(b) Directions arise only in the vector momentum-conservation equations. Conserving $p_y$, which is initially 0, would merely tell us that $\theta_1$ must equal $\theta_2$, as symmetry demands. This, in turn, tells us that the deuterons have equal $p_x$. Thus

$$p_x \text{ conserved: } \frac{h}{\lambda} = 2 \times (\gamma_{0.6c} m_D\, 0.6c) \cos\theta$$

Multiplying by $c$ and again using $\gamma_{0.6c} = 1.25$ enables us to use our earlier result on the left side.

$$\frac{hc}{\lambda} = 1.5 m_D c^2 \cos\theta$$
$$1.54 \times 10^{-10}\ \text{J} = 1.5(2.01355 \times 1.66 \times 10^{-27}\ \text{kg})(3 \times 10^{8}\ \text{m/s})^2 \cos\theta$$
$$\Rightarrow\quad \theta = 70^\circ$$

**Figure 11** In pair production, a gamma-ray photon becomes an electron, which curves one way in a magnetic field, and a positron, which curves the other way.




## 5 Pair Production

The photoelectric effect and the Compton effect are two important ways in which electromagnetic radiation interacts as a particle with matter. We now discuss a third.

In 1932, a revolutionary new particle was discovered. Carl D. Anderson (Nobel Prize, 1936) was studying the effects of cosmic rays, energetic particles bombarding Earth, when he noticed something behaving like an electron but of positive charge. It curved the right amount but the "wrong" way in a magnetic field. This *posi*tively charged elec*tron* was termed the **positron**. We now know that high-energy photons are constantly creating positrons all around us (fortunately not in dangerous numbers) through **pair production**. This process can be revealed, as depicted in Figure 11, by a **bubble chamber** detector immersed in a magnetic field, in which charged particles leave visible trails of bubbles as they curve. From apparently nothing, there suddenly appear two charged particles deflecting in opposite directions. The energy to produce the massive electron-positron pair comes from a high-energy photon, which, being uncharged, leaves no trail. Charge is conserved because the total charge of the pair is zero. Figure 12 shows actual bubble chamber trails of two electron-positron pairs.

The revolutionary aspect of the positron is that it is antimatter. For the positron, life is short and its end dramatic. It quickly finds an electron—any will do—and after a brief quantum dance, they annihilate together: erased, their entire energy suddenly transformed to two photons. The process, **pair annihilation**, is addressed in Exercises 41 and 42.

**Figure 12** Bubble chamber trails showing electron-positron pair production. Photons, uncharged and invisible, are incident from the top. The lower, slowly diverging arcs are one pair. The electron and positron in the upper pair have less energy and spiral faster because much of the photon energy goes to a freed atomic electron (the straighter trail).

## EXAMPLE 5

Calculate the energy and wavelength of the least energetic photon capable of producing an electron-positron pair.

**SOLUTION**

The photon's energy goes to the massive particles as mass/internal energy plus kinetic. The least energetic one must still create the particles but would leave them no kinetic energy. Thus, the photon energy must equal twice the particle mass energy.

$$2m_e c^2 = 2(9.11 \times 10^{-31}\,\text{kg})(3 \times 10^8\,\text{m/s})^2$$
$$= 1.64 \times 10^{-13}\,\text{J} \quad \cong 1\,\text{MeV}$$

Knowing the photon's energy, we may solve for its wavelength.

$$h\frac{c}{\lambda} = 2m_e c^2 \;\Rightarrow\; \lambda = \frac{hc}{2m_e c^2}$$
$$= \frac{(6.63 \times 10^{-34}\,\text{J}\cdot\text{s})(3 \times 10^8\,\text{m/s})}{1.64 \times 10^{-13}\,\text{J}}$$
$$= 1.21 \times 10^{-12}\,\text{m}$$

The term often used for electromagnetic radiation of wavelength shorter than the X-ray range is **gamma rays**. Here, a gamma-ray photon becomes an electron-positron pair.

Judging from the example, it might seem that pair production isn't really a way in which electromagnetic radiation *interacts* with existing matter, but rather a way of *producing* matter. It does involve an interaction, however, for a photon cannot become an electron-positron pair in a vacuum. Conspicuously missing from the example was any consideration of momentum. Momentum isn't conserved if a (moving) photon becomes two stationary massive particles. Even if the photon were more energetic, allowing the pair some kinetic energy after their creation, momentum could not be conserved. We can always choose to consider the process from a reference frame where the newly created particles have opposite velocities. The final momentum would again be zero, but we would still have a nonzero initial momentum—a single photon. Impossible!

What actually happens is that the gamma ray passes by a massive particle, such as an atomic nucleus; they interact via the electromagnetic force; and then a pair is created and some momentum is transferred to the nucleus. Although momentum can't be conserved without it, the nucleus isn't affected much—it "steals" little energy. To show this, suppose we add to the situation in Example 5 a stationary lead nucleus of mass $3.5 \times 10^{-25}$ kg, to which the photon transfers all its momentum. The speed of the nucleus would be

$$v = \frac{p_{\text{photon}}}{m_{\text{nucleus}}} = \frac{h/\lambda}{m_{\text{nucleus}}} = \frac{6.63 \times 10^{-34}\,\text{J}\cdot\text{s}/1.21 \times 10^{-12}\,\text{m}}{3.5 \times 10^{-25}\,\text{kg}} \cong 1600\,\text{m/s}$$

Its kinetic energy would be approximately $4 \times 10^{-19}$ J, or about 6 orders of magnitude less than the total energy involved in the process. Thus, we see that the nucleus can indeed ensure momentum conservation without significantly affecting the energy.

The preceding discussion raises a point that crops up often in physics. Whenever "small" particles interact with "large" ones, the small ones tend to have nearly all the kinetic energy. Why? The kinetic energy of a nonrelativistic particle may be written as $p^2/2m$, so if the interacting particles have comparable momenta (numerator), then a particle with a much larger mass (denominator) will have a much smaller kinetic energy. The rule also tends to hold when massless and massive particles interact, with the massless one filling the small role. For instance, when an atom emits a photon, the recoiling atom and photon have equal momenta, but almost all the kinetic energy goes to the photon.

## 6 Is It a Wave or a Particle?

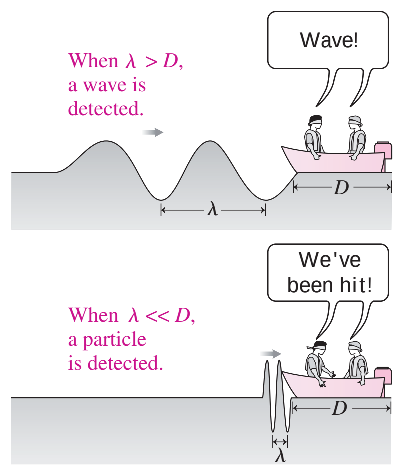
This may be the most perplexing question for the student of quantum mechanics. The simplest answer is that "it" has no predetermined nature. The observation itself—whether the experimenter bounces light off "it," places something in its path, or interacts with it in any way—determines whether "it" will exhibit a wave or a particle nature. Much of what we discuss here applies not only to electromagnetic radiation but also to massive objects. To cover case, let us refer simply to "the phenomenon."

### The Wavelength and the Experiment

We advance the answer a step further with a rough criterion: The behavior a phenomenon exhibits depends on how its wavelength $\lambda$ compares with the "relevant dimensions" of the experimental apparatus, which we represent simply as $D$. All phenomena may, in principle, be described by a **wave function**. The familiar sinusoidal functions for the electric and magnetic fields in a plane wave of light are one example. A massive object might also be described by a sinusoidal wave function. If its wavelength is much smaller than $D$, the phenomenon will exhibit a particle behavior; if comparable to or larger than $D$, it will exhibit a wave behavior.

Figure 13 depicts how the behavior a phenomenon exhibits might depend on a relevant dimension of the apparatus. The boaters are conducting an experiment—blindfolded. If the wavelength of the approaching disturbance (the phenomenon) is larger than $D$, the width of the boat, these experimenters will certainly proclaim it a wave, because the boat is small enough to respond to its crests and troughs separately. But if the wavelength is much smaller than $D$, the boat responds to the whole thing at once, and the experimenters might conclude that a particle, not a diffuse wave, has struck. By the same logic, passengers on an immense ocean liner would see even the top disturbance as a

**Figure 13** An "experiment" in which a disturbance behaves as a wave or a particle, depending on the relative size of the wavelength and the relevant dimension $D$ of the apparatus.




$\lambda \ll D$: particle
$\lambda \gtrsim D$: wave

particle, because $\lambda \ll D$, while ants floating on a popcorn kernel would see the bottom one as a wave, because $\lambda > D$.

A simple application of the criterion to electromagnetic radiation is the case of light passing through a single slit. To see the obvious wave phenomenon of diffraction, the slit width must be narrower than or comparable to the wavelength. If it is much greater, the wave passes straight through, as would particles.

Useful though the criterion may be, it is dangerous to view wave-particle duality too rigidly, as wave *versus* particle. The natures are not incompatible but complementary, just two faces of the same phenomenon. We can't fully explain the behavior of a given phenomenon by either nature alone, and there is a close relationship between the two natures. The famous $E = hf$ and $p = h/\lambda$, which *quantitatively* link the particle properties $E$ and $p$ to the wave properties $f$ and $\lambda$, are one part of the relationship. We now discuss another.

## A Double-Slit Experiment

There is no more direct way to see the link between wave and particle natures than the double-slit experiment. Suppose light is directed at a double slit and is thereafter detected on photographic film beyond the slits. At high intensities, we observe a typical interference pattern, the intensity varying from a maximum (constructive) at the center of the film, to zero (destructive), then back to a maximum, and so on. The light is exhibiting the wave nature we attribute to it classically. Particles don't interfere or cancel; waves do.

Now suppose the intensity is greatly reduced, so that a pattern is no longer visible on the screen. The film still registers the arrival of light, but sporadically at scattered locations. Apparently, the light is being detected one photon—one *particle*—at a time. Figure 14 shows the result if the film registers an arrival with a spot. Although the locations of the spots at first seem almost completely unpredictable, we eventually begin to discern a regular pattern. Understanding the link between wave and particle rests on two key observations: First, the exact location where the next photon will be found evidently can't be known, but logically, the probability of detecting it in a given region should be proportional to the density of spots there—high where density is high, low where density is low. Therefore, if the density of spots assumes a pattern, the *probability* assumes a pattern. Second, careful study reveals that the density of spots in a region is directly proportional to what wave theory (physical optics) predicts should be the relative intensity in that region, which is, in turn, proportional to the square of the amplitude of the electromagnetic wave. In particular, no photons are ever detected at the locations where wave theory says there should be points of destructive interference. *Although the light is being detected one particle at a time, its wave nature is still apparent*.

Combining the two observations, we conclude that because both are proportional to the density of spots, there is a proportionality between the probability of detecting the particle and the square of the amplitude of the wave. This connection between particle and wave natures is a cornerstone of quantum mechanics:

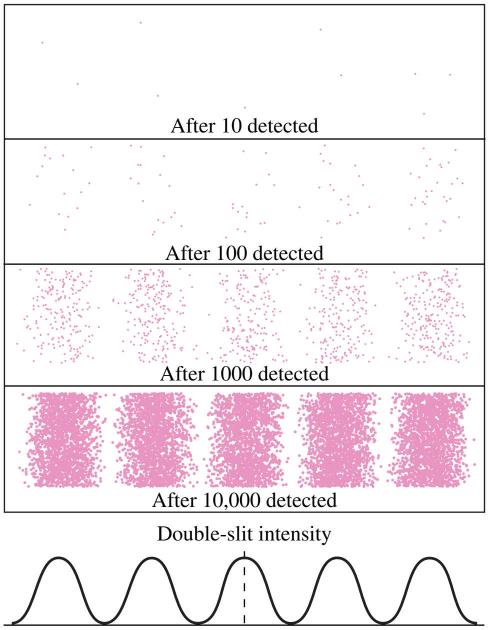



**Figure 14** Photons producing a double-slit interference pattern—one *particle* at a time.

When a phenomenon is detected as *particles*, we cannot predict with certainty where a given particle will be found. The most we can determine is a probability of finding it in a given region, which is proportional to the square of the amplitude of the associated *wave* in that region.

$$\begin{matrix}\text{probability of finding} \\ \textit{particle} \text{ in a region}\end{matrix} \propto \begin{pmatrix}\text{amplitude of } \textit{wave} \\ \text{in that region}\end{pmatrix}^2$$

To clarify the nomenclature, if the wave in question is the electromagnetic field, its "associated particle" is the photon. Equivalently, if the particle in question is the photon, its "associated wave," its alter ego, is the electromagnetic field. In electricity and magnetism, we learn that electromagnetic fields exert forces on charges, and we now claim that they also measure the probability of finding the associated particle—the photon. We point this out partly to prepare the reader for a possible shock: The wave associated with a massive object is indeed a measure of the probability of finding the particle, but it doesn't appear to have any other side to its personality.

## EXAMPLE 6

Light of wavelength 633 nm is directed at a double slit, and the interference pattern is viewed on a screen. The intensity at the center of the pattern is 4.0 W/m$^2$. (a) At what rate are photons detected at the pattern's center? (b) At what rate are photons detected at the first interference minimum? (c) At what rate are photons detected at a point on the screen where the waves from the two sources are out of phase by one-third of a cycle? (*Note:* From physical optics, the double-slit intensity varies according to $I = I_0 \cos^2(\frac{1}{2}\phi)$, where $\phi$ is the phase difference between the waves from the two slits and $I_0$ is the intensity when $\phi = 0$, that is, at the center of the pattern.)

**SOLUTION**

(a) Each photon has energy

$$h\frac{c}{\lambda} = (6.63 \times 10^{-34}\ \text{J}\cdot\text{s})\frac{3 \times 10^{8}\ \text{m/s}}{633 \times 10^{-9}\ \text{m}} = 3.14 \times 10^{-19}\ \text{J}$$

We must therefore have

$$\frac{4.0\ \text{J/s}\cdot\text{m}^2}{3.14 \times 10^{-19}\ \text{J/photon}} = 1.27 \times 10^{19}\ \text{photons/s}\cdot\text{m}^2$$

(b) The first interference minimum is a point of destructive interference, because the difference in distances traveled to the screen creates a phase difference of $\pi$ (one-half cycle) between the waves from the two sources. If the net wave (electric field) is zero, its square is zero and so is the intensity. No photons are detected here.

(c) Somewhere between the center of the pattern and the first minimum, the waves will be out of phase by one-third cycle: $\phi = \frac{1}{3}2\pi$. We know that $I_0$ is 4.0 W/m$^2$, so we have

$$I = (4.0\ \text{W/m}^2)\cos^2[\tfrac{1}{2}(\tfrac{1}{3}2\pi)] = 1.0\ \text{W/m}^2$$

Because this is one-fourth the value at the center, the probability of detecting photons here is one-fourth as large, so only one-fourth as many will be detected per unit time per unit area around this point.

$$3.18 \times 10^{18}\ \text{photons/s}\cdot\text{m}^2$$

Although we detect *particles*, the probability of their detection is governed by the behavior of the associated electromagnetic wave.

Finally, we address a point central to quantum mechanics. In the double slit, it is not proper to ask, “Through which slit did the 17th photon pass?” We haven’t allowed for this, and by altering the experiment so as to observe a particle passing through a slit—an experiment in itself, requiring some interaction with light in one slit alone—we would alter the very behavior we wish to observe. Because interference requires two coherent waves, the pattern would be disturbed. We can’t have it both ways. Wavelike interference can only be observed by allowing each “particle” to behave as a wave—*passing through both slits simultaneously*.

Is it a wave or a particle? It may behave as either, depending on the situation, but the two natures are inextricably related.

