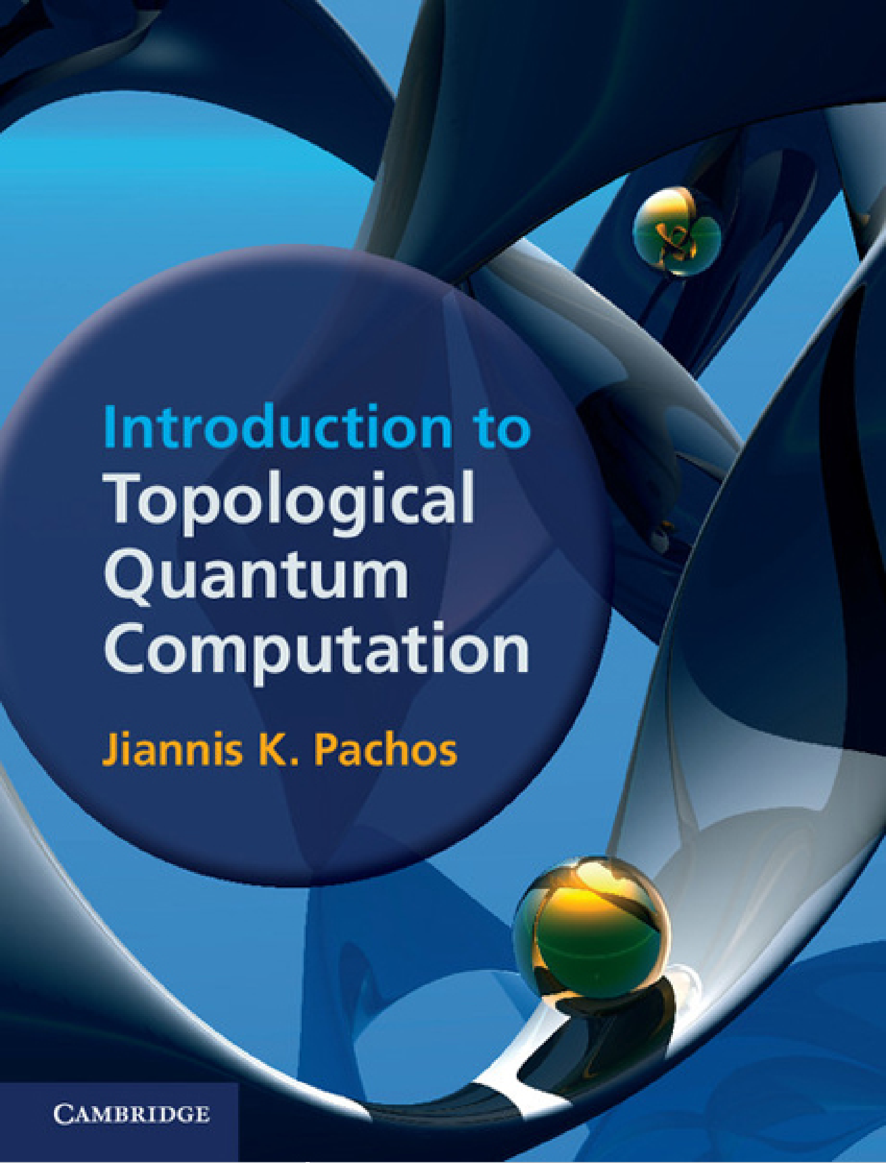
Introduction to Topological Quantum Computation

Jiannis K. Pachos



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Introduction to Topological Quantum Computation

Combining physics, mathematics and computer science, topological quantum computation is a rapidly expanding research area focused on the exploration of quantum evolutions that are immune to errors. In this book, the author presents a variety of different topics developed together for the first time, forming an excellent introduction to topological quantum computation.

The makings of topological systems, their properties and their computational power are presented in a pedagogical way. Relevant calculations are fully explained, and numerous worked examples and exercises support and aid understanding. Special emphasis is given to the motivation and physical intuition behind every mathematical concept.

Demystifying difficult topics by using accessible language, this book has broad appeal and is ideal for graduate students and researchers from various disciplines who want to get into this new and exciting research field.

