CAMBRIDGE

Advanced Level Mathematics

Pure Mathematics 1

Hugh Neill and Douglas Quadling

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Introduction

Cambridge International Examinations (CIE) Advanced Level Mathematics has been created especially for the new CIE mathematics syllabus. There is one book corresponding to each syllabus unit, except that units P2 and P3 are contained in a single book. This book covers the first Pure Mathematics unit, P1.

The syllabus content is arranged by chapters which are ordered so as to provide a viable teaching course. The early chapters develop the foundations of the syllabus; students may already be familiar with some of these topics. Later chapters, however, are largely independent of each other, and teachers may wish to vary the order in which they are used.

Some chapters, particularly Chapters 2, 3 and the first four sections of Chapter 8, contain material which is not in the examination syllabus for P1, and which therefore cannot be the direct focus of examination questions. Some of this is necessary background material, such as indices and surds; some is useful knowledge, such as graphs of powers of x, the use and meaning of modulus, and work on sequences.

A few sections include important results which are difficult to prove or outside the syllabus. These sections are marked with an asterisk (*) in the section heading, and there is usually a sentence early on explaining precisely what it is that the student needs to know.

Occasionally within the text paragraphs appear in *this type style*. These paragraphs are usually outside the main stream of the mathematical argument, but may help to give insight, or suggest extra work or different approaches.

Graphic calculators are not permitted in the examination, but they are useful aids in learning mathematics. In the book the authors have noted where access to a graphic calculator would be especially helpful but have not assumed that they are available to all students.

Numerical work is presented in a form intended to discourage premature approximation. In ongoing calculations inexact numbers appear in decimal form like 3.456..., signifying that the number is held in a calculator to more places than are given. Numbers are not rounded at this stage; the full display could be, for example, 3.456 123 or 3.456 789. Final answers are then stated with some indication that they are approximate, for example '1.23 correct to 3 significant figures'.

There are plenty of exercises, and each chapter ends with a Miscellaneous exercise which includes some questions of examination standard. Three Revision exercises consoliate work in preceeding chpaters. The book concludes with two Practice examination papers.

In some exercises a few of the later questions may go beyond the likely requirements of the P1 examination, either in difficulty or in length or both. Some questions are marked with an asterisk, which indicates that they require knowledge of results outside the syllabus.

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1 Coordinates, points and lines

This chapter uses coordinates to describe points and lines in two dimensions. When you have completed it, you should be able to

- find the distance between two points
- find the mid-point of a line segment, given the coordinates of its end points
- find the gradient of a line segment, given the coordinates of its end points
- find the equation of a line though a given point with a given gradient
- find the equation of the line joining two points
- recognise lines from different forms of their equations
- find the point of intersection of two lines
- tell from their gradients whether two lines are parallel or perpendicular.

1.1 The distance between two points

When you choose an origin, draw an x-axis to the right on the page and a y-axis up the page and choose scales along the axes, you are setting up a coordinate system. The coordinates of this system are called **cartesian coordinates** after the French mathematician René Descartes, who lived in the 17th century.

In Fig. 1.1, two points A and B have cartesian coordinates $(4,3)$ and $(10,7)$. The part of the line AB which lies between A and B is called a **line segment**. The length of the line segment is the distance between the points.

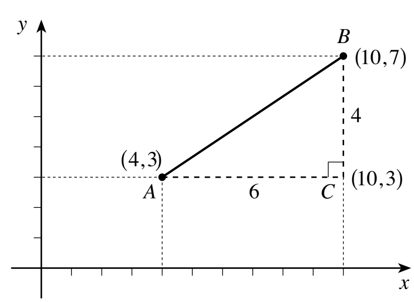


Fig. 1.1

A third point C has been added to Fig. 1.1 to form a right-angled triangle. You can see that C has the same x-coordinate as B and the same y-coordinate as A; that is, C has coordinates $(10,3)$.

It is easy to see that AC has length $10-4=6$, and CB has length $7-3=4$. Using Pythagoras' theorem in triangle ABC shows that the length of the line segment AB is

$$\sqrt{(10-4)^2+(7-3)^2}=\sqrt{6^2+4^2}=\sqrt{36+16}=\sqrt{52}.$$

You can use your calculator to give this as $7.21\ldots$, if you need to, but often it is better to leave the answer as $\sqrt{52}$.

The idea of coordinate geometry is to use algebra so that you can do calculations like this when A and B are any points, and not just the particular points in Fig. 1.1. It often helps to use a notation which shows at a glance which point a coordinate refers to. One way of doing this is with suffixes, calling the coordinates of the first point (x_1, y_1), and the coordinates of the second point (x_2, y_2). Thus, for example, x_1 stands for 'the x-coordinate of the first point'.

Fig. 1.2 shows this general triangle. You can see that C now has coordinates (x_2, y_1), and that $AC = x_2 - x_1$ and $CB = y_2 - y_1$. Pythagoras' theorem now gives

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

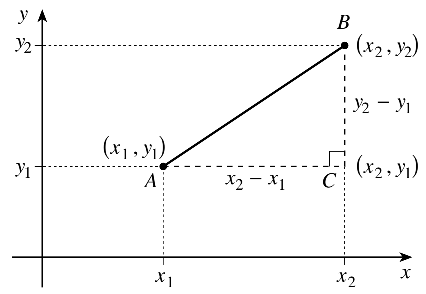


Fig. 1.2

An advantage of using algebra is that this formula works whatever the shape and position of the triangle. In Fig. 1.3, the coordinates of A are negative, and in Fig. 1.4 the line slopes downhill rather than uphill as you move from left to right. Use Figs. 1.3 and 1.4 to work out for yourself the length of AB in each case. You can then use the formula to check your answers.

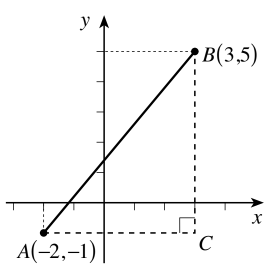


Fig. 1.3

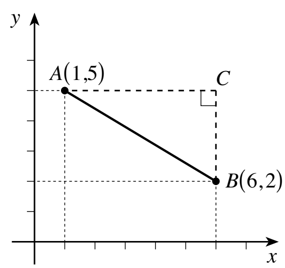


Fig. 1.4

In Fig. 1.3,

$$x_2 - x_1 = 3 - (-2) = 3 + 2 = 5 \quad \text{and} \quad y_2 - y_1 = 5 - (-1) = 5 + 1 = 6,$$

so $$AB = \sqrt{(3-(-2))^2 + (5-(-1))^2} = \sqrt{5^2 + 6^2} = \sqrt{25 + 36} = \sqrt{61}.$$

And in Fig. 1.4,

$$x_2 - x_1 = 6 - 1 = 5 \quad \text{and} \quad y_2 - y_1 = 2 - 5 = -3,$$

so $$AB = \sqrt{(6-1)^2 + (2-5)^2} = \sqrt{5^2 + (-3)^2} = \sqrt{25 + 9} = \sqrt{34}.$$

Also, it doesn't matter which way round you label the points A and B. If you think of B as 'the first point' (x_1, y_1) and A as 'the second point' (x_2, y_2), the formula doesn't change. For Fig. 1.1, it would give

$$BA = \sqrt{(4-10)^2 + (3-7)^2} = \sqrt{(-6)^2 + (-4)^2} = \sqrt{36 + 16} = \sqrt{52}, \text{ as before.}$$

The distance between the points (x_1, y_1) and (x_2, y_2) (or the length of the line segment joining them) is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\ .$$

1.2 The mid-point of a line segment

You can also use coordinates to find the mid-point of a line segment.

Fig. 1.5 shows the same line segment as in Fig. 1.1, but with the mid-point M added. The line through M parallel to the y-axis meets AC at D. Then the lengths of the sides of the triangle ADM are half of those of triangle ACB, so that

$$AD = \tfrac{1}{2}AC = \tfrac{1}{2}(10-4) = \tfrac{1}{2}(6) = 3,$$
$$DM = \tfrac{1}{2}CB = \tfrac{1}{2}(7-3) = \tfrac{1}{2}(4) = 2.$$

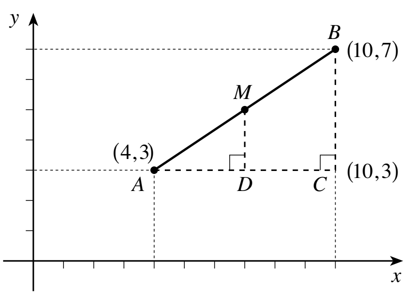


Fig. 1.5

The x-coordinate of M is the same as the x-coordinate of D, which is

$$4 + AD = 4 + \tfrac{1}{2}(10-4) = 4 + 3 = 7.$$

The y-coordinate of M is

$$3 + DM = 3 + \tfrac{1}{2}(7-3) = 3 + 2 = 5.$$

So the mid-point M has coordinates $(7,5)$.

In Fig. 1.6 points M and D have been added in the same way to Fig. 1.2. Exactly as before,

$$AD = \tfrac{1}{2}AC = \tfrac{1}{2}(x_2 - x_1), \qquad DM = \tfrac{1}{2}CB = \tfrac{1}{2}(y_2 - y_1).$$

So the x-coordinate of M is

$$x_1 + AD = x_1 + \tfrac{1}{2}(x_2 - x_1) = x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_1$$
$$= \tfrac{1}{2}x_1 + \tfrac{1}{2}x_2 = \tfrac{1}{2}(x_1 + x_2).$$

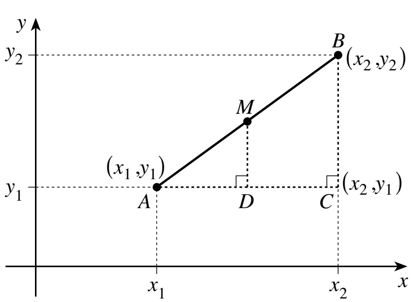


Fig. 1.6

The y-coordinate of M is

$$y_1 + DM = y_1 + \tfrac{1}{2}(y_2 - y_1) = y_1 + \tfrac{1}{2}y_2 - \tfrac{1}{2}y_1$$
$$= \tfrac{1}{2}y_1 + \tfrac{1}{2}y_2 = \tfrac{1}{2}(y_1 + y_2).$$

The mid-point of the line segment joining (x_1, y_1) and (x_2, y_2) has coordinates

$$\left(\tfrac{1}{2}(x_1 + x_2), \tfrac{1}{2}(y_1 + y_2)\right).$$

Now that you have an algebraic form for the coordinates of the mid-point M you can use it for any two points. For example, for Fig. 1.3 the mid-point of AB is

$$\left(\tfrac{1}{2}((-2)+3),\tfrac{1}{2}((-1)+5)\right)=\left(\tfrac{1}{2}(1),\tfrac{1}{2}(4)\right)=\left(\tfrac{1}{2},2\right).$$

And for Fig. 1.4 it is $\left(\tfrac{1}{2}(1+6),\tfrac{1}{2}(5+2)\right)=\left(\tfrac{1}{2}(7),\tfrac{1}{2}(7)\right)=\left(3\tfrac{1}{2},3\tfrac{1}{2}\right)$.