## PROGRESS AND APPLICATIONS

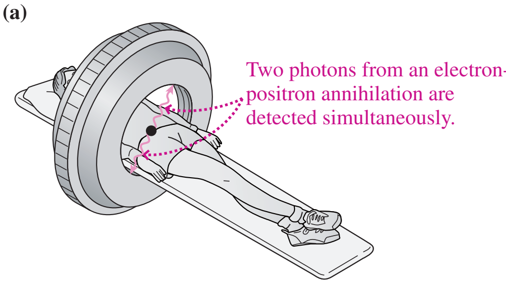
**Ejecting Electrons with X-Rays** Section 2 discusses the photoelectric effect in which photons roughly in the visible wavelength range eject the least tightly bound electrons in a metal. Logically, shorter-wavelength, more-energetic photons should be able to free electrons that are more tightly bound. Today this effect is being exploited to reveal much about how electrons are bound to atoms in materials and how the atoms themselves are arranged. At the forefront of the work is the University of California’s Advanced Light Source (ALS). The “light” in this case is X-rays whose wavelength can be tuned from 0.1 nm to 100 nm. To produce the powerful beam needed, the simple method of smashing electrons into a target gives way to a stream of electrons circulating in a synchrotron. Any charged particle emits electromagnetic radiation when it accelerates, and at the ALS, special magnets, called undulators, wiggle the circulating electrons so as to produce a very bright narrow beam of X-rays. When the beam is directed at a sample, photoelectrons are produced. Studying their kinetic *energies* when free, known as **photoelectron spectroscopy**, tells us about their energies when bound in the material, and studying their motion when free tells us about their locations when bound and, thus, the locations of the atoms. The latter technique is termed **photoelectron diffraction**. The diffraction isn't that of the X-rays but of photoelectrons. As noted in Section 6, objects with mass also exhibit wave-particle duality, and these photoelectrons behave as waves of such short wavelengths as to provide a very high-resolution picture of the atomic landscape from which they were ejected.

**A New Way to Produce X-Rays** Because of their penetrating abilities, X-rays have found a great host of uses in learning what is inside things without breaking them open, probing everything from human bodies to construction materials to superconductors. While the century-old tried-and-true method of Section 3 is still the leader, the conventional X-ray machine, with its hot filament sealed in a tube, is rather unwieldy, and the tube's lifetime is often short. The huge and growing field of nanotechnology—applications in which some crucial element is measured in nanometers—may provide a new method. In work conducted at the University of North Carolina, carbon nanotubes are allied with the quantum-mechanical effect of "field emission" to produce a beam of X-rays strong enough to replace the conventional X-ray machine in several uses. A nanotube is a regular meshwork of carbon atoms, forming a cylinder of only about 1 nm radius, and is closely related to many other all-carbon structures discovered in the 1980s. These continue to surprise us with new, remarkable properties and are a very hot topic of physics research. In the X-ray source application, bundles of nanotubes are deposited in a thin layer on a metal disk. Electrons are coaxed from the layer toward a target not by heating a filament, which wastes power and produces electrical noise, but by field emission, a room-temperature way of producing a flow of electrons that relies on the quantum-mechanical effect of tunneling. The resolution of the new technique is excellent, and another potential advantage is faster response time for tracking moving objects. (See Yue et al., *Applied Physics Letters*, 8 July 2002.)

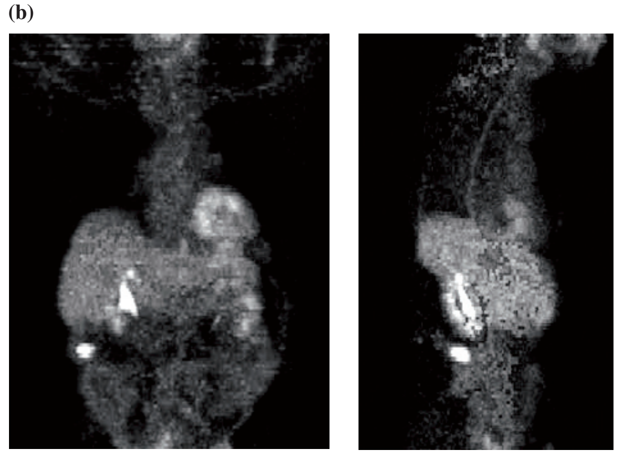
**Medical Imaging with Positrons** As noted in Section 5, once a positron is produced, it soon engages in pair annihilation, simultaneously yielding *two* photons of a characteristic energy (see Exercise 42). This trait is exploited in an increasingly common medical imaging procedure known as positron emission tomography (PET). A tracer material containing a radioisotope that emits positrons at a safely low rate is introduced into the patient via the bloodstream, where it collects in certain tissues. The patient is then placed inside a machine surrounded by a ring of detectors that "look" for photons of the characteristic energy. A positron emitted by the tracer—after no more than about a millimeter of bouncing around—finds an electron with which to annihilate. Each pair is nearly stationary when annihilation occurs, so the two photons created in a given annihilation move in opposite directions. When two photons are detected *simultaneously*, pair annihilation must have occurred along the line connecting the two detectors. Many intersecting lines indicate a high density of annihilations at a point—a concentration of the tracer—in the two-dimensional "slice" of the patient viewed by the detector ring. As the ring passes along the patient's body, looking at each slice, a three-dimensional image of density versus position emerges. Figure 15 shows (a) the basic layout of a PET machine and (b) an actual image.

**Figure 15** Positron emission tomography. (a) A tracer emits a positron that annihilates with a nearby electron, yielding two photons that place the annihilation along a line between the detectors. Many lines combine to produce a two-dimensional image of a slice. (b) Multiple slices produce a three-dimensional image—in this case, showing high tracer uptake in the liver and kidneys.

**(a)**




**(b)**



## Chapter Summary

Electromagnetic radiation behaves in some situations as a collection of particles—photons—having the particlelike properties of discrete energy and momentum, which are related to the wave properties of frequency and wavelength:

$$E = hf \quad (2)$$

$$p = \frac{h}{\lambda} \quad (7)$$

where $h$ is Planck's constant, $6.63 \times 10^{-34}\ \text{J} \cdot \text{s}$. Electromagnetic radiation is more likely to exhibit a particle nature when the wavelength is small compared with the relevant dimensions of the experimental apparatus. For a given intensity, a short wavelength corresponds to a large energy per particle and a correspondingly small particle flux—number per unit time per unit area. The radiation is thus relatively particlelike. For long wavelengths, it is more likely to behave as a continuous wave.

Electromagnetic radiation may exhibit its particle nature in several ways. In the photoelectric effect, a photon gives up a discrete amount of energy to an electron in a metal. In the Compton effect, a short-wavelength photon scatters from an essentially free electron, and both electron and photon obey the usual conservation laws for particles with discrete momentum and energy. In pair production, these conservation laws are also obeyed as a high-energy photon disappears, transferring its discrete energy to a pair of massive particles.

When electromagnetic radiation is detected as particles, it is uncertain where a given photon will be found. The most that can be determined is a probability of finding it in a given region, which is proportional to the square of the amplitude of the associated wave—the oscillating electromagnetic field—in that region.

* indicates advanced questions

## Conceptual Questions

1. Consider two separate objects of unequal temperature. What would you do with them and what would have to happen thereafter to enable them to reach the same common temperature? Use this idea to explain why the electromagnetic radiation enclosed in a cavity has a temperature that is the same as that of the cavity walls.
2. The charge on a piece of metal can be "watched" fairly easily by connecting it to an electroscope, a device with thin leaves that repel when a net charge is present. You place a large excess negative charge on a piece of metal, then separately shine light sources of two pure but different colors at it. The first source is extremely bright, but the electroscope shows no change in the net charge. The second source is feeble, but the charge disappears. Appealing to as few fundamental claims as possible, explain to your friend what evidence this provides for the particle nature of light.
3. You are conducting a photoelectric effect experiment by shining light of 500 nm wavelength at a piece of metal and determining the stopping potential. If, unbeknownst to you, your 500 nm light source actually contained a small amount of ultraviolet light, would it throw off your results by a small amount or by quite a bit? Explain.
4. Suppose we produce X-rays not by smashing *electrons* into targets but by smashing protons, which are far more massive. If the same accelerating potential difference were used for both, how would the cutoff wavelengths of the two X-ray spectra compare? Explain.
5. In the Compton effect, we choose the electron to be at the origin and the initial photon's direction of motion to be in the $+x$ direction. (a) We may also choose the $xy$-plane so that it contains the velocities of the outgoing electron and photon. Why? (b) The incoming photon's wavelength $\lambda$ is assumed to be known. The unknowns after the collision are the outgoing photon's wavelength and direction, $\lambda'$ and $\theta$, and the speed and direction of the electron, $u_e$ and $\phi$. With only three equations—two components of momentum conservation and one of energy—we can't find all four. Equation (8) gives $\lambda'$ *in terms of* $\theta$. Our lack of knowledge of $\theta$ *after* the collision (without an experiment) is directly related to a lack of knowledge of something *before* the collision. What is it? (Imagine the two objects are hard spheres.) (c) Is it reasonable to suppose that we *could* know this? Explain.
6. An isolated atom can emit a photon, and the atom's internal energy drops. In fact, the process has a name: spontaneous emission. Can an isolated electron emit a photon? Why or why not?
7. We analyze the photoelectric effect using photon energy alone. Why isn't the photon momentum a consideration? (It may help to reread the discussion of momentum and energy in connection with pair production.)
8. A ball rebounds elastically from the floor. What does this situation share with the ideas of momentum conservation discussed in connection with pair production?
9. A low-intensity beam of light is sent toward a narrow single slit. On the far side, individual flashes are seen sporadically at detectors over a broad area that is orders of magnitude wider than the slit width. What aspects of the experiment suggest a wave nature for light, and what aspects suggest a particle nature?
10. A coherent beam of light strikes a single slit and produces a spread-out diffraction pattern beyond. The number of photons detected per unit time at a detector in the very center of the pattern is $X$. Now two more slits are opened nearby, the same width as the original, equally spaced on either side of it, and equally well

illuminated by the beam. How many photons will be detected per unit time at the center detector now? Why?

## Exercises

### Section 1

**11.** For small $z$, $e^z$ is approximately $1 + z$. (a) Use this to show that Planck's spectral energy density (1) agrees with the result of classical wave theory in the limit of small frequencies. (b) Show that, whereas the classical formula diverges at high frequencies—the so-called **ultraviolet catastrophe** of this theory—Planck's formula approaches 0.

**12.** At what wavelength does the human body emit the maximum electromagnetic radiation? Use Wien's law from Exercise 14 and assume a skin temperature of 70°F.

**13.** Equation (1) expresses Planck's spectral energy density as an energy per range $df$ of frequencies. Quite often, it is more convenient to express it as an energy per range $d\lambda$ of wavelengths. By differentiating $f = c/\lambda$, we find that $df = -c/\lambda^2\, d\lambda$. Ignoring the minus sign (we are interested only in relating the magnitudes of the ranges $df$ and $d\lambda$), show that, in terms of wavelength, Planck's formula is

$$\frac{dU}{d\lambda} = \frac{8\pi Vhc}{e^{hc/\lambda k_B T} - 1}\frac{1}{\lambda^5}$$

**14.** According to **Wien's law**, the wavelength $\lambda_{max}$ of maximum thermal emission of electromagnetic energy from a body of temperature $T$ obeys

$$\lambda_{max}T = 2.898 \times 10^{-3}\ \text{m} \cdot \text{K}$$

Show that this law follows from the spectral energy density $dU/d\lambda$ obtained in Exercise 13. Obtain an expression that, when solved, would yield the wavelength at which this function is maximum. The transcendental equation cannot be solved exactly, so it is enough to show that $\lambda = 2.898 \times 10^{-3}\ \text{m} \cdot \text{K}/T$ solves it to a reasonable degree of precision.

**15.** The electromagnetic intensity of all wavelengths thermally radiated by a body of temperature $T$ is given by

$$I = \sigma T^4 \text{ where } \sigma = 5.67 \times 10^{-8}\ \text{W/m}^2 \cdot \text{K}^4$$

This is the **Stefan-Boltzmann law**. To derive it, show that the total energy of the radiation in a volume $V$ at temperature $T$ is $U = 8\pi^5 k_B^4 V T^4/15h^3c^3$, by integrating Planck's spectral energy density over all frequencies. Note that

$$\int_0^\infty \frac{x^3}{e^x - 1}dx = \frac{\pi^4}{15}$$

Intensity, or power per unit area, is then the product of energy per unit volume and distance per unit time. But because intensity is a flow in a given direction away from the blackbody, $c$ is not the correct speed. For radiation moving uniformly in all directions, the average *component* of velocity in a given direction is $\frac{1}{4}c$.

### Section 2

**16.** In the photoelectric effect, photoelectrons begin leaving the surface at essentially the instant that light is introduced. If light behaved as a diffuse wave and an electron at the surface of a material could be assumed localized to roughly the area of an atom, it would take far longer. Estimate the time lag, assuming a work function of 4 eV, an atomic radius of approximately 0.1 nm, and a reasonable light intensity of 0.01 W/m$^2$.

**17.** Light of 300 nm wavelength strikes a metal plate, and photoelectrons are produced moving as fast as $0.002c$. (a) What is the work function of the metal? (b) What is the threshold wavelength for this metal?

**18.** What is the stopping potential when 250 nm light strikes a zinc plate?

**19.** What wavelength of light is necessary to produce photoelectrons of speed $2 \times 10^6$ m/s with a magnesium target?

**20.** What is the wavelength of a 2.0 mW laser from which $6 \times 10^{15}$ photons emanate every second?

**21.** A 940 kHz radio station broadcasts 40 kW of power. How many photons emanate from the transmitting antenna every second?

**22.** To expose photographic film, photons of light dissociate silver bromide (AgBr) molecules, which requires an energy of 1.2 eV. What limit does this impose on the wavelengths that may be recorded by photographic film?

**23.** Light of wavelength 590 nm is barely able to eject electrons from a metal plate. What would be the speed of the fastest electrons ejected by light of one-third the wavelength?

**24.** With light of wavelength 520 nm, photoelectrons are ejected from a metal surface with a maximum speed of $1.78 \times 10^5$ m/s. (a) What wavelength would be needed to give a maximum speed of $4.81 \times 10^5$ m/s? (b) Can you guess what metal it is?

**25.** You are an early 20th-century experimental physicist and do not know the value of Planck's constant. By a suitable plot of the following data, and using Einstein's explanation of the photoelectric effect (KE $= hf - \phi$, where $h$ is *not* known), determine Planck's constant.

| Wavelength of Light (nm) | Stopping Potential (V) |
|---|---|
| 550 | 0.060 |
| 500 | 0.286 |
| 450 | 0.563 |
| 400 | 0.908 |

**26.** A sodium vapor light emits 10 W of light energy. Its wavelength is 589 nm, and it spreads in all directions. How many photons pass through your pupil, diameter 4 mm, in 1 s if you stand 10 m from the light?

**27.** Using the high-precision values of $h$, $c$, and $e$, show that the product $hc$ can be expressed as 1240 eV · nm.

## Section 3

**28.** A television picture tube accelerates electrons through a potential difference of 30,000 V. Find the minimum wavelength to be expected in X-rays produced in this tube. (Picture tubes incorporate shielding to control X-ray emission.)

**29.** When a beam of monoenergetic electrons is directed at a tungsten target, X-rays are produced with wavelengths no shorter than 0.062 nm. How fast are the electrons in the beam moving?

## Section 4

**30.** A typical ionization energy—the energy needed to remove an electron—for the elements is 10 eV. Explain why the energy binding the electron to its atom can be ignored in Compton scattering involving an X-ray photon with wavelength about one-tenth of a nanometer.

**31.** A 0.057 nm X-ray photon “bounces off” an initially stationary electron and scatters with a wavelength of 0.061 nm. Find the directions of scatter of (a) the photon and (b) the electron.

**32.** A 0.065 nm X-ray source is directed at a sample of carbon. Determine the maximum speed of scattered electrons.

**33.** Compton used X-rays of 0.071 nm wavelength. Some of carbon’s electrons are too tightly bound to be stripped away by these X-rays, which accordingly interact essentially with the atom as a whole. In effect, $m_e$ in equation (8) is replaced by carbon’s atomic mass. Show that this explains why some X-rays of the incident wavelength were scattered at all angles.

* **34.** An X-ray source of unknown wavelength is directed at a carbon sample. An electron is scattered with a speed of $4.5 \times 10^7$ m/s at an angle of 60°. Determine the wavelength of the X-ray source.

**35.** Determine the wavelength of an X-ray photon that can impart, at most, 80 keV of kinetic energy to a free electron.

**36.** A photon scatters off of a free electron. (a) What is the maximum possible change in wavelength? (b) Suppose a photon scatters off of a free proton. What is the maximum possible change in wavelength now? (c) Which more clearly demonstrates the particle nature of electromagnetic radiation—collision with an electron or collision with a proton?

**37.** Verify that the formula $\Delta \text{KE} = -\Delta mc^2$ applies in Example 4.

* **38.** From equations (4) to (6) obtain equation (8). It is easiest to start by eliminating $\phi$ between equations (4) and (5), using $\cos^2\phi + \sin^2\phi = 1$. The electron speed $u$ may then be eliminated between the remaining equations.