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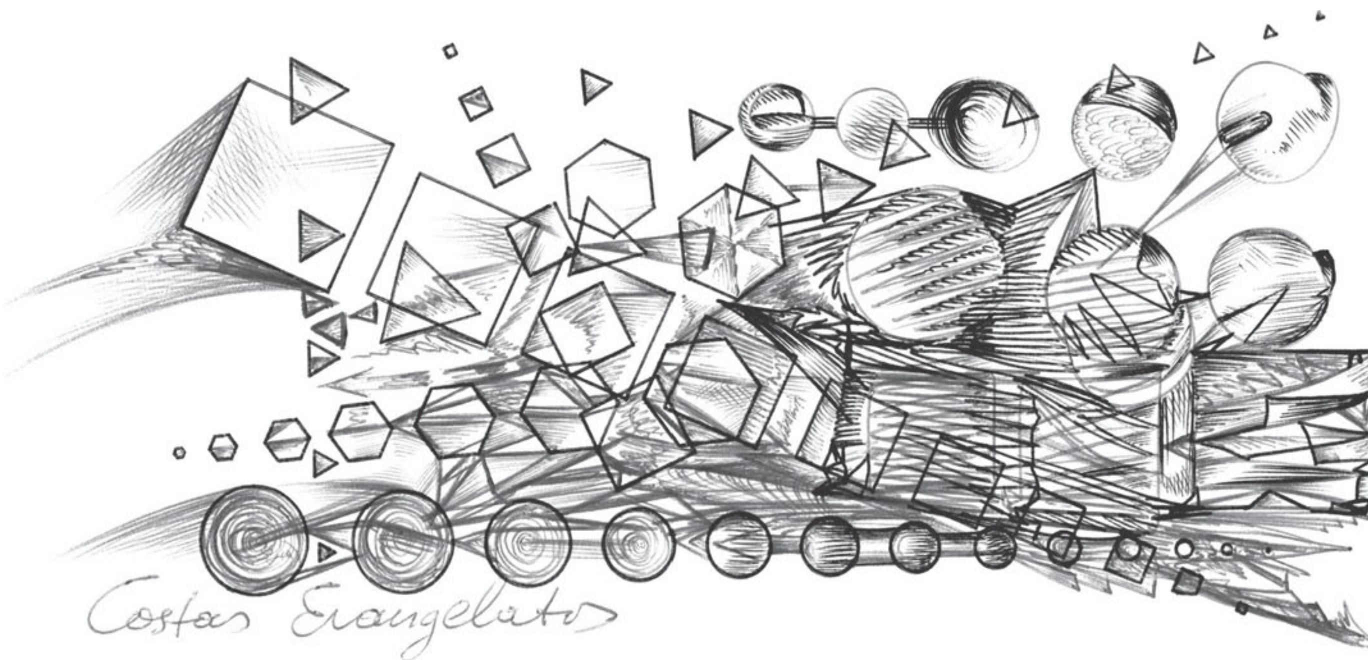
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To Almut and Sevi

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PART I

PRELIMINARIES

1 Introduction

Symmetries play a central role in physics. They dictate what one can change in a physical system without affecting any of its properties. You might have encountered symmetries like translational symmetry, where a system remains unchanged if it is spatially translated by an arbitrary distance. A system with rotational symmetry, however, is invariant under rotations. Some symmetries, like the ones mentioned above, give information about the structure of the system. Others have to do with the more fundamental physical framework that we adopt. An example for this is the invariance under Lorentz transformations in relativistic physics.

Other types of symmetries can be even more subtle. For example, it is rather self-evident that physics should remain unchanged if we exchange two identical point-like particles. Nevertheless, this fundamental property that we call statistical symmetry gives rise to rich and beautiful physics. In three spatial dimensions it dictates the existence of bosons and fermions. These are particles with very different quantum mechanical properties. Their wave function acquires a $+1$ or a -1 phase, respectively, whenever two particles are exchanged. A direct consequence of this is that bosons can actually occupy the same state. In contrast, fermions can only be stacked together with each particle occupying a different state.