Again, it doesn't matter which you call the first point and which the second. In Fig. 1.5, if you take (x_1,y_1) as $(10,7)$ and (x_2,y_2) as $(4,3)$, you find that the mid-point is $\left(\tfrac{1}{2}(10+4),\tfrac{1}{2}(7+3)\right)=(7,5)$, as before.

1.3 The gradient of a line segment

The gradient of a line is a measure of its steepness. The steeper the line, the larger the gradient.

Unlike the distance and the mid-point, the gradient is a property of the whole line, not just of a particular line segment. If you take any two points on the line and find the increases in the x- and y-coordinates as you go from one to the other, as in Fig. 1.7, then the value of the fraction

$$\frac{y\text{-step}}{x\text{-step}}$$

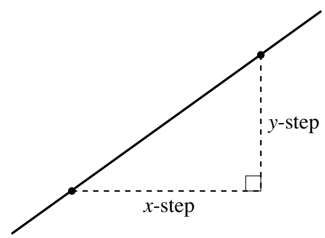


Fig. 1.7

is the same whichever points you choose. This is the **gradient** of the line.

In Fig. 1.2 on page 2 the x-step and y-step are x_2-x_1 and y_2-y_1, so that:

> The gradient of the line joining (x_1,y_1) to (x_2,y_2) is $\dfrac{y_2-y_1}{x_2-x_1}$.

This formula applies whether the coordinates are positive or negative. In Fig. 1.3, for example, the gradient of AB is $\dfrac{5-(-1)}{3-(-2)}=\dfrac{5+1}{3+2}=\tfrac{6}{5}$.

But notice that in Fig. 1.4 the gradient is $\dfrac{2-5}{6-1}=\dfrac{-3}{5}=-\tfrac{3}{5}$; the negative gradient tells you that the line slopes downhill as you move from left to right.

As with the other formulae, it doesn't matter which point has the suffix 1 and which has the suffix 2. In Fig. 1.1, you can calculate the gradient as either $\dfrac{7-3}{10-4}=\tfrac{4}{6}=\tfrac{2}{3}$, or $\dfrac{3-7}{4-10}=\dfrac{-4}{-6}=\tfrac{2}{3}$.

Two lines are **parallel** if they have the same gradient.

Example 1.3.1
The ends of a line segment are $(p-q, p+q)$ and $(p+q, p-q)$. Find the length of the line segment, its gradient and the coordinates of its mid-point.

For the length and gradient you have to calculate

$$x_2 - x_1 = (p+q)-(p-q) = p+q-p+q = 2q$$

and $y_2 - y_1 = (p-q)-(p+q) = p-q-p-q = -2q.$

The length is $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} = \sqrt{(2q)^2+(-2q)^2} = \sqrt{4q^2+4q^2} = \sqrt{8q^2}$.

The gradient is $\dfrac{y_2-y_1}{x_2-x_1} = \dfrac{-2q}{2q} = -1.$

For the mid-point you have to calculate

$$x_1 + x_2 = (p-q)+(p+q) = p-q+p+q = 2p$$

and $y_1 + y_2 = (p+q)+(p-q) = p+q+p-q = 2p.$

The mid-point is $\left(\frac{1}{2}(x_1+x_2), \frac{1}{2}(y_1+y_2)\right) = \left(\frac{1}{2}(2p), \frac{1}{2}(2p)\right) = (p, p).$

Try drawing your own figure to illustrate the results in this example.

Example 1.3.2
Prove that the points $A(1,1)$, $B(5,3)$, $C(3,0)$ and $D(-1,-2)$ form a parallelogram.

You can approach this problem in a number of ways, but whichever method you use, it is worth drawing a sketch. This is shown in Fig. 1.8.

Method 1 (using distances) In this method, find the lengths of the opposite sides. If both pairs of opposite sides are equal, then $ABCD$ is a parallelogram.

$$AB = \sqrt{(5-1)^2+(3-1)^2} = \sqrt{20}.$$
$$DC = \sqrt{(3-(-1))^2+(0-(-2))^2} = \sqrt{20}.$$
$$CB = \sqrt{(5-3)^2+(3-0)^2} = \sqrt{13}.$$
$$DA = \sqrt{(1-(-1))^2+(1-(-2))^2} = \sqrt{13}.$$

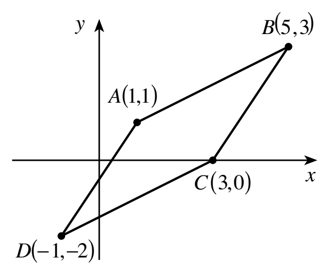


Fig. 1.8

Therefore $AB = DC$ and $CB = DA$, so $ABCD$ is a parallelogram.

Method 2 (using mid-points) In this method, begin by finding the mid-points of the diagonals AC and BD. If these points are the same, then the diagonals bisect each other, so the quadrilateral is a parallelogram.

The mid-point of AC is $\left(\frac{1}{2}(1+3), \frac{1}{2}(1+0)\right)$, which is $\left(2, \frac{1}{2}\right)$. The mid-point of BD is $\left(\frac{1}{2}(5+(-1)), \frac{1}{2}(3+(-2))\right)$, which is also $\left(2, \frac{1}{2}\right)$. So $ABCD$ is a parallelogram.

Method 3 (using gradients) In this method, find the gradients of the opposite sides. If both pairs of opposite sides are parallel, then *ABCD* is a parallelogram.

The gradients of *AB* and *DC* are $\frac{3-1}{5-1}=\frac{2}{4}=\frac{1}{2}$ and $\frac{0-(-2)}{3-(-1)}=\frac{2}{4}=\frac{1}{2}$ respectively, so *AB* is parallel to *DC*. The gradients of *DA* and *CB* are both $\frac{3}{2}$, so *DA* is parallel to *CB*. As the opposite sides are parallel, *ABCD* is a parallelogram.

Exercise 1A

Do not use a calculator. Where appropriate, leave square roots in your answers.

1 Find the lengths of the line segments joining these pairs of points. In parts (e) and (h) assume that $a>0$; in parts (i) and (j) assume that $p>q>0$.

(a) $(2,5)$ and $(7,17)$ (b) $(-3,2)$ and $(1,-1)$

(c) $(4,-5)$ and $(-1,0)$ (d) $(-3,-3)$ and $(-7,3)$

(e) $(2a,a)$ and $(10a,-14a)$ (f) $(a+1,2a+3)$ and $(a-1,2a-1)$

(g) $(2,9)$ and $(2,-14)$ (h) $(12a,5b)$ and $(3a,5b)$

(i) (p,q) and (q,p) (j) $(p+4q,p-q)$ and $(p-3q,p)$

2 Show that the points $(1,-2)$, $(6,-1)$, $(9,3)$ and $(4,2)$ are vertices of a parallelogram.

3 Show that the triangle formed by the points $(-3,-2)$, $(2,-7)$ and $(-2,5)$ is isosceles.

4 Show that the points $(7,12)$, $(-3,-12)$ and $(14,-5)$ lie on a circle with centre $(2,0)$.

5 Find the coordinates of the mid-points of the line segments joining these pairs of points.

(a) $(2,11),(6,15)$ (b) $(5,7),(-3,9)$

(c) $(-2,-3),(1,-6)$ (d) $(-3,4),(-8,5)$

(e) $(p+2,3p-1),(3p+4,p-5)$ (f) $(p+3,q-7),(p+5,3-q)$

(g) $(p+2q,2p+13q),(5p-2q,-2p-7q)$ (h) $(a+3,b-5),(a+3,b+7)$

6 $A(-2,1)$ and $B(6,5)$ are the opposite ends of the diameter of a circle. Find the coordinates of its centre.

7 $M(5,7)$ is the mid-point of the line segment joining $A(3,4)$ to B. Find the coordinates of B.

8 $A(1,-2)$, $B(6,-1)$, $C(9,3)$ and $D(4,2)$ are the vertices of a parallelogram. Verify that the mid-points of the diagonals *AC* and *BD* coincide.

9 Which one of the points $A(5,2)$, $B(6,-3)$ and $C(4,7)$ is the mid-point of the other two? Check your answer by calculating two distances.

10 Find the gradients of the lines joining the following pairs of points.

(a) $(3,8),(5,12)$ (b) $(1,-3),(-2,6)$

(c) $(-4,-3),(0,-1)$ (d) $(-5,-3),(3,-9)$

(e) $(p+3,p-3),(2p+4,-p-5)$ (f) $(p+3,q-5),(q-5,p+3)$

(g) $(p+q-1,q+p-3),(p-q+1,q-p+3)$ (h) $(7,p),(11,p)$

11 Find the gradients of the lines *AB* and *BC* where *A* is $(3,4)$, *B* is $(7,6)$ and *C* is $(-3,1)$. What can you deduce about the points *A*, *B* and *C*?

12 The point $P(x,y)$ lies on the straight line joining $A(3,0)$ and $B(5,6)$. Find expressions for the gradients of *AP* and *PB*. Hence show that $y = 3x - 9$.

13 A line joining a vertex of a triangle to the mid-point of the opposite side is called a median. Find the length of the median *AM* in the triangle $A(-1,1)$, $B(0,3)$, $C(4,7)$.

14 A triangle has vertices $A(-2,1)$, $B(3,-4)$ and $C(5,7)$.

(a) Find the coordinates of *M*, the mid-point of *AB*, and *N*, the mid-point of *AC*.

(b) Show that *MN* is parallel to *BC*.

15 The points $A(2,1)$, $B(2,7)$ and $C(-4,-1)$ form a triangle. *M* is the mid-point of *AB* and *N* is the mid-point of *AC*.

(a) Find the lengths of *MN* and *BC*.

(b) Show that $BC = 2MN$.

16 The vertices of a quadrilateral *ABCD* are $A(1,1)$, $B(7,3)$, $C(9,-7)$ and $D(-3,-3)$. The points *P*, *Q*, *R* and *S* are the mid-points of *AB*, *BC*, *CD* and *DA* respectively.