* **39.** Show that the angles of scatter of the photon and electron in the Compton effect are related by the following formula:

$$\cot\frac{\theta}{2} = \left(1 + \frac{h}{mc\lambda}\right)\tan\phi$$

## Section 5

**40.** A gamma-ray photon changes into a proton-antiproton pair. Ignoring momentum conservation, what must have been the wavelength of the photon (a) if the pair is stationary after creation, and (b) if each moves off at $0.6c$, perpendicular to the motion of the photon? (c) Assume that these interactions occur as the photon encounters a lead plate and that a lead nucleus participates in momentum conservation. In each case, what fraction of the photon’s energy must be absorbed by a lead nucleus?

**41.** A stationary muon $\mu^-$ annihilates with a stationary antimuon $\mu^+$ (same mass, $1.88 \times 10^{-28}$ kg, but opposite charge). The two disappear, replaced by electromagnetic radiation. (a) Why is it not possible for a single photon to result? (b) Suppose two photons result. Describe their possible directions of motion and wavelengths.

**42.** In positron emission tomography (PET), discussed in Progress and Applications, an electron and positron annihilate, and two photons of a characteristic energy are detected (see also Exercise 41). What is this energy, and what is the corresponding wavelength? The pair can be assumed to be essentially stationary before annihilation.

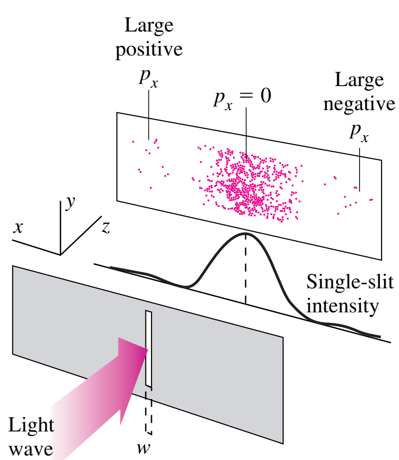
**43.** As shown in Section 5, a lead nucleus can ensure momentum conservation in electron-positron pair production without affecting the energy balance. But roughly what is the limit on the mass of such a “detached participant”? Assume again that it acquires all the momentum of the photon, whose wavelength is $1.21 \times 10^{-12}$ m, but the energy it “steals” is less insignificant, 0.01% of the photon’s energy. What is the mass of this less-detached participant?

## Section 6

**44.** A beam of 500 nm light strikes a barrier in which there is a narrow single slit. At the very center of a screen beyond the single slit, $10^{12}$ photons are detected per square millimeter per second. (a) What is the intensity of the light at the center of the screen? (b) A second slit is now added very close to the first. How many photons will be detected per square millimeter per second at the center of the screen now?

**45.** Electromagnetic "waves" strike a single slit of 1 μm width. Determine the *angular full width* (angle from first minimum on one side of the center to first minimum on the other) in degrees of the central diffraction maximum if the waves are (a) visible light of wavelength 500 nm and (b) X-rays of wavelength 0.05 nm. (c) Which more clearly demonstrates a wave nature?

**46.** A bedrock topic in quantum mechanics is the uncertainty principle. It is discussed mostly for massive objects, but the idea also applies to light: Increasing certainty in knowledge of photon position implies increasing *un*certainty in knowledge of its momentum, and vice versa. A single-slit pattern that is developed (like the double-slit pattern of Section 6) one photon at a time provides a good example. Depicted in the accompanying figure, the pattern shows that photons emerging from a narrow slit are spread all over; a photon's $x$-component of momentum can be any value over a broad range and is thus uncertain. On the other hand, the $x$-coordinate of *position* of an emerging photon covers a fairly small range, for $w$ is small. Using the single-slit diffraction formula $n\lambda = w \sin\theta$, show that the range of likely values of $p_x$, which is roughly $p \sin\theta$, is inversely proportional to the range $w$ of likely position values. Thus, an inherent wave nature implies that the precisions with which the particle properties of position and momentum can be known are inversely proportional.




## Comprehensive Exercises

**47.** A photon has the same momentum as an electron moving at $10^6$ m/s. (a) Determine the photon's wavelength. (b) What is the ratio of the kinetic energies of the two? (*Note:* A photon is *all* kinetic energy.)

**48.** A photon and an object of mass $m$ have the same momentum $p$.

(a) Assuming that the massive object is moving slowly, so that nonrelativistic formulas are valid, find in terms of $m$, $p$, and $c$ the ratio of the massive object's kinetic energy to the photon's kinetic energy, and argue that it is small.

(b) Find the same ratio found in part (a), but using relativistically correct formulas for the massive object. (*Note:* $E^2 = p^2c^2 + m^2c^4$ may be helpful.)

(c) Show that the low-speed limit of the ratio of part (b) agrees with part (a) and that the high-speed limit is 1.

(d) Show that at *very* high speed, the kinetic energy of a massive object approaches $pc$.

**49.** Radiant energy from the Sun arrives at Earth with an intensity of 1.5 kW/m$^2$. Making the rough approximation that all photons are absorbed, find (a) the radiation pressure and (b) the total force experienced by Earth due to this "solar wind."

**50.** A flashlight beam produces 2.5 W of electromagnetic radiation in a narrow beam. Although the light it produces is white (all visible wavelengths), make the simplifying assumption that the wavelength is 550 nm, the middle of the visible spectrum. (a) How many photons per second emanate from the flashlight? (b) What force would the beam exert on a "perfect" mirror (i.e., one that reflects all light completely)?

**51.** The average intensity of an electromagnetic wave is $\frac{1}{2}\varepsilon_0 c E_0^2$, where $E_0$ is the amplitude of the electric-field portion of the wave. Find a general expression for the photon flux $j$ (measured in photons/s · m$^2$) in terms of $E_0$ and wavelength $\lambda$.

**52.** Show that the laws of momentum and energy conservation forbid the complete *absorption* of a photon by a free electron. (*Note:* This is not the photoelectric effect. In the photoelectric effect, the electron is not free; the metal participates in momentum and energy conservation.)

**53.** An electron moving to the left at $0.8c$ collides with an incoming photon moving to the right. After the collision, the electron is moving to the right at $0.6c$ and an outgoing photon moves to the left. What was the wavelength of the incoming photon?

**54.** An object moving to the right at $0.8c$ is struck head-on by a photon of wavelength $\lambda$ moving to the left. The object absorbs the photon (i.e., the photon disappears) and is afterward moving to the right at $0.6c$. (a) Determine the ratio of the object's mass after the collision to its mass before the collision. (*Note:* The object is not a "fundamental particle," and its mass is therefore subject to change.) (b) Does kinetic energy increase or decrease?

**55.** Photons from space are bombarding your laboratory and smashing massive objects to pieces! Your detectors indicate that two fragments, each of mass $m_0$, depart such a collision moving at $0.6c$ at 60° to the photon's original direction of motion. In terms of $m_0$, what are the energy of the cosmic-ray photon and the mass $M$ of the particle being struck (assumed initially stationary)?

## Answers to Selected Exercises

**17.** 3.12 eV, 399 nm
**19.** 82.4 nm
**21.** $6.42 \times 10^{31}$ photons per sec
**23.** $1.22 \times 10^{6}$ m/s
**29.** $8.15 \times 10^{7}$ m/s
**31.** 130.5°, 23.9°
**35.** 0.00659 nm
**41.** (b) opposite, $1.18 \times 10^{-14}$ m
**43.** $9.1 \times 10^{-27}$ kg
**45.** 60°, $5.73 \times 10^{-3}$ degrees, visible light
**47.** $7.38 \times 10^{-10}$ m, 600
**49.** $5 \times 10^{-6}$ Pa, $6.37 \times 10^{8}$ N
**51.** $\frac{1}{2}\varepsilon_0 E^2 \lambda/h$
**53.** $2.91 \times 10^{-12}$ m
**55.** $\frac{3}{4}m_0c^2$, $\frac{7}{4}m_0$

## Credits

**12:** LBNL/Photo Researchers; **(both):** Positron Emission Tomography Department, NIH Clinical Center, National Institutes of Health

# Waves and Particles II: Matter Behaving as Waves

**Chapter Outline**



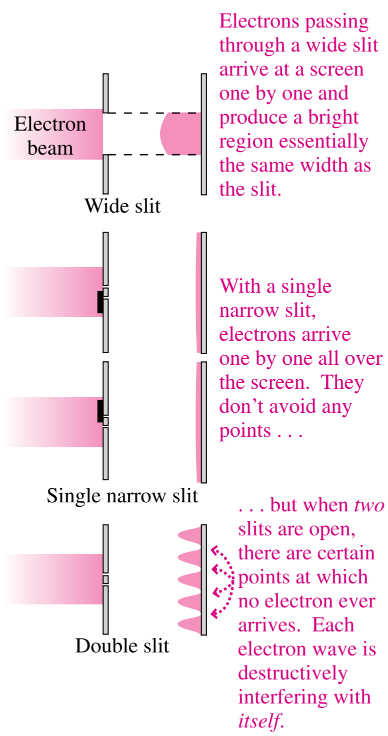
Electromagnetic radiation, classically a wave, has a particle nature. We now begin our study of the complementary and fascinating truth that matter, classically particlelike, has a wave nature. Although a challenging notion, it is the key to understanding behaviors in the submicroscopic world, and thus it lies at the heart of much of modern science.

To force light to show its wave nature, we need an apparatus with a dimension comparable to its wavelength. For instance, light of 0.6 μm wavelength, such as a sodium streetlight, seems to pass straight through a 1 m wide doorway but visibly diffracts when passing through a 1 μm aperture—which isn't hard to make. The wave nature of *matter* would seem a much less foreign concept if we could obtain similar evidence so easily, but we can't, and the reason is that the wavelength of **matter waves**, as a rule, is even smaller than that of light—much smaller. Before we begin our quantitative study of this "hidden" nature, it may be helpful to point out perhaps the most widely known idea in science that rests directly on it: Electrons orbiting atoms can have only certain energies. Why? The atom confines its electrons to *very* small dimensions, in which case their wave nature should predominate. We know from studying waves and sound that when a wave is confined, as on a stretched string or in an organ pipe, only certain discrete standing waves are possible. Why are only certain energies allowed in the atom? The electrons, confined to dimensions less than a 1 nm, are behaving as standing waves.

Being diffuse, waves are analyzed differently from discrete particles. In classical mechanics, objects are particles, and the equations governing their behavior—kinematic equations, $\mathbf{F} = d\mathbf{p}/dt$, and so forth—are relatively simple. Waves, on the other hand, obey equations involving more sophisticated calculus. For instance, waves on a string obey the "wave equation," a partial differential equation in position and time, and light obeys Maxwell's equations, similarly involving calculus with both position and time variables. The distinction between particle and wave natures also governs what questions we ask. For particles, it is often, "Where is it going? When will it get there?" For waves, we ask, "What is its amplitude? What is its wavelength? How spread out is it? Where is it zero?"

We consider these questions soon, and afterward we introduce the Schrödinger equation, the equation obeyed by matter waves. First, however, is the Big Question: Just what is a matter wave, and how do we know it even exists? Nothing illustrates the point better than the double-slit experiment.

**Figure 1** Intensity patterns when an electron beam strikes various slits.




## 1 A Double-Slit Experiment

Imagine a beam of monoenergetic electrons[1] striking a barrier with a slit, beyond which is a screen that registers each electron's arrival by producing a small flash. When the slit is "wide," as shown at the top of Figure 1, the beam passes straight through and produces—electron by electron—a stripe on the screen essentially the same width as the slit. But with a narrow slit, we find electrons registering sporadically *over the entire screen*. Although this spreading alone is hard to reconcile with the notion of electrons as strictly particles, if we add a second slit, the conclusion is inescapable.

Suppose, then, that we add a second narrow slit. Again, with either slit open alone, electrons are detected sooner or later at *all* points on the screen. But when both are open together, we see certain places, where electrons *had been detected* with either slit open separately, where electrons are now never detected. Opening a second "door" decreases to zero the number of electrons arriving per unit time at specific, regularly spaced locations on the screen, even at such low intensity that they must pass through one at a time! This is impossible to explain if electrons are simply particles passing through one slit or the other. A particle passing through one slit would not suddenly have reason to avoid specific locations on the screen just because another slit had been opened elsewhere. On the contrary, this is destructive interference. Because interference requires multiple coherent waves, *each* electron must be behaving as a wave passing through *both* slits at once.

Figure 2 shows how the electron flashes accumulate with both slits open, and Figure 3 shows the final pattern in an actual electron double-slit experiment. Figure 2 should look familiar—it is the double-slit photon-detection pattern of Figure 14. The point is that both electromagnetic radiation and massive objects exhibit the same kind of wave-particle duality. Associated with the particle of light (photon) is a wave of oscillating electromagnetic field. We must now accept that there is also a wave associated with a massive particle. So what is analogous to light's electromagnetic fields? In a matter wave, what is oscillating?

To identify at least one property of this wave, we return to the same two observations we made about light: (1) Although it is apparently impossible to say where the next electron will be detected, the probability that it will be

[1]The electron is our preferred massive object of study because of its relatively small mass. As we will soon see, this makes its wavelength not too small, so its wave nature is more easily revealed.

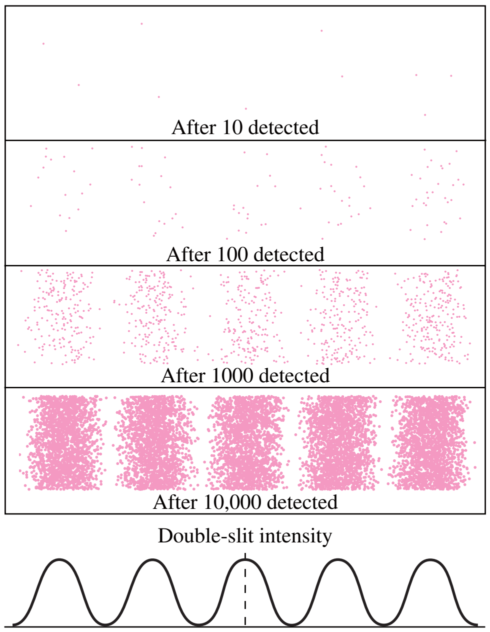
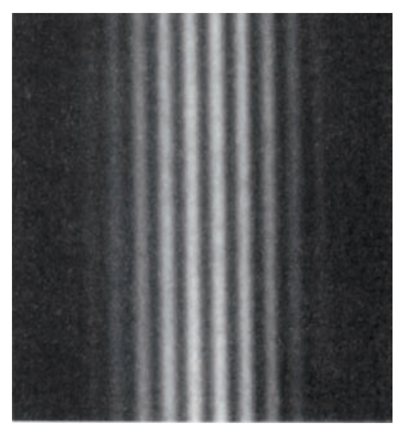



**Figure 2** Electrons producing a double-slit interference pattern—one particle at a time.

found in a given region should be proportional to the density of spots there. (2) If we concede that coherent waves of equal amplitude emerge from the two slits, we would expect an interference pattern. In particular, we would expect the amplitude at the pattern's center, where the interference is constructive, to be twice what it would be if only one slit were open, and the square of the amplitude, proportional to intensity, should be four times as large. And what does the experiment show? It shows a density of spots that is four times larger at the center with both open than with either alone. In fact, for the entire pattern, there is a proportionality between the density of spots and the intensity—the square of the wave—arising from the standard analysis of the double slit. Combining these observations—that the probability of finding the particle and the square of the wave's amplitude are proportional to the same thing (the density of spots)—we reach the same conclusion as for light:

**Figure 3** Actual double-slit pattern produced by electrons.



When a phenomenon is detected as *particles*, we cannot predict with certainty where a given particle will be found. The most we can determine is a probability of finding it in a given region, which is proportional to the square of the amplitude of the associated *wave* in that region.

$$\begin{matrix}\text{probability of finding} \\ \textit{particle} \text{ in a region}\end{matrix} \propto \begin{pmatrix}\text{amplitude of } \textit{wave} \\ \text{in that region}\end{pmatrix}^2$$

Although electromagnetic radiation and massive objects share this fundamental wave-particle relationship, matter waves differ in an important way: They cannot be directly detected. Electric and magnetic fields can be isolated and caused to exert forces on objects—we can directly detect them. But we haven't found any analogous way to directly detect matter waves. No one has ever "seen" one. So how do we answer the question, What is oscillating? Apparently, we have only a single candidate—*probability* is oscillating.[2]

## Matter Wave Interference: Evidence

The double slit is conceptually the simplest experiment for verifying matter wave interference, but unfortunately it isn't easy to do. The first evidence that matter has a wave nature was obtained by Clinton J. Davisson and Lester H. Germer in 1927. Investigating properties of metal surfaces by observing how a nickel crystal scatters a beam of electrons, Davisson and Germer were surprised to find that the electrons seemed to scatter preferentially at only certain discrete angles. Particles should not do this.

