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A Mathematical Introduction to String Theory

Variational problems, geometric and probabilistic methods

Sergio Albeverio, Jürgen Jost,
Sylvie Paycha & Sergio Scarlatti

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Preface

This book is intended as an introduction to certain global analytic and probabilistic aspects of string theory. Nowadays string theory is a domain where mathematics and physics meet, and proceed together concerning certain aspects. However, the theory itself is far from being complete, in fact it is suspended between purely heuristic Ansätze with little hope of mathematical justification and very advanced mathematical ideas. Our aim has been to bring together as far as presently possible the differential-geometric aspects (related to theory of harmonic maps, infinite dimensional differential geometry, Riemann surfaces) and the measure theoretical and probabilistic aspects one encounters when trying to give a sense to the heuristic "Feynman path integrals", so often used not only by physicists but also by mathematicians "to get started".

One of us (J. Jost) worked out a theory of strings with boundary as a quantization of Plateau's problem for minimal surfaces and lectured at several conferences on the geometric aspects of the theory. Two of us, S. Paycha and S. Scarlatti, have been working on relating these aspects with probabilistic ones, in connection with Ph.D. theses in Bochum/Paris and Rome respectively, under the direction of S. Albeverio [Pa1], [Sc]. The probabilistic aspects are connected with the study of a mass zero Høegh-Krohn model, and the first basic study of these aspects was undertaken by S. Albeverio, S. Paycha and S. Scarlatti in collaboration with the late R. Høegh-Krohn.

It was then natural to join efforts and to produce a book which unifies the approaches. We hope our endeavour will be appreciated by the reader. We stress once more that the book presents only a small portion of all aspects of string theory – but we have strived to present this portion as much as possible as a coherent mathematical theory.

Bochum, December 1994

Dedication

This book is dedicated to the dear memory of Raphael Høegh-Krohn (1938–1988). He was a great mathematician and a natural philosopher who on so many occasions was able to show us the correct way, foreseeing so many of the new developments.

I.0 Introduction

In recent years string theory has attracted great interest in physics and mathematics and has become one of the main sources of mutual stimulation and cross fertilization in these areas. Many reasons for this fact can be mentioned:
(a) Classical string theory is concerned with the propagation of classical 1-dimensional curves, "strings", open or closed, in some ambient space, usually $\mathbb{R}^d$, under a dynamics which is relativistic and given by a variational principle; see e.g. [GGRT], [N], [Go], [Gr].
This gives interesting connections with the classical calculus of variations, the theory of minimal surfaces (Plateau problem) and the theory of harmonic maps, as developed e.g. in [J2] and [JS].

To explain the basic idea, let us first consider a relativistic point particle of mass m moving freely (i.e. without any acting forces) in 4-dimensional Minkowski space time $M := M^4$. Its trajectory γ_c is a critical point of the action functional

$$A(\gamma) = -m \int_S ds(\gamma) \tag{0.1}$$

where ds is the line element, $m > 0$ is a constant and S is the parameter space (e.g. $S = [0, t]$).

Observing that $A(\gamma)$ is invariant under reparametrizations, it is natural to seek by analogy a classical dynamics for a relativistic string moving in M by looking for critical points of an action functional of the form

$$A(X) = -C \int_S \sqrt{\det \gamma_X}\, d\eta \tag{0.2}$$

where S is the 2-dimensional parameter space for the string, taken to be a 2-dimensional surface embedded in M through $X : S \mapsto M$, $\mu \mapsto X^\mu(\eta)$ $\mu = 1, \ldots, 4$, $\sqrt{\det \gamma_X}\, d\eta$ being the infinitesimal area element of the string as embedded in M, and X being the embedding map from S into M. $\det \gamma_X$ is the determinant of the matrix

$$\frac{\partial X^\mu}{\partial \eta^\alpha} \frac{\partial X^\nu}{\partial \eta \beta} \mu_{\mu\nu},$$

$\mu_{\mu\nu}$, $\mu, \nu = 1, \ldots, 4$ being the Minkowski metric on M (with signature $(-1, +1, +1, +1)$), and $\eta^\alpha, \alpha = 1, 2$ a parametrization of S. C is a positive constant.

In fact an extension of great relevance in string theory is the one where M is replaced by a d-dimensional Minkowski space or rather its Euclidean version. Interesting connections have been found with problems in algebraic geometry, the theory of Riemann surfaces and number theory; see e.g. [Ma1,2], [Sm], [MoP].