(a) Find the gradient of each side of *PQRS*.

(b) What type of quadrilateral is *PQRS*?

17 The origin *O* and the points $P(4,1)$, $Q(5,5)$ and $R(1,4)$ form a quadrilateral.

(a) Show that *OR* is parallel to *PQ*.

(b) Show that *OP* is parallel to *RQ*.

(c) Show that $OP = OR$.

(d) What shape is *OPQR*?

18 The origin *O* and the points $L(-2,3)$, $M(4,7)$ and $N(6,4)$ form a quadrilateral.

(a) Show that $ON = LM$.

(b) Show that *ON* is parallel to *LM*.

(c) Show that $OM = LN$.

(d) What shape is *OLMN*?

19 The vertices of a quadrilateral *PQRS* are $P(1,2)$, $Q(7,0)$, $R(6,-4)$ and $S(-3,-1)$.

(a) Find the gradient of each side of the quadrilateral.

(b) What type of quadrilateral is *PQRS*?

20 The vertices of a quadrilateral are $T(3,2)$, $U(2,5)$, $V(8,7)$ and $W(6,1)$. The mid-points of *UV* and *VW* are *M* and *N* respectively. Show that the triangle *TMN* is isosceles.

21 The vertices of a quadrilateral *DEFG* are $D(3,-2)$, $E(0,-3)$, $F(-2,3)$ and $G(4,1)$.

(a) Find the length of each side of the quadrilateral.

(b) What type of quadrilateral is *DEFG*?

22 The points $A(2,1)$, $B(6,10)$ and $C(10,1)$ form an isosceles triangle with *AB* and *BC* of equal length. The point *G* is $(6,4)$.

(a) Write down the coordinates of *M*, the mid-point of *AC*.

(b) Show that $BG = 2GM$ and that *BGM* is a straight line.

(c) Write down the coordinates of *N*, the mid-point of *BC*.

(d) Show that *AGN* is a straight line and that $AG = 2GN$.

1.4 What is meant by the equation of a straight line or of a curve?

How can you tell whether or not the points $(3,7)$ and $(1,5)$ lie on the curve $y = 3x^2 + 2$? The answer is to substitute the coordinates of the points into the equation and see whether they fit; that is, whether the equation is **satisfied** by the coordinates of the point.

For $(3,7)$: the right side is $3 \times 3^2 + 2 = 29$ and the left side is 7, so the equation is not satisfied. The point $(3,7)$ does not lie on the curve $y = 3x^2 + 2$.

For $(1,5)$: the right side is $3 \times 1^2 + 2 = 5$ and the left side is 5, so the equation is satisfied. The point $(1,5)$ lies on the curve $y = 3x^2 + 2$.

> The equation of a line or curve is a rule for determining whether or not the point with coordinates (x, y) lies on the line or curve.

This is an important way of thinking about the equation of a line or curve.

1.5 The equation of a line

Example 1.5.1
Find the equation of the line with gradient 2 which passes through the point $(2,1)$.

Fig. 1.9 shows the line of gradient 2 through $A(2,1)$, with another point $P(x, y)$ lying on it. P lies on the line if (and only if) the gradient of AP is 2.

The gradient of AP is $\frac{y-1}{x-2}$. Equating this to 2 gives $\frac{y-1}{x-2} = 2$, which is $y - 1 = 2x - 4$, or $y = 2x - 3$.

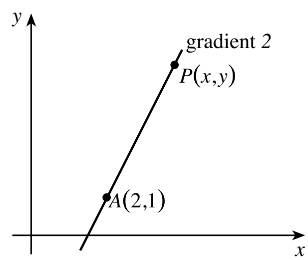


Fig. 1.9

In the general case, you need to find the equation of the line with gradient m through the point A with coordinates (x_1, y_1). Fig. 1.10 shows this line and another point P with coordinates (x, y) on it. The gradient of AP is $\frac{y-y_1}{x-x_1}$.

Equating to m gives $\frac{y-y_1}{x-x_1} = m$, or $y - y_1 = m(x - x_1)$.

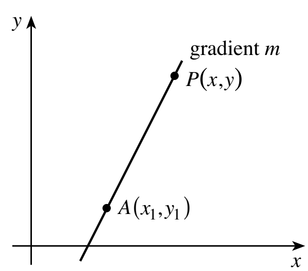


Fig. 1.10

> The equation of the line through (x_1, y_1) with gradient m is $y - y_1 = m(x - x_1)$.

Notice that the coordinates of $A(x_1, y_1)$ *satisfy this equation.*

Example 1.5.2
Find the equation of the line through the point $(-2,3)$ with gradient -1.

Using the equation $y - y_1 = m(x - x_1)$ gives the equation $y - 3 = -1(x - (-2))$, which is $y - 3 = -x - 2$ or $y = -x + 1$. As a check, substitute the coordinates $(-2,3)$ into both sides of the equation, to make sure that the given point does actually lie on the line.

Example 1.5.3
Find the equation of the line joining the points $(3,4)$ and $(-1,2)$.

To find this equation, first find the gradient of the line joining $(3,4)$ to $(-1,2)$. Then you can use the equation $y - y_1 = m(x - x_1)$.

The gradient of the line joining $(3,4)$ to $(-1,2)$ is $\frac{2-4}{(-1)-3} = \frac{-2}{-4} = \frac{1}{2}$.

The equation of the line through $(3,4)$ with gradient $\frac{1}{2}$ is $y - 4 = \frac{1}{2}(x - 3)$. After multiplying out and simplifying you get $2y - 8 = x - 3$, or $2y = x + 5$.

Check this equation mentally by substituting the coordinates of the other point.

1.6 Recognising the equation of a line

The answers to Examples 1.5.1–1.5.3 can all be written in the form $y = mx + c$, where m and c are numbers.

It is easy to show that any equation of this form is the equation of a straight line. If $y = mx + c$, then $y - c = m(x - 0)$, or

$$\frac{y-c}{x-0} = m \qquad \text{(except when } x = 0\text{)}.$$

This equation tells you that, for all points (x,y) whose coordinates satisfy the equation, the line joining $(0,c)$ to (x,y) has gradient m. That is, (x,y) lies on the line through $(0,c)$ with gradient m.

The point $(0,c)$ lies on the y-axis. The number c is called the ***y*-intercept** of the line.

To find the x-intercept, put $y = 0$ in the equation, which gives $x = -\frac{c}{m}$. But notice that you can't do this division if $m = 0$. In that case the line is parallel to the x-axis, so there is no x-intercept.

When $m = 0$, all the points on the line have coordinates of the form $(\text{something}, c)$. Thus the points $(1,2)$, $(-1,2)$, $(5,2)$, … all lie on the straight line $y = 2$, shown in Fig. 1.11. As a special case, the x-axis has equation $y = 0$.

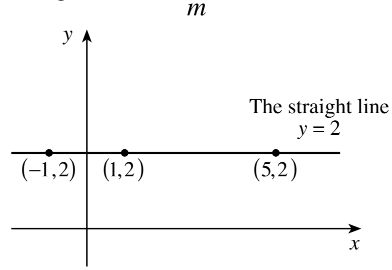


Fig. 1.11

Similarly, a straight line parallel to the y-axis has an equation of the form $x = k$. All points on it have coordinates $(k, \text{something})$. Thus the points $(3,0)$, $(3,2)$, $(3,4)$, … all lie on the line $x = 3$, shown in Fig. 1.12. The y-axis itself has equation $x = 0$.

The line $x = k$ does not have a gradient; its gradient is undefined. Its equation cannot be written in the form $y = mx + c$.

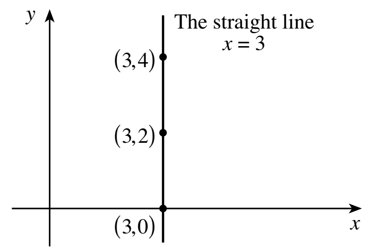


Fig. 1.12

1.7 The equation $ax + by + c = 0$

Suppose you have the equation $y = \frac{2}{3}x + \frac{4}{3}$. It is natural to multiply by 3 to get $3y = 2x + 4$, which can be rearranged to get $2x - 3y + 4 = 0$. This equation is in the form $ax + by + c = 0$ where a, b and c are constants.

Notice that the straight lines $y = mx + c$ and $ax + by + c = 0$ both contain the letter c, but it doesn't have the same meaning. For $y = mx + c$, c is the y-intercept, but there is no similar meaning for the c in $ax + by + c = 0$.

A simple way to find the gradient of $ax + by + c = 0$ is to rearrange it into the form $y = \ldots$. Here are some examples.

Example 1.7.1
Find the gradient of the line $2x + 3y - 4 = 0$.

Write this equation in the form $y = \ldots$, and then use the fact that the straight line $y = mx + c$ has gradient m.

From $2x + 3y - 4 = 0$ you find that $3y = -2x + 4$ and $y = -\frac{2}{3}x + \frac{4}{3}$. Therefore, comparing this equation with $y = mx + c$, the gradient is $-\frac{2}{3}$.

Example 1.7.2
One side of a parallelogram lies along the straight line with equation $3x - 4y - 7 = 0$. The point $(2,3)$ is a vertex of the parallelogram. Find the equation of one other side.

The line $3x - 4y - 7 = 0$ is the same as $y = \frac{3}{4}x - \frac{7}{4}$, so its gradient is $\frac{3}{4}$. The line through $(2,3)$ with gradient $\frac{3}{4}$ is $y - 3 = \frac{3}{4}(x - 2)$, or $3x - 4y + 6 = 0$.

1.8 The point of intersection of two lines

Suppose that you have two lines with equations $2x - y = 4$ and $3x + 2y = -1$. How do you find the coordinates of the point of intersection of these lines?

You want the point (x, y) which lies on both lines, so the coordinates (x, y) satisfy both equations. Therefore you need to solve the equations simultaneously.

From these two equations, you find $x = 1$, $y = -2$, so the point of intersection is $(1,-2)$.

This argument applies to straight lines with any equations provided they are not parallel. To find points of intersection, solve the equations simultaneously. The method can also be used to find the points of intersection of two curves.