Microscopically, a crystal is an arrangement of regularly spaced atoms. If a wave is incident, each atom reflects the wave in all directions. In essence, each becomes a point source of waves. These sources produce an interference pattern, just as do the multiple slits of a diffraction grating, with sharp interference maxima separated by broad regions of low intensity. In the Davisson-Germer experiment, whose apparatus is shown in Figure 4, the experimental detection rate versus angle agrees perfectly with a theoretical prediction (see Exercise 23) based on the assumption that each electron behaves as a diffuse wave reflecting from many atoms and *interfering with itself*.

The key to the experiment's success in revealing the electron's wave nature is that the relevant dimension of the apparatus, the atomic spacing, is very small. A crystal is thus an excellent testbed to verify a wave nature. But nowadays, it is the other way around—the electron's wave nature, taken for granted, is exploited to learn about the crystal. In essence, the diffraction pattern maps out the microscopic geometry. Figure 5 is a good

**Figure 4** Davisson and Germer's original electron diffraction apparatus, showing the mechanism for varying a sample's angle.

[2]A common misconception is that the electron's *mass* is oscillating, perhaps that bits of it somehow jiggle back and forth. The wave isn't the particle. Mass doesn't oscillate in a matter wave any more than photons oscillate in an electromagnetic wave. It may be helpful to view both phenomena consistently as essentially waves of oscillating probability, addressing other possible traits only as the need arises.

example. Produced by electrons diffracting from a single grain of aluminum-manganese alloy, it reveals atoms arranged in a five-sided geometry. (Actually, such a geometry came as a surprise when first seen in the 1980s, and study of "quasicrystals" with such symmetries has been quite active ever since.)

## The Bragg Law

In a crystal, matter waves often penetrate to many atomic planes deeper than the surface, and the most commonly used quantitative relationship for constructive interference, which we now obtain, thus requires a bit more work than for a simple grating. Suppose, as depicted in Figure 6, a beam is directed at an angle $\theta$ with respect to a surface atomic plane, and the detector is positioned to receive waves reflecting at the same angle with the plane. An atom in the top plane reflects a small portion of the wave, ray 1, which scatters in all directions. But much of the wave penetrates deeper, so an atom in the second atomic plane also scatters a portion in all directions. As shown in the figure, ray 2 has $2d \sin \theta$ farther to travel than ray 1 to reach the detector, where $d$ is the spacing between atomic planes. Ray 3 travels the same distance farther than ray 2, and so on with each deeper atomic plane. With the incident beam and detector at equal angles, the waves scattering from any atoms in the *same* plane—for instance, rays 1 and 4—always have the same distance to travel from source to detector. So we view interference only between atomic planes at different

**Figure 5** Interference pattern of electron waves diffracted by aluminum manganese alloy.

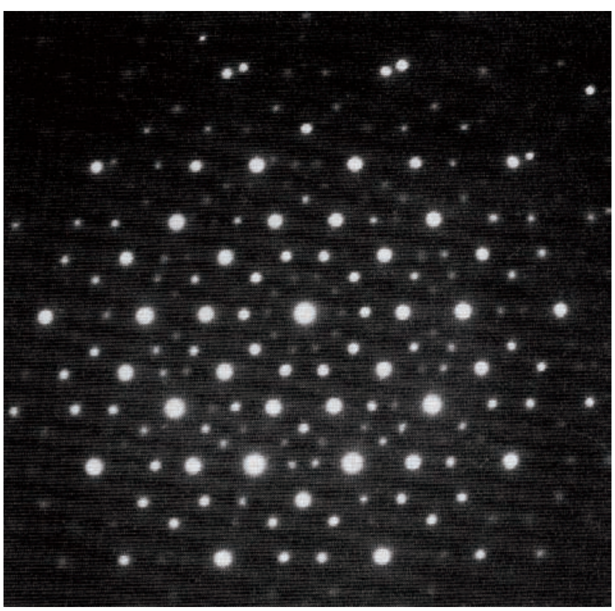
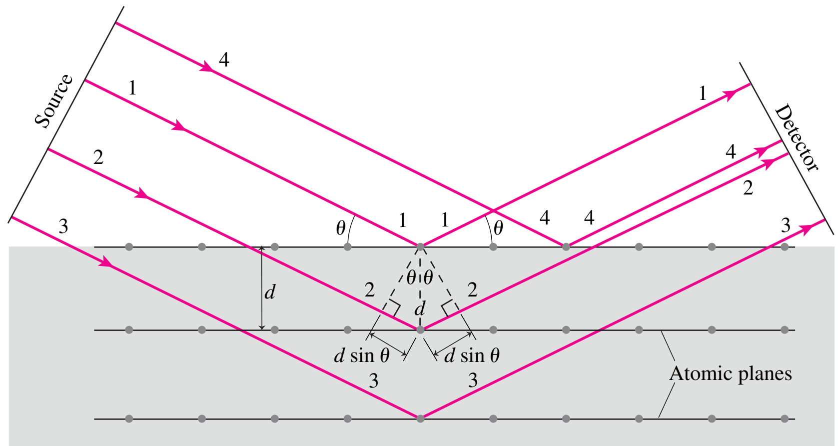


**Figure 6** Diffraction of a beam from multiple atomic planes.




depths in the crystal—very nice! Constructive interference occurs at angles where

$$2d \sin \theta = m\lambda \tag{1}$$

This is known as the **Bragg law**, for father-son team W. H. Bragg and W. L. Bragg, who won the 1915 Nobel Prize for their diffraction work. Their source wasn't electrons, however—it was X-rays. But the same equation applies, because a wave is a wave. We noted in Section 4 that X-rays may behave as particles where longer-wavelength visible light would behave as waves. The spacing *d* of atomic planes in a crystal is so small that even X-rays behave as waves. Anything will be wavelike in small enough dimensions, and diffraction patterns have been produced by beams of neutrons and even whole atoms. Of course, to exploit interference condition (1) in the case of matter waves—to determine an actual plane spacing *d*—we would need the wavelength. So let us now turn to the quantitative properties of matter waves.

## 2 Properties of Matter Waves

What properties characterize a wave? It should have a wavelength, a frequency, and a speed. It should also have an amplitude that varies with position and time. The generic term for the function giving the amplitude is **wave function**. Because what actually oscillates depends on the kind of wave, we use different symbols for the wave functions of different kinds of waves. For a transverse wave on a string, we often use the symbol $y(x, t)$. The string's transverse displacement $y$ varies as a function of the position $x$ along the string and time $t$. For an electromagnetic plane wave moving along the $x$-axis, we have two wave functions, $\mathbf{E}(x, t)$ and $\mathbf{B}(x, t)$, which describe how the oscillating electric and magnetic fields vary with position and time. For a matter wave, the symbol we choose for the wave function is $\Psi(x, t)$. Strictly speaking, it should be referred to as the **probability amplitude**, for it is the amplitude of the wave that (when squared) tells us probability. In practice, however, we usually just call it the wave function. In the next section, we will encounter the "wave equation" that this wave function must obey. But first let us study some properties and behaviors that don't require us to have an explicit formula for the wave function.

Matter wave function: $\Psi(x, t)$
Probability amplitude

## Wavelength

In our discussions so far, wavelength seems to play the pivotal role. What is it? In 1924, Louis de Broglie submitted the following hypothesis: The wavelength of the matter wave associated with a massive object depends on its momentum $p$ and is given by

$$\lambda = \frac{h}{p} \tag{2}$$

Wavelength of matter wave

This relationship has been confirmed beyond any doubt, even for relativistic speeds, by experiments such as crystal diffraction, in which the momentum of the electrons in a beam is known, and analysis of the pattern establishes the wavelength. The contribution won for de Broglie the 1929 Nobel Prize, and in recognition, we often refer to the wavelength of a matter wave as the **de Broglie wavelength**. *This relationship between wavelength and momentum is universal*—true for all phenomena.

### EXAMPLE 1

If moving at 900 m/s, what would be the wavelength of (a) an electron and (b) a 25,000 kg airplane? (c) Which is more likely to exhibit a wave nature?

**SOLUTION**

(a) $\lambda_{\text{electron}} = \dfrac{6.63 \times 10^{-34}\,\text{J}\cdot\text{s}}{(9.11 \times 10^{-31}\,\text{kg})(900\,\text{m/s})} = 8.1 \times 10^{-7}\,\text{m}$

(b) $\lambda_{\text{airplane}} = \dfrac{6.63 \times 10^{-34}\,\text{J}\cdot\text{s}}{(25{,}000\,\text{kg})(900\,\text{m/s})} = 2.9 \times 10^{-41}\,\text{m}$

(c) An apparatus with a relevant dimension on the micrometer scale might coax the electron to show its wave side. Actually, this is very slow as electron speeds go, so considerably smaller dimensions are usually needed. The airplane's wavelength, however, is a good 25 orders of magnitude smaller than the atomic nucleus ($\sim 10^{-15}$ m). Although "composite" objects much bigger than electrons—whole atoms, for instance—have been shown to behave in experiments as simple waves with a wavelength obeying equation (2), where $p$ is the object's center-of-mass momentum, in no conceivable experiment would something so big as an airplane ever behave as a wave.

In ordinary situations, the wavelengths of matter waves are short enough to ensure particlelike behavior, because Planck's constant is so small. However, as an object's momentum approaches zero, wouldn't its wavelength become arbitrarily large? We might draw the unsettling conclusion that any stationary object should behave as a wave. We shall face this predicament in Example 5, but the upshot is that it is hard to be sure that something is indeed stationary.

With our quantitative grasp of wavelength, let us return to the simplest example of interference.

## EXAMPLE 2

Suppose a beam of electrons moving at $3 \times 10^6$ m/s strikes a barrier that has two narrow slits separated by 0.020 μm, beyond which are electron detectors.[3] At the center detector, directly in the path the beam would follow if unobstructed, 100 electrons per second are detected. As the detector angle varies, the number per unit time varies in a typical double-slit pattern between the maximum of 100 $s^{-1}$ and the minimum of 0. The first minimum occurs at detector X, an angle $\theta_X$ from the center. (a) Find $\theta_X$. (b) How many electrons would be detected per second at the center detector if one of the slits were blocked? (c) How many would be detected per second at the center detector and at detector X if one slit were narrowed so that it *alone* would give a count rate 36% of its original value?

**SOLUTION**

(a) At detector X, the first point of destructive interference, the wave from one slit has $\frac{1}{2}\lambda$ farther to travel than the wave from the other. From physical optics, we know that the difference in distances traveled by the two waves in a double-slit experiment is $d \sin\theta$, where $d$ is the slit separation. Thus,

$$d \sin\theta_X = \frac{1}{2}\lambda$$

According to de Broglie's hypothesis, equation (2),

$$\lambda = \frac{h}{p} = \frac{6.63 \times 10^{-34}\,\text{J}\cdot\text{s}}{(9.11 \times 10^{-31}\,\text{kg})(3 \times 10^6\,\text{m/s})} = 2.43 \times 10^{-10}\,\text{m}$$

[3] As noted earlier, it is hard to do the electron double slit. In fact, the first success, shown in Figure 3, didn't come until the 1960s—and even then, it required a few tricks. The problem is that the smallest achievable slit separations are still much larger than the electron wavelength, even if the electron moves rather slowly, as assumed in Example 2.

Reinserting,

$$(0.020 \times 10^{-6}\,\text{m})\sin\theta_X = \frac{1}{2}(2.43 \times 10^{-10}\,\text{m}) \Rightarrow \theta_X = 0.35°$$

(b) With both slits open, the detection rate is 100 $s^{-1}$. This is proportional to the particle detection probability and, thus, to the *square of the amplitude of the matter wave*, the total wave arriving from both slits.

$$|\Psi_T|^2 \propto 100\,\text{s}^{-1} \Rightarrow |\Psi_T| \propto 10$$

(*Note:* To avoid distractions, we omit the units on $\Psi$. As discussed in Exercise 29, including them would merely introduce a proportionality constant that would cancel in the end.) The waves from the slits add equally at this point of constructive interference, so the amplitude of either wave alone must be half the total.

$$|\Psi_1| \propto 5 \Rightarrow |\Psi_1|^2 \propto 25\,\text{s}^{-1}$$

The electron detection rate would be 25 $s^{-1}$ at the center detector. Note that without a second slit/wave to interfere, 25 electrons per second would arrive at *all* detectors. With two slits, twice as many *should* be detected each second, but only *on average.* At points of constructive interference, 100 are detected per second, and at points of destructive interference, none. The average *is* 50, but its distribution in such a pattern cannot be understood by a strict particle view.

(c) If only the single narrowed slit were open, all detectors would register a detection rate 0.36 times 25 $s^{-1}$, or 9 $s^{-1}$:

$$|\Psi_1'|^2 = 0.36 \times |\Psi_1|^2 \propto 0.36 \times 25\,\text{s}^{-1} = 9\,\text{s}^{-1}$$

The amplitude of the wave follows:

$$|\Psi_1'|^2 \propto 9\,\text{s}^{-1} \Rightarrow |\Psi_1'| \propto 3$$

This is 60% of the original amplitude. Sensibly, an amplitude only 60% of the original implies a *square* of the amplitude of only 36%.

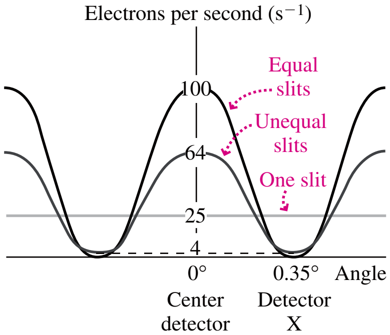
With both slits open, we have two waves of different amplitudes, one proportional to 5 (original) and one to 3 (narrowed). At the center detector, where they *add* constructively, the total amplitude is proportional to 5 + 3:

$$|\Psi_T'|_{\text{constr}} \propto 8 \Rightarrow |\Psi_T'|^2_{\text{constr}} \propto 64\,\text{s}^{-1}$$

At detector X, where the waves are 180° out of phase, the interference would no longer be totally destructive—their amplitudes aren't equal. Still, they *subtract*, so the total amplitude would be proportional to 5 – 3:

$$|\Psi_T'|_{\text{destr}} \propto 2 \Rightarrow |\Psi_T'|^2_{\text{destr}} \propto 4\,\text{s}^{-1}$$

**Figure 7** Electron detection rate versus angle for equal-width slits, unequal-width slits, and one slit.




The average is $\frac{1}{2}(64\ s^{-1} + 4\ s^{-1}) = 34\ s^{-1}$, which is the sum of the 9 $s^{-1}$ and 25 $s^{-1}$ expected from each slit alone. But to find the probability, or detection rate, at a location, we do not add the *probabilities* from each slit. Probabilities are always positive, so they can't cancel. Rather, we add the *waves*, which may add constructively or destructively, to find the total wave, and then square that to find the probability.

Figure 7 shows how the electron detection rate in Example 2 varies with angle for the equal-width, one-slit, and unequal-width cases. Note that, except for part (a), Example 2 could have involved light. We would simply replace the term *matter wave* with "electric field" and *electron detection rate* or *probability* with "light intensity." We have analyzed electron behavior via standard wave theory.

Having gained some quantitative feel for matter waves, let us take a look at producing and using a matter wave beam.

## EXAMPLE 3

To put the wave nature of electrons to use, an accelerating potential is often the start. Obtain a formula for the potential difference $V$ required to give a particle of mass $m$ and charge $q$ a wavelength of $\lambda$. Assume that the potential difference is insufficient to accelerate the particle to relativistic speeds.

### SOLUTION

The accelerating potential gives the electron kinetic energy, and velocity is related to the wavelength via the de Broglie formula.

$$qV = \frac{1}{2}mv^2 \qquad v = \frac{p}{m} = \frac{h}{m\lambda}$$

Eliminating $v$ between the two gives

$$V = \frac{h^2}{2mq\lambda^2}$$

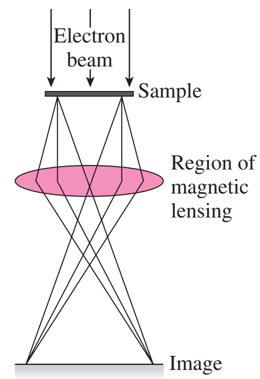
We see that a shorter wavelength requires a higher accelerating potential. As noted in Section 1, the electron's small wavelength is now routinely exploited. This result is an important element.