(b) The quantization of string theory gives rise to problems in different areas, according to the method used. E.g. in the case $d = 26$ (resp. $d = 10$ for "fermionic strings") the representation theory of certain infinite-dimensional Lie algebras, Kac–Moody and Virasoro algebras has been used for quantization. This also yields very interesting connections with 2-dimensional conformal fields (and conformal models of statistical mechanics); see e.g. [Ca], [Ka], [Mi], [AHKMTT]. Another approach to quantization of strings, which also works for $d < 26$ (resp. $d < 10$), has been provided by Polyakov ([P1]), who used a heuristic functional integration method, starting from a heuristic Gibbs-like measure †). Since in this monograph we shall basically follow Polyakov's approach, trying to give mathematical meaning to its geometric, functional-analytic and probabilistic aspects, let us describe it shortly at this point.

The basic idea actually goes back to Deser–Zumino and Brink–Di Vecchia–Howe and consists in replacing the relativistic action (0.2) by a relativistic action of the form

$$A(X,g) = -C \int_S \sqrt{\det g}\, g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu \; d\eta \tag{0.3}$$

with g a pseudo-Riemannian locally Minkowski metric on S. Heuristically the critical points of (0.3), in a suitable variational principle, jointly in g and X, are seen to yield the same dynamics as the one obtained from (0.2); see e.g. [GrSW], [BrH].

Polyakov's idea consists in replacing the relativistic action (0.3) by the corresponding Euclidean action $A_E(X,g)$, defined in the same way as $A(X,g)$ but with d-dimensional Minkowski space M^d replaced by Euclidean space $\mathbb{R}^d$, and the pseudo-Riemannian metric g by a Riemannian metric, also denoted by g, on S. Let us remark that a similar substitution of relativistic actions by "Euclidean" actions was actually familiar through the Euclidean approach to quantum field theory (see e.g. [S1], [GJ], [AFHKL]) which had permitted a successful use of methods of statistical mechanics. The recovery of the relativistic quantities is then, with strings as well as with quantum fields, to be undertaken at the end, by a suitable "analytic continuation" (exploiting the fact that the usual Euclidean metric in $\mathbb{R}^d$ is obtained from the Minkowski metric of signature $(-1,+1,+1,+1,\ldots+1)$ by "analytic continuation" of the first coordination x_0, the time, to "imaginary time" ix_0).

The critical points of A_E are then harmonic maps, and A_E can be looked upon as the classical action of a Euclidean "σ-model" (describing fields X

†) In this book we shall use equivalently the adjectives "heuristic" and "formal" for objects which are (or are not yet) defined in rigorous mathematical terms.

on S with values in M^d). Polyakov's approach to quantization of strings is by formulating interesting quantities associated with the strings essentially as integrals with respect to a heuristic probability measure of the form

$$Z^{-1}\exp[-A_E(X,g)]d_g[X]d[g], \tag{0.4}$$

with Z a normalization constant and $d_g[X], d[g]$ heuristic "volume measures" on the spaces of all embeddings $X = (X^\mu, \mu = 1,\dots,d)$ resp. the space of all Riemannian metrics g.

Polyakov's idea was then to exploit a heuristic (infinite-dimensional) differential geometry to "change integration variables" (X,g) to (X,f,φ,t), with f in the space of diffeomorphisms of S and φ in the space of smooth real functions over S (describing "conformal changes of metrics"), and t in the Teichmüller space of S.

The mathematical description of this change of variables in fact involves a series of mathematical problems which connect with complex analysis and certain parts of infinite-dimensional differential geometry. It is one of the aims of this monograph to present this mathematical description and the tools required to perform it. These involve, besides geometric, complex-analytic and global-analytic methods, the control of certain infinite-dimensional determinants.

After these transformations, following Polyakov, the integration with respect to Polyakov's measure (0.4) is heuristically reduced, for $d \le 26$, after adding a renormalization term to the action $A_E(X,g)$ (which corresponds to taking an ε-regularization $A_E^\varepsilon(X,g)$ of it, adding a term $\mu_\varepsilon^2 \int_S \sqrt{\det g}\, d\eta$ and taking the limit $\varepsilon \downarrow 0$, $\mu_\varepsilon^2 \to \infty$), to an integration with respect to a probability measure, a "Liouville measure", associated with a certain 2-dimensional Euclidean quantum field model (Høegh-Krohn's model with mass zero or Liouville's model) and a finite-dimensional measure (on the moduli space for S). This heuristic reduction is described in [P1] and various subsequent publications, e.g. [A], [AN], [D], [DHP1,2], [DNOP], [F], [Gi], [Hab], [Ja], [MN], [Pol,2], [W]. For $d < 26$, the presence of the Liouville measure is essential.
A mathematical description of this measure has been given in [AHKPS1,2,3], adapting methods used before in the massive Høegh-Krohn's model in [AHK1,3] (see also e.g. [AFHKL]). The application of this well-defined measure to Polyakov's heuristic formulation of string theory reduces the range of the embedding dimension to $d \le 13$ (resp. $d \le 19$, when using newer results by Kusuoka [Ku]).