When one considers two spatial dimensions, a wide variety of statistical behaviours is possible. Apart from bosonic and fermionic behaviours, arbitrary phase factors, or even non-trivial unitary evolutions, can be obtained when two particles are exchanged (Leinaas and Myrheim, 1977). Particles with such exotic statistics have been named anyons by Frank Wilczek (1982). The transformation of the anyonic wave function is consistent with the exchange symmetry. Indeed, similarly to the fermionic case, the anyonic exchange transformations are not detectable by local measurements on the particles. This 'indirect' nature of the statistical transformations of anyons is at the core of their intellectual appeal. It also provides the technological advantage of anyonic systems in performing quantum computation that is protected from a malicious environment.

1.1 Particle exchange and quantum physics

Statistics, as arising from indistinguishability of particles, is a quantum mechanical property. Classical particles are always distinguishable as we can keep track of their position at all times. Quantum mechanically, the position of a particle is determined via a spatially extended wave function. The wave functions of two particles might overlap even if they are not peaked at exactly the same position. Hence the position is, in general, not a good property for identifying particles, thereby making it impossible to define distinguishability in a fundamental way. This suggests adopting a common wave function to describe the system of the two particles.

Indistinguishable particles in quantum mechanics should have all their intrinsic properties, such as mass, charge, spin and any other quantum number, exactly the same. This seemingly innocent property has far-reaching consequences. It allows us to construct universal theories to describe elementary particles based on simple statistical rules. More dramatically, it forces us to adopt the new framework of statistical physics that abandons the distinguishability of particles.

Exchange statistics describe the change in the wave function of two identical particles, when they are exchanged. Its properties need to be compatible with the symmetry imposed by indistinguishability. As an important consequence, these changes are independent of many details of the system. Consider, for example, the case where the exchange is not a mathematical procedure, but a physical process of moving two particles along an exchange path. The effect of this transport on the wave function should not depend on the particular shape of the path taken by the particles when they are exchanged or the speed the path is traversed. Nevertheless, the evolution might still depend on some global, topological characteristics of the path, such as the number of times the particles are exchanged. Statistical evolutions are hence topological in their nature.

In three spatial dimensions the indistinguishability of particles allows for the possibility of having bosons and fermions. Bosons satisfy the Bose–Einstein distribution (Bose, 1924; Einstein, 1924) and fermions the Fermi–Dirac distribution (Fermi, 1926; Dirac, 1926). These distributions emerge from the general requirement that an ensemble of indistinguishable particles is described either by completely symmetric or completely antisymmetric wave functions with respect to particle exchanges. The first case corresponds to bosons and the second to fermions. In particular, when two fermions are positioned on top of each other their state should be simultaneously symmetric and antisymmetric, giving zero as the only possible solution. This gives rise to the Pauli exclusion principle that assigns zero probability to such configurations. However, there is no such restriction for the case of bosons which can freely occupy the same position.

Another surprising consequence of indistinguishability is the relation between spin and statistics. Pauli (1940) proved that bosons have integer spin and fermions half-integer spin. This is a rather surprising relation as spin is an intrinsic property that can be determined by considering an isolated particle. Contrarily, to determine the statistics we need to consider an ensemble of at least two particles. We shall visit this relation again later on and we shall generalise it to the case of anyons where exotic statistical behaviours give rise to equally exotic values of spins.

1.2 Anyons and topological systems

Statistics is spectacularly manifested in two-dimensional systems. There, exotic wave functions of particles can be realised that give rise to anyons. The study of anyons started as a theoretical curiosity in two-dimensional models (Wilczek, 1982). However, it was soon realised that they can be encountered in physical systems with effective two-dimensional behaviour. For example, gases of electrons confined on thin films in the presence of sufficiently strong magnetic field and at a sufficiently low temperature give rise to the fractional quantum Hall effect (Camino *et al.*, 2005; Laughlin, 1983; Tsui *et al.*, 1982). The low-energy excitations of these systems are localised quasiparticle excitations that exhibit anyonic statistics. Beyond the fractional quantum Hall effect, other two-dimensional systems have emerged which theoretically support anyons (Volovik, 2003). These range from superconductors (Chamon *et al.*, 2001) and topological insulators (Hasan and Kane, 2010) to spin lattice models.