## REAL-WORLD EXAMPLE USEFUL WAVELENGTHS AND ACCELERATING POTENTIAL

The **transmission electron microscope (TEM)** is a workhorse in biological sciences and several other fields. The TEM replaces the optical microscope's illuminating light beam with a beam of electrons accelerated through a potential difference, while magnetic "lenses" take the place of glass ones, as depicted schematically in Figure 8. After passing through the sample and a series of "lenses," the electron beam produces an image on a screen. As we learn in optics, a microscope's resolution is limited by diffraction; features smaller than the wavelength are blurred. Light has wavelengths measured in hundreds of nanometers. The TEM's electrons have much shorter wavelength, diffract less, and thus reveal much finer detail.

In **low-energy electron diffraction (LEED)**, another application relying on an accelerating potential, the fact that electrons diffract as waves is, as the name suggests, the whole point. Like the Davisson-Germer experiment, LEED reflects an electron beam from a crystal. By using "low" accelerating potentials, electrons in LEED don't penetrate far, and the resulting diffraction peaks reveal the geometrical structure of the atoms on the surface. Let us take a look at the accelerating potentials in these applications.

**Figure 8** In an electron microscope, electrons replace light rays and magnetic fields replace lenses.




### Applying the Physics

(a) To produce a good diffraction pattern in a technique like LEED, an incident beam should have a wavelength comparable to the separation between the "slits"—the atoms that scatter the beam. A typical atomic spacing in a crystal is 0.2 nm. Approximately what potential difference is appropriate? (b) The accelerating potential in a particular TEM is 50 kV. If this were the only factor governing resolution, how small a detail could be seen?

**SOLUTION**

(a) Using the relationship derived in Example 3, we have

$$V = \frac{(6.63 \times 10^{-34}\,\mathrm{J \cdot s})^2}{2(9.11 \times 10^{-31}\,\mathrm{kg})(1.6 \times 10^{-19}\,\mathrm{C})(0.2 \times 10^{-9}\mathrm{m})^2} = 38\ \mathrm{V}$$

Accelerating voltages in LEED are typically 20–200 V, so our estimate is quite good.

(b) By the same formula, the accelerating potential determines the wavelength

$$50 \times 10^3\ \mathrm{V} = \frac{(6.63 \times 10^{-34}\,\mathrm{J \cdot s})^2}{2(9.11 \times 10^{-31}\mathrm{kg})(1.6 \times 10^{-19}\,\mathrm{kg})\lambda^2}$$

$$\Rightarrow \lambda = 0.0055\ \mathrm{nm}$$

We conclude that such fast electrons, much less wavelike than in the LEED application, could, in principle, resolve details of about 0.01 nm. As it turns out, aberrations in the magnetic lenses limit even the best TEMs, which use nearly 10 times this accelerating potential, to resolutions no finer than about 0.1 nm. Still, this is far better than the approximate 100 nm limit of optical microscopes.

## Frequency

Interference patterns give clear evidence for the wavelength of matter waves. The evidence for the frequency isn't so direct, but again, the relationship is the same as for electromagnetic radiation:

$$f = \frac{E}{h} \qquad (3)$$

Frequency of matter waves

It is often more convenient to express equations (2) and (3) in terms of the **wave number** and **angular frequency**, defined as follows:

$$k \equiv \frac{2\pi}{\lambda} \qquad\qquad \omega \equiv \frac{2\pi}{T}$$

Wave number Angular frequency

Note that wave number is a “spatial frequency.” Just as angular frequency $\omega$ is inversely proportional to the temporal period $T$, $k$ is inversely proportional to the spatial period $\lambda$. Another very convenient definition is

$$\hbar \equiv \frac{h}{2\pi} = 1.055 \times 10^{-34}\ \text{J} \cdot \text{s}$$

With these definitions, we express the fundamental wave-particle relationships as

Fundamental wave-particle relationships

$$p = \frac{h}{\lambda} = \hbar k \tag{4}$$

$$E = hf = \hbar\omega \tag{5}$$

## Velocity

We have left matter wave velocity for last because we simply won’t use it much, but it is worthwhile to see why. The famous $v = f\lambda$ does correctly give the *wave* speed. Using equations (4) and (5), it is

$$v_{\text{wave}} = f\lambda = \frac{E}{h}\frac{h}{p} = \frac{E}{p}$$

However, this may or may not be the speed of the *particle*. We saw that massless particles, such as photons, move at $c$, and their *particle* properties $E$ and $p$ are related by $E = pc$. The above relationship then confirms that electromagnetic *waves* also move at $c$. *Matter waves* would move at $c$ if $E = pc$ also held for massive particles—but it doesn’t. They would at least move at the *particle* speed if $E$ were equal to $pv_{\text{particle}}$, but this too is not the case. (Exercise 31 discusses the point further.)

Wave and particle velocities, known respectively as phase and group velocities. The main point here is that the formula $v = f\lambda$ is of rather limited use for massive particles, because $v$ is neither the speed of the particle nor the speed of light. The usual relationships between *strictly* particle properties ($p = mv$, KE $= \frac{1}{2}mv^2$, etc.) are fine, and equations (4) and (5) are universal. But for massive particles, $E$ is not $hc/\lambda$ nor $hv_{\text{particle}}/\lambda$, and $p$ is neither $hf/c$ nor $hf/v_{\text{particle}}$ (Exercise 32 focuses on the correct way to relate energy to wavelength and momentum to frequency.)

# 3 The Free-Particle Schrödinger Equation

How do we determine the wave function $\Psi(x, t)$ of a matter wave? In one sense, all types of waves are the same. For each, there is an underlying **wave equation**, of which the **wave function** (don't confuse these terms!) must be a solution. Let us look at two familiar cases.

## Waves on a String

For transverse waves on a stretched string, the wave equation is

$$v^2 \frac{\partial^2 y(x, t)}{\partial x^2} = \frac{\partial^2 y(x, t)}{\partial t^2}$$

where $v$ is the wave speed. The wave function is the solution $y(x, t)$, which gives the string's transverse amplitude as a function of position and time. All wave equations ultimately rest on fundamental laws, and this one comes (after some clever adaptations) from $\mathbf{F}_{\text{net}} = m\mathbf{a}$. A basic sinusoidal solution of this wave equation, illustrated in Figure 9, is the wave function

$$y(x, t) = A \sin(kx - \omega t) \quad \text{where} \quad \frac{\omega}{k} = v$$

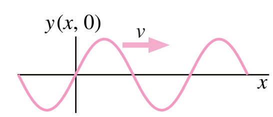



**Figure 9** A wave disturbance on a string.

## Electromagnetic Waves

The fundamental laws governing electromagnetic phenomena are Maxwell's equations. In vacuum, where charges and currents are absent, they are

$$\oint \mathbf{E} \cdot d\mathbf{A} = 0 \quad \text{(6a)} \qquad \oint \mathbf{B} \cdot d\mathbf{A} = 0 \quad \text{(6b)}$$

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t}\int \mathbf{B} \cdot d\mathbf{A} \quad \text{(6c)} \qquad \oint \mathbf{B} \cdot d\mathbf{l} = \frac{1}{c^2}\frac{\partial}{\partial t}\int \mathbf{E} \cdot d\mathbf{A} \quad \text{(6d)}$$

Although we omit the details, these, like $\mathbf{F}_{\text{net}} = m\mathbf{a}$, can be rearranged into wave equations—one for **E**, one for **B**—involving partial derivatives in space and time. A primary difference is that electromagnetic waves *by nature* have *two* parts, **E** and **B**. The basic sinusoidal solution/wave function in this case is a **plane wave**. By definition, a plane wave moves in one direction and has a constant amplitude—it doesn't spread out. Figure 10 depicts a plane wave moving in the $x$ direction, and the solutions of Maxwell's equations that describe it are

$$\begin{aligned} \mathbf{E}(x, t) &= A \sin(kx - \omega t)\hat{\mathbf{y}} \\ \mathbf{B}(x, t) &= \frac{1}{c} A \sin(kx - \omega t)\hat{\mathbf{z}} \end{aligned} \quad \text{where} \quad \frac{\omega}{k} = c \qquad (7)$$

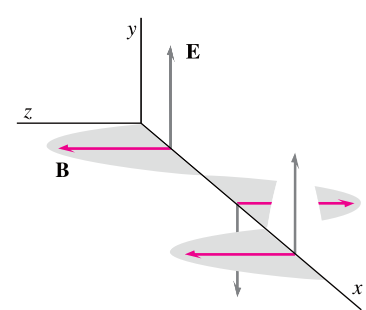



**Figure 10** An electromagnetic plane wave.

## Matter Waves

The wave equation obeyed by matter waves is the **Schrödinger equation**. In this chapter, we consider only the special case of free particles. In the absence of external forces, the Schrödinger equation is

Schrödinger equation (free particle)

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} = i\hbar\frac{\partial\Psi(x,t)}{\partial t} \tag{8}$$

This equation can trouble beginning students of quantum mechanics on two accounts: First, because its form is certainly not intuitively obvious, we might hope for a derivation from first principles. However, there simply is no more basic physical principle on which it is built. Its acceptance as, in essence, a "law" rests on its rendering correct predictions—probabilities of finding particles, for instance. Although we can't derive it, we will soon argue that at least it has a plausible foundation.

The second concern is that the Schrödinger equation is complex, involving $i$; that is, $\sqrt{-1}$. It might be incorrectly concluded that a matter wave is not "real." Unfortunately, this would also seem to fit perfectly with the point we made earlier: The wave function is not directly observable. (Why should it be, since it isn't real anyway?) The reason for the $i$ is not that matter waves are unreal, but that they can't be represented by a single real function. Like electromagnetic waves, they, by nature, have two parts, and a complex function, carrying twice the information of a real one, enables us to handle them together. In analyzing an electromagnetic wave, we could treat **E** and **B** as a single complex unit by including an $i$ (see Exercise 35), without making either field "unreal." We don't do this, because **E** and **B** have different personalities, and we like to keep them separate. But we haven't found similar reasons to keep the two parts of a matter wave separate, so we use a single complex function. It is simply a matter of convenience (see Exercise 34).

It may be convenient, but can we *physically interpret* a complex wave function? At the risk of sounding impertinent, there is no need, for the wave function itself cannot be physically detected. What *is* open to experimental scrutiny is the wave function's "square," which is a real, nonnegative quantity: the probability density.

## Probability Density

We have noted that for all phenomena, the probability of detecting the particle is proportional to the square of the wave's amplitude. But what does this mean if the wave has two parts, an **E** and a **B**, or real and imaginary parts of $\Psi(x, t)$? For electromagnetic waves, experiment verifies that the probability is proportional to $E^2 + (cB)^2$, which, not coincidentally, is proportional to the total electromagnetic intensity. Adding the squares seems to be the natural way.

Experiment verifies the same for matter waves—add the squares of the real and imaginary parts.

$$[\text{Re}\,\Psi(x, t)]^2 + [\text{Im}\,\Psi(x, t)]^2 = \Psi^*(x, t)\Psi(x, t) = |\Psi(x, t)|^2$$

where $\Psi(x, t) = \text{Re}\,\Psi(x, t) + i\,\text{Im}\,\Psi(x, t)$. Note that $\Psi^*$ signifies the complex conjugate of $\Psi$ and, conveniently, gives the expression the appearance of a simple square. Fortunately, we will usually work with just a single real wave function, a piece of the full complex one, and when we say "the square of the wave function," it is a simple square. But when we do use the full complex function, this expression is understood to mean the product of the function and its complex conjugate.

Until now, we have been careful to say that the probability and square of the wave's amplitude are proportional. Suppose we look for the particle in a region of width $\delta$ surrounding a certain point. For a given wave amplitude, there is a given probability of finding the particle there. But if $\delta$ is so small that the wave's amplitude is essentially constant around that region, then the probability of finding the particle in a region of width $2\delta$ must be twice as large. Thus, the square of the wave function's amplitude must give a probability *per unit length*. In three dimensions, it is probability *per unit volume*. The generic term for probability per unit length or volume is **probability density**.

$$\text{probability density} = |\Psi(x, t)|^2 \tag{9}$$

## The Plane Wave

Let us now reveal the foundation of the free-particle Schrödinger equation by considering its most basic solution. A plane-wave solution is the **complex exponential**

$$\Psi(x, t) = Ae^{i(kx-\omega t)} \tag{10}$$

where $A$ is a constant. To verify that this is a solution of equation (8), the question is

$$-\frac{\hbar^2}{2m}\frac{\partial^2 Ae^{i(kx-\omega t)}}{\partial x^2} \overset{?}{=} i\hbar\frac{\partial Ae^{i(kx-\omega t)}}{\partial t}$$

Taking the partial derivatives on both sides, we have

$$-\frac{\hbar^2}{2m}(ik)^2 Ae^{i(kx-\omega t)} = i\hbar(-i\omega)Ae^{i(kx-\omega t)}$$

and canceling,

$$\frac{\hbar^2 k^2}{2m} = \hbar\omega \tag{11a}$$

That the functional dependence on position and time cancels means that function (10) obeys the Schrödinger equation for all values of $x$ and $t$, provided only that $k$ and $\omega$ are related as in condition (11a). It is left as an exercise to show that the more familiar $A\sin(kx - \omega t)$ and $A\cos(kx - \omega t)$—applicable to waves on a string, for example—simply don't work the same way. They are not solutions. It is true by the Euler formula that $Ae^{i(kx - \omega t)}$ is equivalent to $A\cos(kx - \omega t) + iA\sin(kx - \omega t)$, so there is a similarity. But the complex exponential has two parts, and they are out of phase by one-quarter cycle.

The requirement that (11a) must hold is the key to seeing how the Schrödinger *wave* equation relates to the classical physics of *particles*. We hope that it isn't at odds with the fundamental wave-particle relationships, $p = \hbar k$ and $E = \hbar\omega$. What happens if we insert them? Condition (11a) becomes

$$\frac{p^2}{2m} = E \tag{11b}$$

Given that $p^2/2m = (mv)^2/2m = \frac{1}{2}mv^2$, this merely says that the particle's kinetic energy must equal its total energy, which is the classical truth, because a free particle has no potential energy. Thus, *the Schrödinger equation is related to a classical accounting of energy*.

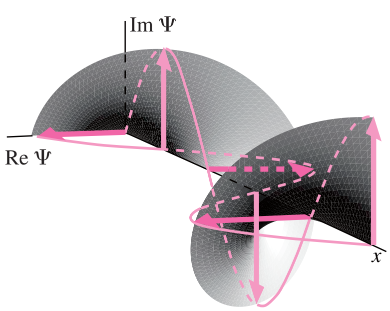
Although the wave function isn't physically detectable, Figure 11, which plots $Ae^{i(kx - \omega t)}$ at $t = 0$, provides some insight into the mathematical nature of a plane wave. The real part of $\Psi$ is a cosine, the imaginary part a sine, and the two parts are out of phase in such a way that the *magnitude is constant*—it varies neither in position nor in time. The direct calculation of the probability density agrees:

$$|\Psi(x, t)|^2 = \Psi^*(x, t)\Psi(x, t) = \left[Ae^{-i(kx - \omega t)}\right]\left[Ae^{+i(kx - \omega t)}\right] = A^2$$

or

$$|\Psi(x, t)| = A$$

**Figure 11** A plane matter wave: The real and imaginary parts of $Ae^{i(kx-\omega t)}$, plotted at $t = 0$.




That the probability per unit length is constant means that if we were to look for it, *a particle represented by a plane wave would be equally likely to be found anywhere*.

Obviously, a plane wave is not very realistic, but it is still quite useful. In physical optics, we speak of plane waves of light, because they are often a sufficiently good approximation of the actual wave. The same is true of matter waves. But the plane wave's importance goes even deeper, for a more general wave can be treated as an algebraic sum of plane waves—they are easily analyzed "building blocks." We will make use of this fact at several points later in the text. Let us now turn to a topic where we need not know the actual $\Psi(x, t)$. It is one of the most profound ideas in all of physics.

# 4 The Uncertainty Principle

The mere fact that a phenomenon has a wave nature implies inherent uncertainties in its particle properties. For example, passing through a single slit causes an electromagnetic plane wave to spread out, so it must also cause uncertainty in the momenta of the particles (photons) detected afterward. The same must apply to an electron plane wave. Figure 12 depicts a single-slit pattern developed (like the earlier double slit) one electron at a time. The $x$-component of momentum of an electron after passing through is obviously uncertain.

