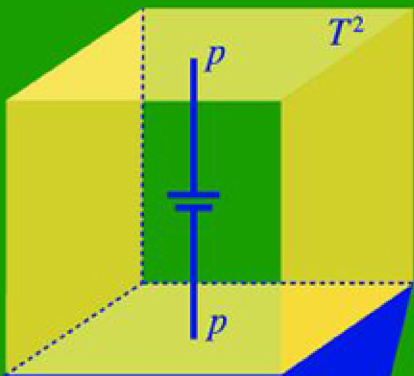
# THE GEOMETRY OF PHYSICS

## AN INTRODUCTION




THEODORE FRANKEL

**THIRD EDITION**

CAMBRIDGE

# The Geometry of Physics

This book is intended to provide a working knowledge of those parts of exterior differential forms, differential geometry, algebraic and differential topology, Lie groups, vector bundles, and Chern forms that are essential for a deeper understanding of both classical and modern physics and engineering. Included are discussions of analytical and fluid dynamics, electromagnetism (in flat and curved space), thermodynamics, elasticity theory, the geometry and topology of Kirchhoff's electric circuit laws, soap films, special and general relativity, the Dirac operator and spinors, and gauge fields, including Yang–Mills, the Aharonov–Bohm effect, Berry phase, and instanton winding numbers, quarks, and the quark model for mesons. Before a discussion of abstract notions of differential geometry, geometric intuition is developed through a rather extensive introduction to the study of surfaces in ordinary space; consequently, the book should be of interest also to mathematics students.

This book will be useful to graduate and advance undergraduate students of physics, engineering, and mathematics. It can be used as a course text or for self-study.

This Third Edition includes a new overview of Cartan's exterior differential forms. It previews many of the geometric concepts developed in the text and illustrates their applications to a single extended problem in engineering; namely, the Cauchy stresses created by a small twist of an elastic cylindrical rod about its axis.

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# The Geometry of Physics

## An Introduction

*Third Edition*

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*For*
*Thom-kat, Mont, Dave*
*and*
*Jonnie*

*and*

*In fond memory of*
*Raoul Bott*
*1923–2005*

*Photograph of Raoul by Montgomery Frankel*

# Contents

# I Manifolds, Tensors, and Exterior Forms

# Preface to the Third Edition

A main addition introduced in this third edition is the inclusion of an Overview

**An Informal Overview of Cartan's Exterior Differential Forms, Illustrated with an Application to Cauchy's Stress Tensor**

which can be read before starting the text. This appears at the beginning of the text, before Chapter 1. The only prerequisites for reading this overview are sophomore courses in calculus and basic linear algebra. Many of the geometric concepts developed in the text are previewed here and these are illustrated by their applications to a single extended problem in engineering, namely the study of the Cauchy stresses created by a small twist of an elastic cylindrical rod about its axis.

The new shortened version of Appendix A, dealing with elasticity, requires the discussion of Cauchy stresses dealt with in the Overview. The author believes that the use of Cartan's vector valued exterior forms in elasticity is more suitable (both in principle and in computations) than the classical tensor analysis usually employed in engineering (which is also developed in the text.)

The new version of Appendix A also contains contributions by my engineering colleague Professor Hidenori Murakami, including his treatment of the Truesdell stress rate. I am also very grateful to Professor Murakami for many very helpful conversations.

# Preface to the Second Edition

This second edition differs mainly in the addition of three new appendices: C, D, and E. Appendices C and D are applications of the elements of representation theory of compact Lie groups.

Appendix C deals with applications to the flavored quark model that revolutionized particle physics. We illustrate how certain observed mesons (pions, kaons, and etas) are described in terms of quarks and how one can "derive" the mass formula of Gell-Mann/Okubo of 1962. This can be read after Section 20.3b.

Appendix D is concerned with isotropic hyperelastic bodies. Here the main result has been used by engineers since the 1850s. My purpose for presenting proofs is that the hypotheses of the Frobenius–Schur theorems of group representations are exactly met here, and so this affords a compelling excuse for developing representation theory, which had not been addressed in the earlier edition. An added bonus is that the group theoretical material is applied to the three-dimensional rotation group $SO(3)$, where these generalities can be pictured explicitly. This material can essentially be read after Appendix A, but some brief excursion into Appendix C might be helpful.

Appendix E delves deeper into the geometry and topology of compact Lie groups. Bott's extension of the presentation of Morse theory that was given in Section 14.3c is sketched and the example of the topology of the Lie group $U(3)$ is worked out in some detail.

# Preface to the Revised Printing

In this reprinting I have introduced a new appendix, Appendix B, Harmonic Chains and Kirchhoff's Circuit Laws. This appendix deals with a finite-dimensional version of Hodge's theory, the subject of Chapter 14, and can be read at any time after Chapter 13. It includes a more geometrical view of cohomology, dealt with entirely by matrices and elementary linear algebra. A bonus of this viewpoint is a systematic "geometrical" description of the Kirchhoff laws and their applications to direct current circuits, first considered from roughly this viewpoint by Hermann Weyl in 1923.

I have corrected a number of errors and misprints, many of which were kindly brought to my attention by Professor Friedrich Heyl.

Finally, I would like to take this opportunity to express my great appreciation to my editor, Dr. Alan Harvey of Cambridge University Press.

# Preface to the First Edition

The basic ideas at the foundations of point and continuum mechanics, electromagnetism, thermodynamics, special and general relativity, and gauge theories are geometrical, and, I believe, should be approached, by both mathematics and physics students, from this point of view.

This is a textbook that develops some of the geometrical concepts and tools that are helpful in understanding classical and modern physics and engineering. The mathematical subject material is essentially that found in a first-year graduate course in differential geometry. This is not coincidental, for the founders of this part of geometry, among them Euler, Gauss, Jacobi, Riemann and Poincaré, were also profoundly interested in "natural philosophy."

Electromagnetism and fluid flow involve line, surface, and volume integrals. Analytical dynamics brings in multidimensional versions of these objects. In this book these topics are discussed in terms of **exterior differential forms**. One also needs to differentiate such integrals with respect to time, especially when the domains of integration are changing (circulation, vorticity, helicity, Faraday's law, etc.), and this is accomplished most naturally with aid of the **Lie derivative**. Analytical dynamics, thermodynamics, and robotics in engineering deal with **constraints**, including the puzzling nonholonomic ones, and these are dealt with here via the so-called Frobenius theorem on differential forms. All these matters, and more, are considered in Part One of this book.

Einstein created the astonishing principle **field strength = curvature** to explain the gravitational field, but if one is not familiar with the classical meaning of surface curvature in ordinary 3-space this is merely a tautology. Consequently I introduce **differential geometry** before discussing general relativity. **Cartan's** version, in terms of exterior differential forms, plays a central role. Differential geometry has applications to more down-to-earth subjects, such as soap bubbles and periodic motions of dynamical systems. Differential geometry occupies the bulk of Part Two.

Einstein's principle has been extended by physicists, and now all the field strengths occurring in elementary particle physics (which are required in order to construct a

**Lagrangian**) are discussed in terms of curvature and **connections**, but it is the curvature of a **vector bundle**, that is, the *field* space, that arises, not the curvature of space–time. The symmetries of the quantum field play an essential role in these **gauge theories**, as was first emphasized by Hermann Weyl, and these are understood today in terms of **Lie groups**, which are an essential ingredient of the vector bundle. Since many quantum situations (charged particles in an electromagnetic field, Aharonov–Bohm effect, Dirac monopoles, Berry phase, Yang–Mills fields, instantons, etc.) have analogues in elementary differential geometry, we can use the geometric methods and pictures of Part Two as a guide; a picture *is* worth a thousand words! These topics are discussed in Part Three.

**Topology** is playing an increasing role in physics. A physical problem is "well posed" if there *exists* a solution and it is *unique*, and the topology of the configuration (spherical, toroidal, etc.), in particular the singular **homology groups**, has an essential influence. The **Brouwer degree**, the **Hurewicz homotopy groups**, and **Morse theory** play roles not only in modern gauge theories but also, for example, in the theory of "defects" in materials.

Topological methods are playing an important role in field theory; versions of the **Atiyah–Singer index theorem** are frequently invoked. Although I do not develop this theorem in general, I *do* discuss at length the most famous and elementary example, the **Gauss–Bonnet–Poincaré** theorem, in two dimensions and also the meaning of the **Chern characteristic classes**. These matters are discussed in Parts Two and Three.

The Appendix to this book presents a nontraditional treatment of the **stress tensors** appearing in continuum mechanics, utilizing exterior forms. In this endeavor I am greatly indebted to my engineering colleague Hidenori Murakami. In particular Murakami has supplied, in Section **g** of the Appendix, some typical computations involving stresses and strains, but carried out with the machinery developed in this book. We believe that these computations indicate the efficiency of the use of forms and Lie derivatives in elasticity. The material of this Appendix could be read, except for some minor points, after Section **9.5**.

Mathematical applications to physics occur in at least two aspects. Mathematics is of course the principal tool for solving technical analytical problems, but increasingly it is also a principal guide in our understanding of the basic structure and concepts involved. Analytical computations with elliptic functions *are* important for certain technical problems in rigid body dynamics, but one could not have begun to understand the dynamics before Euler's introducing the moment of inertia tensor. I am very much concerned with the basic concepts in physics. A glance at the Contents will show in detail what mathematical and physical tools are being developed, but frequently physical applications appear also in Exercises. My main philosophy has been to attack physical topics as soon as possible, but only after effective mathematical tools have been introduced. By analogy, one *can* deal with problems of velocity and acceleration after having learned the definition of the derivative as the limit of a quotient (or even before, as in the case of Newton), but we all know how important the *machinery* of calculus (e.g., the power, product, quotient, and chain rules) is for handling specific problems. In the same way, it is a mistake to talk seriously about thermodynamics before understanding that a total differential equation in more than two dimensions need not possess an integrating factor.

In a sense this book is a "final" revision of sets of notes for a year course that I have given in La Jolla over many years. My goal has been to give the reader a *working* knowledge of the tools that are of great value in geometry and physics and (increasingly) engineering. For this it is *absolutely essential* that the reader work (or at least attempt) the Exercises. *Most of the problems are simple and require simple calculations. If you find calculations becoming unmanageable, then in all probability you are not taking advantage of the machinery developed in this book.*

This book is intended primarily for two audiences, first, the physics or engineering student, and second, the mathematics student. My classes in the past have been populated mostly by first-, second-, and third-year graduate students in physics, but there have also been mathematics students and undergraduates. The only real *mathematical* prerequisites are *basic* linear algebra and some familiarity with calculus of several variables. Most students (in the United States) have these by the beginning of the third undergraduate year.

All of the physical subjects, with two exceptions to be noted, are preceded by a brief introduction. The two exceptions are analytical dynamics and the quantum aspects of gauge theories.

Analytical (Hamiltonian) dynamics appears as a problem set in Part One, with very little motivation, for the following reason: the problems form an ideal application of exterior forms and Lie derivatives and involve no knowledge of physics. Only in Part Two, after geodesics have been discussed, do we return for a discussion of analytical dynamics from first principles. (Of course most physics and engineering students will already have seen *some* introduction to analytical mechanics in their course work anyway.) The significance of the Lagrangian (based on special relativity) is discussed in Section **16.4** of Part Three when changes in dynamics are required for discussing the effects of electromagnetism.

An introduction to quantum mechanics would have taken us too far afield. Fortunately (for me) only the simplest quantum ideas are needed for most of our discussions. I would refer the reader to Rabin's article [R] and Sudbery's book [Su] for excellent introductions to the quantum aspects involved.

Physics and engineering readers would profit *greatly* if they would form the habit of translating the vectorial and tensorial statements found in their customary reading of physics articles and books into the language developed in this book, and using the newer methods developed here in their own thinking. (By "newer" I mean methods developed over the last one hundred years!)

As for the mathematics student, I feel that this book gives an overview of a large portion of differential geometry and topology that should be helpful to the mathematics graduate student in this age of very specialized texts and absolute rigor. The student preparing to specialize, say, in differential geometry *will* need to augment this reading with a more rigorous treatment of some of the subjects than that given here (e.g., in Warner's book [Wa] or the five-volume series by Spivak [Sp]). The mathematics student should also have exercises devoted to showing what can go wrong if hypotheses are weakened. I make no pretense of worrying, for example, about the differentiability classes of mappings needed in proofs. (Such matters are studied more carefully in the book [A, M, R] and in the encyclopedia article [T, T]. This latter article (and the accompanying one by Eriksen) are also excellent for questions of historical priorities.) I hope that mathematics students will enjoy the discussions of the physical subjects even if they know very little physics; after all, physics is *the* source of interesting vector fields. Many of the "physical" applications are useful even if they are thought of as simply giving explicit examples of rather abstract concepts. For example, Dirac's equation in *curved space* can be considered as a nontrivial application of the method of connections in associated bundles!

This *is* an introduction and there is much important mathematics that is not developed here. Analytical questions involving existence theorems in partial differential equations, Sobolev spaces, and so on, are missing. Although complex manifolds are defined, there is no discussion of Kaehler manifolds nor the algebraic–geometric notions used in string theory. Infinite dimensional manifolds are not considered. On the physical side, topics are introduced usually only if I felt that geometrical ideas would be a great help in their understanding or in computations.

I have included a small list of references. Most of the articles and books listed have been referred to in this book for specific details. The reader will find that there are many good books on the subject of "geometrical physics" that are not referred to here, primarily because I felt that the development, or sophistication, or notation used was sufficiently different to lead to, perhaps, more confusion than help in the first stages of their struggle. A book that I feel is in very much the same spirit as my own is that by Nash and Sen [N, S]. The standard reference for differential geometry is the two-volume work [K, N] of Kobayashi and Nomizu.

Almost every section of this book begins with a question or a quotation which may concern anything from the main thrust of the section to some small remark that should not be overlooked.

A term being defined will usually appear in **bold type**.