Exercise 1B

1 Test whether the given point lies on the straight line (or curve) with the given equation.

(a) $(1,2)$ on $y = 5x - 3$
(b) $(3,-2)$ on $y = 3x - 7$
(c) $(3,-4)$ on $x^2 + y^2 = 25$
(d) $(2,2)$ on $3x^2 + y^2 = 40$
(e) $\left(1,1\frac{1}{2}\right)$ on $y = \dfrac{x+2}{3x-1}$
(f) $\left(5p,\dfrac{5}{p}\right)$ on $y = \dfrac{5}{x}$
(g) $\left(p,(p-1)^2+1\right)$ on $y = x^2 - 2x + 2$
(h) $\left(t^2,2t\right)$ on $y^2 = 4x$

2 Find the equations of the straight lines through the given points with the gradients shown. Your final answers should not contain any fractions.

(a) $(2,3)$, gradient 5
(b) $(1,-2)$, gradient -3
(c) $(0,4)$, gradient $\frac{1}{2}$
(d) $(-2,1)$, gradient $-\frac{3}{8}$
(e) $(0,0)$, gradient -3
(f) $(3,8)$, gradient 0
(g) $(-5,-1)$, gradient $-\frac{3}{4}$
(h) $(-3,0)$, gradient $\frac{1}{2}$
(i) $(-3,-1)$, gradient $\frac{3}{8}$
(j) $(3,4)$, gradient $-\frac{1}{2}$
(k) $(2,-1)$, gradient -2
(l) $(-2,-5)$, gradient 3
(m) $(0,-4)$, gradient 7
(n) $(0,2)$, gradient -1
(o) $(3,-2)$, gradient $-\frac{5}{8}$
(p) $(3,0)$, gradient $-\frac{3}{5}$
(q) $(d,0)$, gradient 7
(r) $(0,4)$, gradient m
(s) $(0,c)$, gradient 3
(t) $(c,0)$, gradient m

3 Find the equations of the lines joining the following pairs of points. Leave your final answer without fractions and in one of the forms $y = mx + c$ or $ax + by + c = 0$.

(a) $(1,4)$ and $(3,10)$
(b) $(4,5)$ and $(-2,-7)$
(c) $(3,2)$ and $(0,4)$
(d) $(3,7)$ and $(3,12)$
(e) $(10,-3)$ and $(-5,-12)$
(f) $(3,-1)$ and $(-4,20)$
(g) $(2,-3)$ and $(11,-3)$
(h) $(2,0)$ and $(5,-1)$
(i) $(-4,2)$ and $(-1,-3)$
(j) $(-2,-1)$ and $(5,-3)$
(k) $(-3,4)$ and $(-3,9)$
(l) $(-1,0)$ and $(0,-1)$
(m) $(2,7)$ and $(3,10)$
(n) $(-5,4)$ and $(-2,-1)$
(o) $(0,0)$ and $(5,-3)$
(p) $(0,0)$ and (p,q)
(q) (p,q) and $(p+3,q-1)$
(r) $(p,-q)$ and (p,q)
(s) (p,q) and $(p+2,q+2)$
(t) $(p,0)$ and $(0,q)$

4 Find the gradients of the following lines.

(a) $2x+y=7$ (b) $3x-4y=8$ (c) $5x+2y=-3$

(d) $y=5$ (e) $3x-2y=-4$ (f) $5x=7$

(g) $x+y=-3$ (h) $y=3(x+4)$ (i) $7-x=2y$

(j) $3(y-4)=7x$ (k) $y=m(x-d)$ (l) $px+qy=pq$

5 Find the equation of the line through $(-2,1)$ parallel to $y=\frac{1}{2}x-3$.

6 Find the equation of the line through $(4,-3)$ parallel to $y+2x=7$.

7 Find the equation of the line through $(1,2)$ parallel to the line joining $(3,-1)$ and $(-5,2)$.

8 Find the equation of the line through $(3,9)$ parallel to the line joining $(-3,2)$ and $(2,-3)$.

9 Find the equation of the line through $(1,7)$ parallel to the x-axis.

10 Find the equation of the line through $(d,0)$ parallel to $y=mx+c$.

11 Find the points of intersection of the following pairs of straight lines.

(a) $3x+4y=33,\ 2y=x-1$ (b) $y=3x+1,\ y=4x-1$

(c) $2y=7x,\ 3x-2y=1$ (d) $y=3x+8,\ y=-2x-7$

(e) $x+5y=22,\ 3x+2y=14$ (f) $2x+7y=47,\ 5x+4y=50$

(g) $2x+3y=7,\ 6x+9y=11$ (h) $3x+y=5,\ x+3y=-1$

(i) $y=2x+3,\ 4x-2y=-6$ (j) $ax+by=c,\ y=2ax$

(k) $y=mx+c,\ y=-mx+d$ (l) $ax-by=1,\ y=x$

12 Let P, with coordinates (p,q), be a fixed point on the 'curve' with equation $y=mx+c$ and let Q, with coordinates (r,s), be any other point on $y=mx+c$. Use the fact that the coordinates of P and Q satisfy the equation $y=mx+c$ to show that the gradient of PQ is m for all positions of Q.

13 There are some values of a, b and c for which the equation $ax+by+c=0$ does not represent a straight line. Give an example of such values.

1.9 The gradients of perpendicular lines

In Section 1.3 it is stated that two lines are parallel if they have the same gradient. But what can you say about the gradients of two lines which are perpendicular?

Firstly, if a line has a positive gradient, then the perpendicular line has a negative gradient, and vice versa. But you can be more exact than this.

In Fig. 1.13, if the gradient of PB is m, you can draw a 'gradient triangle' PAB in which PA is one unit and AB is m units.

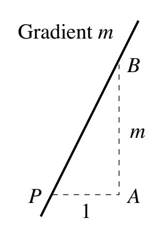


Fig. 1.13

In Fig 1.14, the gradient triangle PAB has been rotated through a right-angle to $PA'B'$, so that PB' is perpendicular to PB. The y-step for $PA'B'$ is 1 and the x-step is $-m$, so

$$\text{gradient of } PB' = \frac{y\text{-step}}{x\text{-step}} = \frac{1}{-m} = -\frac{1}{m}.$$

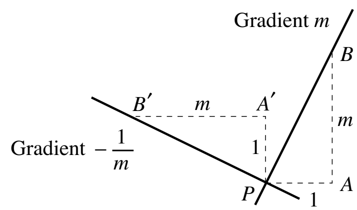


Fig. 1.14

Therefore the gradient of the line perpendicular to PB is $-\frac{1}{m}$.

Thus if the gradients of the two perpendicular lines are m_1 and m_2, then $m_1 m_2 = -1$. It is also true that if two lines have gradients m_1 and m_2, and if $m_1 m_2 = -1$, then the lines are perpendicular. To prove this, see Miscellaneous exercise 1 Question 22.

> Two lines with gradients m_1 and m_2 are perpendicular if
>
> $m_1 m_2 = -1$, or $m_1 = -\frac{1}{m_2}$, or $m_2 = -\frac{1}{m_1}$.

Notice that the condition does not work if the lines are parallel to the axes. However, you can see that a line $x = \text{constant}$ is perpendicular to one of the form $y = \text{constant}$.

Example 1.9.1
Show that the points $(0,-5)$, $(-1,2)$, $(4,7)$ and $(5,0)$ form a rhombus.

You could tackle this question in several ways. This solution shows that the points form a parallelogram, and then that its diagonals are perpendicular.

The mid-points of the diagonals are $\left(\frac{1}{2}(0+4), \frac{1}{2}(-5+7)\right)$, or $(2,1)$, and $\left(\frac{1}{2}((-1)+5), \frac{1}{2}(2+0)\right)$, or $(2,1)$. As these are the same point, the quadrilateral is a parallelogram.

The gradients of the diagonals are $\frac{7-(-5)}{4-0} = \frac{12}{4} = 3$ and $\frac{0-2}{5-(-1)} = \frac{-2}{6} = -\frac{1}{3}$. As the product of the gradients is -1, the diagonals are perpendicular. Therefore the parallelogram is a rhombus.

Example 1.9.2
Find the coordinates of the foot of the perpendicular from $A(-2,-4)$ to the line joining $B(0,2)$ and $C(-1,4)$.

Always draw a diagram, like Fig. 1.15; it need not be to scale. The foot of the perpendicular is the point of intersection, P, of BC and the line through A perpendicular to BC. First find the gradient of BC and its equation.

The gradient of BC is $\dfrac{4-2}{-1-0}=\dfrac{2}{-1}=-2$.

The equation of BC is $y-2=-2(x-0)$, which simplifies to $2x+y=2$.

The gradient of the line through A perpendicular to BC is $-\dfrac{1}{-2}=\frac{1}{2}$.

The equation of this line is

$$y-(-4)=\tfrac{1}{2}(x-(-2)),$$

or $\quad x-2y=6$.

These lines meet at the point P, whose coordinates satisfy the simultaneous equations $2x+y=2$ and $x-2y=6$. This is the point $(2,-2)$.

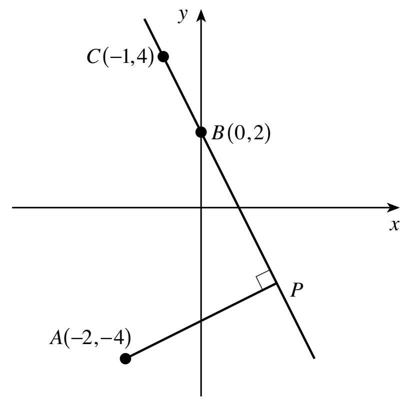


Fig. 1.15

Exercise 1C

1 In each part write down the gradient of a line which is perpendicular to one with the given gradient.

(a) 2 (b) -3 (c) $\frac{3}{4}$ (d) $-\frac{5}{6}$

(e) -1 (f) $1\frac{3}{4}$ (g) $-\dfrac{1}{m}$ (h) m

(i) $\dfrac{p}{q}$ (j) 0 (k) $-m$ (l) $\dfrac{a}{b-c}$

2 In each part find the equation of the line through the given point which is perpendicular to the given line. Write your final answer so that it doesn't contain fractions.

(a) $(2,3)$, $y=4x+3$ (b) $(-3,1)$, $y=-\frac{1}{2}x+3$

(c) $(2,-5)$, $y=-5x-2$ (d) $(7,-4)$, $y=2\frac{1}{2}$

(e) $(-1,4)$, $2x+3y=8$ (f) $(4,3)$, $3x-5y=8$

(g) $(5,-3)$, $2x=3$ (h) $(0,3)$, $y=2x-1$

(i) $(0,0)$, $y=mx+c$ (j) (a,b), $y=mx+c$

(k) (c,d), $ny-x=p$ (l) $(-1,-2)$, $ax+by=c$

3 Find the equation of the line through the point $(-2,5)$ which is perpendicular to the line $y=3x+1$. Find also the point of intersection of the two lines.