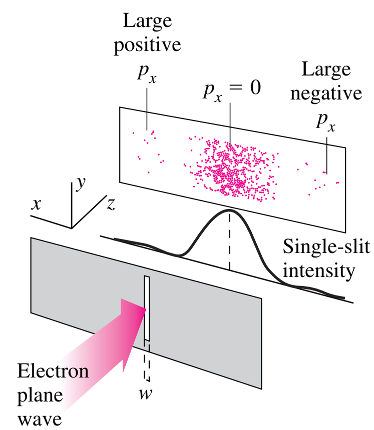



**Figure 12** Single-slit diffraction of an electron plane wave.

What we mean by "uncertainty" in momentum is that if the experiment is repeated many times *identically*, the momentum detected after passing through the slit still varies over a range of values. But how do we quantify it? Suppose that the $p_x$-values we record fall within the range $-1$ kg · m/s to $+1$ kg · m/s, except for one at $+50$ kg · m/s. What value do we assign to the uncertainty? 1 kg · m/s? 25 kg · m/s? 51 kg · m/s?

The definition of uncertainty is an arbitrary choice, but it obviously should measure how far deviations are from the mean (average) value. In physics, we define it as **standard deviation**. For example, suppose repeated experiments are carried out to determine a quantity $Q$, where $Q$ might represent position $x$, a component of momentum $p_x$, or any other measurable quantity. The value $Q_1$ is obtained $n_1$ times, the value $Q_2$ is obtained $n_2$ times, and so on. We find the **mean** $\overline{Q}$ by multiplying a particular value $Q_i$ by the number of times it is obtained, $n_i$, summing over all values, then dividing by the total number of times for all values.

$$\overline{Q} = \frac{\sum_i Q_i n_i}{\sum_i n_i} \tag{12}$$

The standard deviation $\Delta Q$ is defined as the square root of the mean of the squares of the values' deviations from the mean (explaining its alternative name, root-mean-square deviation). Here we merely present the formula that goes with the words:

$$\Delta Q = \sqrt{\frac{\sum_i (Q_i - \overline{Q})^2 n_i}{\sum_i n_i}} \tag{13}$$

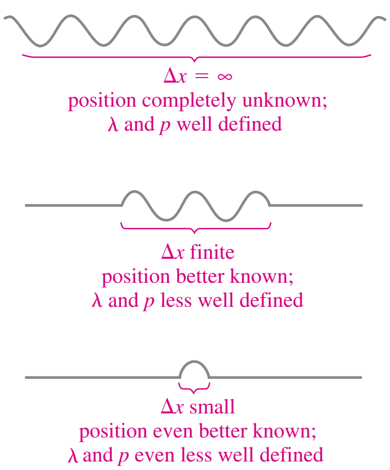
This definition is very well suited to its role. It is the most tractable one that is zero *if and only if* there is only one value ever obtained, which would automatically be $\overline{Q}$, and when values do vary, it gets larger as they become more spread out.

Although it is important to know that uncertainty has a logical definition, as we continue to investigate the uncertainty principle in this section, we won't actually *use* the definition. The point is that when we say, for instance, that there is "an uncertainty in the electrons' momentum," we aren't speaking of something nebulous but a specific value following from a concrete definition.

So let us return to the single slit. As Figure 12 shows, there is an uncertainty in the $x$-component of momentum of electrons detected beyond the slit, for which the symbol is $\Delta p_x$. (The symbol $\Delta$ often means "change," but not here. Here it means uncertainty, or standard deviation.) On the other hand, were we to conduct a different experiment, designed to establish the position of electrons exiting the slit, there would be an uncertainty in this quantity, too—$\Delta x$. The electron wave front is spread over the entire width $w$, so there would be a probability of finding the particle anywhere in this range, and the narrower the slit, the smaller would be this uncertainty. Above all, there is a link between the uncertainties in $p_x$ and $x$. Because the width of a diffraction pattern, related to $\Delta p_x$, is inversely proportional to the slit width, related to $\Delta x$, *the uncertainties are inversely proportional*.

$$\Delta p_x \propto \frac{1}{\Delta x}$$

**Figure 13** As a wave becomes more compact, its overall wavelength and momentum are less well defined.




Although we have used the familiar single slit as a vehicle, the particular experiment is not to "blame" for the conclusion. Regardless of the circumstances, it is an inescapable consequence of matter's wave nature—whether obvious or not—that increased precision in the knowledge of position implies decreased precision in the knowledge of momentum and vice versa. Figure 13 illustrates a simplified, qualitative argument. The top wave is infinite and regular. While there is no doubt of its wavelength, there would be a probability of finding the particle at places along the entire infinite $x$-axis. Wavelength and, thus, momentum $h/\lambda$ are certain ($\Delta p = 0$), but essentially nothing is known about the particle's position ($\Delta x = \infty$). The center wave is regular over only a finite region. The cost of obtaining a wave for which the particle's probable whereabouts are narrowed down ($\Delta x \neq \infty$) is that the wave is not regular everywhere. In any fair way of taking into account all of space, the wavelength cannot be said to be simply $\lambda$, so neither can we claim that the momentum is precisely $h/\lambda$ ($\Delta p \neq 0$). The bottom wave gives an even better known position, but only by further restricting the region over which the wave is regular. Accordingly, it is even less fair to say that the wavelength of this wave as a whole is $\lambda$, so $\Delta p$ is larger still. The relationship between $\Delta p$ and $\Delta x$ is developed quantitatively in Section 7. Here we concern ourselves only with the conclusion, known as the **uncertainty principle**, and its ramifications.

Momentum-position uncertainty principle

Because of a particle's wave nature, it is theoretically impossible to know precisely both its position along an axis and its momentum component along that axis; $\Delta x$ and $\Delta p_x$ cannot be zero simultaneously. There is a strict theoretical lower limit on their product:

$$\Delta p_x \Delta x \geq \frac{\hbar}{2} \tag{14}$$

Often referred to as the Heisenberg uncertainty principle, for its discoverer Werner Heisenberg (Nobel Prize 1932), it is a shocking revelation. There is a *theoretical limit* on the precision with which some familiar quantities can be known simultaneously. If we know a particle's position *exactly*, we can know *nothing* about its momentum ($\Delta x = 0 \Rightarrow \Delta p_x = \infty$). If momentum is known exactly, position is completely unknown. The plane wave is a good example of the latter case. This fundamental matter wave has a wavelength perfectly regular throughout space, giving it a perfectly precise momentum, but it represents a particle equally likely to be found anywhere. A property in which there is no uncertainty is said to be **well defined**. For the plane wave, momentum is well defined ($\Delta p_x = 0$), but position couldn't be more *un*defined ($\Delta x = \infty$).

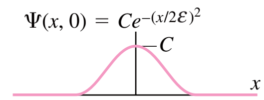



**Figure 14** A Gaussian wave form, $\Delta x\, \Delta p = \frac{1}{2}\hbar$.

Don't be troubled by the inequality in (14)—there is no uncertainty about the uncertainty principle. The $\geq$ reflects the simple fact that there is a particular wave shape, called a **Gaussian**, also known as a bell curve, for which the product of uncertainties is a minimum. Figure 14 shows a Gaussian wave form, a constant $C$ times a "Gaussian factor" $e^{-(x/2\varepsilon)^2}$ where $\varepsilon$ is a constant. It is maximum at $x = 0$, falls off toward 0 symmetrically as $x$ becomes large, and the rate of fall-off depends on $\varepsilon$. If $\varepsilon$ is large, the wave form is broad, falling off very slowly; whereas if $\varepsilon$ is small, the wave form is narrow. We leave the actual calculation of uncertainties from wave functions until later, but it shouldn't be surprising that the position uncertainty $\Delta x$ is proportional to $\varepsilon$. It is also true that $\Delta p$ is *inversely* proportional to $\varepsilon$, and it is only for a particle whose matter wave function is of this form that the product of the two uncertainties is the minimum theoretically possible, $\frac{1}{2}\hbar$. For any other shape, simultaneous knowledge of the two is less precise: $\Delta p_x \Delta x > \frac{1}{2}\hbar$.

The whole idea behind the uncertainty principle is rather upsetting to a student of classical physics. Classically, we claim that we can calculate a particle's position and velocity for all time via $\mathbf{F} = m\mathbf{a}$ and kinematics. We need only know the forces acting and the initial position and velocity. But now we see that even starting such a calculation is problematic, for precise knowledge of position and velocity simultaneously is impossible. Fortunately, as we soon see, the uncertainty principle is of little consequence for "large" things.

On the other hand, it is of great consequence for the small things we study in quantum mechanics. As we saw in Example 1, while wavelengths of macroscopic objects are ridiculously small, an electron might well have a wavelength measured in nanometers or larger. Undoubtedly, it would behave as a particle in a situation where distances are measured in meters, as in a television's cathode-ray tube. But it should definitely show its wave side when confined in a system measured in fractions of a nanometer. Such a system is the atom, probably the most logical test of quantum mechanics, and the simplest atom is hydrogen—essentially an electron orbiting a stationary proton. In such small confines, the electron must be treated not as an orbiting particle but as a bound three-dimensional wave surrounding the proton. Because the wave is spread diffusely, the probability of finding the electron is spread diffusely. Our knowledge of the atom's approximate size sets a rather small *maximum* possible value for the position uncertainty, and there is, correspondingly, a rather large *minimum* theoretical uncertainty in momentum.

## EXAMPLE 4

The hydrogen atom is known to be about 0.1 nm in radius. That is, the electron's orbit, whatever may be its shape, extends to about this far from the proton. Accordingly, the uncertainty in the electron's position is no larger than about 0.1 nm. What is the minimum theoretical uncertainty in its velocity?

**SOLUTION**

An electron in an atom moves in three dimensions, but considering components of motion along just one of the axes should give us a pretty good approximation.

$$\Delta p_x \Delta x \geq \frac{\hbar}{2} \rightarrow \Delta p_x (0.1 \times 10^{-9}\text{ m}) \geq \frac{1.055 \times 10^{-34}\text{ J}\cdot\text{s}}{2}$$

$$\Rightarrow \quad \Delta p_x \geqslant 5.3 \times 10^{-25}\text{ kg}\cdot\text{m/s}$$

Now using $p = mv$,

$$\Delta v_x = \frac{\Delta p_x}{m} \geq \frac{5.3 \times 10^{-25}\text{ kg}\cdot\text{m/s}}{9.11 \times 10^{-31}\text{ kg}} = 5.8 \times 10^5\text{ m/s}$$

From Example 4, we conclude that an experiment designed to determine the hydrogen electron's speed, if repeated identically, must produce a range of values covering more than $\frac{1}{10}$% of $c$. In fact, careful study of the hydrogen atom reveals slight relativistic effects. Actually, the theoretical minimum $\Delta p_x$ of $5.3 \times 10^{-25}$ kg · m/s would apply only if the wave function were a Gaussian, which it is not, so the true uncertainty is somewhat larger. But to be too concerned with this point is to overlook much of the power of the uncertainty principle. It governs all phenomena, for all have an underlying wave nature, and it may be used for order-of-magnitude calculations *in complete ignorance* of the wave function. We shall see just how useful this can be in Example 6, which gives us an excellent estimate of the hydrogen electron's *energy*. First, let us revisit the problem of reconciling the uncertainty principle with classical mechanics.

## The Classical Limit

The uncertainty principle places no significant limitation on the use of classical mechanics in classical situations. In the following example, we justify this claim and also confront our earlier predicament: A stationary object, with a corresponding infinite wavelength, should behave as a wave.

## EXAMPLE 5

By simple visual inspection, we can establish the location of an object within an uncertainty of about 550 nm, the wavelength of visible light. Suppose the object is a 1 mg grain of sand, apparently stationary. (a) What is the minimum uncertainty in its velocity, and if moving at this speed, how long would it take to travel the smallest distance perceivable, about 1 μm? (b) A wavelength of 1 nm would be small enough to ensure particle behavior in everyday circumstances. How fast would the grain of sand have to move to have such a wavelength?

**SOLUTION**

(a)
$$\Delta p_x \Delta x \geq \frac{\hbar}{2} \;\rightarrow\; \Delta p_x\,(5.5 \times 10^{-7}\,\text{m}) \geq \frac{1.055 \times 10^{-34}\,\text{J}\cdot\text{s}}{2}$$
$$\Rightarrow \quad \Delta p_x \geq 9.59 \times 10^{-29}\,\text{kg}\cdot\text{m/s}$$

In essence, $\Delta p_x$ is small because $\hbar$ is small.

$$\Delta v = \frac{\Delta p}{m} \geq \frac{9.59 \times 10^{-29}\,\text{kg}\cdot\text{m/s}}{10^{-6}\,\text{kg}} = 9.59 \times 10^{-23}\,\text{m/s}$$

In Example 4, the electron's velocity uncertainty was nearly relativistic. It is a relief to find it so much smaller here. Large mass is the reason. Quantum-mechanically speaking, a grain of sand is huge.

$$t = \frac{\text{distance}}{\text{speed}} = \frac{10^{-6}\,\text{m}}{9.59 \times 10^{-23}\,\text{m/s}} = 1.04 \times 10^{16}\,\text{s}$$
$$= 3.3 \text{ million centuries}$$

Clearly, we can know both the position and the velocity of this object precisely enough to apply classical mechanics.

(b) Using the de Broglie formula,

$$v = \frac{p}{m} = \frac{h/\lambda}{m} = \frac{6.63 \times 10^{-34}\,\text{J}\cdot\text{s}}{(10^{-6}\,\text{kg})(10^{-9}\,\text{m})} = 6.63 \times 10^{-19}\,\text{m/s}$$

Part (a) calculates the *theoretical* minimum velocity uncertainty—more precise knowledge of $v$ is impossible—but measurements in classical situations don't even come close to such a ridiculously small speed. Here we see that even a (classically) tiny object could appear absolutely stationary yet have a wavelength so short as to behave particlelike for all classical purposes.

## A Practical Application

The uncertainty principle alone explains a behavior for which classical arguments fail. Classically, there is no lower limit on the energy a small particle may have as it orbits a large body. For instance, a satellite may be positioned at any distance from a planet and, if given the proper velocity, maintain a circular orbit. The smaller the orbit radius, the lower would be the energy, though there is a *practical* lower limit in which the satellite simply rests on the planet's surface. By analogy, if an electron orbits a proton in a hydrogen atom, there should be no lower limit on the energy it may have. But there *is* a minimum energy, called the ground-state energy, and it is inconsistent with the electron simply resting on the proton. The electron's wave nature—specifically, the inverse relationship between its momentum and position uncertainties—is the answer to the mystery.

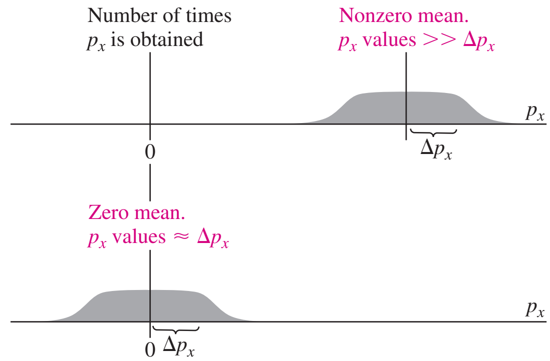



**Figure 15** If $p_x$ has nonzero mean, a typical value of $p_x$ differs considerably from $\Delta p_x$, but if the mean is zero, a typical value cannot be much larger than $\Delta p_x$.

Before we show this, we discuss a point that bears on various applications of the uncertainty principle: If the mean of a quantity is *zero*, then the *uncertainty* in the quantity and *typical values* of the quantity are comparable. If the mean is nonzero, this need not hold. For instance, a position mean of 100.0 m with an uncertainty of 0.1 m implies typical position values in the range of 99.9 m to 100.1 m—obviously not comparable to 0.1 m. However, the same uncertainty with a mean of zero would imply positions between −0.1 m and +0.1 m (or 0 and 0.1 m, if negative values aren't allowed), which *are* comparable to the uncertainty.

Figure 15 illustrates the point in the context of momentum. The top diagram shows results of a hypothetical experiment in which measured $p_x$ values have a mean that is large and positive. Essentially all $p_x$ values are much larger than $\Delta p_x$. The bottom diagram shows a case with the same $\Delta p_x$ but a mean of zero. Any $p_x$ value within about $\Delta p_x$ of the mean is likely. The chance of its being $10\Delta p_x$ is clearly quite small, and even the chance of its being within, say, $\frac{1}{10}\Delta p_x$ of zero is rather small. Therefore, the mostly likely values of $p_x$ are those roughly the same order of magnitude as $\Delta p_x$.

## EXAMPLE 6

An electron is held in orbit about a proton by electrostatic attraction. Its total mechanical energy is the sum of its kinetic energy and the electrostatic potential energy between charges $+e$ and $-e$ that are a distance $r$ apart.

$$E = \frac{1}{2}mv^2 + \frac{1}{4\pi\varepsilon_0}\frac{(+e)(-e)}{r}$$

(a) If the electron behaves as a classical particle, it must obey $F = ma$. Assuming a circular orbit, apply $F = ma$ to eliminate $v$ in favor of $r$ in the energy expression, and demonstrate that the energy has no minimum.