I wish to express my gratitude to Harley Flanders, who introduced me long ago to exterior forms and de Rham's theorem, whose superb book [Fl] was perhaps the first to awaken scientists to the use of exterior forms in their work. I am indebted to my chemical colleague John Wheeler for conversations on thermodynamics and to Donald Fredkin for helpful criticisms of earlier versions of my lecture notes. I have already expressed my deep gratitude to Hidenori Murakami. Joel Broida made many comments on earlier versions, and also prevented my Macintosh from taking me over. I've had many helpful conversations with Bruce Driver, Jay Fillmore, and Michael Freedman. Poul Hjorth made many helpful comments on various drafts and also served as "beater," herding physics students into my course. Above all, my colleague Jeff Rabin used my notes as the text in a one-year graduate course and made many suggestions and corrections. I have also included corrections to the 1997 printing, following helpful remarks from Professor Meinhard Mayer.

Finally I am grateful to the many students in my classes on geometrical physics for their encouragement and enthusiasm in my endeavor. Of course none of the above is responsible for whatever inaccuracies undoubtedly remain.

OVERVIEW

# An Informal Overview of Cartan's Exterior Differential Forms, Illustrated with an Application to Cauchy's Stress Tensor

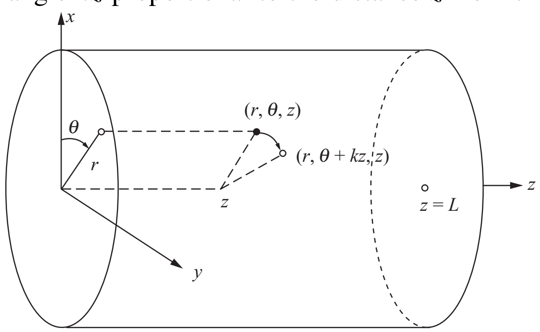
## Introduction

### 0.a. Introduction

My goal in this overview is to introduce **exterior calculus** in a *brief* and *informal* way that leads directly to their use in engineering and physics, both in basic physical concepts and in specific engineering calculations. The presentation will be very informal. Many times a proof will be omitted so that we can get quickly to a calculation. In some "proofs" we shall look only at a typical term.

The chief mathematical prerequisites for this overview are sophomore courses dealing with basic linear algebra, partial derivatives, multiple integrals, and tangent vectors to parameterized curves, but not necessarily "vector calculus," i.e., curls, divergences, line and surface integrals, Stokes' theorem, . . . . These last topics will be sketched here using Cartan's "exterior calculus."

We shall take advantage of the fact that most engineers live in euclidean 3-space $\mathbb{R}^3$ with its everyday metric structure, but we shall try to use methods that make sense in much more general situations. Instead of including exercises we shall consider, in the section **Elasticity and Stresses**, one main example and illustrate *everything* in terms of this example but hopefully the general principles will be clear. This engineering example will be the following. Take an elastic circular cylindrical rod of radius $a$ and length $L$, described in cylindrical coordinates $r$, $\theta$, $z$, with the ends of the cylinder at $z = 0$ and $z = L$. Look at this same cylinder except that it has been axially twisted through an angle $kz$ proportional to the distance $z$ from the fixed end $z = 0$.



We shall *neglect gravity* and investigate the **stresses** in the cylinder in its final twisted state, in the first approximation, i.e., where we put $k^2 = 0$. Since "stress" and "strain" are "tensors" (as Cauchy and I will show) this is classically treated via "tensor analysis." The final equilibrium state involves surface integrals and the tensor divergence of the Cauchy stress tensor. Our main tool will *not* be the usual *classical* tensor analysis (Christoffel symbols $\Gamma^i_{jk}$ ..., etc.) but rather **exterior differential forms** (first used in the nineteenth century by Grassmann, Poincaré, Volterra, ..., and developed especially by **Elie Cartan**), which, I believe, is a far more appropriate tool.

We are very much at home with cartesian coordinates but curvilinear coordinates play a very important role in physical applications, and the fact that there are *two distinct types of vectors* that arise in curvilinear coordinates (and, even more so, in *curved spaces*) that appear identical in cartesian coordinates *must* be understood, not only when making calculations but also in our understanding of the basic ingredients of the physical world. We shall let $x^i$, and $u^i$, $i = 1, 2, 3$, be **general** (curvilinear) coordinates, in euclidean 3 dimensional space $\mathbb{R}^3$. *If cartesian coordinates are wanted, I will say so explicitly.*

## Vectors, 1-Forms, and Tensors

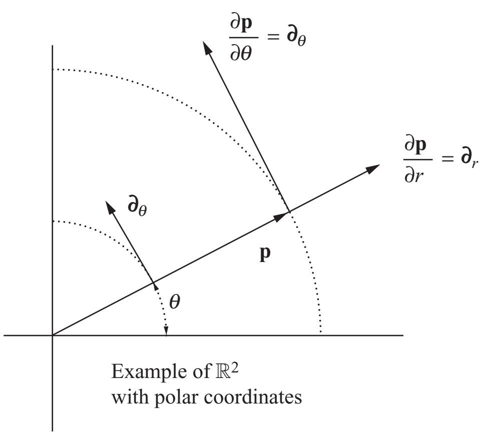
### 0.b. Two Kinds of Vectors

There are two kinds of vectors that appear in physical applications and it is important that we distinguish between them. First there is the familiar "arrow" version.

Consider $n$ dimensional euclidean space $\mathbb{R}^n$ with cartesian coordinates $x^1, \ldots, x^n$ and local (perhaps curvilinear) coordinates $u^1, \ldots, u^n$.

**Example:** $\mathbb{R}^2$ with cartesian coordinates $x^1 = x$, $x^2 = y$, and with polar coordinates $u^1 = r$, $u^2 = \theta$.

**Example:** $\mathbb{R}^3$ with cartesian coordinates $x, y, z$ and with cylindrical coordinates $R, \Theta, Z$.




Example of $\mathbb{R}^2$ with polar coordinates

Let $\mathbf{p}$ be the position vector from the origin of $\mathbb{R}^n$ to the point $p$. In the curvilinear coordinate system $u$, the coordinate curve $C_i$ through the point $p$ is the curve where all

$u^j$, $j \neq i$, are constants, and where $u^i$ is used as parameter. Then the tangent vector to this curve in $\mathbb{R}^n$ is

$$\partial \mathbf{p}/\partial u^i \quad \text{which we shall abbreviate to} \quad \boldsymbol{\partial}_i \quad \text{or} \quad \boldsymbol{\partial}/\partial u^i$$

At the point $p$ these $n$ vectors $\boldsymbol{\partial}_1, \ldots, \boldsymbol{\partial}_n$ form a **basis** for all vectors in $\mathbb{R}^n$ based at $p$. Any vector $\mathbf{v}$ at $p$ has a unique expansion with curvilinear coordinate *components* $(v^1, \ldots, v^n)$

$$\mathbf{v} = \boldsymbol{\Sigma}_i v^i \boldsymbol{\partial}_i = \boldsymbol{\Sigma}_i \boldsymbol{\partial}_i v^i$$

We prefer the last expression with the components to the *right* of the basis vectors since it is traditional to put the vectorial *components* in a *column* matrix, and we can then form the matrices

$$\boldsymbol{\partial} = (\boldsymbol{\partial}_1, \ldots, \boldsymbol{\partial}_n) \quad \text{and} \quad v = \begin{pmatrix} v^1 \\ \cdot \\ \cdot \\ \cdot \\ v^n \end{pmatrix} = (v^1 \ldots v^n)^T$$

($T$ denotes transpose) and then we can write the matrix expression (with $\mathbf{v}$ a $1 \times 1$ matrix)

$$\mathbf{v} = \boldsymbol{\partial} v \tag{0.1}$$

Please *beware* though that in $\boldsymbol{\partial}_i v^i$ or $(\boldsymbol{\partial}/\partial u^i)v^i$ or $\mathbf{v} = \boldsymbol{\partial} v$, the bold $\boldsymbol{\partial}$ does not differentiate the component term to the right; it is merely the symbol for a basis vector. Of course we can still differentiate a function $f$ along a vector $\mathbf{v}$ by *defining*

$$\mathbf{v}(f) := (\boldsymbol{\Sigma}_i \boldsymbol{\partial}_i v^i)(f) = \boldsymbol{\Sigma}_i \partial/\partial u^i (f) v^i := \boldsymbol{\Sigma}_i (\partial f/\partial u^i) v^i$$

replacing the basis vector $\boldsymbol{\partial}/\partial u^i$ with bold $\boldsymbol{\partial}$ by the partial differential operator $\partial/\partial u^i$ and then applying to the function *f*. A vector *is* a first order differential operator on functions!

In cylindrical coordinates $R$, $\Theta$, $Z$ in $\mathbb{R}^3$ we have the basis vectors $\boldsymbol{\partial}_R = \boldsymbol{\partial}/\partial R$, $\boldsymbol{\partial}_\Theta = \boldsymbol{\partial}/\partial\Theta$, and $\boldsymbol{\partial}_Z = \boldsymbol{\partial}/\partial Z$.

Let $\mathbf{v}$ be a vector at a point $p$. We can always find a curve $u^i = u^i(t)$ through $p$ whose velocity vector there is $\mathbf{v}$, $v^i = du^i/dt$. Then if $u'$ is a second coordinate system about $p$, we then have $v'^j = du'^j/dt = (\partial u'^j/\partial u^i) du^i/dt = (\partial u'^j/\partial u^i) v^i$. Thus the **components** of a vector transform under a change of coordinates by the rule

$$v'^j = \boldsymbol{\Sigma}_i (\partial u'^j/\partial u^i) v^i \quad \text{or as matrices} \quad v' = (\partial u'/\partial u) v \tag{0.2}$$

where $(\partial u'/\partial u)$ is the Jacobian matrix. This is the **transformation law** for the components of a **contravariant** vector, or **tangent** vector, or simply **vector**.

There is a second, different, type of vector. In linear algebra we learn that to each vector space $V$ (in our case the space of all vectors at a point $p$) we can associate its

**dual** vector space $V^*$ of all real **linear** functionals $\alpha : V \to \mathbb{R}$ . In coordinates, $\alpha(\mathbf{v})$ is a number

$$\alpha(\mathbf{v}) = \Sigma_i a_i v^i$$

for unique numbers $(a_i)$. We shall explain why $i$ is a *sub*script in $a_i$ shortly.

The most familiar linear functional is the **differential** of a function $df$. As a function on vectors it is defined by the derivative of $f$ along $v$

$$df(\mathbf{v}) := \mathbf{v}(f) = \Sigma_i (\partial f/\partial u^i) v^i \quad \text{and so} \quad (df)_i = \partial f/\partial u^i$$

Let us write $df$ in a much more familiar form. In elementary calculus there is mumbo-jumbo to the effect that $du^i$ is a function of pairs of points: it gives you the difference in the $u^i$ coordinates between the points, and the points do not need to be close together. What is *really* meant is

$du^i$ is the **linear functional** that reads off the $i$th component of any vector $\mathbf{v}$ with respect to the basis vectors of the coordinate system $u$

$$du^i(\mathbf{v}) = du^i(\Sigma_j \boldsymbol{\partial}_j v^j) := v^i$$

Note that this agrees with $du^i(\mathbf{v}) = \mathbf{v}(u^i)$ since $\mathbf{v}(u^i) = (\Sigma_j \boldsymbol{\partial}_j v^j)(u^i) = \Sigma_j (\partial u^i/\partial u^j) v^j = \Sigma_j \delta^i_j v^j = v^i$.

Then we can write

$$df(\mathbf{v}) = \Sigma_i (\partial f/\partial u^i) v^i = \Sigma_i (\partial f/\partial u^i) du^i(\mathbf{v})$$

i.e.,

$$df = \Sigma_i (\partial f/\partial u^i) du^i$$

as usual, except that now both sides have meaning as linear functionals on vectors.

**Warning:** We shall see that this is *not* the gradient vector of $f$!

It is very easy to see that $du^1, \ldots, du^n$ form a basis for the space of linear functionals at each point of the coordinate system $u$, since they are linearly independent. In fact, this basis of $V^*$ is the **dual basis** to the basis $\boldsymbol{\partial}_1, \ldots, \boldsymbol{\partial}_n$, meaning

$$du^i(\boldsymbol{\partial}_j) = \delta^i_j$$

Thus in the coordinate system $u$, every linear functional $\alpha$ is of the form

$$\alpha = \Sigma_i a_i(u) du^i \quad \text{where} \qquad \alpha(\boldsymbol{\partial}_j) = \Sigma_i a_i(u) du^i(\boldsymbol{\partial}_j) = \Sigma_i a_i(u) \delta^i_j = a_j$$

is the $j$th **component** of $\alpha$.

We shall see in Section $\mathbb{O}$.i that it is *not* true that every $\alpha$ is equal to $df$ for some $f$!

Corresponding to ($\mathbb{O}$.1) we can write the matrix expansion for a linear functional as

$$\alpha = (a_1, \ldots, a_n)(du^1, \ldots, du^n)^T = a\, du \qquad (\mathbb{O}.3)$$

i.e., $a$ is a **row** matrix and $du$ is a column matrix!

If $V$ is the space of contravariant vectors at $p$, then $V^*$ is called the space of **covariant** vectors, or covectors, or **1-forms** at $p$. Under a change of coordinates, using the chain rule, $\alpha = a' \, du' = a \, du = (a)(\partial u/\partial u')(du')$, and so

$$a' = a(\partial u/\partial u') = a(\partial u'/\partial u)^{-1} \qquad \text{i.e.,} \quad a'_j = \Sigma_i a_i (\partial u^i/\partial u'^j) \tag{0.4}$$

which should be compared with (0.2). This is the law of transformation of components of a covector.

Note that by definition, if $\alpha$ is a covector and $\mathbf{v}$ is a vector, then the value

$$\alpha(\mathbf{v}) = av = \Sigma_i a_i v^i$$

is **invariant**, i.e., independent of the coordinates used. This also follows, from (0.2) and (0.4)

$$\alpha(\mathbf{v}) = a'v' = a(\partial u/\partial u')(\partial u'/\partial u)v = a(\partial u'/\partial u)^{-1}(\partial u'/\partial u)v = av$$

Note that a vector can be considered as a linear functional on covectors,

$$\mathbf{v}(\alpha) := \alpha(\mathbf{v}) = \Sigma_i a_i v^i$$

### 0.c. Superscripts, Subscripts, Summation Convention

First the **summation convention**. Whenever we have a single term of an expression with any number of indices up and down, e.g., $T^{abc}{}_{de}$, if we rename one of the **lower** indices, say $d$ so that it becomes the same as one of the **upper** indices, say $b$, and if we then sum over this index, the result, call it $S$,

$$\Sigma_b T^{abc}{}_{be} = S^{ac}{}_e$$

is called a **contraction** of $T$. The index $b$ has disappeared (it was a summation or "dummy" index on the left expression; you could have called it anything). This process of summing over a repeated index *that occurs as both a subscript and a superscript* occurs so often that we shall omit the summation sign and merely write, for example, $T^{abc}{}_{be} = S^{ac}{}_e$. This "Einstein convention" does *not* apply to two upper or two lower indices. Here is why.