4 Find the equation of the line through the point $(1,1)$ which is perpendicular to the line $2x-3y=12$. Find also the point of intersection of the two lines.

5 A line through a vertex of a triangle which is perpendicular to the opposite side is called an altitude. Find the equation of the altitude through the vertex A of the triangle ABC where A is the point $(2,3)$, B is $(1,-7)$ and C is $(4,-1)$.

6 $P(2,5)$, $Q(12,5)$ and $R(8,-7)$ form a triangle.

(a) Find the equations of the altitudes (see Question 5) through (i) R and (ii) Q.

(b) Find the point of intersection of these altitudes.

(c) Show that the altitude through P also passes through this point.

Miscellaneous exercise 1

1 Show that the triangle formed by the points $(-2,5)$, $(1,3)$ and $(5,9)$ is right-angled.

2 Find the coordinates of the point where the lines $2x + y = 3$ and $3x + 5y - 1 = 0$ meet.

3 A triangle is formed by the points $A(-1,3)$, $B(5,7)$ and $C(0,8)$.

(a) Show that the angle ACB is a right angle.

(b) Find the coordinates of the point where the line through B parallel to AC cuts the x-axis.

4 $A(7,2)$ and $C(1,4)$ are two vertices of a square $ABCD$.

(a) Find the equation of the diagonal BD.

(b) Find the coordinates of B and of D.

5 A quadrilateral $ABCD$ is formed by the points $A(-3,2)$, $B(4,3)$, $C(9,-2)$ and $D(2,-3)$.

(a) Show that all four sides are equal in length.

(b) Show that $ABCD$ is not a square.

6 P is the point $(7,5)$ and l_1 is the line with equation $3x + 4y = 16$.

(a) Find the equation of the line l_2 which passes through P and is perpendicular to l_1.

(b) Find the point of intersection of the lines l_1 and l_2.

(c) Find the perpendicular distance of P from the line l_1.

7 Prove that the triangle with vertices $(-2,8)$, $(3,20)$ and $(11,8)$ is isosceles. Find its area.

8 The three straight lines $y = x$, $7y = 2x$ and $4x + y = 60$ form a triangle. Find the coordinates of its vertices.

9 Find the equation of the line through $(1,3)$ which is parallel to $2x + 7y = 5$. Give your answer in the form $ax + by = c$.

10 Find the equation of the perpendicular bisector of the line joining $(2,-5)$ and $(-4,3)$.

11 The points $A(1,2)$, $B(3,5)$, $C(6,6)$ and D form a parallelogram. Find the coordinates of the mid-point of AC. Use your answer to find the coordinates of D.

12 The point P is the foot of the perpendicular from the point $A(0,3)$ to the line $y = 3x$.

(a) Find the equation of the line AP.

(b) Find the coordinates of the point P.

(c) Find the perpendicular distance of A from the line $y = 3x$.

13 Points which lie on the same straight line are called collinear. Show that the points $(-1,3)$, $(4,7)$ and $(-11,-5)$ are collinear.

14 Find the equation of the straight line that passes through the points $(3,-1)$ and $(-2,2)$, giving your answer in the form $ax+by+c=0$. Hence find the coordinates of the point of intersection of the line and the x-axis. (OCR)

15 The coordinates of the points A and B are $(3,2)$ and $(4,-5)$ respectively. Find the coordinates of the mid-point of AB, and the gradient of AB.

Hence find the equation of the perpendicular bisector of AB, giving your answer in the form $ax+by+c=0$, where a, b and c are integers. (OCR)

16 The curve $y=1+\dfrac{1}{2+x}$ crosses the x-axis at the point A and the y-axis at the point B.

(a) Calculate the coordinates of A and of B.

(b) Find the equation of the line AB.

(c) Calculate the coordinates of the point of intersection of the line AB and the line with equation $3y=4x$. (OCR)

17 The straight line p passes through the point $(10,1)$ and is perpendicular to the line r with equation $2x+y=1$. Find the equation of p.

Find also the coordinates of the point of intersection of p and r, and deduce the perpendicular distance from the point $(10,1)$ to the line r. (OCR)

18 Show by calculation that the points $P(0,7)$, $Q(6,5)$, $R(5,2)$ and $S(-1,4)$ are the vertices of a rectangle.

19 The line $3x-4y=8$ meets the y-axis at A. The point C has coordinates $(-2,9)$. The line through C perpendicular to $3x-4y=8$ meets it at B. Calculate the area of the triangle ABC.

20 The points $A(-3,-4)$ and $C(5,4)$ are the ends of the diagonal of a rhombus $ABCD$.

(a) Find the equation of the diagonal BD.

(b) Given that the side BC has gradient $\frac{5}{3}$, find the coordinates of B and hence of D.

21 Find the equations of the medians (see Exercise 1A Question 13) of the triangle with vertices $(0,2)$, $(6,0)$ and $(4,4)$. Show that the medians are concurrent (all pass through the same point).

22 Two lines have equations $y=m_1x+c_1$ and $y=m_2x+c_2$, and $m_1m_2=-1$. Prove that the lines are perpendicular.

2 Surds and indices

The first part of this chapter is about expressions involving square and cube roots. The second part is about index notation. When you have completed it, you should

- be able to simplify expressions involving square, cube and other roots
- know the rules of indices
- know the meaning of negative, zero and fractional indices
- be able to simplify expressions involving indices.

2.1 Different kinds of number

At first numbers were used only for counting, and 1, 2, 3, … were all that was needed. These are **natural numbers**, or **positive integers**.

Then it was found that numbers could also be useful for measurement and in commerce. For these purposes fractions were also needed. Integers and fractions together make up the **rational numbers**. These are numbers which can be expressed in the form $\frac{p}{q}$ where p and q are integers, and q is not 0.

One of the most remarkable discoveries of the ancient Greek mathematicians was that there are numbers which cannot be expressed in this way. These are called **irrational numbers**. The first such number to be found was $\sqrt{2}$, which is the length of the diagonal of a square with side 1 unit, by Pythagoras' theorem. The argument that the Greeks used to prove that $\sqrt{2}$ cannot be expressed as a fraction can be adapted to show that the square root, cube root, … of any positive integer is either an integer or an irrational number. Many other numbers are now known to be irrational, of which the most well known is π.

Rational and irrational numbers together make up the **real numbers**. Integers, rational and irrational numbers, and real numbers can be either positive, negative or zero.

When rational numbers are written as decimals, they either come to a stop after a number of places, or the sequence of decimal digits eventually starts repeating in a regular pattern. For example,

$$\tfrac{7}{10} = 0.7,\quad \tfrac{7}{11} = 0.6363\ldots,\quad \tfrac{7}{12} = 0.5833\ldots,\quad \tfrac{7}{13} = 0.538\,461\,538\,461\,53\ldots,$$
$$\tfrac{7}{14} = 0.5,\quad \tfrac{7}{15} = 0.466\ldots,\quad \tfrac{7}{16} = 0.4375,\quad \tfrac{7}{17} = 0.411\,764\,705\,882\,352\,941\,176\ldots$$

The reverse is also true: if a decimal number stops or repeats indefinitely then it is a rational number. So if an irrational number is written as a decimal, the pattern of the decimal digits never repeats however long you continue the calculation.

2.2 Surds and their properties

When you met expressions such as $\sqrt{2}$, $\sqrt{8}$ and $\sqrt{12}$ before, it is likely that you used a calculator to express them in decimal form. You might have written

$$\sqrt{2} = 1.414\ldots \quad \text{or} \quad \sqrt{2} = 1.414 \text{ correct to 3 decimal places} \quad \text{or} \quad \sqrt{2} \approx 1.414.$$

Why is the statement '$\sqrt{2} = 1.414$' incorrect?

Expressions like $\sqrt{2}$ or $\sqrt[3]{9}$ are called **surds**. This section is about calculating with surds. You need to remember that $\sqrt{x}$ always means the *positive* square root of x (or zero when $x = 0$).

The main properties of surds that you will use are:

$$\sqrt{xy} = \sqrt{x} \times \sqrt{y} \quad \text{and} \quad \sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}.$$

You can see that as $\left(\sqrt{x} \times \sqrt{y}\right) \times \left(\sqrt{x} \times \sqrt{y}\right) = \left(\sqrt{x} \times \sqrt{x}\right) \times \left(\sqrt{y} \times \sqrt{y}\right) = x \times y = xy$, and as $\sqrt{x} \times \sqrt{y}$ is positive, it is the square root of xy. Therefore $\sqrt{xy} = \sqrt{x} \times \sqrt{y}$. Similar reasoning will convince you that $\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$.

The following examples illustrate these properties:

$$\sqrt{8} = \sqrt{4 \times 2} = \sqrt{4} \times \sqrt{2} = 2\sqrt{2}; \quad \sqrt{12} = \sqrt{4 \times 3} = \sqrt{4} \times \sqrt{3} = 2\sqrt{3};$$

$$\sqrt{18} \times \sqrt{2} = \sqrt{18 \times 2} = \sqrt{36} = 6; \quad \frac{\sqrt{27}}{\sqrt{3}} = \sqrt{\frac{27}{3}} = \sqrt{9} = 3.$$

It is well worth checking some or all of the calculations above on your calculator.

Example 2.2.1
Simplify (a) $\sqrt{28} + \sqrt{63}$, (b) $\sqrt{5} \times \sqrt{10}$.

Alternative methods of solution may be possible, as in part (b).

(a) $\sqrt{28} + \sqrt{63} = \left(\sqrt{4} \times \sqrt{7}\right) + \left(\sqrt{9} \times \sqrt{7}\right) = 2\sqrt{7} + 3\sqrt{7} = 5\sqrt{7}$.

(b) **Method 1** $\sqrt{5} \times \sqrt{10} = \sqrt{5 \times 10} = \sqrt{50} = \sqrt{25 \times 2} = 5\sqrt{2}$.
Method 2 $\sqrt{5} \times \sqrt{10} = \sqrt{5} \times \left(\sqrt{5} \times \sqrt{2}\right) = \left(\sqrt{5} \times \sqrt{5}\right) \times \sqrt{2} = 5\sqrt{2}$.

It is sometimes useful to be able to remove a surd from the denominator of a fraction such as $\frac{1}{\sqrt{2}}$. You can do this by multiplying top and bottom by $\sqrt{2}$: $\frac{1 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{\sqrt{2}}{2}$.