(b) Suppose now that the electron behaves as an orbiting wave. The energy expression decreases as both position $r$ and momentum $mv$ approach zero. Assume that each is very small and thus comparable to its respective uncertainty: $r \approx \Delta r$ and $p \approx \Delta p$. The uncertainty principle then implies that $pr \approx \hbar$. Use this to eliminate $v$ in favor of $r$ in the energy expression.

(c) Sketch on the same axes the energy expressions from parts (a) and (b) versus $r$.

(d) Find the minimum possible energy for the orbiting electron wave and the corresponding value of $r$.

**SOLUTION**

(a) Coulomb's law gives us the electrostatic force, and the acceleration of a particle in circular motion is $v^2/r$.

$$F = ma \;\rightarrow\; \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2} = m\frac{v^2}{r} \;\Rightarrow\; v^2 = \frac{e^2}{4\pi\varepsilon_0 mr}$$

Thus,

$$E_{\substack{\text{classical}\\\text{particle}}} = \frac{1}{2}m\left(\frac{e^2}{4\pi\varepsilon_0 mr}\right) - \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r} = \underbrace{\frac{e^2}{8\pi\varepsilon_0 r}}_{\text{Kinetic}} - \underbrace{\frac{e^2}{4\pi\varepsilon_0 r}}_{\text{Potential}} = -\frac{e^2}{8\pi\varepsilon_0 r}$$

We see that the negative potential energy is of greater magnitude than the positive kinetic, and the total strictly decreases (becomes more negative) as $r$ decreases. There is no minimum energy.

(b) Assuming $pr = \hbar$, we have $p = \hbar/r$ or $v = \hbar/mr$. Thus,

$$E_{\substack{\text{matter}\\\text{wave}}} = \frac{1}{2}m\left(\frac{\hbar}{mr}\right)^2 - \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r} = \frac{\hbar^2}{2mr^2} - \frac{e^2}{4\pi\varepsilon_0 r}$$

Now, as $r$ decreases and the wave becomes more compact, likely values of the speed increase inversely, and so must the kinetic energy.

(c) The two plots are shown in Figure 16. While the energy of a classical particle decreases monotonically as $r$ decreases, the energy of the matter wave reaches a minimum and then increases.

(d) To find the minimum, we set the derivative to 0.

$$\frac{dE_{\substack{\text{matter}\\\text{wave}}}}{dr} = -\frac{\hbar^2}{mr^3} + \frac{e^2}{4\pi\varepsilon_0 r^2} = 0$$

$$\Rightarrow \quad r = \frac{4\pi\varepsilon_0\hbar^2}{me^2} = \frac{4\pi(8.85\times10^{-12}\,\text{N}\cdot\text{m}^2/\text{C}^2)(1.055\times10^{-34}\,\text{J}\cdot\text{s})^2}{(9.11\times10^{-31}\,\text{kg})(1.6\times10^{-19}\,\text{C})^2}$$

$$= 5.3\times10^{-11}\,\text{m}$$

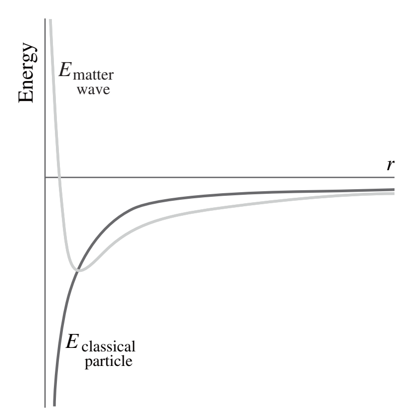
Reinserting,

$$E_{\substack{\text{matter}\\\text{wave}}} = \frac{\hbar^2}{2m}\left(\frac{me^2}{4\pi\varepsilon_0\hbar^2}\right)^2 - \frac{e^2}{4\pi\varepsilon_0}\left(\frac{me^2}{4\pi\varepsilon_0\hbar^2}\right) = -\frac{me^4}{32\pi^2\,\varepsilon_0^2\hbar^2}$$

$$= -\frac{(9.11\times10^{-31}\,\text{kg})(1.6\times10^{-19}\,\text{C})^4}{32\pi^2(8.85\times10^{-12}\,\text{N}\cdot\text{m}^2/\text{C}^2)^2(1.055\times10^{-34}\,\text{J}\cdot\text{s})^2}$$

$$= -2.2\times10^{-18}\,\text{J} = -13.6\,\text{eV}$$

**Figure 16** As the radius of an orbiting matter wave approaches zero, its momentum uncertainty and, thus, kinetic energy approach infinity.



This value happens to equal the experimentally verified minimum energy, and the radius is also the correct most probable radius at which to find the electron (which *doesn't* rest on the proton, whose radius is 10,000 times smaller). That they agree *so closely* is an accident—we have made many approximations—but it is no accident that they are of the correct order of magnitude. The uncertainty principle is a powerful tool.

## The Uncertainty Principle in Three Dimensions

The qualitative idea behind the uncertainty principle is the same in multiple dimensions as in one. The more compact the wave along a given axis, the less well we can specify the wavelength and therefore the momentum component *along that axis*. The result is a logical generalization of the one-dimensional result:

$$\Delta p_x \Delta x \geq \frac{\hbar}{2} \qquad \Delta p_y \Delta y \geq \frac{\hbar}{2} \qquad \Delta p_z \Delta z \geq \frac{\hbar}{2}$$

Note that the dimensions are independent. The single-slit pattern of Figure 12 bears this out. Passing through the slit, narrow along $x$ only, produces a large uncertainty in $p_x$, indicated by the subsequent detections being spread over a large region of the screen. In the $y$-direction, the aperture is wide, so less is known about this component of position, and there is correspondingly little spreading of the pattern in that dimension. Thus, $\Delta p_y$ and $\Delta x$ can be small simultaneously.

## The Energy-Time Uncertainty Principle

The momentum-position uncertainty relation is, at heart, a mathematical relationship. A width in space is inversely proportional to a "width" in the spatial frequency $k = 2\pi/\lambda$ (see Section 7). It is the fundamental wave-particle physics, $p = \hbar k$, that takes it the final step. The same math relates a width in time to a width in the *temporal* frequency $\omega = 2\pi/T$. With $E = \hbar\omega$, the corresponding physical consequence is

Energy-time uncertainty principle

$$\Delta E \, \Delta t \geq \frac{\hbar}{2} \tag{15}$$

How do we interpret this? If a state, or even a particle, exists for only a limited span of time, its energy is uncertain. One example is the fleeting life of certain exotic subatomic particles. Their lifetimes can be quite short—less than $10^{-20}$ seconds—and this leads to considerable uncertainty in their mass/energy. Another example is the state temporarily occupied by an electron as it jumps down through energy levels in an atom. Because the state is occupied for a finite time interval $\Delta t$, its energy is uncertain by an amount $\Delta E$ inversely related to $\Delta t$, which in turn gives rise to an uncertainty in the energy of the photon produced when the electron drops down. This effect contributes to the broadening of atomic spectral lines (see Exercise 72).

## 5 The Not-Unseen Observer

Let us spend a little time summarizing the limitations that quantum mechanics places on our knowledge. If the forces are known, the Schrödinger equation may, in principle, be solved for the wave function of a massive object, which contains all information that can be known. But this isn't everything we might expect classically. The uncertainty principle, for instance, says that a wave function of simultaneously precise momentum and position is a theoretical impossibility. It follows that any experiment or measurement that precisely determines position must result in a state in which nothing is known about the momentum and vice versa.

Suppose we carry out an experiment on a particle, experiment A, applying external forces in such a way as to determine both its position and its momentum as precisely as possible, such that $\Delta x\, \Delta p = \hbar/2$. Assume, for the sake of discussion, that $\Delta x$ is 100 μm, and call the wave function $\Psi_A$. We have found the wave function, but we aren't satisfied, for we haven't really "found" the particle—its "location." All we have is this mysterious probability amplitude.

We conduct another experiment, experiment B, in which the particle registers its presence at a detector at a definite location. We rejoice—we have found the particle. However, there are no "point detectors." If the detector's width is smaller than the 100 μm position uncertainty in $\Psi_A$, then we have indeed narrowed down the possible locations, but we haven't established a location with complete certainty. Yes, we have reduced the uncertainty in position, but if this is so, experiment B has changed the wave function. At the very least, it has increased $\Delta p$.

If we repeated this pair of experiments many times—experiment A to establish the initial wave function $\Psi_A$ and experiment B to "find" the particle—experiment B would find it at various locations within the 100 μm uncertainty of wave function $\Psi_A$, and the number detected at a given location would be proportional to $|\Psi_A|^2$. In essence, we would simply verify that $|\Psi_A|^2$ is proportional to the probability of finding the particle after experiment A. But because experiment B changes the wave function, we can't "watch"—repeatedly find—the *same* particle while preserving a single wave function $\Psi_A$.

The double-slit experiment, depicted in Figure 17, is a good example of these ideas. In effect, the slits are an experiment A, establishing an initial wave function $\Psi_A$ beyond them, and experiment B is the detection of a particle at

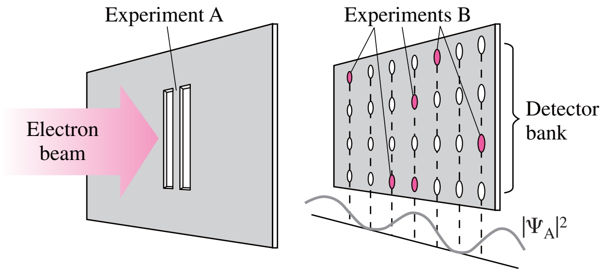



**Figure 17** Experiment A establishes $\Psi_A$, which repeated experiments B verify.

the screen. By sending in a beam of particles one at a time, we are carrying out experiment A then experiment B repeatedly. Where $\Psi_A$ is large, experiment B registers particles in abundance; where $\Psi_A$ is zero, experiment B registers no particles. We cannot conduct an intermediate experiment, determining which slit a given particle passes through, and yet hope to observe the interference pattern exhibited by $\Psi_A$, for this intermediate experiment would itself alter the wave function. (A recent confirmation is discussed in Progress and Applications.) To observe interference at the screen, we must allow each particle's wave function to pass through both slits simultaneously—otherwise there would not be two coherent waves to interfere.

The discussion raises an interesting point: If we cannot know the location of a particle *until* we actually look for it, it is hard to justify the claim that it even *has* a location before we look for it. Early in the quantum age, many eminent physicists, most notably Albert Einstein himself, asserted that theories of wave-particle duality must be incomplete, that some modification is needed to allow "real" quantities, such as position, to have definite values at all times. However, the modern consensus, known as the **Copenhagen interpretation**, is that until an experiment actually localizes it, a particle simply does not have a location.

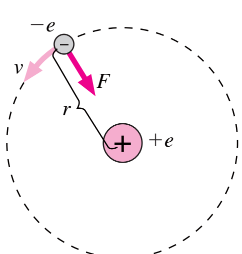
In summary, rather than the classical ideas of position and velocity, quantum mechanics allows us to know only probabilities and corresponding uncertainties based on the most recent observation of the "particle," and a determination of one property is liable to alter another. (*Note:* There is a way to precisely determine an uncertain property without in any way upsetting the particle. It involves "entangled" particles.

## OPTIONAL 6 The Bohr Model of the Atom

It is instructive to take a look at an early attempt to solve the mysteries of the atom. This work, for which Niels Bohr won the 1922 Nobel Prize, is known as the **Bohr model of the atom**, or simply the Bohr atom.

When the fundamental workings of something are so obscure as to defy formulation of a comprehensive theory, we construct a model. We observe that a system behaves in a certain way, and the model is a simplified theory that tries to explain the behavior. If it agrees with further experimental observation, we cautiously take it as evidence that its basic assumptions are valid, and we move forward. If, on the other hand, the model's predictions are at odds with further experiment, the model must be changed. Still, we learn something worthwhile: At least one of the model's assumptions is invalid.

**Figure 18** A classical picture of an electron orbiting a proton.




The Bohr model predicted that the electron orbiting the proton in a hydrogen atom may take on only certain, discrete energies, and the predicted values agreed with the experimental evidence. Combining simple quantum principles with classical physics, it is based on (1) the classical second law of motion, applied to an electron assumed to be held in circular orbit by its electrostatic attraction to a proton, as shown in Figure 18; (2) a classical expression for the energy of the orbiting electron; and (3) a postulate involving the quantization of the electron's angular momentum.

The classical second law of motion sets the Coulomb force between electron and proton equal to the mass times the centripetal acceleration for circular motion:

$$\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2} = m\frac{v^2}{r} \quad \text{or} \quad v^2 = \frac{e^2}{4\pi\varepsilon_0 m}\frac{1}{r} \tag{16}$$

The work of combining this with the classical expression for energy was done in Example 6. The result is

$$E_{\substack{\text{classical}\\ \text{particle}}} = -\frac{e^2}{8\pi\varepsilon_0 r} \tag{17}$$

It is negative because the negative potential energy exceeds the positive kinetic. In this classical expression, energy varies continuously, for $r$ may take on any of a continuum of values.

Now we add Bohr's main postulate: The electron's angular momentum $L$ may take on only the values

$$L = n\hbar \qquad \text{where } n = 1, 2, 3, \ldots$$

Because $L = mvr$ in a circular orbit, this condition may also be written

$$mvr = n\hbar \qquad n = 1, 2, 3, \ldots \tag{18}$$

Figure 19 illustrates a plausible basis for Bohr's postulate. If we assume that the orbiting electron behaves as a wave wrapped around a circle and that it must meet itself smoothly, so that the circumference is an integral number of wavelengths, then $\lambda = h/p$ implies that the product $mvr$ may take on only the values $n\hbar$.

Between equations (16) and (18), we may eliminate $v$ and obtain a condition restricting $r$ only to certain values.

$$r = \frac{(4\pi\varepsilon_0)\hbar^2}{me^2}n^2 \qquad n = 1, 2, 3, \ldots \tag{19}$$

or

$$r = a_0 n^2 \qquad \text{where } a_0 \equiv \frac{(4\pi\varepsilon_0)\hbar^2}{me^2} = 0.0529\ \text{nm}$$

Bohr model radii for hydrogen

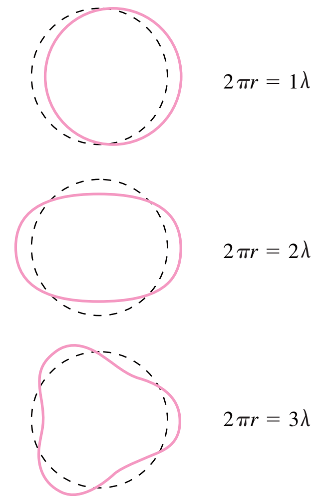
According to Bohr's theory, the electron orbits at certain radii that are multiples of the **Bohr radius** $a_0$. Energy, in turn, is also quantized. Inserting equation (19) into (17),

$$E = -\frac{me^4}{2(4\pi\varepsilon_0)^2\hbar^2}\frac{1}{n^2} = -13.6\ \text{eV}\frac{1}{n^2} \qquad n = 1, 2, 3, \ldots \tag{20}$$

Bohr model energies for hydrogen

The allowed values of the electron's energy depend on the integer $n$, known as the **principal quantum number**. As noted, these agree with the experimental evidence.

**Figure 19** Fitting whole waves around a circumference.




$$2\pi r = n\lambda \ \text{but}\ \lambda = \frac{h}{p} = \frac{h}{mv}$$

$$\text{Thus}\ 2\pi r = n\frac{h}{mv}$$

$$mvr = n\frac{h}{2\pi}$$

$$L = n\hbar$$

The Bohr model of the atom is an excellent example of working with the knowledge at hand—the Schrödinger equation hadn't even been developed yet! However, it *is* flawed. Although orbiting electrons are *most likely* to be found at certain distances from the proton, they must really be treated as diffuse waves spread over a broad range of radii. This casts doubt on (19), which, in turn, calls into question the model's predicted energies. Furthermore, in reality, orbiting electron waves have not only *rotational* kinetic energy, due to motion about the origin, but also *radial* kinetic energy due to motion toward and away from the origin.