We have seen that if $\alpha$ is a covector, and if $\mathbf{v}$ is a vector then $\alpha(\mathbf{v}) = a_i v^i$ is an invariant, independent of coordinates. But if we have another vector, say $\mathbf{w} = \boldsymbol{\partial} w$ then $\Sigma_i v^i w^i$ *will not be invariant*

$$\Sigma_i v'^i w'^i = v'^T w' = [(\partial u'/\partial u)v]^T (\partial u'/\partial u)w = v^T (\partial u'/\partial u)^T (\partial u'/\partial u)w$$

will not be equal to $v^T w$, for all **v, w** unless $(\partial u'/\partial u)^T = (\partial u'/\partial u)^{-1}$, i.e., unless the coordinate change matrix is an **orthogonal** matrix, as it is when $u$ and $u'$ are cartesian coordinate systems.

Our **conventions** regarding the **components** of vectors and covectors

$$(\text{contravariant} \Rightarrow \text{ index up}) \text{ and } (\text{covariant} \Rightarrow \text{ index down}) \tag{**}$$

*help us avoid errors*! For example, in calculus, the differential equations for curves of **steepest ascent** for a function $f$ are written in cartesian coordinates as

$$dx^i/dt = \partial f/\partial x^i$$

but these equations cannot be correct, say, in spherical coordinates, since we cannot equate the *contravariant* components $v^i$ of the velocity vector with the *covariant* components of the differential $df$; they transform in different ways under a (nonorthogonal) change of coordinates. We shall see the correct equations for this situation in Section 0.d.

**Warning:** Our convention (**) applies only to the **components** of vectors and covectors. In $\alpha = a_i dx^i$, the $a_i$ are the components of a single covector $\alpha$, while each individual $dx^i$ is itself a basis covector, *not* a component. The summation convention, however, always holds.

I cringe when I see expressions like $\Sigma_i v^i w^i$ in noncartesian coordinates, for the notation is informing me that I have misunderstood the "variance" of one of the vectors.

## 0.d. Riemannian Metrics

One *can* identify vectors and covectors by introducing an *additional* structure, but the identification will depend on the structure chosen. The metric structure of ordinary euclidean space $\mathbb{R}^3$ is based on the fact that we can measure angles and lengths of vectors and scalar products $\langle , \rangle$. The arc length of a curve $C$ is

$$\int_C ds$$

where $ds^2 = dx^2 + dy^2 + dz^2$ in *cartesian* coordinates. In curvilinear coordinates $u$ we have, putting $dx^k = (\partial x^k/\partial u^i)du^i$, and then

$$ds^2 = \Sigma_k (dx^k)^2 = \Sigma_{i,j} g_{ij} du^i du^j = g_{ij} du^i du^j \qquad (0.5)$$

where

$$\begin{aligned} g_{ij} &= \Sigma_k (\partial x^k/\partial u^i)(\partial x^k/\partial u^j) \\ &= \langle \partial \mathbf{p}/\partial u^i, \partial \mathbf{p}/\partial u^j \rangle \text{ (since the } x \text{ coordinates are cartesian)} \end{aligned}$$

$$g_{ij} = \langle \boldsymbol{\partial}_i, \boldsymbol{\partial}_j \rangle = g_{ji}$$

and generally

$$\langle \mathbf{v}, \mathbf{w} \rangle = g_{ij} v^i w^j \qquad (0.6)$$

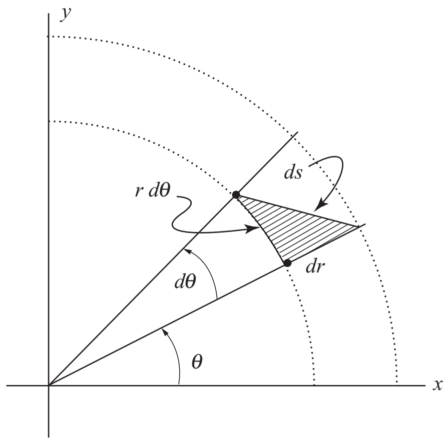
For example, consider the plane $\mathbb{R}^2$ with cartesian coordinates $x^1 = x$, $x^2 = y$, and polar coordinates $u^1 = r$, $u^2 = \theta$. Then

$$\begin{bmatrix} g_{xx} = 1 & g_{xy} = 0 \\ g_{yx} = 0 & g_{yy} = 1 \end{bmatrix} \quad \text{i.e.,} \quad \begin{bmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then, from $x = r\cos\theta$, $dx = dr\cos\theta - r\sin\theta\, d\theta$, etc., we get $ds^2 = dr^2 + r^2\, d\theta^2$,

$$\begin{bmatrix} g_{rr} = 1 & g_{r\theta} = 0 \\ g_{\theta r} = 0 & g_{\theta\theta} = r^2 \end{bmatrix} \quad \text{i.e.,} \quad \begin{bmatrix} g_{rr} & g_{r\theta} \\ g_{\theta r} & g_{\theta\theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} \tag{0.7}$$

which is "evident" from the picture




In **spherical** coordinates a picture shows $ds^2 = dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\varphi^2$, where $\theta$ is co-latitude and $\varphi$ is co-longitude, so $(g_{ij}) = \operatorname{diag}(1, r^2, r^2\sin^2\theta)$. In **cylindrical** coordinates, $ds^2 = dR^2 + R^2\, d\Theta^2 + dZ^2$, with $(g_{ij}) = \operatorname{diag}(1, R^2, 1)$.

Let us look again at the expression (0.5). If $\alpha$ and $\beta$ are 1-forms, i.e., linear functionals, *define* their **tensor product** $\alpha \otimes \beta$ to be the function of (ordered) **pairs** of vectors defined by

$$\alpha \otimes \beta(\mathbf{v}, \mathbf{w}) := \alpha(\mathbf{v})\beta(\mathbf{w}) \tag{0.8}$$

In particular

$$(du^i \otimes du^k)(\mathbf{v}, \mathbf{w}) := v^i w^k$$

Likewise $(\boldsymbol{\partial}_i \otimes \boldsymbol{\partial}_j)(\alpha, \beta) = a_i b_j$ (why?).

$\alpha \otimes \beta$ is a *bilinear* function of **v** and **w**, i.e., it is linear in each vector when the other is unchanged. A **second rank covariant tensor** is just such a bilinear function and in the coordinate system $u$ it can be expressed as

$$\boldsymbol{\Sigma}_{i,j} a_{ij} du^i \otimes du^j$$

where the coefficient matrix $(a_{ij})$ is written with indices down. Usually the tensor product sign $\otimes$ is omitted (in $du^i \otimes du^j$ but *not* in $\alpha \otimes \beta$). For example, the metric

$$ds^2 = g_{ij} du^i \otimes du^j = g_{ij} du^i du^j \tag{0.5'}$$

is a second rank covariant tensor that is **symmetric**, i.e., $g_{ji} = g_{ij}$. We may write

$$ds^2(\mathbf{v}, \mathbf{w}) = \langle \mathbf{v}, \mathbf{w} \rangle$$

It is easy to see that under a change of coordinates $u' = u'(u)$, demanding that $ds^2$ be independent of coordinates, $g'_{ab} du'^a du'^b = g_{ij} du^i du^j$, yields the transformation rule

$$g'_{ab} = (\partial u^i / \partial u'^a) g_{ij} (\partial u^j / \partial u'^b) \tag{0.9}$$

for the components of a second rank *covariant* tensor.

**Remark:** We have been using the euclidean metric structure to construct $(g_{ij})$ in any coordinate system, but there are times when other structures are more appropriate. For example, when considering some delicate astronomical questions, a metric from Einstein's general relativity yields more accurate results. When dealing with complex analytic functions in the upper half plane $y > 0$, Poincaré found that the planar metric $ds^2 = (dx^2 + dy^2)/y^2$ was very useful. In general, when some second rank covariant tensor $(g_{ij})$ is used in a metric $ds^2 = g_{ij} dx^i dx^j$ (in which case it must be symmetric and positive definite), this metric is called a **Riemannian metric**, after Bernhard Riemann, who was the first to consider this generalization of **Gauss**' thoughts.

Given a Riemannian metric, one can associate to each (contravariant) vector $\mathbf{v}$ a covector $v$ by

$$v(\mathbf{w}) = \langle \mathbf{v}, \mathbf{w} \rangle$$

for *all* vectors $\mathbf{w}$, i.e.,

$$v_j w^j = v^k g_{kj} w^j \quad \text{and so} \quad v_j = v^k g_{kj} = g_{jk} v^k$$

In *components*, it is traditional to use the same letter for the covector as for the vector

$$v_j = g_{jk} v^k$$

there being no confusion since the covector has the subscript. We say that "we lower the contravariant index" by means of the covariant metric tensor $(g_{jk})$.

Similarly, since $(g_{jk})$ is the matrix of a positive definite quadratic form $ds^2$, it has an inverse matrix, written $(g^{jk})$, which can be shown to be a **contravariant** second rank symmetric tensor (a bilinear function of pairs of covectors given by $g^{jk} a_j b_k$). Then for each covector $\alpha$ we can associate a vector $\mathbf{a}$ by $a^i = g^{ij} a_j$, i.e., we *raise the covariant index* by means of the contravariant metric tensor $(g^{jk})$.

The **gradient vector** of a function $f$ is defined to be the vector **grad** $f = \nabla f$ associated to the covector $df$, i.e., $df(\mathbf{w}) = \langle \quad \mathbf{f}, \mathbf{w} \rangle$

$$(\nabla f)^i := g^{ij} \partial f / \partial u^j$$

Then the correct version of the equation of steepest ascent considered at the end of section 0.c is

$$du^i/dt = (\nabla f)^i = g^{ij} \partial f / \partial u^j$$

in *any* coordinates. For example, in polar coordinates, from (0.7), we see $g^{rr} = 1$, $g^{\theta\theta} = 1/r^2$, $g^{r\theta} = 0 = g^{\theta r}$.

### 0.e. Tensors

We shall consider examples rather than generalities.

(i) A tensor of the third rank, twice contravariant, once covariant, is locally of the form

$$A = \boldsymbol{\partial}_i \otimes \boldsymbol{\partial}_j A^{ij}{}_k \otimes du^k$$

It is a trilinear function of pairs of covectors $\alpha = a_i du^i$, $\beta = b_j du^j$, and a single vector $\mathbf{v} = \boldsymbol{\partial}_k v^k$

$$A(\alpha, \beta, \mathbf{v}) = a_i b_j A^{ij}{}_k v^k$$

summed, of course, on all indices. Its components transform as

$$A'^{\,ef}{}_g = (\partial u'^e/\partial u^i)(\partial u'^f/\partial u^j)A^{ij}{}_k(\partial u^k/\partial u'^g)$$

(When I was a lad I learned the mnemonic "*co low, primes below*.")

If we **contract** on $i$ and $k$, the result $B^j := A^{ij}{}_i$ are the components of a contravariant **vector**

$$\begin{aligned} B'^f &= A'^{\,ef}{}_e = A^{ij}{}_k(\partial u'^f/\partial u^j)(\partial u^k/\partial u'^e)(\partial u'^e/\partial u^i) \\ &= A^{ij}{}_k(\partial u'^f/\partial u^j)\delta^k{}_i = A^{ij}{}_i(\partial u'^f/\partial u^j) = (\partial u'^f/\partial u^j)B^j \end{aligned}$$

(ii) A **linear transformation** is a second rank ("mixed") tensor $P = \boldsymbol{\partial}_i P^i{}_j \otimes du^j$. Rather than thinking of this as a real valued bilinear function of a covector and a vector, we usually consider it as a *linear function taking vectors into vectors* (called a vector valued 1-form in Section 0.n)

$$P(\mathbf{v}) = [\boldsymbol{\partial}_i P^i{}_j \otimes du^j](\mathbf{v}) := \boldsymbol{\partial}_i P^i{}_j\{du^j(\mathbf{v})\} = \boldsymbol{\partial}_i P^i{}_j v^j$$

i.e., the usual

$$[P(\mathbf{v})]^i = P^i_j v^j$$

Under a coordinate change, $(P^i{}_j)$ transforms as $P' = (\partial u'/\partial u)P(\partial u'/\partial u)^{-1}$, as usual. If we contract we obtain a scalar (invariant), tr $P := P^i{}_i$ , the **trace** of $P$. tr $P' =$ tr $P(\partial u'/\partial u)^{-1}(\partial u'/\partial u) =$ tr $P$.

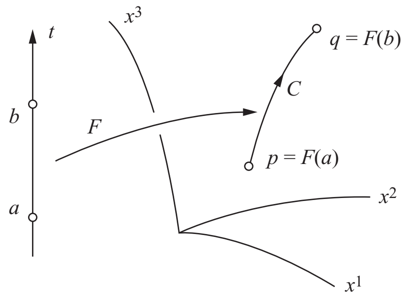
**Beware:** If we have a twice covariant tensor $G$ (a "bilinear form"), for example, a metric $(g_{ij})$, then $\boldsymbol{\Sigma}_k g_{kk}$ is *not a scalar*, although it is the trace of the matrix; see for example, equation (0.7). This is because the transformation law for the matrix $G$ is, from (0.9), $G' = (\partial u/\partial u')^T G(\partial u/\partial u')$ and tr $G' \neq$ tr $G$ generically.