Systems that support anyons are called topological as they inherit the topological properties of the anyonic statistical evolutions. Topological systems are usually many-particle systems that support localised excitations, so-called quasiparticles, that can exhibit anyonic behaviour. In general, they have highly entangled degenerate ground states. As a consequence local order parameters, such as the magnetisation, are not able to describe topological phases. So we need to employ non-local order parameters. Various characteristics exist that identify topological order, such as ground state degeneracy, edge states in the presence of a gapped bulk, topological entanglement entropy or the explicit detection of anyons. As topological order comes in various forms, the study and characterisation of topological systems in their generality is complex and still an open problem. Over the last years the richness in the behaviour of two-dimensional topological systems has inspired many scientists. One of the most thought-provoking ideas is to use topological systems for quantum computation.

1.3 Quantum computation with anyons

In the last decades progress in physics and the understanding of nature has advanced the way we perceive information. Quantum physics has opened the possibility of yet another way of storing, manipulating and transmitting information. Importantly, quantum computers have been proposed with the ability to outperform their classical counterparts, thereby promising far-reaching consequences. Quantum computation requires the encoding of quantum information and its efficient manipulation with quantum gates (Nielsen and Chuang, 2000). Qubits, the quantum version of classical bits, provide an elementary encoding space. Quantum gates manipulate the qubits to eventually perform a computation. A universal quantum computer employs a sufficiently large set of gates in order to perform arbitrary quantum algorithms. In recent years, there have been two main quests for quantum computation. First, to find new algorithms, that go beyond the already discovered algorithms of searching (Grover, 1996) and factorising (Shor, 1997). Second, to perform quantum computation that is resilient to errors.

In the 1990s a surprising connection was made. It was argued by Castagnoli and Rasetti (1993) that anyons could be employed to perform quantum computation. Kitaev (2003) demonstrated that anyons could actually be used to perform fault-tolerant quantum computation. This was a very welcome advance as errors infest any physical realisation of quantum computation, coming from the environment or from control imperfections. Shor (1995) and Steane (1996) independently demonstrated that for sufficiently isolated quantum systems and for sufficiently precise quantum gates, quantum error correction can allow fault-tolerant computation. However, the required thresholds are too stringent and demand a large overhead in qubits and quantum gates for error correction to be realised. In contrast to this, anyonic quantum computation promises to resolve the problem of errors from the hardware level.

Topological systems can serve as quantum memories or as quantum computers. They can encode information in a way that is protected from environmental perturbations. In fact, topological systems have already proven to be a serious candidate for constructing fault-tolerant quantum hard disks. The intertwining of anyons and quantum information in topological systems is performed in an unusual way. Information is encoded in the possible outcomes when bringing two anyons together. This information is not accessible when the anyons are kept apart, and hence it is protected. The exchange of anyons gives rise to statistical logical gates. In this way anyons can manipulate information with very accurate quantum gates, while keeping the information hidden at all times. If the statistical evolutions are complex enough then they can realise arbitrary quantum algorithms. Fundamental properties of anyonic quasiparticles can thus become the means to perform quantum computation. Fault-tolerance simply stems from the ability to keep these quasiparticles intact. The result is a surprisingly effective and aesthetically appealing method for performing fault-tolerant quantum computation.

1.4 Abelian and non-Abelian anyonic statistics

It is commonly accepted that in three spatial dimensions indistinguishable particles, elementary or not, come in two species: bosons or fermions. The possibility for these statistical behaviours can be obtained from a simple thought experiment. Consider two identical particles in three dimensions, where one of them circulates the other via the path C_1, as shown in Figure 1.1(a). As we are only interested in the statistical behaviour of these particles, we focus on the topological characteristics of this process. These characteristics should be independent of details such as the particular geometry of the path or direct interactions between the particles. Hence, we can continuously deform the path C_1 to the path C_2. This involves only local deformations of the evolution without cutting or otherwise drastically changing the nature of the path. In its turn, path C_2 can be continuously deformed to a trivial path, C_0, that keeps the particle at its initial position at all times. As a consequence, the wave function, $\Psi(C_1)$, of the system after the circulation has to be exactly the same as the original one $\Psi(C_0)$, i.e.,