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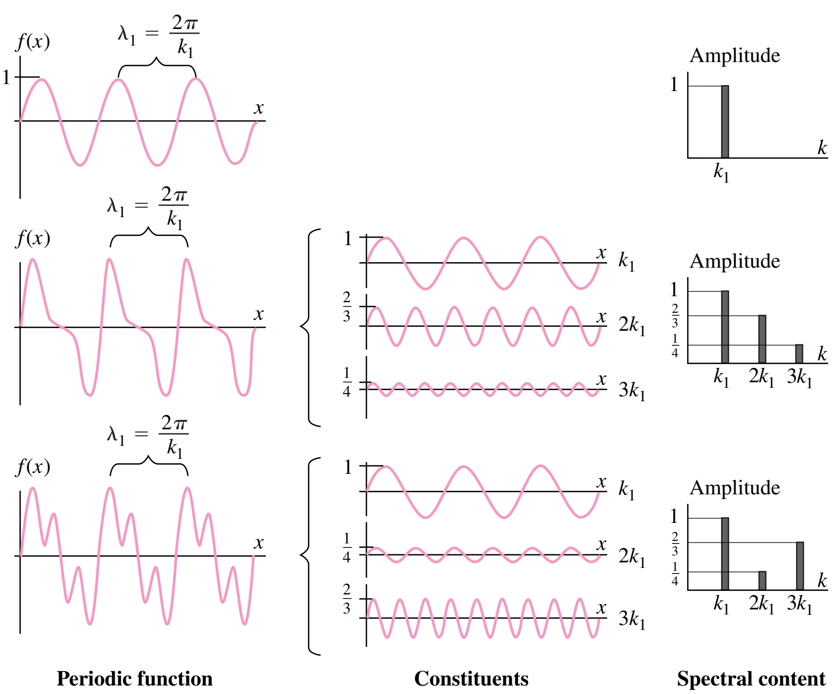
## 7 Mathematical Basis of the Uncertainty Principle—The Fourier Transform

The uncertainty principle rests on a mathematical relationship completely independent of any physical application: The more spatially compact a wave is, the less well its wavelength may be specified.

To begin, consider the left-hand plots in Figure 20, which show periodic functions of position. All repeat within the same interval along the $x$-axis, indicated by $\lambda_1$. The top waveform is a pure sinusoidal wave. The other

**Figure 20** Three functions of the same fundamental wavelength but different spectral (harmonic) contents.



two are not pure, but they are algebraic sums of pure waveforms of different wavelengths, as shown in the figure's center plots. These sums have the same fundamental (longest) wavelength as the top waveform, plus differing amounts of wavelengths half as long and one-third as long. Waveforms rich in such harmonics are common in musical instruments. A wave on a guitar string, for example, consists of a large amplitude of the fundamental wavelength, coexisting with shorter-wavelength harmonics of various amplitudes. The important point here is that a complicated periodic waveform can often be treated as an algebraic sum of pure sine waves of different wavelengths. To know the amplitudes of the different wavelengths is to know the waveform's "spectral content," and these are the right-hand plots in Figure 20. Such plots may be familiar, for they are what the graphic equalizer on a stereo displays, with big spikes at one end when the bass is loud and at the other when the treble is loud. And what does this have to do with the uncertainty principle? Making a wave very compact makes it a sum of pure waves covering a huge range of wavelengths, and this means a huge range of *momenta*.

For what follows, it is more convenient to refer to wave number than to wavelength. Don't forget its definition, $k = 2\pi/\lambda$, and that it is a spatial frequency. For instance, if $\lambda$ were 1.0 m, $k$ would be $2\pi\ \mathrm{m}^{-1}$, a whole cycle (radians is understood) per meter; if $\lambda$ is 0.5 m, $k$ is $4\pi\ \mathrm{m}^{-1}$, two cycles per meter. As a simple but useful example of its convenience, the harmonics shown in Figure 20 have wavelengths obeying $\lambda_n = \lambda_1/n$. The corresponding wave numbers are $2\pi/(\lambda_1/n) = n\ 2\pi/\lambda_1$, integral multiples of the fundamental wave number. It is usually more convenient to speak of something that can be integral *multiples* of a basic value than a basic value *over* an integer.

Now, adding pure sine waves whose wave numbers are multiples of a fundamental wave number always yields a *periodic* function. But what of a nonperiodic waveform? A wave *pulse*, like that shown in Figure 21, is of great interest in quantum mechanics, for it approximates a well-localized particle. It isn't periodic, but can it be considered as a sum of pure sine waves? The answer is yes, but not if the sum is restricted to multiples of a fundamental wave number.

**Figure 21** A wave pulse.

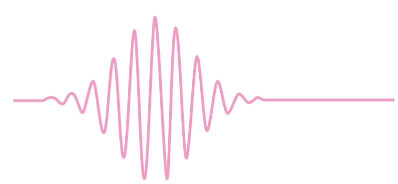
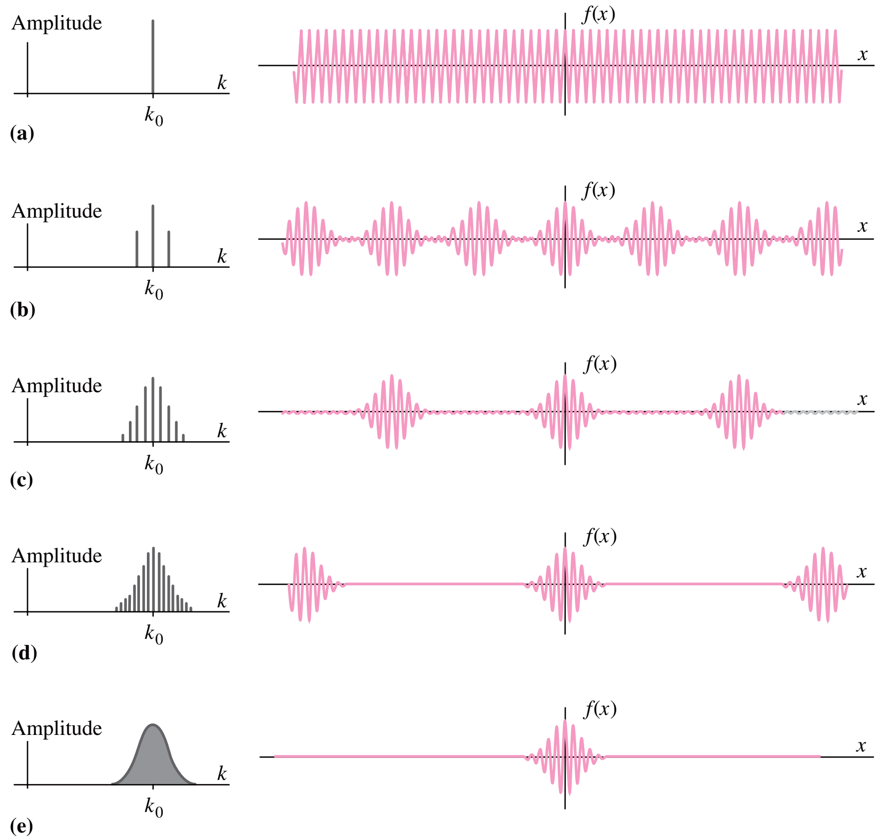


Figure 22 shows why. Waveform (a) is a pure sine wave whose wave number is the pulse's *apparent* wave number $k_0$. Obviously, it is a poor approximation of the pulse. Waveform (b), a sum of just three sine waves of different amplitudes and wave numbers, does considerably better. Waveforms (c) and (d) add wave numbers more densely spaced and covering a greater range above and below $k_0$, and here we begin to see a trend. The periodic "impostors" retreat from our desired waveform. We can eliminate them completely only by including an *infinite* number of waves in the sum. We will soon see how we knew what amplitudes and wave numbers to add together. But, as illustrated by waveform (e), the main point is this: A nonperiodic wave can be treated as a sum of sine waves of different amplitudes covering a *continuous* range of wave numbers. It isn't a sum, but an integral.[4]

In quantum mechanics, our basic function is the pure sinusoidal plane wave describing a free particle, given in equation (10): $\Psi(x, t) = Ae^{i(kx-\omega t)}$. We aren't interested here in how things behave in time, so we choose the convenient time of zero. Thus, our "building block" is $e^{ikx}$. Now we claim that any

[4]In some sense, an integral over $k$ is like a sum over all multiples of a fundamental wave number that is infinitesimal, $dk$. Its infinite wavelength allows the wave's overall period to be infinite. A pulse *never* repeats.

**Figure 22** Building a single isolated pulse from pure sine waves requires a continuum of wave numbers.




general, nonperiodic wave function $\psi(x)$ can be expressed as a sum/integral of these building blocks over the continuum of wave numbers:[5]

Function $\psi(x)$ as a sum of plane waves of amplitude $A(k)$

$$\psi(x) = \int_{-\infty}^{+\infty} A(k)\, e^{ikx}\, dk \qquad (21)$$

The amplitude $A(k)$ of the plane wave is naturally a function of $k$, for it tells us how much of each different wave number goes into the sum. As we will soon see, it is the key to the whole idea. Although we can't, of course, pull it out of the integral, the equation can be "solved" for $A(k)$. The result is

Fourier transform $A(k)$ of function $\psi(x)$

$$A(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \psi(x)\, e^{-ikx}\, dx \qquad (22)$$

[5]It is conventional to use $\Psi(x, t)$ when time is a factor, and $\psi(x)$ otherwise.

The proper name for $A(k)$ is the **Fourier transform** of the function $\psi(x)$. Fourier analysis is the technique of treating general functions as sums of basic ones, and the word *transform* suggests doing something with a function, such as throwing $\psi(x)$ in an integral with $e^{ikx}$ and obtaining something else, $A(k)$, related to that function—which is what we're doing. Before applying it, two points are worth reiterating. First, although the building block $e^{ikx}$ is not as easy to visualize as a real sinusoidal function, it is just two such functions linked in a special way, $\cos(kx) + i\sin(kx)$. Second, equation (21) defines $A(k)$ as the amplitude of each building block, which means that, although continuous, it is really the same quantity shown in the "amplitude" plots in Figures 20 and 22. Now let us use equation (22) to see what it really tells us.

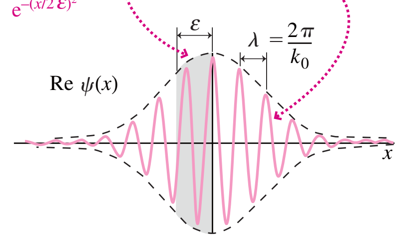
## Gaussian Wave Packet

A plausible wave function for a reasonably well-localized particle is the **Gaussian wave packet**:

$$\psi(x) = Ce^{-(x/2\varepsilon)^2} e^{ik_0x} \tag{23}$$

Let us inspect this one piece at a time. The $C$ is just a constant setting the function's "height" and related to the total probability, which doesn't concern us much here. The $e^{-(x/2\varepsilon)^2}$ is what qualifies the whole function as "Gaussian." As noted in Section 4, it is a maximum at $x = 0$ and falls off on either side. The fall-off is fast when $\varepsilon$ is large and slow when it is small. In other words, the width of the bell curve is proportional to $\varepsilon$. Multiplying the Gaussian factor is $e^{ik_0x}$, a plane wave of wave number $k_0$. A Gaussian is just a bump, but multiplying it by this sinusoidal function gives the product an oscillatory character. Figure 23 shows the real part of the product. From the $\cos(kx_0)$—the real part of $e^{ik_0x}$—it gets its wavelength $\lambda = 2\pi/k_0$, and from the Gaussian factor, its fall-off. Figure 24 represents the entire complex Gaussian wave packet in the same way that Figure 11 does a pure plane wave. The difference here is that our plane wave is modulated by the Gaussian factor. This certainly looks more particlelike than a plane wave alone, but what else does this imply?

**Figure 23** The real part of a Gaussian matter wave packet.

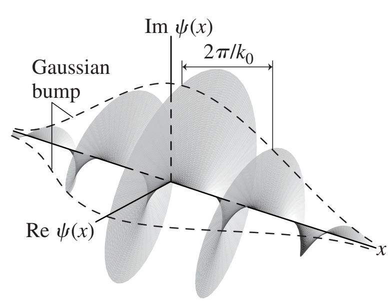



To answer this, we calculate $A(k)$. Using (22),

$$A(k) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} Ce^{-(x/2\varepsilon)^2} e^{ik_0x} e^{-ikx}\,dx$$

$$= \frac{C}{2\pi}\int_{-\infty}^{+\infty} e^{-(1/4\varepsilon^2)x^2 + i(k_0-k)x}\,dx$$

**Figure 24** A Gaussian wave packet: A plane wave $Ce^{ik_0x}$ of wavelength $2\pi/k_0$, modulated by a Gaussian bump, $e^{-(x/2\varepsilon)^2}$.




The integral is a standard form, known not coincidentally as a **Gaussian integral**,

$$\int_{-\infty}^{+\infty} e^{-az^2+bz}\,dz = \sqrt{\frac{\pi}{a}}\,e^{b^2/4a}$$

We have $a = 1/(4\varepsilon^2)$ and $b = i(k_0 - k)$, so

$$A(k) = \frac{C}{2\pi} e^{-\varepsilon^2(k-k_0)^2} \sqrt{4\varepsilon^2\pi} \tag{24}$$

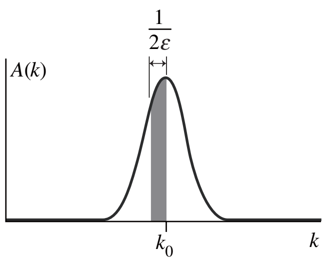



**Figure 25** The Fourier transform $A(k)$ of a Gaussian wave packet.

Now we reach the pivotal question: What does $A(k)$ tell us? It is plotted in Figure 25, confirming what we see in the formula itself—that $A(k)$ also happens to be a Gaussian function (which explains the shape chosen for the "amplitudes" in Figure 22). This one, however, is a function of $k$ and is centered at $k = k_0$. Equation (21) says that we can create our Gaussian $\psi(x)$ by adding pure $e^{ikx}$ plane waves, each multiplied by the coefficient $A(k)$ given in (24). We conclude that in this sum, we would need large amplitudes of plane waves whose wave number is near $k_0$ and smaller amplitudes for other wave numbers. Furthermore, as $\varepsilon$ gets small, $\psi(x)$ gets narrow, but $A(k)$ gets wide, for $\varepsilon$ is in the *numerator* of the Gaussian's argument in $A(k)$ and in the *denominator* in $\psi(x)$. *The width of $\psi(x)$ is inversely proportional to the width of $A(k)$*. (Why the width of $A(k)$ is shown as $1/(2\varepsilon)$ rather than just $1/\varepsilon$ we address a bit later.)

Figure 26 plots $A(k)$ and the real part of $\psi(x)$ for different $\varepsilon$ values. When $\varepsilon$ is very large, the Gaussian factor in $\psi(x)$ falls off slowly, and $\psi(x)$ is

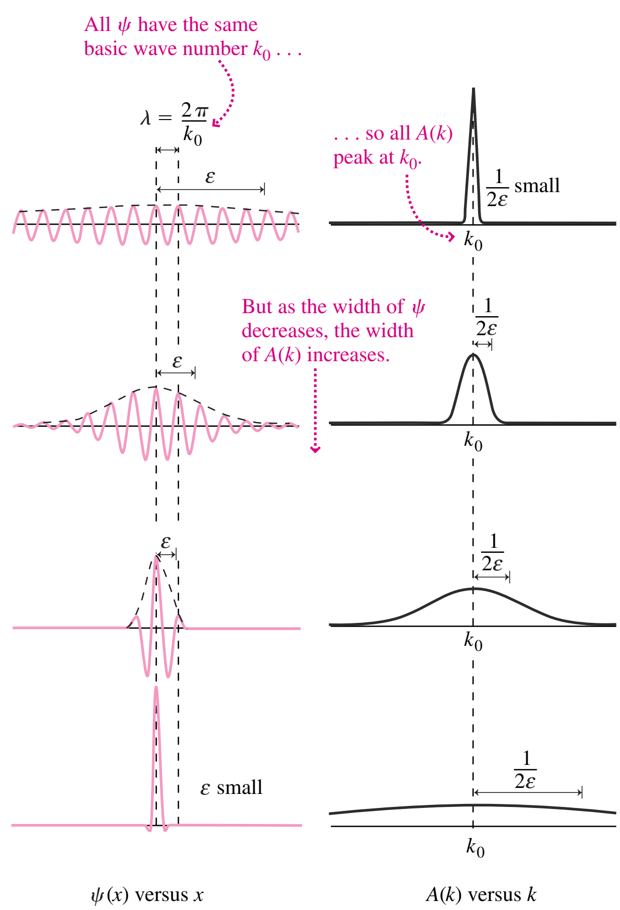



**Figure 26** The spatial width of $\psi(x)$ is inversely proportional to the wave number width of $A(k)$.