## Integrals and Exterior Forms

### 0.f. Line Integrals

We illustrate in $\mathbb{R}^3$ with any coordinates $x$. For simplicity, let $C$ be a smooth "oriented" or "directed" curve, the image under $F : [a,b] \subset \mathbb{R}^1 \to C \subset \mathbb{R}^3$ (which is read

"$F$ maps the interval $[a,b]$ on $\mathbb{R}^1$ into the curve $C$ in $\mathbb{R}^3$") with $F(a)$ for some $p$ and $F(b)$ for some $q$.




If $\alpha = \alpha^1 = a_i(x)dx^i$ is a 1-form, a covector, in $\mathbb{R}^3$, we define the line integral $\int_C \alpha$ as follows.

Using the parameterization $x^i = F^i(t)$ of $C$, we define

$$\int_C \alpha^1 = \int_C a_i(x)dx^i := \int_a^b a_i(x(\mathrm{t}))(dx^i/dt)dt = \int_a^b \alpha(d\mathbf{x}/dt)dt \qquad (0.10)$$

We say that we *pull back* the form $\alpha^1$ (that lives in $\mathbb{R}^3$ ) to a 1-form on the parameter space $\mathbb{R}^1$, called the **pull-back** of $\alpha$, denoted by $F^*(\alpha)$

$$F^*(\alpha) = \alpha(d\mathbf{x}/dt)dt = a_i(x(t))(dx^i/dt)dt$$

and then take the *ordinary* integral $\int_a^b \alpha(d\mathbf{x}/dt)dt$. It is a classical theorem that the result is *independent of the parameterization* of $C$ chosen, so long as the resulting curve has the same orientation. This will become "apparent" from the usual geometric interpretation that we now present.

In the definition there has been no mention of *arc length* or *scalar product*. Suppose now that a Riemannian metric (e.g., the usual metric in $\mathbb{R}^3$) is available. Then to $\alpha$ we may associate its contravariant vector $\mathbf{A}$. Then $\alpha(d\mathbf{x}/dt) = \langle \mathbf{A}, d\mathbf{x}/dt\rangle = \langle \mathbf{A}, d\mathbf{x}/ds\rangle(ds/dt)$ where $s = s(t)$ is the arclength parameter along $C$. Then $F^*(\alpha) = \alpha(d\mathbf{x}/dt)dt = \langle \mathbf{A}, d\mathbf{x}/ds\rangle ds$. But $\mathbf{T} := d\mathbf{x}/ds$ is the **unit** tangent vector to $C$ since $g_{ij}(dx^i/ds)(dx^j/ds) = (g_{ij}dx^i dx^j)/(ds^2) = 1$. Thus

$$F^*(\alpha) = \langle \mathbf{A}, \mathbf{T}\rangle ds = \|\mathbf{A}\|\,\|\mathbf{T}\| \cos\angle(\mathbf{A}, \mathbf{T})ds$$

and so

$$\int_C \alpha = \int_C A_{\tan} ds \qquad (0.11)$$

is geometrically the integral of the tangential component of $\mathbf{A}$ with respect to the arc length parameter along $C$. This "shows" independence of the parameter $t$ chosen, but to evaluate the integral one would *usually* just use (0.10) which involves no metric at all!

**Moral:** The integrand in a line integral is naturally a **1-form**, *not* a vector.

For example, in *any* coordinates, force is often a 1-form $f^1$ since a basic measure of force is given by a line integral $W = \int_C f^1 = \int_C f_k dx^k$ which measures the **work** done by the force along the curve $C$, and this does not require a metric. Frequently there is a force **potential** $V$ such that $f^1 = dV$, exhibiting $f$ *explicitly* as a covector. (In this case, from (0.10), $W = \int_C f^1 = \int_C dV = \int_a^b dV(d\mathbf{x}/dt)dt = \int_a^b (\partial V/\partial x^i)(dx^i/dt)dt =$

$\int_a{}^b\{dV(\mathbf{x}(t)/dt\}dt = V[\,\mathbf{x}(b)] - V[\,\mathbf{x}(a)] = V(q) - V(p)$.) Of course metrics do play a large role in mechanics. In Hamiltonian mechanics, a particle of mass $m$ has a kinetic energy $T = mv^2/2 = mg_{ij}\dot{x}^i\dot{x}^j/2$ (where $\dot{x}^i$ is $dx^i/dt$) and its **momentum** is defined by $p_k = \partial(T - V)/\partial\dot{x}^k$. When the potential energy is independent of $\dot{\mathbf{x}} = d\mathbf{x}/dt$, we have $p_k = \partial T/\partial\dot{x}^k = (1/2)mg_{ij}(\delta^i{}_k\dot{x}^j + \dot{x}^i\delta^j{}_k) = (m/2)(g_{kj}\dot{x}^j + g_{ik}\dot{x}^i) = mg_{kj}\dot{x}^j$. Thus in this case $p$ is $m$ times the *covariant version* of the velocity vector $d\mathbf{x}/dt$.

The momentum 1-form "$p_i dx^i$" on the "phase space" with coordinates $(x, p)$ plays a *central role* in all of Hamiltonian mechanics.

## 0.g. Exterior 2-Forms

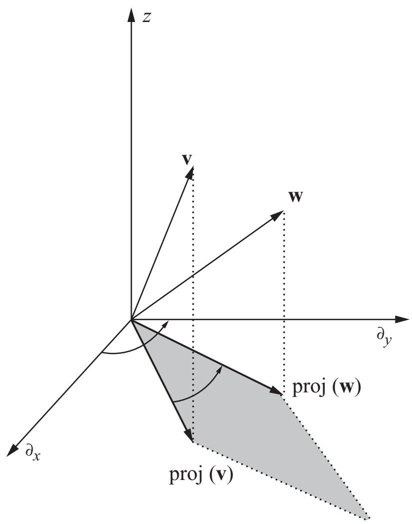
We have already defined the **tensor** product $\alpha^1 \otimes \beta^1$ of two 1-forms to be the bilinear form $\alpha^1 \otimes \beta^1(\mathbf{v}, \mathbf{w}) = \alpha^1(\mathbf{v})\beta^1(\mathbf{w})$. We now define a more *geometrically* significant **wedge** or **exterior product** $\alpha \wedge \beta$ to be the *skew symmetric* bilinear form

$$\alpha^1 \wedge \beta^1 := \alpha^1 \otimes \beta^1 - \beta^1 \otimes \alpha^1$$

and thus

$$du^j \wedge du^k(\mathbf{v}, \mathbf{w}) = v^j w^k - v^k w^j = \begin{vmatrix} du^j(\mathbf{v}) & du^j(\mathbf{w}) \\ du^k(\mathbf{v}) & du^k(\mathbf{w}) \end{vmatrix} \qquad (0.12)$$

In **cartesian** coordinates $x, y, z$ in $\mathbb{R}^3$, see the figure below, $dx \wedge dy(\mathbf{v}, \mathbf{w})$ is $\pm$ the area of the parallelogram spanned by the projections of $\mathbf{v}$ and $\mathbf{w}$ into the $x, y$ plane, the plus sign used only if proj($\mathbf{v}$) and proj($\mathbf{w}$) describe the same orientation of the plane as the basis vectors $\boldsymbol{\partial}_x$ and $\boldsymbol{\partial}_y$.




Let now $x^i, i = 1, 2, 3$ be *any* coordinates in $\mathbb{R}^3$. Note that

$$dx^j \wedge dx^k = -dx^k \wedge dx^j \quad \text{and} \quad dx^k \wedge dx^k = 0 \quad \text{(no sum!)} \qquad (0.13)$$

The most general **exterior 2-form** is of the form $\beta^2 = \boldsymbol{\Sigma}_{i<j} b_{ij}\, dx^i \wedge dx^j$ where $b_{ji} = -b_{ij}$. In $\mathbb{R}^3$, $\beta^2 = b_{12}\, dx^1 \wedge dx^2 + b_{23} dx^2 \wedge dx^3 + b_{13}\, dx^1 \wedge dx^3$, or, as we prefer, for

reasons soon to be evident,

$$\beta^2 = b_{23} dx^2 \wedge dx^3 + b_{31} dx^3 \wedge dx^1 + b_{12}\, dx^1 \wedge dx^2 \qquad (\mathbb{O}.14)$$

An exterior 2-form is a skew symmetric covariant tensor of the second rank in the sense of Section $\mathbb{O}$.d. We frequently will omit the *term* "exterior," but *never* the wedge $\wedge$.

## $\mathbb{O}$.h. Exterior *p*-Forms and Algebra in $\mathbb{R}^n$

The **exterior algebra** has the following properties. We have already discussed 1-forms and 2-forms. An (exterior) $p$-form $\alpha^p$ in $\mathbb{R}^n$ is a completely skew symmetric multilinear function of $p$-tuples of vectors $\alpha(\mathbf{v}_1, \ldots, \mathbf{v}_p)$ that changes sign whenever two vectors are interchanged. In any coordinates $x$, for example, the 3-form $dx^i \wedge dx^j \wedge dx^k$ in $\mathbb{R}^n$ is defined by

$$dx^i \wedge dx^j \wedge dx^k(\mathbf{A}, \mathbf{B}, \mathbf{C}) := \begin{vmatrix} dx^i(\mathbf{A}) & dx^i(\mathbf{B}) & dx^i(\mathbf{C}) \\ dx^j(\mathbf{A}) & dx^j(\mathbf{B}) & dx^j(\mathbf{C}) \\ dx^k(\mathbf{A}) & dx^k(\mathbf{B}) & dx^k(\mathbf{C}) \end{vmatrix} = \begin{vmatrix} A^i & B^i & C^i \\ A^j & B^j & C^j \\ A^k & B^k & C^k \end{vmatrix} \qquad (\mathbb{O}.15)$$

When the coordinates are cartesian the interpretation of this is similar to that in ($\mathbb{O}$.12). Take the three vectors at a given point $x$ in $\mathbb{R}^n$, project them down into the 3 dimensional affine subspace of $\mathbb{R}^n$ spanned by $\boldsymbol{\partial}_i$, $\boldsymbol{\partial}_j$, and $\boldsymbol{\partial}_k$ at $x$, and read off $\pm$ the 3-volume of the parallelopiped spanned by the projections, the $+$ used only if the projections define the same orientation as $\boldsymbol{\partial}_i$, $\boldsymbol{\partial}_j$, and $\boldsymbol{\partial}_k$.

Clearly any interchange of a single pair of $dx$ will yield the negative, and thus *if the same $dx^i$ appears twice the form will vanish*, just as in ($\mathbb{O}$.12), similarly for a $p$-form. The most general 3-form is of the form $\alpha^3 = \boldsymbol{\Sigma}_{i<j<k} a_{ijk} dx^i \wedge dx^j \wedge dx^k$. In $\mathbb{R}^3$ there is only one nonvanishing 3-form, $dx^1 \wedge dx^2 \wedge dx^3$ and its multiples. In *cartesian* coordinates this is the **volume form** $\mathrm{vol}^3$, but in spherical coordinates we know that $dr \wedge d\theta \wedge d\phi$ does *not* yield the euclidean volume element, which is $r^2 \sin\theta\, dr \wedge d\theta \wedge d\phi$. We will discuss this soon. Note further that all $p > n$ forms in $\mathbb{R}^n$ vanish since there are always repeated $dx$ in each term.

We take the **exterior product** of a $p$-form $\alpha$ and a $q$-form $\beta$, yielding a $p + q$ form $\alpha \wedge \beta$ by expressing them in terms of the $dx$, using the usual algebra (including the associative law), except that the product of $dx$ is anticommutative, $dx \wedge dy = -dy \wedge dx$. For examples in $\mathbb{R}^3$ with any coordinates

$$\begin{aligned} \alpha^1 \wedge \gamma^1 &= (a_1\, dx^1 + a_2\, dx^2 + a_3\, dx^3) \wedge (c_1\, dx^1 + c_2\, dx^2 + c_3\, dx^3) \\ &= \cdots (a_2\, dx^2) \wedge (c_1\, dx^1) + \cdots + (a_1\, dx^1) \wedge (c_2\, dx^2) + \cdots \\ &= (a_2 c_3 - a_3 c_2)\, dx^2 \wedge dx^3 + (a_3 c_1 - a_1 c_3)\, dx^3 \wedge dx^1 \\ &\quad + (a_1 c_2 - a_2 c_1)\, dx^1 \wedge dx^2 \end{aligned}$$

which in *cartesian* coordinates has the components of the vector product $\mathbf{a} \times \mathbf{c}$. Also we have

$$\alpha^1 \wedge \beta^2 = (a_1\,dx^1 + a_2\,dx^2 + a_3\,dx^3) \wedge (b_{23}\,dx^2 \wedge dx^3 + b_{31}\,dx^3 \wedge dx^1 + b_{12}\,dx^1 \wedge dx^2)$$
$$= (a_1b_1 + a_2b_2 + a_3b_3)dx^1 \wedge dx^2 \wedge dx^3$$

(where we use the notation $b_1 := b_{23}$, $b_2 = b_{31}$, $b_3 = b_{12}$, but *only* in cartesian coordinates) with component $\mathbf{a} \cdot \mathbf{b}$ in cartesian coordinates. The $\wedge$ product in cartesian $\mathbb{R}^3$ yields both the dot $\cdot$ and the cross $\times$ products of vector analysis!! The $\cdot$ and $\times$ products of vector analysis have strange expressions when curvilinear coordinates are used in $\mathbb{R}^3$, but the form expressions $\alpha^1 \wedge \beta^2$ and $\alpha^1 \wedge \gamma^1$ are always the same. Furthermore, the $\times$ product is nasty since it is not associative, $\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) \neq (\mathbf{i} \times \mathbf{i}) \times \mathbf{j}$.