$$\Psi(C_1) = \Psi(C_2) = \Psi(C_0). \tag{1.1}$$

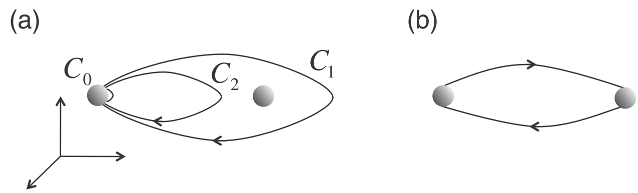

Fig. 1.1 (a) A particle spans a loop around another one. In three dimensions it is always possible to continuously deform the path C_1 to the path C_2, which is equivalent to a trivial path, C_0. (b) Two successive exchanges between two particles are equivalent to a circulation of one particle around the other and a translation.

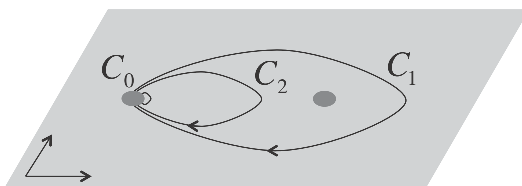

Fig. 1.2 In two dimensions the two paths C_1 and C_2 are topologically distinct. This gives the possibility of having non-trivial phase factors appearing when one particle circulates around the other. In other words, one can assign a non-trivial unitary to the evolution corresponding to path C_1.

Figure 1.1(b) depicts a single exchange of two particles. If we perform two of these exchanges in succession then we obtain a full circulation of one particle around the other accompanied by an irrelevant spatial translation. Thus, a single exchange can result in a phase factor $e^{i\varphi}$ that has to square to unity in order to be consistent with (1.1). This has two solutions, $\varphi = 0$ and $\varphi = \pi$, corresponding to the bosonic and fermionic statistics, respectively. These are the only statistical behaviours that can exist in three spatial dimensions.

When we restrict ourselves to two spatial dimensions, then we are faced with a wealth of possible statistical evolutions. If the particle circulation C_1 is performed on a plane, as shown in Figure 1.2, then it is not possible to continuously deform it to the path C_2, as we do not have access to an extra dimension to lift the loop and undo the linking. To do that would necessitate cutting the path, passing it over the circulated particle and glueing it again, thus changing in-between its topological characteristics. Still, the evolution that corresponds to C_2 is equivalent to the trivial evolution. As we are not able to deform the evolution of path C_1 to the trivial one, the argument we employed in the three-dimensional case does not apply any more. Actually, now, it is possible to assign an arbitrary phase factor, or even a whole unitary matrix, to the evolution corresponding to C_1 that is equivalent to two successive exchanges. Thus, particles in two dimensions can have rich statistical behaviours.

We would like now to analyse the difference between phase factors and unitary matrices as statistical evolutions. In the former case the anyons are known as Abelian, and the statistical phase factor can take any value between the bosonic case of $e^{i\varphi_\text{b}} = 1$ and the fermionic case of $e^{i\varphi_\text{f}} = -1$. In fact, it is the possibility of these particles having any statistics that led to the name 'anyon' (Wilczek, 1982). Nevertheless, the statistics of a specific anyon type is always well defined and a given pair will always yield the same statistical phase. This phase therefore characterises the species of the exchanged anyons.

Beyond a phase factor it is possible to have a statistical evolution that is more complex. Certain species of anyons called non-Abelian give rise to an exchange evolution that can actually lead to a higher-dimensional unitary matrix. In contrast to phase factors, matrices do not in general commute, which motivates the name ‘non-Abelian’. For a matrix statistical evolution to emerge, the wave function that describes the particles needs to be part of a degenerate subspace of states. The particle exchange then transforms between states in this subspace without changing the energy of the system. Nevertheless, there is an important constraint we need to impose on the statistical evolution in order to be in agreement with the exchange symmetry. To preserve the physics when two identical non-Abelian anyons are exchanged, we require that the degenerate states should be non-distinguishable if one looks at each anyon individually. As a result, interchanging these anyons causes a transformation within this state subspace that is not detectable by local measurements, giving a valid statistical transformation. One would need to perform non-local operations, like bringing these anyons close together, in order to distinguish between these states and observe the effect of statistics. It is rather surprising that consistent particle theories exist that have such exotic behaviours as the non-Abelian statistics. Before characterising these theories we shall first investigate the physical principles that allow this behaviour to emerge.