Results which it is often helpful to use are:

$$\frac{x}{\sqrt{x}} = \sqrt{x}\text{, and its reciprocal } \frac{1}{\sqrt{x}} = \frac{\sqrt{x}}{x}.$$

Removing the surd from the denominator is called **rationalising the denominator**.

Example 2.2.2

Rationalise the denominator in the expressions (a) $\frac{6}{\sqrt{2}}$, (b) $\frac{3\sqrt{2}}{\sqrt{10}}$.

(a) $\frac{6}{\sqrt{2}} = \frac{3\times 2}{\sqrt{2}} = 3\times\frac{2}{\sqrt{2}} = 3\sqrt{2}$.

(b) $\frac{3\sqrt{2}}{\sqrt{10}} = \frac{3\times\sqrt{2}}{\sqrt{5}\times\sqrt{2}} = \frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{5}$.

Similar rules to those for square roots also apply to cube roots and higher roots.

Example 2.2.3

Simplify (a) $\sqrt[3]{16}$, (b) $\sqrt[3]{12}\times\sqrt[3]{18}$.

(a) $\sqrt[3]{16} = \sqrt[3]{8\times 2} = \sqrt[3]{8}\times\sqrt[3]{2} = 2\times\sqrt[3]{2}$.

(b) $\sqrt[3]{12}\times\sqrt[3]{18} = \sqrt[3]{12\times 18} = \sqrt[3]{216} = 6$.

Example 2.2.4

Fig. 2.1 shows the vertical cross-section of a roof of a building as a right-angled triangle ABC, with $AB = 15$ m. The height of the roof, BD, is 10 m. Calculate x and y.

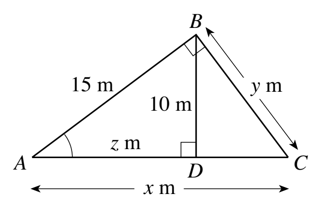


Fig. 2.1

Starting with triangle ADB, by Pythagoras' theorem $z^2 + 10^2 = 15^2$, so $z^2 = 225 - 100 = 125$ and

$$z = \sqrt{125} = \sqrt{25\times 5} = 5\sqrt{5}.$$

Now notice that the triangles ADB and ABC are similar. You can see the similarity more clearly by flipping triangle ADB over to make the side AB horizontal, as in Fig. 2.2. The sides of triangles ADB and ABC must therefore be in the same proportion, so $\frac{x}{15} = \frac{y}{10} = \frac{15}{z}$. Since $\frac{15}{z} = \frac{15}{5\sqrt{5}} = \frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{5}$,

$$x = 15\times\frac{3\sqrt{5}}{5} = 9\sqrt{5} \quad\text{and}\quad y = 10\times\frac{3\sqrt{5}}{5} = 6\sqrt{5}.$$

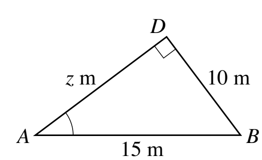


Fig. 2.2

Use Pythagoras' theorem in triangle ABC to check that $x^2 = 15^2 + y^2$.

Exercise 2A

1 Simplify the following without using a calculator.

(a) $\sqrt{3}\times\sqrt{3}$ (b) $\sqrt{10}\times\sqrt{10}$ (c) $\sqrt{16}\times\sqrt{16}$ (d) $\sqrt{8}\times\sqrt{2}$

(e) $\sqrt{32}\times\sqrt{2}$ (f) $\sqrt{3}\times\sqrt{12}$ (g) $5\sqrt{3}\times\sqrt{3}$ (h) $2\sqrt{5}\times3\sqrt{5}$

(i) $3\sqrt{6}\times4\sqrt{6}$ (j) $2\sqrt{20}\times3\sqrt{5}$ (k) $\left(2\sqrt{7}\right)^2$ (l) $\left(3\sqrt{3}\right)^2$

(m) $\sqrt[3]{5}\times\sqrt[3]{5}\times\sqrt[3]{5}$ (n) $\left(2\sqrt[4]{3}\right)^4$ (o) $\left(2\sqrt[3]{2}\right)^6$ (p) $\sqrt[4]{125}\times\sqrt[4]{5}$

2 Simplify the following without using a calculator.

(a) $\sqrt{18}$ (b) $\sqrt{20}$ (c) $\sqrt{24}$ (d) $\sqrt{32}$

(e) $\sqrt{40}$ (f) $\sqrt{45}$ (g) $\sqrt{48}$ (h) $\sqrt{50}$

(i) $\sqrt{54}$ (j) $\sqrt{72}$ (k) $\sqrt{135}$ (l) $\sqrt{675}$

3 Simplify the following without using a calculator.

(a) $\sqrt{8}+\sqrt{18}$ (b) $\sqrt{3}+\sqrt{12}$ (c) $\sqrt{20}-\sqrt{5}$

(d) $\sqrt{32}-\sqrt{8}$ (e) $\sqrt{50}-\sqrt{18}-\sqrt{8}$ (f) $\sqrt{27}+\sqrt{27}$

(g) $\sqrt{99}+\sqrt{44}+\sqrt{11}$ (h) $8\sqrt{2}+2\sqrt{8}$ (i) $2\sqrt{20}+3\sqrt{45}$

(j) $\sqrt{52}-\sqrt{13}$ (k) $20\sqrt{5}+5\sqrt{20}$ (l) $\sqrt{48}+\sqrt{24}-\sqrt{75}+\sqrt{96}$

4 Simplify the following without using a calculator.

(a) $\dfrac{\sqrt{8}}{\sqrt{2}}$ (b) $\dfrac{\sqrt{27}}{\sqrt{3}}$ (c) $\dfrac{\sqrt{40}}{\sqrt{10}}$ (d) $\dfrac{\sqrt{50}}{\sqrt{2}}$

(e) $\dfrac{\sqrt{125}}{\sqrt{5}}$ (f) $\dfrac{\sqrt{54}}{\sqrt{6}}$ (g) $\dfrac{\sqrt{3}}{\sqrt{48}}$ (h) $\dfrac{\sqrt{50}}{\sqrt{200}}$

5 Rationalise the denominator in each of the following expressions, and simplify them.

(a) $\dfrac{1}{\sqrt{3}}$ (b) $\dfrac{1}{\sqrt{5}}$ (c) $\dfrac{4}{\sqrt{2}}$ (d) $\dfrac{6}{\sqrt{6}}$

(e) $\dfrac{11}{\sqrt{11}}$ (f) $\dfrac{2}{\sqrt{8}}$ (g) $\dfrac{12}{\sqrt{3}}$ (h) $\dfrac{14}{\sqrt{7}}$

(i) $\dfrac{\sqrt{6}}{\sqrt{2}}$ (j) $\dfrac{\sqrt{2}}{\sqrt{6}}$ (k) $\dfrac{3\sqrt{5}}{\sqrt{3}}$ (l) $\dfrac{4\sqrt{6}}{\sqrt{5}}$

(m) $\dfrac{7\sqrt{2}}{2\sqrt{3}}$ (n) $\dfrac{4\sqrt{2}}{\sqrt{12}}$ (o) $\dfrac{9\sqrt{12}}{2\sqrt{18}}$ (p) $\dfrac{2\sqrt{18}}{9\sqrt{12}}$

6 Simplify the following, giving each answer in the form $k\sqrt{3}$.

(a) $\sqrt{75}+\sqrt{12}$ (b) $6+\sqrt{3}\left(4-2\sqrt{3}\right)$ (c) $\dfrac{12}{\sqrt{3}}-\sqrt{27}$

(d) $\dfrac{2}{\sqrt{3}}+\dfrac{\sqrt{2}}{\sqrt{6}}$ (e) $\sqrt{2}\times\sqrt{8}\times\sqrt{27}$ (f) $\left(3-\sqrt{3}\right)\left(2-\sqrt{3}\right)-\sqrt{3}\times\sqrt{27}$

7 $ABCD$ is a rectangle in which $AB=4\sqrt{5}$ cm and $BC=\sqrt{10}$ cm. Giving each answer in simplified surd form, find

(a) the area of the rectangle, (b) the length of the diagonal AC.

8 Solve the following equations, giving each answer in the form $k\sqrt{2}$.

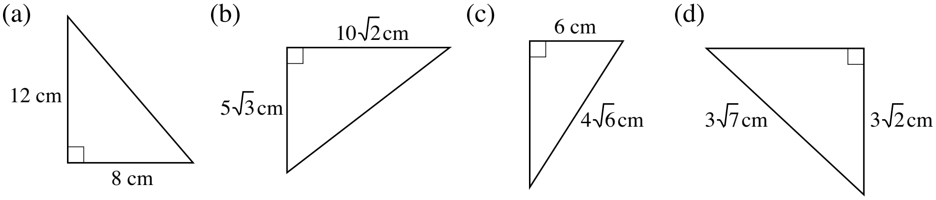
(a) $x\sqrt{2} = 10$ (b) $2y\sqrt{2} - 3 = \dfrac{5y}{\sqrt{2}} + 1$ (c) $z\sqrt{32} - 16 = z\sqrt{8} - 4$

9 Express in the form $k\sqrt[3]{3}$

(a) $\sqrt[3]{24}$, (b) $\sqrt[3]{81} + \sqrt[3]{3}$, (c) $\left(\sqrt[3]{3}\right)^4$, (d) $\sqrt[3]{3000} - \sqrt[3]{375}$.

10 Find the length of the third side in each of the following right-angled triangles, giving each answer in simplified surd form.

(a) (b) (c) (d)



11 You are given that, correct to 12 decimal places, $\sqrt{26} = 5.099\,019\,513\,593$.

(a) Find the value of $\sqrt{104}$ correct to 10 decimal places.

(b) Find the value of $\sqrt{650}$ correct to 10 decimal places.

(c) Find the value of $\dfrac{13}{\sqrt{26}}$ correct to 10 decimal places.