By counting the number of interchanges of pairs of $dx$ one can see the **commutation rule**

$$\alpha^p \wedge \beta^q = (-1)^{pq}\beta^q \wedge \alpha^p \tag{$\mathcal{O}$.16}$$

## $\mathcal{O}$.i. The Exterior Differential *d*

First a remark. If $\mathbf{v} = \boldsymbol{\partial}_a v^a$ is a contravariant vector field, then generically $(\partial v^a/\partial x^b) = Q^a{}_b$ *do not yield the components of a tensor* in curvilinear coordinates, as is easily seen from looking at the transformation of $Q$ under a change of coordinates and using ($\mathcal{O}$.2). It is, however, always possible, in $\mathbb{R}^n$ and in any coordinates, to take a very important **exterior** derivative $d$ of $p$-forms. We define $d\alpha^p$ to be a $p+1$ form, as follows; $\alpha$ is a sum of forms of the type $a(x)dx^i \wedge dx^j \wedge \cdots \wedge dx^k$. Define

$$d[a(x)dx^i \wedge dx^j \wedge \ldots \wedge dx^k] = da \wedge dx^i \wedge dx^j \wedge \ldots \wedge dx^k$$
$$= \boldsymbol{\Sigma}_r(\partial a/\partial x^r)dx^r \wedge dx^i \wedge dx^j \wedge \ldots \wedge dx^k \tag{$\mathcal{O}$.17}$$

(in particular $d[dx^i \wedge dx^j \wedge \ldots \wedge dx^k] = 0$), and then sum over all the terms in $\alpha^p$. In particular, in $\mathbb{R}^3$ in *any* coordinates

$$df^0 = df = (\partial f/\partial x^1)dx^1 + (\partial f/\partial x^2)dx^2 + (\partial f/\partial x^3)dx^3$$
$$d\alpha^1 = d(a_1\,dx^1 + a_2\,dx^2 + a_3\,dx^3) = (\partial a_1/\partial x^2)dx^2 \wedge dx^1 + (\partial a_1/\partial x^3)dx^3 \wedge dx^1 + \cdots$$
$$= [(\partial a_3/\partial x^2) - (\partial a_2/\partial x^3)]dx^2 \wedge dx^3 + [(\partial a_1/\partial x^3) - (\partial a_3/\partial x^1)]dx^3 \wedge dx^1$$
$$+ [(\partial a_2/\partial x^1) - (\partial a_1/\partial x^2)]dx^1 \wedge dx^2 \tag{$\mathcal{O}$.18}$$
$$d\beta^2 = d(b_{23}\,dx^2 \wedge dx^3 + b_{31}\,dx^3 \wedge dx^1 + b_{12}\,dx^1 \wedge dx^2)$$
$$= [(\partial b_{23}/\partial x^1) + (\partial b_{31}/\partial x^2) + (\partial b_{12}/\partial x^3)]dx^1 \wedge dx^2 \wedge dx^3$$

In *cartesian* coordinates we then have correspondences with vector analysis, using again $b_1 := b_{23}$ etc.,

$$df^0 \Leftrightarrow \quad \mathbf{f} \cdot \mathbf{dx} \quad d\alpha^1 \Leftrightarrow (\text{curl }\mathbf{a}) \cdot \text{“}\mathbf{dA}\text{”} \quad d\beta^2 \Leftrightarrow \text{div }\mathbf{B}\text{ “dvol”} \tag{$\mathcal{O}$.19}$$

the quotes, for example, “**dA**” being used since this is not really the differential of a 1-form. We shall make this correspondence precise, in any coordinates, later. Exterior differentiation of **exterior forms** does essentially grad, curl and divergence with a *single general formula* (0.17)!! Also, this machinery works in $\mathbb{R}^n$ as well. Furthermore, *d does not require a metric*. On the other hand, without a metric (and hence without cartesian coordinates), one *cannot* take the curl of a contravariant vector field. Also to take the *divergence* of a vector field requires at least a specified "volume form." These will be discussed in more detail later in section 0.n.

There are two fairly easy but very important properties of the differential $d$:

$$\begin{aligned} d^2\alpha^p &:= d\, d\alpha^p = 0 \text{ (which says curl grad} = 0 \text{ and div curl} = 0 \text{ in } \mathbb{R}^3) \\ d(\alpha^p \wedge \beta^q) &= d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta \end{aligned} \qquad (0.20)$$

For example, in $\mathbb{R}^3$ with function (0-form) $f$, $df = (\partial f/\partial x)dx + (\partial f/\partial y)dy + (\partial f/\partial z)dz$, and then $d^2 f = (\partial^2 f/\partial x\, \partial y)dy \wedge dx + \cdots + (\partial^2 f/\partial y\, \partial x)dx \wedge dy + \cdots = 0$, since $(\partial^2 f/\partial y\, \partial x) = (\partial^2 f/\partial x\, \partial y)$.

Note then that a *necessary* condition for a $p$-form $\beta^p$ to be the differential of some $(p-1)$-form, $\beta^p = d\alpha^{p-1}$, is that $d\beta = d\, d\alpha = 0$. (What does this say in vector analysis in $\mathbb{R}^3$ ?)

Also, we know that in cartesian $\mathbb{R}^3$, $\alpha^1 \wedge \beta^1 \Leftrightarrow \mathbf{a} \times \mathbf{b}$ is a 2-form, $d(\alpha \wedge \beta) \Leftrightarrow$ div $\mathbf{a} \times \mathbf{b}$ (from (0.19)), and $d\alpha \Leftrightarrow$ curl $\mathbf{a}$, and we know $\alpha^1 \wedge \gamma^2 = \gamma^2 \wedge \alpha^1 \Leftrightarrow \mathbf{a} \cdot \mathbf{c}$. Then (0.20), in cartesian coordinates, says *immediately* that $d(\alpha \wedge \beta) = d\alpha \wedge \beta - \alpha \wedge d\beta$, i.e.,

$$\text{div } \mathbf{a} \times \mathbf{b} = (\text{curl } \mathbf{a}) \cdot \mathbf{b} - \mathbf{a} \cdot (\text{curl } \mathbf{b}) \qquad (0.21)$$

## 0.j. The Push-Forward of a Vector and the Pull-Back of a Form

Let $F$: $\mathbb{R}^k \to \mathbb{R}^n$ be any differentiable map of $k$-space into $n$-space, where any values of $k$ and $n$ are permissible. Let $(u^1, \ldots, u^k)$ be any coordinates in $\mathbb{R}^k$, let $(x^1, \ldots, x^n)$ be any coordinates in $\mathbb{R}^n$. Then $F$ is described by $n$ functions $x^i = F^i(u) = F^i(u^1, \ldots, u^r, \ldots, u^k)$ or briefly $x^i = x^i(u)$.

The "pull-back" of a **function** (0-form) $\phi = \phi(x)$ on $\mathbb{R}^n$ is the function $F^*\phi = \phi(x(u))$ on $\mathbb{R}^k$, i.e., the function on $\mathbb{R}^k$ whose value at $u$ is simply the value of $\phi$ at $x = F(u)$.

Given a vector $\mathbf{v}_0$ at the point $u_0 \in \mathbb{R}^k$ we can "push forward" the vector to the point $x_0 = F(u_0) \in \mathbb{R}^n$ by means of the so-called "differential of $F$," written $F_*$, as follows. Let $u = u(t)$ be any curve in $\mathbb{R}^k$ with $u(0) = u_0$ and velocity at $u_0 = [du/dt]_0$ equal to the given $\mathbf{v}_0$. (For example, in terms of the coordinates $u$, you may use the curve defined by $u^r(t) = u_0{}^r + v_0{}^r t$ .) Then the image curve $x(t) = x(u(t))$ will have velocity vector at $t = 0$ called $F_*[\mathbf{v}_0]$ given by the chain rule,

$$[F_*(\mathbf{v}_0)]^i := dx^i(u(t))/dt]_0 = [\partial x^i/\partial u^r]_{u(0)}[du^r/dt]_0 = [\partial x^i/\partial u^r]_{u(0)}\mathbf{v}_0{}^r$$

Briefly

$$[F_*(\mathbf{v})]^i = (\partial x^i/\partial u^r)v^r$$

Then

$$F_*[v^r \boldsymbol{\partial}/\partial u^r] = v^r\ \boldsymbol{\partial}/\partial x^i(\partial x^i/\partial u^r), \qquad (0.22)_*$$

and so

$$F_* \boldsymbol{\partial}_r = F_*[\partial/\partial u^r] = [\boldsymbol{\partial}/\partial x^i](\partial x^i/\partial u^r) = \boldsymbol{\partial}_i(\partial x^i/\partial u^r)$$

*is again simply the chain rule.*

Given any $p$-form $\alpha$ at $x \in \mathbb{R}^n$, we define the **pull-back** $F^*(\alpha)$ to be the $p$-form at each pre-image point $u \in F^{-1}(x)$ of $\mathbb{R}^k$ by

$$(F^*\alpha)(\mathbf{v}, \dots, \mathbf{w}) := \alpha(F_*\mathbf{v}, \dots, F_*\mathbf{w}) \tag{0.23}$$

For the 1-form $dx^i$, $F^*dx^i$ must be of the form $a_s du^s$; using $dx^i(\boldsymbol{\partial}_j) = \delta^i{}_j$ we get

$$(F^*dx^i)(\boldsymbol{\partial}_r) = dx^i[\boldsymbol{\partial}_j(\partial x^j/\partial u^r)] = \partial x^i/\partial u^r = (\partial x^i/\partial u^s)du^s(\boldsymbol{\partial}_r)$$

and so

$$F^*dx^i = (\partial x^i/\partial u^s)du^s \tag{0.22$^*$}$$

is again simply the chain rule.

It can be shown in general that $F^*$ operating on forms satisfies

$$F^*(\alpha^p \wedge \beta^q) = (F^*\alpha) \wedge (F^*\beta)$$

and

$$F^*d\alpha = dF^*\alpha \tag{0.24}$$

For example, $F^*dx^i = dF^*(x^i) = dx^i(u) = (\partial x^i/\partial u^s)du^s$, as we have just seen.

For $p$-forms we shall use the same procedure but also use the fact that $F^*$ commutes with exterior product, $F^*(\alpha \wedge \beta) = (F^*\alpha) \wedge (F^*\beta)$. For simplicity we shall just illustrate the idea for the case when $\beta^2$ is a 2-form in $\mathbb{R}^n$ and $F\colon \mathbb{R}^3 \to \mathbb{R}^n$. For more simplicity we just consider a typical term $b_{23}(x)dx^2 \wedge dx^3$ of $\beta$.

$$\begin{aligned} F^*[b_{23}(x)dx^2 \wedge dx^3] &:= [F^*b_{23}(x)][F^*dx^2] \wedge [F^*dx^3] \\ &:= b_{23}(x(u))[(\partial x^2/\partial u^a)du^a] \\ &\quad \wedge [(\partial x^3/\partial u^c)du^c] \quad \text{(summed on } a \text{ and } c) \end{aligned}$$

Now $(\partial x^2/\partial u^a)du^a = (\partial x^2/\partial u^1)du^1 + (\partial x^2/\partial u^2)du^2 + (\partial x^2/\partial u^3)du^3$ with a similar expression for $(\partial x^3/\partial u^c)du^c$. Taking their $\wedge$ product and using (0.13)

$$\begin{aligned} &[(\partial x^2/\partial u^1)du^1 + (\partial x^2/\partial u^2)du^2 + (\partial x^2/\partial u^3)du^3] \wedge [(\partial x^3/\partial u^1)du^1 \\ &\quad + (\partial x^3/\partial u^2)du^2 + (\partial x^3/\partial u^3)du^3] \\ &\quad = (\partial x^2/\partial u^1)du^1 \wedge (\partial x^3/\partial u^2)du^2 + (\partial x^2/\partial u^1)du^1 \wedge (\partial x^3/\partial u^3)du^3 \\ &\qquad + (\partial x^2/\partial u^2)du^2 \wedge (\partial x^3/\partial u^1)du^1 + (\partial x^2/\partial u^2)du^2 \wedge (\partial x^3/\partial u^3)du^3 \\ &\qquad + (\partial x^2/\partial u^3)du^3 \wedge (\partial x^3/\partial u^1)du^1 + (\partial x^2/\partial u^3)du^3 \wedge (\partial x^3/\partial u^2)du^2 \\ &\quad = [(\partial x^2/\partial u^2)(\partial x^3/\partial u^3) - (\partial x^2/\partial u^3)(\partial x^3/\partial u^2)]du^2 \wedge du^3 \\ &\qquad + [(\partial x^2/\partial u^1)(\partial x^3/\partial u^3) - (\partial x^2/\partial u^3)(\partial x^3/\partial u^1)]du^1 \wedge du^3 \\ &\qquad + [(\partial x^2/\partial u^1)(\partial x^3/\partial u^2) - (\partial x^2/\partial u^2)(\partial x^3/\partial u^1)]du^1 \wedge du^2 \end{aligned}$$

and so

$$F^*[b_{23}(x)dx^2 \wedge dx^3] = b_{23}(x(u))\boldsymbol{\Sigma}_{a<c}[\partial(x^2, x^3)/\partial(u^a, u^c)]du^a \wedge du^c$$

where

$$\partial(x, y)/\partial(u, v) = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$$

is the usual **Jacobian determinant**. In general, for pulling back a $p$-form on $\mathbb{R}^n$ to $\mathbb{R}^k$ via $F$: $\mathbb{R}^k \to \mathbb{R}^n$ we use

$$F^*(dx^i \wedge \ldots \wedge dx^j) = \Sigma_{a<\ldots<_r}[\partial(x^i, \ldots \partial x^j)/\partial(u^a, \ldots, u^r)]du^a \wedge \ldots \wedge du^r \tag{$\mathcal{O}$.22}$$ **

This procedure will play a key role in our discussion of surface integrals, see ($\mathcal{O}$.25).

($\mathcal{O}$.20) and ($\mathcal{O}$.24) contribute to what makes forms so powerful and useful, compared to vector fields. The push-forward $F_*$ associated to a map $F$: $\mathbb{R}^k \to \mathbb{R}^n$ will map a vector $\mathbf{v}$ at $u \in \mathbb{R}^k$ to a vector $F_*\mathbf{v}$ at $x = F(u)$. But let $\mathbf{v}$ be a vector **field**, say on all of $\mathbb{R}^k$ and suppose $F$ is not 1:1. Let $u' \neq u$ and $F(u') = x = F(u)$. Then generically $F_*\mathbf{v}(u')$ will not agree with $F_*\mathbf{v}(u)$, and so $F_*\mathbf{v}$ will *not* be a well defined vector **field** on $\mathbb{R}^n$. On the other hand, if $\alpha$ is a $p$-form at $x$, then $F^*\alpha$ will define a unique form at $u$ and another form at $u'$. If $\alpha^p$ is a well defined $p$-form field on $\mathbb{R}^n$ then $F^*\alpha$ is a well defined $p$-form field on $\mathbb{R}^k$. For fields the tools ($\mathcal{O}$.24) are then available.