1.5 What are anyonic systems?

The study of anyons becomes even more exciting when the possibility arises to realise them in the laboratory. To date it is believed that Abelian anyons have already been detected in the laboratory (Camino *et al*., 2005), and there is strong evidence for the existence of non-Abelian anyons (Willett *et al*., 2009). But how is it possible to construct a purely two-dimensional world, where the exotic properties of anyons can emerge? In order to determine how plausible this is, we need to identify the main characteristics of anyons. Only then can we decide whether we can physically realise topological systems that can support anyons.

1.5.1 Two-dimensional wave functions and quasiparticles

Admittedly, our physical world appears to be three- and not two-dimensional. This is also well manifested in the statistical properties of the elementary particles accounted for in nature, bosons and fermions. The natural question then arises: how is it possible to obtain a two-dimensional world where anyons can emerge? Even if we make a system arbitrarily thin, it is impossible to trick nature into believing that it is actually reduced to two dimensions. To the rescue comes quantum mechanics. It is possible to construct a quantum system with a wave function that splits, via the separation of variables method, to a purely two-dimensional and a one-dimensional part. Let us analyse this in more detail.

To determine the behaviour of a particle in three spatial dimensions with position $\mathbf{r} = (x, y, z)$, subject to a potential of the form

$$V(\mathbf{r}) = V_{xy}(x, y) + V_z(z), \tag{1.2}$$

we can employ the separation of variables method. In this case the wave function can be written as

$$\Psi(\mathbf{r}) = \Psi_{xy}(x, y)\Psi_z(z), \tag{1.3}$$

where $\Psi_{xy}(x, y)$ satisfies the two-dimensional Schrödinger equation subject to the potential $V_{xy}(x, y)$ and $\Psi_z(z)$ satisfies a one-dimensional Schrödinger equation subject to $V_z(z)$. Hence, the wave function $\Psi_{xy}(x, y)$ is purely two-dimensional with its dynamics decoupled from the third direction z.

Consider now the system being homogeneously confined along the z direction. The low energy levels corresponding to this trapping are discrete. For strongly confining potentials the typical energy splitting, ΔE, between these levels is large. Let us take the particle to be initially prepared in the ground state. If the particle is subject to additional dynamics like a perturbation, beyond the trapping potential, with a scale much smaller than ΔE, then the particle will remain in the same energy level. This is an important mechanism for reducing the dimensionality of the system from three to two by suppressing the motion in the third direction. We also demand the presence of an energy gap that separates $\Psi_{xy}(x, y)$ from the two-dimensional excited states. This gap protects the characteristics of $\Psi_{xy}(x, y)$ against external perturbations. Under these conditions the behaviour of the system is essentially given by the two-dimensional wave function $\Psi_{xy}(x, y)$.

It is important to notice that the finite energy scales that either isolate the anyonic behaviour of the reduced state $\Psi_{xy}(x, y)$ from spurious excitations or suppress the motion in the third direction are the Achilles' heel that makes anyonic systems fragile. Indeed, when perturbations or temperature are strong enough compared to these energy scales then either the anyonic characteristics are washed out or the state of the system stops being two-dimensional. Hence, we need to keep track of such spurious effects in order to ensure reliable anyonic behaviour. Needless to say, if we had a truly two-dimensional system then anyons would be fundamental particles and they would be robust even at much higher energies. This sensitivity of effective anyonic models is a main challenge for topological quantum computation.

The particles that are subject to the above conditions do not actually see only two dimensions, but their wave function becomes effectively two-dimensional. Hence, we cannot expect the constituent particles to automatically acquire anyonic properties. Nevertheless, we could expect that effective particles, so-called quasiparticles, emerge from the properties of many-particle wave functions that are truly two-dimensional. In Figure 1.3 a many-particle system is shown and the possible emergence of quasiparticles is described.

Quasiparticles are entities defined through the wave function of a many-particle system. They behave like particles, i.e., they have local properties and they respond to their local