12 Solve the simultaneous equations $7x - \left(3\sqrt{5}\right)y = 9\sqrt{5}$ and $\left(2\sqrt{5}\right)x + y = 34$.

13 Simplify the following.

(a) $\left(\sqrt{2}-1\right)\left(\sqrt{2}+1\right)$ (b) $\left(2-\sqrt{3}\right)\left(2+\sqrt{3}\right)$

(c) $\left(\sqrt{7}+\sqrt{3}\right)\left(\sqrt{7}-\sqrt{3}\right)$ (d) $\left(2\sqrt{2}+1\right)\left(2\sqrt{2}-1\right)$

(e) $\left(4\sqrt{3}-\sqrt{2}\right)\left(4\sqrt{3}+\sqrt{2}\right)$ (f) $\left(\sqrt{10}+\sqrt{5}\right)\left(\sqrt{10}-\sqrt{5}\right)$

(g) $\left(4\sqrt{7}-\sqrt{5}\right)\left(4\sqrt{7}+\sqrt{5}\right)$ (h) $\left(2\sqrt{6}-3\sqrt{3}\right)\left(2\sqrt{6}+3\sqrt{3}\right)$

14 In Question 13, every answer is an integer. Copy and complete each of the following.

(a) $\left(\sqrt{3}-1\right)(\qquad) = 2$ (b) $\left(\sqrt{5}+1\right)(\qquad) = 4$

(c) $\left(\sqrt{6}-\sqrt{2}\right)(\qquad) = 4$ (d) $\left(2\sqrt{7}+\sqrt{3}\right)(\qquad) = 25$

(e) $\left(\sqrt{11}+\sqrt{10}\right)(\qquad) = 1$ (f) $\left(3\sqrt{5}-2\sqrt{6}\right)(\qquad) = 21$

The examples in Questions 15 and 16 indicate a method for rationalising the denominator in cases which are more complicated than those in Question 5.

15 (a) Explain why $\dfrac{1}{\sqrt{3}-1} = \dfrac{1}{\sqrt{3}-1} \times \dfrac{\sqrt{3}+1}{\sqrt{3}+1}$ and hence show that $\dfrac{1}{\sqrt{3}-1} = \dfrac{\sqrt{3}+1}{2}$.

(b) Show that $\dfrac{1}{2\sqrt{2}+\sqrt{3}} = \dfrac{2\sqrt{2}-\sqrt{3}}{5}$.

16 Rationalise the denominators and simplify these fractions.

(a) $\dfrac{1}{2-\sqrt{3}}$ (b) $\dfrac{1}{3\sqrt{5}-5}$ (c) $\dfrac{4\sqrt{3}}{2\sqrt{6}+3\sqrt{2}}$

2.3 Working with indices

In the 16th century, when mathematics books began to be printed, mathematicians were finding how to solve cubic and quartic equations. They found it was more economical to write and to print the products *xxx* and *xxxx* as x^3 and x^4.

This is how index notation started. But it turned out to be much more than a convenient shorthand. The new notation led to important mathematical discoveries, and mathematics as it is today would be inconceivable without index notation.

You will already have used simple examples of this notation. In general, the symbol a^m stands for the result of multiplying m as together:

$$a^m = \overbrace{a \times a \times a \times \ \dots \ \times a}^{m \text{ of these}}.$$

The number a is called the **base**, and the number m is the **index** (plural 'indices'). Notice that, although a can be any kind of number, m must be a positive integer. Another way of describing this is `a raised to the mth power', or more shortly 'a to the power m'. Expressions in index notation can often be simplified by using a few simple rules.

One of these is the **multiplication rule**,

$$a^m \times a^n = \overbrace{a \times a \times \ \dots \ \times a}^{m \text{ of these}} \times \overbrace{a \times a \times \ \dots \ \times a}^{n \text{ of these}} = \overbrace{a \times a \times \ \dots \ \times a}^{m+n \text{ of these}} = a^{m+n}.$$

This is used, for example, in finding the volume of a cube of side a:

$$\text{volume} = \text{base area} \times \text{height} = a^2 \times a = a^2 \times a^1 = a^{2+1} = a^3.$$

Closely linked with this is the **division rule**,

$$\begin{aligned} a^m \div a^n &= \overbrace{(a \times a \times \ \dots \ \times a)}^{m \text{ of these}} \div \overbrace{(a \times a \times \ \dots \ \times a)}^{n \text{ of these}} \\ &= \overbrace{a \times a \times \ \dots \ \times a}^{m-n \text{ of these}} \quad (\text{since } n \text{ of the } a\text{s cancel out}) \\ &= a^{m-n}, \quad \text{provided that } m > n. \end{aligned}$$

Another rule is the **power-on-power rule**,

$$\begin{aligned} (a^m)^n &= \overbrace{\overbrace{a \times a \times \ \dots \ \times a}^{m \text{ of these}} \times \overbrace{a \times a \times \ \dots \ \times a}^{m \text{ of these}} \times \dots \times \overbrace{a \times a \times \ \dots \ \times a}^{m \text{ of these}}}^{n \text{ of these brackets}} \\ &= \overbrace{a \times a \times \ \dots \ \times a}^{m \times n \text{ of these}} = a^{m \times n}. \end{aligned}$$

One further rule, the **factor rule**, has two bases but just one index:

$$(a \times b)^m = \overbrace{(a \times b) \times (a \times b) \times \dots \times (a \times b)}^{m \text{ of these brackets}} = \overbrace{a \times a \times \ \dots \ \times a}^{m \text{ of these}} \times \overbrace{b \times b \times \ \dots \ \times b}^{m \text{ of these}} = a^m \times b^m.$$

In explaining these rules multiplication signs have been used. But, as in other parts of algebra, they are usually omitted if there is no ambiguity. For completeness, here are the rules again.

Multiplication rule:	$a^m \times a^n = a^{m+n}$
Division rule:	$a^m \div a^n = a^{m-n}$, provided that $m > n$
Power-on-power rule:	$(a^m)^n = a^{m\times n}$
Factor rule:	$(a\times b)^m = a^m \times b^m$

Example 2.3.1
Simplify $(2a^2b)^3 \div (4a^4b)$.

$$\begin{aligned} (2a^2b)^3 \div (4a^4b) &= \left(2^3(a^2)^3b^3\right) \div (4a^4b) && \text{factor rule} \\ &= (8a^{2\times 3}b^3) \div (4a^4b) && \text{power-on-power rule} \\ &= (8\div 4)\times(a^6 \div a^4)\times(b^3 \div b^1) && \text{rearranging} \\ &= 2a^{6-4}b^{3-1} && \text{division rule} \\ &= 2a^2b^2. \end{aligned}$$

2.4 Zero and negative indices

The definition of a^m in Section 2.3, as the result of multiplying m as together, makes no sense if m is zero or a negative integer. You can't multiply -3 as or 0 as together. But extending the meaning of a^m when the index is zero or negative is possible, and useful, since it turns out that the rules still work with such index values.

Look at this sequence: $2^5 = 32,\ 2^4 = 16,\ 2^3 = 8,\ 2^2 = 4,\ \ldots$.

On the left sides, the base is always 2, and the indices go down by 1 at each step. On the right, the numbers are halved at each step. So you might continue the process

$$\ldots,\ 2^2 = 4,\ 2^1 = 2,\ 2^0 = 1,\ 2^{-1} = \tfrac{1}{2},\ 2^{-2} = \tfrac{1}{4},\ 2^{-3} = \tfrac{1}{8},\ \ldots$$

and you can go on like this indefinitely. Now compare

$$2^1 = 2 \text{ with } 2^{-1} = \tfrac{1}{2}, \quad 2^2 = 4 \text{ with } 2^{-2} = \tfrac{1}{4}, \quad 2^3 = 8 \text{ with } 2^{-3} = \tfrac{1}{8}.$$

It looks as if 2^{-m} should be defined as $\dfrac{1}{2^m}$, with the special value in the middle $2^0 = 1$.

This observation, extended to any base a (except 0), and any positive integer m, gives the **negative power rule**.

Negative power rule: $a^{-m} = \dfrac{1}{a^m}$ and $a^0 = 1$.

Here are some examples to show that, with these definitions, the rules established in Section 2.3 for positive indices still work with negative indices. Try making up other examples for yourself.

Multiplication rule:
$$a^3 \times a^{-7} = a^3 \times \frac{1}{a^7} = \frac{1}{a^7 \div a^3}$$
$$= \frac{1}{a^{7-3}} \qquad \text{using the division rule for positive indices}$$
$$= \frac{1}{a^4} = a^{-4} = a^{3+(-7)}.$$

Power-on-power rule:
$$(a^{-2})^{-3} = \left(\frac{1}{a^2}\right)^{-3} = \frac{1}{(1/a^2)^3} = \frac{1}{1/(a^2)^3}$$
$$= \frac{1}{1/a^6} \qquad \text{using the power-on-power rule for positive indices}$$
$$= a^6 = a^{(-2)\times(-3)}.$$

Factor rule:
$$(ab)^{-3} = \frac{1}{(ab)^3} = \frac{1}{a^3b^3} \qquad \text{using the factor rule for positive indices}$$
$$= \frac{1}{a^3} \times \frac{1}{b^3} = a^{-3}b^{-3}.$$

Example 2.4.1
If $a = 5$, find the value of $4a^{-2}$.

The important thing to notice is that the index -2 goes only with the a and not with the 4. So $4a^{-2}$ means $4 \times \frac{1}{a^2}$. When $a = 5$, $4a^{-2} = 4 \times \frac{1}{25} = 0.16$.

Example 2.4.2
Simplify (a) $4a^2b \times (3ab^{-1})^{-2}$, (b) $\left(\frac{\text{MLT}^{-2}}{\text{L}^2}\right) \div \left(\frac{\text{LT}^{-1}}{\text{L}}\right)$.

(a) **Method 1** Turn everything into positive indices.
$$4a^2b \times (3ab^{-1})^{-2} = 4a^2b \times \frac{1}{(3a \times 1/b)^2} = 4a^2b \times \frac{1}{9a^2 \times 1/b^2} = 4a^2b \times \frac{b^2}{9a^2}$$
$$= \tfrac{4}{9}b^{1+2} = \tfrac{4}{9}b^3.$$

Method 2 Use the rules directly with positive and negative indices.
$$4a^2b \times (3ab^{-1})^{-2} = 4a^2b \times \left(3^{-2}a^{-2}(b^{-1})^{-2}\right) \qquad \text{factor rule}$$
$$= 4a^2b \times (3^{-2}a^{-2}b^2) \qquad \text{power-on-power rule}$$
$$= \left(4 \times \frac{1}{3^2}\right) \times (a^2a^{-2}) \times (bb^2) = \tfrac{4}{9}a^0b^3 = \tfrac{4}{9}b^3.$$

(b) This is an application in mechanics: M, L, T stand for dimensions of mass, length and time in the measurement of viscosity. Taking the brackets separately,

$$\left(\frac{\text{MLT}^{-2}}{\text{L}^2}\right) = \text{ML}^{1-2}\text{T}^{-2} = \text{ML}^{-1}\text{T}^{-2} \quad \text{and} \quad \left(\frac{\text{LT}^{-1}}{\text{L}}\right) = \text{L}^{1-1}\text{T}^{-1} = \text{L}^0\text{T}^{-1} = \text{T}^{-1},$$

$$\text{so} \quad \left(\frac{\text{MLT}^{-2}}{\text{L}^2}\right) \div \left(\frac{\text{LT}^{-1}}{\text{L}}\right) = \left(\text{ML}^{-1}\text{T}^{-2}\right) \div \text{T}^{-1} = \text{ML}^{-1}\text{T}^{-2-(-1)} = \text{ML}^{-1}\text{T}^{-1}.$$

One application of negative indices is in writing down very small numbers. You probably know how to write very large numbers in standard form, or scientific notation. For example, it is easier to write the speed of light as 3.00×10^8 m s^{-1} than as 300 000 000 m s^{-1}. Similarly, the wavelength of red light, about 0.000 000 75 metres, is more easily appreciated written as 7.5×10^{-7} metres.