Note that when $F : \mathbb{R}^n \to \mathbb{R}^n$ is the **identity** map, using two sets of coordinates, for example, $(r, \theta)$ and $(x, y)$ in the plane, and where the identity map $F = I$ is $x = r\cos\theta$, $y = r\sin\theta$ in $\mathbb{R}^2$, then the pull-back $F^*\alpha$ is simply expressing the form $\alpha$, given in coordinates $x$ in terms of the new coordinates $u$.

Finally note that ($\mathcal{O}$.23) makes sense when $\alpha$ is a **covariant** $p$-tensor even if it is not an exterior form, i.e., even if $\alpha$ is not completely skew symmetric. The pull-back of the Riemannian metric tensor $g$, $g(\mathbf{v}, \mathbf{w}) = g_{ij}v^i w^j$ plays a central role in elasticity, as will be seen in Section $\mathcal{O}$.p. The pull-back of the quadratic form $g_{ij}\, dx^i dx^j$ is again just the application of the chain rule. Of course ($\mathcal{O}$.24) does not make sense if $\alpha$ is not an exterior form.

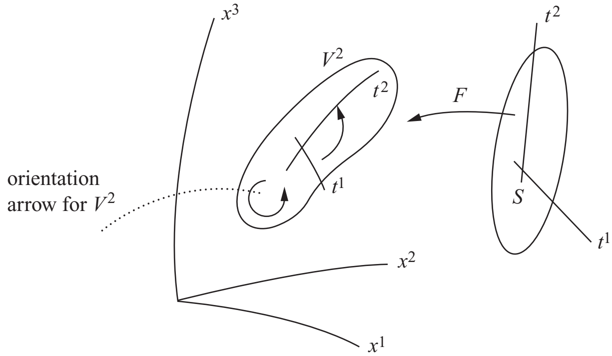
## $\mathcal{O}$.k. Surface Integrals and "Stokes' theorem"

We illustrate with a surface $V^2$ in $\mathbb{R}^3$. Assume, for example, that $\mathbb{R}^3$ has the "right handed orientation." Assume that $V^2$ is also "oriented" meaning that at each point $p$ of $V$ there is a preferred sense of rotation of the tangent plane at $p$ (indicated in the figure below by a circular arrow), and this sense varies continuously on $V$. For example, if $V$ has a continuous choice of normal vector everywhere (unlike a Möbius band) then the right hand rule for $\mathbb{R}^3$ will yield an orientation for $V$.

We are going to define $\int_V \beta^2$ for any 2-form $\beta$ on $\mathbb{R}^3$. If $V$ is sufficiently small we may choose a parameterization of all of $V$ that yields the same orientation as $V$, i.e., we ask for a smooth 1:1 map

$$F : \text{ region } S^2 \subset \text{ some } \mathbb{R}^2 \to \text{ onto } V^2 \subset \mathbb{R}^3 \qquad x^i = x^i(t^1, t^2)$$

(If $V$ is too large for such a parameterization, break it up into smaller pieces and add up the individual resulting integrals.) We picture the resulting $t^1, t^2$ coordinate curves on $V$ as engraved on $V$ just as latitude and longitude curves are engraved on globes of the Earth. We demand that the sense of rotation from the engraved $t^1$ curve to the $t^2$ curve on $V$ (i.e., from $F_*\boldsymbol{\partial}_1$ to $F_*\boldsymbol{\partial}_2$) is the same as the given orientation arrow on $V$. We say $V = F(S)$.




We now define

$$\int_V b_{23}\,dx^2 \wedge dx^3 + b_{31}\,dx^3 \wedge dx^1 + b_{12}\,dx^1 \wedge dx^2 = \int_V \beta^2 = \int_{F(S)} \beta^2 := \int_S F^*\beta$$

reducing the problem to defining the integral of the pull-back of $\beta$ over $S$. First write this out, but for simplicity we just look at the term $b_{31}(x)dx^3 \wedge dx^1$. As in (0.22)**

$$\begin{aligned}\int_S F^*(b_{31}(x)dx^3 \wedge dx^1) &:= \int_S b_{31}(x(t))[(\partial x^3/\partial t^a)dt^a \wedge (\partial x^1/\partial t^b)dt^b] \\ &= \int_S b_{31}(x(t))[\partial(x^3, x^1)/\partial(t^1, t^2)]dt^1 \wedge dt^2 \\ &:= \int_S b_{31}(x(t))[\partial(x^3, x^1)/\partial(t^1, t^2)]dt^1 dt^2\end{aligned}$$

and where the very last integral, with no $\wedge$ , is the **usual double integral** over a region $S$ in the $t^1, t^2$ plane. Thus

$$\begin{aligned}\int_V \beta^2 &= \int_{F(S)} \beta^2 = \int_S F^*\beta^2 \\ &:= \int_S \{b_{23}(x(t))[\partial(x^2, x^3)/\partial(t^1, t^2)] + b_{31}(x(t))[\partial(x^3, x^1)/\partial(t^1, t^2)] \\ &\quad + b_{12}(x(t))[\partial(x^1, x^2)/\partial(t^1, t^2)]\}dt^1 dt^2 \qquad (0.25)\end{aligned}$$

Note that one does not need to commit this to memory. One merely uses the chain rule in calculus and $dt^1 \wedge dt^2 = -dt^2 \wedge dt^1$ to get an integral over a region in the $t^1, t^2$ plane, then omit the $\wedge$ and evaluate the resulting double integral.

**Interpretation:** In *cartesian* coordinates with the usual metric in $\mathbb{R}^3$, associate to $\beta^2$ the vector

$$\mathbf{B} = (B^1 = b_{23}, B^2 = b_{31}, B^3 = b_{12})^T$$

$\mathbf{n} = [\partial\mathbf{x}/\partial t^1] \times [\partial\mathbf{x}/\partial t^2]$ is a normal to the surface with components

$$([\partial(x^2, x^3)/\partial(t^1, t^2)], [\partial(x^3, x^1)/\partial(t^1, t^2)], [\partial(x^1, x^2)/\partial(t^1, t^2)])^T$$

Just as in the case of a curve, where $\|d\mathbf{x}/dt\|dt$ is the element of arc length $ds$, so in the case of a surface, where $\partial\mathbf{x}/\partial t^1$ and $\partial\mathbf{x}/\partial t^2$ span a parallelogram of area $\|(\partial\mathbf{x}/\partial t^1) \times (\partial\mathbf{x}/\partial t^2)\| = \|\mathbf{n}\|$, we have the area element "$dA$" $= \|\mathbf{n}\|dt^1 dt^2$. Our integral ($\mathcal{O}$.25) then becomes

$$\int_V \beta^2 = \iint_S \langle \mathbf{B}, \mathbf{n}\rangle dt^1 dt^2 = \iint_S \|\mathbf{B}\|\|\mathbf{n}\| \cos\angle(\mathbf{B}, \mathbf{n}) dt^1 dt^2$$
$$= \iint_V B_{\text{normal}}\text{“}dA\text{”} \quad \text{(classically)}$$

and this shows further that the integral $\int_V \beta$ is in fact independent of the parameterization $F$ used.

Note again that our form version ($\mathcal{O}$. 25) requires no metric or area element.

**Moral:** The integrand in a surface integral is naturally a 2-form, not a vector.

One integrates exterior $p$-forms over oriented $p$ dimensional "surfaces" $V^p$. If $V^p$ is not a "closed" surface it will generically have a $(p-1)$ dimensional oriented boundary, written $\partial V$. For example, if $V^2$ is oriented, then the circular orientation arrow near the boundary curve of $V$ will yield a "direction" for $\partial V$ ( see the surface integral picture above)

$$\textbf{Stokes' Theorem} \quad \int_V d\beta^{p-1} = \oint_{\partial V} \beta^{p-1} \tag{$\mathcal{O}$.26}$$

is perhaps the World's Most Beautiful Formula. The vector analysis versions, using ($\mathcal{O}$.19), include not only Stokes' theorem (really due to **William Thomson, Lord Kelvin**) when $p = 2$ and $V^2$ is an oriented surface and $\partial V$ is its closed curve boundary, but also **Gauss**' divergence theorem when $p = 3$, $V^3$ is a bounded region in space and $\partial V$ is its closed surface boundary. For a proof see Chapter 3.

## $\mathcal{O}$.l. Electromagnetism, or, Is it a Vector or a Form?

For simplicity we consider electric and magnetic fields caused by charges, currents, and magnets in a vacuum (without polarizations, . . .)

**Electric field intensity E:** The work done in moving a particle with charge $q$ along a curve $C$ is classically $W = \int_C q\, \mathbf{E} \cdot d\mathbf{r}$ but really $w = q\int_C \mathcal{E}^1 = q\int_C E_1\, dx^1 + E_2\, dx^2 + E_3\, dx^3$. The electric field intensity is a 1-form $\mathcal{E}^1 = E_1\, dx^1 + E_2\, dx^2 + E_3\, dx^3$.

**Electric field D:** The charge $Q$ contained in a region $V^3$ with boundary $\partial V$ is classically given by $4\pi Q(V^3) = \iint_{\partial V} \mathbf{D} \cdot d\mathbf{A} = \iiint_V \text{div } \mathbf{D} \text{ vol}$, but really

$$\iint_{\partial V} \mathcal{D}^2 = \iiint_V d\mathcal{D} = 4\pi Q(V^3) = 4\pi \iiint_V \rho\text{vol}^3$$

where $\rho$ is the charge density. Stokes' theorem thus yields **Gauss' law**

$$d\mathcal{D}^2 = 4\pi\ \rho\text{vol}^3$$

$\mathcal{D}^2$ is a 2-form version of $\mathcal{E}^1$ . In *cartesian* coordinates $\mathcal{D}^2 = E_1\, dx^2 \wedge dx^3 + E_2\, dx^3 \wedge dx^1 + E_3\, dx^1 \wedge dx^2$.

**Magnetic field intensity B:** Faraday's law says classically, for a *fixed* surface $V^2$, $\oint_{\partial V} \mathbf{E} \cdot d\mathbf{r} = -d/dt \iint_V \mathbf{B} \cdot d\,\mathbf{A}$. Really $\oint_{\partial V} \mathcal{E}^1 = -d/dt \iint_V \mathcal{B}^2$. The magnetic field intensity is a 2-form $\mathcal{B}^2$ and **Faraday's law** says

$$d\mathcal{E}^1 = -\partial\mathcal{B}^2/\partial t$$

where $\partial\mathcal{B}^2/\partial t$ means take the time derivative of the components of $\mathcal{B}^2$. Another axiom states that

$$\text{div } \mathbf{B} = 0 = d\mathcal{B}^2$$

**Magnetic field H: Ampère–Maxwell** says classically $\oint_{C=\partial V} \mathbf{H} \cdot d\mathbf{r} = 4\pi \iint_V \mathbf{j} \cdot d\mathbf{A} + d/dt \iint_V \mathbf{D} \cdot d\mathbf{A}$ where $V^2$ is **fixed** and **j** is the current vector. Really

$$\oint_{C=\partial V} \mathcal{H}^1 = 4\pi \iint_V \mathcal{j}^2 + d/dt \iint_V \mathcal{D}^2$$

and thus

$$d\mathcal{H}^1 = 4\pi \mathcal{j}^2 + \partial\mathcal{D}^2/\partial t$$

where $\mathcal{j}^2$ is the current 2-form whose integral over $V^2$ (with a preferred normal direction) measures the time rate of charge passing through $V^2$ in that direction. $\mathcal{H}^1$ is a 1-form version of $\mathcal{B}^2$. In *cartesian* coordinates

$$\mathcal{H}^1 = B_{23}dx^1 + B_{31}dx^2 + B_{12}dx^3$$

**Heaviside–Lorentz force:** Classically the electromagnetic force acting on a particle of charge $q$ moving with velocity **v** is given by $\mathbf{f} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. We have seen that force and the electric field should be 1-forms, $\mathcal{f}^1 = q(\mathcal{E}^1 + ??)$. **v** is definitely a vector, and $\mathcal{B}$ is a 2-form! We now discuss this dilemma raised by the vector product $\times$ and its resolution will play a large role in our discussion of elasticity also.

## 0.m. Interior Products

We are at home with the fact $\alpha^1 \wedge \beta^1$ is a 2-form replacement for a $\times$ product of vectors in $\mathbb{R}^3$, but if we had started out with two vectors **A** and **B** it would require a metric to change them to 1-forms. It turns out there is also a 1-form replacement that is frequently more useful, and will resolve the Lorentz force problem.

In $\mathbb{R}^n$, if **v** is a vector and $\beta^p$ is a $p$-form, $p > 0$, we define the **interior product** of **v** and $\beta$ to be the $(p-1)$-form $i_\mathbf{v}\beta$ (sometimes we write $i(\mathbf{v})\beta$) with values

$$i_\mathbf{v}\beta^p(\mathbf{A}_2, \ldots, \mathbf{A}_p) := \beta^p(\mathbf{v}, \mathbf{A}_2, \ldots, \mathbf{A}_p) \tag{0.27}$$

(It can be shown that this is a contraction, $(i_\mathbf{v}\beta)_{bc\ldots} = v^i\beta_{ibc\ldots}$). This is a form since it clearly is multilinear in $\mathbf{A}_2, \ldots, \mathbf{A}_p$, since $\beta$ is, and changes sign under each interchange of the $A$, and is defined independent of any coordinates. In the case of a 1-form $\beta$, $i_\mathbf{v}\beta$ is the 0-form (function)

$$i_\mathbf{v}\beta^1 = \beta^1(\mathbf{v}) = b_i v^i$$

which is equal to $\langle \mathbf{v}, \mathbf{b} \rangle$ in any Riemannian metric. Look at $i_{\mathbf{v}}(\alpha^1 \wedge \beta^1)$:

$$i_{\mathbf{v}}(\alpha^1 \wedge \beta^1)(\mathbf{C}) = (\alpha^1 \wedge \beta^1)(\mathbf{v}, \mathbf{C}) = \alpha(\mathbf{v})\beta(\mathbf{C}) - \alpha(\mathbf{C})\beta(\mathbf{v})$$
$$= (i_{\mathbf{v}}\alpha)\beta(\mathbf{C}) - (i_{\mathbf{v}}\beta)\alpha(\mathbf{C}) = [(i_{\mathbf{v}}\alpha)\beta - (i_{\mathbf{v}}\beta)\alpha](\mathbf{C})$$

A more tedious calculation shows the general product rule

$$i_{\mathbf{v}}(\alpha^p \wedge \beta^q) = [i_{\mathbf{v}}(\alpha^p)] \wedge \beta^q + (-1)^p \alpha^p \wedge [i_{\mathbf{v}}\beta^q] \tag{0.28}$$

just as for the differential $d$ (see (0.20)).