Heuristically for the case $d = 26$, called "critical dimension" in the physical literature, the integration with respect to the heuristic Liouville measure is absent and the whole heuristic Polyakov measure (0.4) reduces to a finite-dimensional measure.

In the present monograph we shall give a mathematical treatment of the cases $d \leq 13$ and $d = 26$, presenting all mathematical instruments which are necessary for this study. We shall discuss the case of open as well as closed strings.

Before we start on a more detailed description of the contents of our work, let us shortly mention some topics which we unfortunately had to leave out of our discussion. We discuss only the case of Riemann surfaces S of a fixed genus: we do not sum over the genus (for heuristic discussions of such sums see e.g. [P3]). Also we only mention shortly amplitudes (see e.g. [BrH], [Kn] for heuristic discussion of these quantities); results concerning the Polyakov path integral over bordered surfaces which was investigated in [A], [Jas3,4], will merely be quoted. We also do not enter the discussion of either string field theory or of fermionic and supersymmetric strings (for these cases see e.g. the discussions in [AGGSIS], [BDVH], [P2], [GrSW]). Finally we do not treat such topics as dimensional reduction, connection with infinite dimensional Lie algebras, light cone quantization, statistical physics, conformal field theory and loop spaces. For these topics we can only refer to the current books or specialized work, e.g. [AHKMTT], [ID], [JM], [Ka], [Le], [Mi], [P3], [PS].

Let us now come back to the description of the content of our work. Chapter I presents all mathematical tools needed for the mathematical description of quantized strings to be achieved in chapter II. In section I.1 we briefly discuss the 2-dimensional variational problem, the Plateau problem, associated with the action functional A_E. In section I.2 we introduce the basic topological and metric structures on the space of mappings and metrics. In section I.3 we discuss harmonic maps and global structures on 2-dimensional differentiable manifolds with boundary. In section I.4 we discuss the Cauchy–Riemann operator on Riemann surfaces.

Section I.5 is devoted to a detailed discussion of zeta-functions and heat-kernel determinants of operators. This discussion is useful for handling various infinite-dimensional determinants which arise in the mathematical implementation of Polyakov's reduction procedure, which is the topic of chapter II of our work. We show in particular that in our case the zeta-function and the heat-kernel definitions of determinants are identical up to some constant.

In section I.6 we give a mathematical description of the so called "Faddeev–Popov-procedure" to "factor out" a certain volume term related to the "gauge group" from heuristic functional integrals constructed with actions having large invariance groups. The mathematical definition we give exploits the techniques of regularized determinants discussed in section I.5.

In section I.7 we show how the notion of regularized determinants for elliptic operators on manifolds arises naturally when equipping determinant bundles with a Hermitian metric. This relates in particular to the work of Quillen and of [BF], [F1,2], which also influences section I.8, where we discuss Chern classes of determinant bundles.

Sections I.9 and I.10 are concerned with a completely different topic, namely probabilistic tools needed for the description of the "quantized Liouville model" on a compact surface. This model describes a two-dimensional quantum field, which arises naturally in the Polyakov reduction studied in the second part of our work (chapter II). In section I.9 we present the Gaussian measures and associated random fields which are useful in handling the free Markov field on a Riemann surface. In section I.10 the construction of the quantized Liouville model (mass zero Høegh-Krohn model) on a Riemann surface is presented.
The first chapter of the book ends with the description of the small time asymptotics. The first part of the book ends with the description of the small time asymptotics for heat-kernel regularized determinants, such as those which enter the "Polyakov reductions" to be discussed in chapter II. All mathematical tools having been presented in chapter I, chapter II concentrates on the mathematical discussion of quantized strings, relating also to heuristic procedures by Polyakov and others.

In section II.1 we describe in general terms the procedures of quantization by functional integrals. The heuristic Polyakov measure and its connections with mathematical objects discussed in chapter I are presented in section II.2. In section II.3 formal Lebesgue measures on Hilbert spaces are introduced and related in section II.4 to integration theory with respect to Gaussian measures on the space of embeddings. The Faddeev–Popov procedure for bosonic strings is explained in section II.5, as an application of the general discussion of section I.6. The connection between the Polyakov measure in noncritical dimension $d < 26$ and the Liouville model discussed in section I.10 is made in section II.6, whereas section II.7 discusses the Polyakov measure in the critical dimension $d = 26$. We conclude with a short discussion of amplitudes, in section II.8.
Clearly much more remains to be done and we hope that our mathematical presentation of string theory will encourage new research and new cross fertilizations between mathematics and physics.