Computers and calculators often give users the option to work in scientific notation, and if numbers become too large (or too small) to be displayed in ordinary numerical form they will switch into standard form, for example 3.00E8 or 7.5E±7. The symbol E stands for 'exponent', which is yet another word for 'index'. You can write this in scientific notation by simply replacing the symbol Em by $\times 10^m$, for any integer m.

Example 2.4.3

Calculate the universal constant of gravitation, G, from $G = \frac{gR^2}{M}$ where, in SI units, $g \approx 9.81$, $R \approx 6.37 \times 10^6$ and $M \approx 5.97 \times 10^{24}$. ($R$ and M are the earth's radius and mass, and g is the acceleration due to gravity at the earth's surface.)

$$G \approx \frac{9.81 \times (6.37 \times 10^6)^2}{5.97 \times 10^{24}} = \frac{9.81 \times (6.37)^2}{5.97} \times \frac{(10^6)^2}{10^{24}}$$

$$\approx 66.7 \times \frac{10^{12}}{10^{24}} = 6.67 \times 10^1 \times 10^{-12} = 6.67 \times 10^{1-12} = 6.67 \times 10^{-11}.$$

Exercise 2B

1 Simplify the following expressions.

(a) $a^2 \times a^3 \times a^7$ (b) $(b^4)^2$ (c) $c^7 \div c^3$

(d) $d^5 \times d^4$ (e) $(e^5)^4$ (f) $(x^3y^2)^2$

(g) $5g^5 \times 3g^3$ (h) $12h^{12} \div 4h^4$ (i) $(2a^2)^3 \times (3a)^2$

(j) $(p^2q^3)^2 \times (pq^3)^3$ (k) $(4x^2y)^2 \times (2xy^3)^3$ (l) $(6ac^3)^2 \div (9a^2c^5)$

(m) $(3m^4n^2)^3 \times (2mn^2)^2$ (n) $(49r^3s^2)^2 \div (7rs)^3$ (o) $(2xy^2z^3)^2 \div (2xy^2z^3)$

2 Simplify the following, giving each answer in the form 2^n.

(a) $2^{11} \times (2^5)^3$ (b) $(2^3)^2 \times (2^2)^3$ (c) 4^3 (d) 8^2

(e) $\dfrac{2^7 \times 2^8}{2^{13}}$ (f) $\dfrac{2^2 \times 2^3}{(2^2)^2}$ (g) $4^2 \div 2^4$ (h) $2 \times 4^4 \div 8^3$

3 Express each of the following as an integer or a fraction.

(a) 2^{-3} (b) 4^{-2} (c) 5^{-1} (d) 3^{-2}

(e) 10^{-4} (f) 1^{-7} (g) $\left(\frac{1}{2}\right)^{-1}$ (h) $\left(\frac{1}{3}\right)^{-3}$

(i) $\left(2\frac{1}{2}\right)^{-1}$ (j) 2^{-7} (k) 6^{-3} (l) $\left(1\frac{1}{3}\right)^{-3}$

4 If $x = 2$, find the value of each of the following.

(a) $4x^{-3}$ (b) $(4x)^{-3}$ (c) $\frac{1}{4}x^{-3}$

(d) $\left(\frac{1}{4}x\right)^{-3}$ (e) $(4 \div x)^{-3}$ (f) $(x \div 4)^{-3}$

5 If $y = 5$, find the value of each of the following.

(a) $(2y)^{-1}$ (b) $2y^{-1}$ (c) $\left(\frac{1}{2}y\right)^{-1}$

(d) $\frac{1}{2}y^{-1}$ (e) $\dfrac{1}{(2y)^{-1}}$ (f) $\dfrac{2}{\left(y^{-1}\right)^{-1}}$

6 Express each of the following in as simple a form as possible.

(a) $a^4 \times a^{-3}$ (b) $\dfrac{1}{b^{-1}}$ (c) $(c^{-2})^3$

(d) $d^{-1} \times 2d$ (e) $e^{-4} \times e^{-5}$ (f) $\dfrac{f^{-2}}{f^3}$

(g) $12g^3 \times (2g^2)^{-2}$ (h) $(3h^2)^{-2}$ (i) $(3i^{-2})^{-2}$

(j) $\left(\frac{1}{2}j^{-2}\right)^{-3}$ (k) $(2x^3y^{-1})^3$ (l) $(p^2q^4r^3)^{-4}$

(m) $(4m^2)^{-1} \times 8m^3$ (n) $(3n^{-2})^4 \times (9n)^{-1}$ (o) $(2xy^2)^{-1} \times (4xy)^2$

(p) $(5a^3c^{-1})^2 \div (2a^{-1}c^2)$ (q) $(2q^{-2})^{-2} \div \left(\dfrac{4}{q}\right)^2$ (r) $(3x^{-2}y)^2 \div (4xy)^{-2}$

7 Solve the following equations.

(a) $3^x = \frac{1}{9}$ (b) $5^y = 1$ (c) $2^z \times 2^{z-3} = 32$

(d) $7^{3x} \div 7^{x-2} = \frac{1}{49}$ (e) $4^y \times 2^y = 8^{120}$ (f) $3^t \times 9^{t+3} = 27^2$

8 The length of each edge of a cube is 3×10^{-2} metres.

(a) Find the volume of the cube. (b) Find the total surface area of the cube.

9 An athlete runs 2×10^{-1} km in 7.5×10^{-3} hours. Find her average speed in km h^{-1}.

10 The volume, V m^3, of l metres of wire is given by $V = \pi r^2 l$, where r metres is the radius of the circular cross-section.

(a) Find the volume of 80 m of wire with radius of cross-section 2×10^{-3} m.

(b) Another type of wire has radius of cross-section 5×10^{-3} m. What length of this wire has a volume of 8×10^{-3} m^3?

(c) Another type of wire is such that a length of 61 m has a volume of 6×10^{-3} m^3. Find the radius of the cross-section.

11 An equation which occurs in the study of waves is $y = \dfrac{\lambda d}{a}$.

(a) Calculate y when $\lambda = 7 \times 10^{-7}$, $d = 5 \times 10^{-1}$ and $a = 8 \times 10^{-4}$.

(b) Calculate λ when $y = 10^{-3}$, $d = 0.6$ and $a = 2.7 \times 10^{-4}$.

12 Solve the equation $\dfrac{3^{5x+2}}{9^{1-x}} = \dfrac{27^{4+3x}}{729}$.

2.5 Fractional indices

In Section 2.4, you saw that the four index rules still work when m and n are integers, but not necessarily positive. What happens if m and n are not necessarily integers?

If you put $m = \frac{1}{2}$ and $n = 2$ in the power-on-power rule, you find that

$$\left(x^{\frac{1}{2}}\right)^2 = x^{\frac{1}{2}\times 2} = x^1 = x.$$

Putting $x^{\frac{1}{2}} = y$, this equation becomes $y^2 = x$, so $y = \sqrt{x}$ or $y = -\sqrt{x}$, which is $x^{\frac{1}{2}} = \sqrt{x}$ or $-\sqrt{x}$. *Defining* $x^{\frac{1}{2}}$ to be the positive square root of x, you get $x^{\frac{1}{2}} = \sqrt{x}$.

Similarly, if you put $m = \frac{1}{3}$ and $n = 3$, you can show that $x^{\frac{1}{3}} = \sqrt[3]{x}$. More generally, by putting $m = \dfrac{1}{n}$, you find that $\left(x^{\frac{1}{n}}\right)^n = x^{\frac{1}{n}\times n} = x$, which leads to the result

$$x^{\frac{1}{n}} = \sqrt[n]{x}.$$

Notice that for the case $x^{\frac{1}{2}} = \sqrt{x}$, you must have $x \geqslant 0$, but for the case $x^{\frac{1}{3}} = \sqrt[3]{x}$ you do not need $x \geqslant 0$, because you can take the cube root of a negative number.

A slight extension of the $x^{\frac{1}{n}} = \sqrt[n]{x}$ rule can show you how to deal with expressions of the form $x^{\frac{2}{3}}$. There are two alternatives:

$$x^{\frac{2}{3}} = x^{\frac{1}{3}\times 2} = \left(x^{\frac{1}{3}}\right)^2 = \left(\sqrt[3]{x}\right)^2 \quad \text{and} \quad x^{\frac{2}{3}} = x^{2\times\frac{1}{3}} = (x^2)^{\frac{1}{3}} = \sqrt[3]{x^2}.$$

(If x has an exact cube root it is usually best to use the first form; otherwise the second form is better.) In general, similar reasoning leads to the **fractional power rule**.

Fractional power rule: $x^{\frac{m}{n}} = \left(\sqrt[n]{x}\right)^m = \sqrt[n]{x^m}$.

Fractional powers can also be written as $x^{1/2}$, $x^{m/n}$ *and so on.*

Example 2.5.1

Simplify (a) $9^{\frac{1}{2}}$, (b) $3^{\frac{1}{2}} \times 3^{\frac{3}{2}}$, (c) $16^{-\frac{3}{4}}$.

(a) $9^{\frac{1}{2}} = \sqrt{9} = 3$. (b) $3^{\frac{1}{2}} \times 3^{\frac{3}{2}} = 3^{\frac{1}{2}+\frac{3}{2}} = 3^2 = 9$.

(c) **Method 1** $16^{-\frac{3}{4}} = (2^4)^{-\frac{3}{4}} = 2^{-3} = \frac{1}{8}$.

Method 2 $16^{-\frac{3}{4}} = \dfrac{1}{16^{\frac{3}{4}}} = \dfrac{1}{(\sqrt[4]{16})^3} = \dfrac{1}{2^3} = \frac{1}{8}$.