## 0.n. Volume Forms and Cartan's Vector Valued Exterior Forms

Let $x$, $y$ be positively oriented cartesian coordinates in $\mathbb{R}^2$. The area 2-form in the cartesian plane is $\text{vol}^2 = dx \wedge dy$, but in polar coordinates we have $\text{vol}^2 = r dr \wedge d\theta$. Looking at (0.7) we note that $r = \sqrt{g}$, where

$$g := \det(g_{ij}) \tag{0.29}$$

In any Riemannian metric, in any oriented $\mathbb{R}^n$, we define the volume $n$-form to be

$$\text{vol}^n := \sqrt{g} dx^1 \wedge \ldots \wedge dx^n \tag{0.30}$$

in any positively oriented curvilinear coordinates. It can be shown that this is indeed an $n$-form (modulo some question of orientation that I do not wish to consider here). In spherical coordinates in $\mathbb{R}^3$ we get, since $(g_{ij}) = \text{diag}(1, r^2, r^2 \sin^2\theta)$, the familiar $\text{vol}^3 = r^2 \sin\theta dr \wedge d\theta \wedge d\phi$.

Note now the following in $\mathbb{R}^3$ in any coordinates. For any vector $\mathbf{v}$

$$i_{\mathbf{v}}\text{vol}^3 = i_{\mathbf{v}}\sqrt{g} dx^1 \wedge dx^2 \wedge dx^3 = \sqrt{g} i_{\mathbf{v}}(dx^1 \wedge dx^2 \wedge dx^3)$$

Now apply the product rule (0.28) repeatedly

$$i_{\mathbf{v}}(dx^1 \wedge dx^2 \wedge dx^3) = v^1 dx^2 \wedge dx^3 - dx^1 \wedge i_{\mathbf{v}}(dx^2 \wedge dx^3)$$
$$= v^1 dx^2 \wedge dx^3 - dx^1 \wedge [v^2 dx^3 - v^3 dx^2]$$
$$= v^1 dx^2 \wedge dx^3 - v^2 dx^1 \wedge dx^3 + v^3 dx^1 \wedge dx^2$$

and so

$$i_{\mathbf{v}}\text{vol}^3 = \sqrt{g}[v^1 dx^2 \wedge dx^3 + v^2 dx^3 \wedge dx^1 + v^3 dx^1 \wedge dx^2] \tag{0.31}$$

is the **2-form version** of a vector $\mathbf{v}$ in $\mathbb{R}^3$ with a volume form $\text{vol}^3$.

**Remark:** For a surface $V^2$ in Riemannian $\mathbb{R}^3$, with **unit** normal vector field $\mathbf{n}$, it is easy to see that $i_{\mathbf{n}}\text{vol}^3$ is the **area 2-form** for $V^2$. Simply look at its value on a pair of vectors $(\mathbf{A}, \mathbf{B})$ **tangent** to $V$; $i_{\mathbf{n}}\text{vol}^3(\mathbf{A}, \mathbf{B}) = \text{vol}^3(\mathbf{n}, \mathbf{A}, \mathbf{B})$ is the area spanned by $\mathbf{A}$ and $\mathbf{B}$.

Comparing (0.31) with (0.14) we see that the most general 2-form $\beta^2$ in $\mathbb{R}^3$ (with $\text{vol}^3$), in any coordinates, is of the form

$$\beta^2 = i_{\mathbf{b}}\text{vol}^3 \quad \text{where } b^1 = b_{23}/\sqrt{g}, \text{etc.} \tag{0.14$'$}$$

In electromagnetism,

$$\mathcal{D}^2 = i_{\mathbf{E}}\mathrm{vol}^3$$

The same procedure works for an $(n-1)$ form in $\mathbb{R}^n$. Note that this does not require an entire metric tensor, we use only the *volume element*. If we have a *distinguished volume form* (i.e., if we have a coordinate independent notion of the volume spanned by a "positively oriented" $n$-tuple of vectors in $\mathbb{R}^n$), even if it is not derived from a metric, we shall use the same notation in positively oriented coordinates, as given in ($\mathcal{O}$.30)

$$\mathrm{vol}^n = \sqrt{g} dx^1 \wedge \ldots \wedge dx^n$$

where $\sqrt{g} > 0$ is now merely some coefficient function dependent on the choice of volume form and the coordinates used. (Warning: this notation is my own and is not standard.)

If we have a volume form, we can define the **divergence of a vector field v** as follows

$$\begin{aligned}(\mathrm{div}\,\mathbf{v})\mathrm{vol}^n :&= d(i_{\mathbf{v}}\mathrm{vol}^n) = d\{\sqrt{g}[v^1\, dx^2 \wedge dx^3 \wedge \ldots \wedge dx^n \\ &\quad - v^2\, dx^1 \wedge dx^3 \wedge \ldots \wedge dx^n + \cdots]\} \\ &= [\partial(v^1\sqrt{g})/\partial x^1 + \partial(v^2\sqrt{g})/\partial x^2 + \cdots]\, dx^1 \wedge \ldots \wedge dx^n\end{aligned}$$

i.e.,

$$\mathrm{div}\,\mathbf{v} = (1/\sqrt{g})\partial/\partial x^i(\sqrt{g}v^i) \tag{$\mathcal{O}$.32}$$

If, furthermore, the volume form comes from a Riemannian metric we can define the **Laplacian** of a **function** $f$ by

$$\nabla^2 f := \Delta f := \mathrm{div}\,\nabla f = (1/\sqrt{g})\partial/\partial x^i(\sqrt{g}g^{ij}\partial f/\partial x^j)) \tag{$\mathcal{O}$.33}$$

We now wish to consider the notion of vector or $\times$ product in more detail. We have seen in Section $\mathcal{O}$.h that in $\mathbb{R}^3$ in *any* coordinates the 2-form

$$\begin{aligned}\alpha^1 \wedge \gamma^1 &= (a_1\, dx^1 + a_2\, dx^2 + a_3\, dx^3) \wedge (c_1\, dx^1 + c_2\, dx^2 + c_3\, dx^3) \\ &= (a_2c_3 - a_3c_2)dx^2 \wedge dx^3 + (a_3c_1 - a_1c_3)dx^3 \wedge dx^1 + (a_1c_2 - a_2c_1)dx^2 \wedge dx^3\end{aligned}$$

corresponds to the cross product $\mathbf{a} \times \mathbf{c}$ in *cartesian* coordinates, and this 2-form version is ideal when considering surface integrals in any coordinates.

We shall now give a 1-form version of $\mathbf{a} \times \mathbf{b}$, we write $(\mathbf{a} \times \mathbf{b})_*$, which will be very useful in line integrals and in our later sections considering electromagnetism and elasticity.

In $\mathbb{R}^3$ with a $\mathrm{vol}^3$, and in any coordinates, we define

$(\mathbf{a} \times \mathbf{b})_*$ is the unique **1-form** defined by $(\mathbf{a} \times \mathbf{b})_*(\mathbf{c}) := \mathrm{vol}^3(\mathbf{a}, \mathbf{b}, \mathbf{c})$

for every vector $\mathbf{c}$. If we have a metric, then $(\mathbf{a} \times \mathbf{b})_*\,(\mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c}) = \mathrm{vol}^3(\mathbf{a}, \mathbf{b}, \mathbf{c})$ gives the usual definition of the **vector a** $\times$ **b**, but clearly the 1-form version is more basic since it does not require a metric. (Question: how would you define a $\times$-product of $n-1$ vectors in an $\mathbb{R}^n$ with a $\mathrm{vol}^n$?)

Note

$$\mathrm{vol}^3(\mathbf{a}, \mathbf{b}, \mathbf{c}) = -\mathrm{vol}^3(\mathbf{b}, \mathbf{a}, \mathbf{c}) = (-i_{\mathbf{b}}\ \mathrm{vol}^3)(\mathbf{a}, \mathbf{c}) = -\beta^2(\mathbf{a}, \mathbf{c}) = (-i_{\mathbf{a}}\beta^2)(\mathbf{c})$$

where $\beta^2 = i_{\mathbf{b}}$ vol is the 2-form version of $\mathbf{b}$. Thus in *any* coordinates with a $\mathrm{vol}^3$

$$(\mathbf{a} \times \mathbf{b})_* = -i_{\mathbf{a}}\beta^2 = -i_{\mathbf{a}}[i_{\mathbf{b}}\ \mathrm{vol}^3] \qquad (0.34)$$

which, from (0.31)

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})_* &= -i(a^1\boldsymbol{\partial}_1 + a^2\boldsymbol{\partial}_2 + a^3\boldsymbol{\partial}_3)\sqrt{g}[b^1\, dx^2 \wedge dx^3 + b^2\, dx^3 \wedge dx^1 + b^3\, dx^1 \wedge dx^2] \\ &= \sqrt{g}[(a^2b^3 - a^3b^2)dx^1 + (a^3b^1 - a^1b^3)dx^2 + (a^1b^2 - a^2b^1)dx^3]\end{aligned}$$

*Now* we can write the Lorentz force law of Section 0.1

$$\mathcal{f}^1 = q(\mathcal{E}^1 - i_{\mathbf{v}}\mathcal{B}^2)$$

Finally, an important restatement of the cross product in $\mathbb{R}^3$. We are going to follow **Elie Cartan** and use 2-forms whose values on pairs of vectors are not numbers but rather vectors or covectors. Let $\chi_*^{(2)} = \chi_*$ be the covector-valued 2-form with value the covector $\chi_*\,(\mathbf{a}, \mathbf{b})\!: = (\mathbf{a} \times \mathbf{b})_*$ . The $\boldsymbol{j}^{\text{th}}$ component of this covector is

$$\chi_*(\mathbf{a}, \mathbf{b})_j = (\mathbf{a} \times \mathbf{b})_j = (\mathbf{a} \times \mathbf{b})_*(\boldsymbol{\partial}_j) = \mathrm{vol}^3(\boldsymbol{\partial}_j, \mathbf{a}, \mathbf{b}) = [i(\boldsymbol{\partial}_j)\mathrm{vol}^3](\mathbf{a}, \mathbf{b})$$

Thus

$$\chi_* = dx^j \otimes \chi_j = dx^j \otimes [i(\boldsymbol{\partial}_j)\mathrm{vol}^3] \qquad (0.35)_*$$

Note the $\otimes$ *not* $\wedge$. By definition, the value of the 2-form $\chi_*$ on the pair of vectors $\mathbf{a}$, $\mathbf{b}$ is not a number, but rather the 1-form

$$\chi_*(\mathbf{a}, \mathbf{b}) = [\mathrm{vol}^3(\boldsymbol{\partial}_j, \mathbf{a}, \mathbf{b})]dx^j$$

With a Riemannian metric, the **contravariant** version is the vector valued 2-form

$$\boldsymbol{\chi}^* = \boldsymbol{\partial}_i \otimes g^{ij} i(\boldsymbol{\partial}_j)\mathrm{vol}^3 \qquad (0.35)^*$$

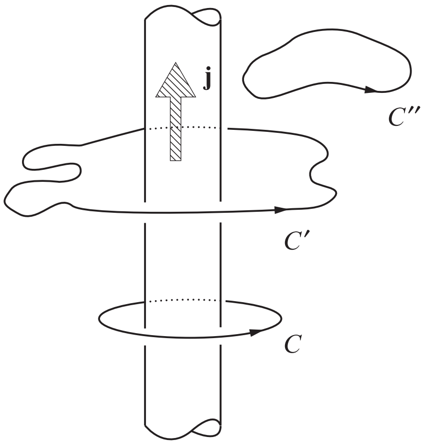
This is the 2-form that, when applied to the pair of vectors, yields $\mathbf{a} \times \mathbf{b}$. In cartesian coordinates we can write it symbolically as the column of 2-forms

$$[dy \wedge dz \ \ dz \wedge dx \ \ dx \wedge dy]^T$$

whose value on a pair of vectors $(\mathbf{a}, \mathbf{b})$ is the column of components of $\mathbf{a} \times \mathbf{b}$.

## 0.o. Magnetic Field for Current in a Straight Wire

This simple example illustrates much of what we have done. Consider a steady current $\mathbf{j}$ in a thin straight wire of infinite length.




Since the current is steady we have Ampère's law $\oint_{C=\partial V} \mathcal{H}^1 = 4\pi \iint_V j^2$. Looking at three surfaces bounded respectively by $C$, $C'$, and $C''$ and the flux of current through them, we have

$$\oint_C \mathcal{H}^1 = 4\pi j = \oint_{C'} \mathcal{H}^1$$

while $\oint_{C''} \mathcal{H}^1 = 0$. Introducing cylindrical coordinates, we can guess immediately that $\mathcal{H}^1 = 2j\, d\theta$ *in the region outside the wire*, for it has the correct integrals. We *require*, however, that div $\mathbf{B} = 0 = d\mathcal{B}^2$. Now $\mathcal{B}^2 = i_{\mathbf{H}}\, \text{vol}^3$ where $\mathbf{H}$ is the contravariant version of the 1-form $\mathcal{H}$. The metric for cylindrical coordinates is diag$(1, r^2, 1)$ and $H_\theta = 2j$ is the only nonzero component of our guess $\mathcal{H}^1$, hence $H^\theta = g^{\theta\theta} H_\theta$ (no sum) $= (1/r^2)2j$. Then $\mathcal{B}^2 = i_{\mathbf{H}}\text{vol}^3$ becomes

$$\mathcal{B}^2 = (2j/r^2)i(\boldsymbol{\partial}_\theta)r\, dr \wedge d\theta \wedge dz = -(2j/r)dr \wedge dz = d[-2j(\ln r)dz]$$

Clearly $d\mathcal{B} = 0$, as required and, in fact, $[-2j(\ln r)dz]$ is a "magnetic **potential**" 1-form $\alpha^1$ outside the wire, $\mathcal{B}^2 = d\alpha^1$. Another choice is $\alpha^1 = 2jz/r\, dr$.

## Elasticity and Stresses

### $\mathcal{O}$.p. Cauchy Stress, Floating Bodies, Twisted Cylinders, and Strain Energy

In learning the sciences examples are of more use than precepts.

Isaac Newton, *Arithmetica Universalis* (1707)

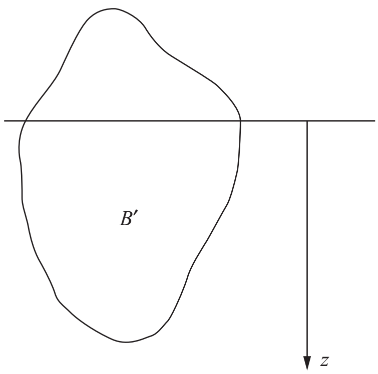
We look at our cylinder $B$ and its twisted version $F(B)$ in Section $\mathcal{O}$.a, but *first* we shall use *cartesian* coordinates $x^i$. Consider any small surface $V$ in $F(B)$ passing through a point $p$ and let $\mathbf{n}$ be a normal to $V$ at $p$. Then because of the twisting, the material on the side of $V$ towards which $\mathbf{n}$ is pointing, exerts a force $\mathbf{f}$ on the material on the other side of $V$. Cauchy's "first theorem" states that this force is reversed if we replace $\mathbf{n}$ by

–**n**, and further this (contravariant) force is given by integrating a vector valued 2-form $\mathfrak{t}$ over $V$ (*not* Cauchy's language)

$$\mathbf{f} \text{ on } V = \boldsymbol{\partial}_a \left[ \int_V t^{ab} i(\boldsymbol{\partial}_b) \text{vol}^3 \right]$$

where **t** is the "Cauchy stress tensor." A sketch of a proof of Cauchy's theorem will be given in Section $\mathfrak{O}$.q. Cauchy's "second theorem" says $t^{ab} = t^{ba}$ and a proof sketch is given in Section $\mathfrak{O}$.r. (The fact that the stress force is reversed if **n** is replaced by $-\mathbf{n}$ informs us (see Section 2.8f) that the stress form is technically a "pseudo-form." )

As a warm-up check of our machinery, let us look first at an example of the simplest type of stress from elementary physics. In the case of a **nonviscous fluid**, given a very small parallelogram spanned by **v** and **w** and with normal $\mathbf{n} = \mathbf{v} \times \mathbf{w}$, the fluid on the side to which **n** is pointing exerts a force on the other side approximated by $-p\mathbf{v} \times \mathbf{w}$, where $p$ is the hydrostatic **pressure**. From ($\mathfrak{O}$.35) the stress vector valued 2-form is given by $\mathfrak{t} = -\boldsymbol{\partial}_i \otimes pg^{ij} i(\boldsymbol{\partial}_j)\text{vol}^3$. In a pool with cartesian coordinates $x$, $y$, $z$, with the origin at the surface and $z$ pointing down, look at a floating body $B$, with portion $B'$ below the water surface, with surface normal pointing out of $B$. While Archimedes knew the result, *we* need to practice with our new tools.




Then the total stress force exerted on $\partial B$ from water of *constant* density $\rho$ outside $B$ is, with $g^{ij} = \delta^{ij}$ and $p = \rho g z$

$$\begin{aligned} \mathbf{f} &= \boldsymbol{\partial}_i \int_{\partial B'} t^{ij} i(\boldsymbol{\partial}_j)\text{vol}^3 = -\boldsymbol{\partial}_i \int_{\partial B'} p\delta^{ij}\, i(\boldsymbol{\partial}_j)\text{vol}^3 \\ &= -\boldsymbol{\partial}_x \int_{\partial B'} \rho g z\, dy \wedge dz - \boldsymbol{\partial}_y \int_{\partial B'} \rho g z\, dz \wedge dx - \boldsymbol{\partial}_z \int_{\partial B'} \rho g z\, dx \wedge dy \end{aligned}$$

where we have included the part of $\partial B'$ at water level $z = 0$, even though there is no water there, since $\rho g z = 0$ there and we get a 0 contribution from it. We shall evaluate the surface integrals by applying Stokes' theorem ($\mathfrak{O}$.26) to $B'$. The three 2-forms $\rho g z\, dy \wedge dz$, etc, apply only to the outside of $B'$ since there is no water inside $B'$. To apply Stokes' theorem to $B'$, we must *extend* these 2-forms from the boundary of $B'$ mathematically to the inside of $B'$, *in any smooth way that we wish*, and we choose the same forms as are given outside $B'$, with $\rho = \rho_{\text{water}}$ again! Then by Stokes

$$\mathbf{f} = -\boldsymbol{\partial}_x \int_{B'} d[\rho g z \, dy \wedge dz] \quad -\boldsymbol{\partial}_y \int_{B'} d[\rho g z \, dz \wedge dx] \quad -\boldsymbol{\partial}_z \int_{B'} d[\rho g z \, dx \wedge dy]$$
$$= -\boldsymbol{\partial}_z \int_{B'} \rho g \, dx \wedge dy \wedge dz = -\boldsymbol{\partial}_z W'$$

where $W'$ is the weight of the water displaced by $B'$. Equilibrium demands this must equal the weight of the whole body $B$. Thus a floating body displaces its own weight in water. EUREKA!

**Back to our twisted cylinder:** Introduce cylindrical coordinates $(X^A) = (R, \Theta, Z)$ for the untwisted cylinder $B$. Next, introduce an *identical* set of coordinates $(x^a) = (r, \theta, z)$ and use the capitalized coordinates for a point in the untwisted body and $r, \theta, z$ for the coordinates of the image point under the twist $F$. Thus $F$ is described by $r = R, \theta = \Theta + kZ$, and $z = Z$, where $k$ is a constant. We need to determine the **Cauchy** vector valued stress 2-form $\mathfrak{t} = \boldsymbol{\partial}_a \otimes \mathfrak{t}_a = \boldsymbol{\partial}_a \otimes t^{ab} i(\boldsymbol{\partial}_b)\text{vol}^3$ on $F(B)$ in terms of the twisting forces and the material from which $B$ is made. We shall do this by first pulling this 2-form back to the untwisted body $B$ by the following procedure; we **pull** back the 2-forms $\mathfrak{t}^a$ by $F^*$ and we **push** the vectors $\boldsymbol{\partial}_a$ back to $B$ by the inverse $(F^{-1})_*$, which exists since $F$ is a 1:1 deformation. The resulting vector valued 2-form on $B$ is

$$\mathbb{S} = [(F^{-1})_*(\boldsymbol{\partial}_a)] \otimes F^*\mathfrak{t}^a = (F^{-1})_*(\boldsymbol{\partial}_a) \otimes F^*[t^{ab} i(\boldsymbol{\partial}_b)\text{vol}^3]$$

which is of the form

$$\mathbb{S} = \boldsymbol{\partial}_A \otimes \mathbb{S}^A = \boldsymbol{\partial}_A \otimes \text{S}^{AB} i(\boldsymbol{\partial}_B)\text{vol}^3 \qquad (\mathcal{C}.36)$$

called the **second Piola–Kirchhoff** vector valued stress 2-form. We shall relate *this* form to the twist $F$ by a generalization of Hooke's law.

We need to know how this twist $F$ has stretched lengths and changed angles in the body, and this is described as follows. The euclidean metric is $dS^2 = (dR^2 + R^2\, d\Theta^2 + dZ^2) = ds^2 = (dr^2 + r^2\, d\theta^2 + dz^2)$. The pull-back (last paragraph of Section $\mathcal{C}$.j) of $ds^2$ under the twist $F$ is given by the chain rule

$$F^*ds^2 = F^*(dr^2 + r^2\, d\theta^2 + dz^2) = dR^2 + R^2[(\partial\theta/\partial\Theta)d\Theta + (\partial\theta/\partial Z)dZ]^2 + dZ^2$$
$$= dR^2 + R^2[d\Theta + k\, dZ]^2 + dZ^2$$
$$= dR^2 + R^2[d\Theta^2 + 2k\, d\Theta dZ + k^2\, dZ^2] + dZ^2$$

Recall what this is saying. At a point $R, \Theta, Z$ of the untwisted body, given two vectors $\mathbf{A}, \mathbf{B}$, we have not only the scalar product $\langle \mathbf{A}, \mathbf{B} \rangle = dS^2(\mathbf{A}, \mathbf{B})$ but also the scalar product of the images after the twist, i.e., from ($\mathcal{C}$.23), $ds^2(F_*\mathbf{A}, F_*\mathbf{B}) =: (F^*ds^2)(\mathbf{A}, \mathbf{B})$. Then *one* measure of how much the twist $F$ is distorting distances and angles is defined by the **Lagrange deformation tensor**

$$E := \tfrac{1}{2}[(F^*ds^2) - dS^2] \qquad (\mathcal{C}.37)$$

The quadratic form (covariant second rank tensor) $E$ is determined by its square matrix.

How do the stresses depend on the deformations? In our twisting case we have $E = kR^2\, d\Theta\, dZ + \frac{1}{2}k^2 R^2\, dZ^2$. We will work only to the *first approximation for small*

$k$, i.e., we shall put $k^2 = 0$, so $E = kR^2\, d\Theta\, dZ = \frac{1}{2}kR^2(d\Theta\, dZ + dZ\, d\Theta)$. We write the components as a symmetric matrix

$$(E_{IJ}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & kR^2/2 \\ 0 & kR^2/2 & 0 \end{bmatrix}$$

The mixed version, using $E^A{}_B = G^{AI} E_{IB}$ and $(G^{KL}) = \mathrm{diag}(1, 1/R^2, 1)$, is the (nonsymmetric)

$$(E^A{}_B) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & k/2 \\ 0 & kR^2/2 & 0 \end{bmatrix}$$

and thus tr $E = E^A{}_A = 0$ "mod $k^2$," i.e., putting $k^2 = 0$. Finally, putting $E^{AB} = E^A{}_I G^{IB}$

$$(E^{AB}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & k/2 \\ 0 & k/2 & 0 \end{bmatrix}$$

**Linear** elasticity assumes a linear, vastly generalized "Hooke's law" relating the stress $S$ to the deformation $E$. Assuming the body is **isotropic** (i.e., the material has no special internal directional structure such as grains in wood), it can then be shown (e.g., equation (D.9)), that there are then only two "elastic constants" $\mu$ and $\lambda$ relating $S$ to $E$

$$S^{AB} = 2\mu E^{AB} + \lambda(\mathrm{tr}\, E)G^{AB} \tag{0.38}$$

and so

$$(S^{AB}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \mu k \\ 0 & \mu k & 0 \end{bmatrix}$$

This gives rise to the second Piola–Kirchhoff vector valued stress 2-form on the *undeformed* body

$$\begin{aligned} \mathcal{S} &:= \boldsymbol{\partial}_I \otimes S^{IJ} i(\boldsymbol{\partial}_J)\mathrm{VOL}^3 = \boldsymbol{\partial}_I \otimes S^{IJ} i(\boldsymbol{\partial}_J) R\, dR \wedge d\Theta \wedge dZ \\ &= [\boldsymbol{\partial}_\Theta \otimes S^{\theta Z} i(\boldsymbol{\partial}_Z) + \boldsymbol{\partial}_Z \otimes S^{Z\theta} i(\boldsymbol{\partial}_\theta)] R\, dR \wedge d\Theta \wedge dZ \\ \mathcal{S} &= \mu k R[\boldsymbol{\partial}_\Theta \otimes dR \wedge d\Theta + \boldsymbol{\partial}_Z \otimes dZ \wedge dR] \end{aligned} \tag{0.38'}$$

Finally, the **Cauchy stress vector valued 2-form** $\mathfrak{t}$ on the "current" deformed body from (0.36), is $\mathfrak{t} = F_* \boldsymbol{\partial}_A \otimes (F^{-1})^* S^A$. Using $F^{-1}$ defined by $R = r$, $\Theta = \theta - kz$, $Z = z$, we get

$$\begin{aligned} \mathfrak{t} &= \mu k r[\boldsymbol{\partial}_\theta \otimes (F^{-1})^*(dR \wedge d\Theta) + \boldsymbol{\partial}_z \otimes (F^{-1})^*(dZ \wedge dR)] \\ &= \mu k r[\boldsymbol{\partial}_\theta \otimes dr \wedge (d\theta - k\, dz) + \boldsymbol{\partial}_z \otimes dz \wedge dr] \quad \text{and discarding } k^2 \\ \mathfrak{t} &= \mu k r[\boldsymbol{\partial}_\theta \otimes dr \wedge d\theta + \boldsymbol{\partial}_z \otimes dz \wedge dr] \end{aligned} \tag{0.39}$$

To get correct "dimensions" for force we use the "physical" components of force, i.e., we normalize the (already orthogonal) basis vectors. Since $g_{rr} = 1 = g_{zz}$, $\boldsymbol{\partial}_r$ and

$\boldsymbol{\partial}_z$ are unit vectors, call them $\mathbf{e}_r$ and $\mathbf{e}_z$. But $g_{\theta\theta} = r^2$, and so $\boldsymbol{\partial}_\theta$, by (©.6), has length $r$, and so we put $\mathbf{e}_\theta = r^{-1}\boldsymbol{\partial}_\theta$. We make no changes to the form parts $dr$, $d\theta$, and $dz$

$$\mathfrak{t} = \mu k r^2 \mathbf{e}_\theta \otimes dr \wedge d\theta + \mu k r \mathbf{e}_z \otimes dz \wedge dr \tag{©.40}$$

We shall now see the consequences of this Cauchy stress. Look first at the lateral surface $r = a$. Then $dr = 0$ there and so $\mathfrak{t} = 0$ on this surface. *This means that no external "traction" on this part of the boundary is needed for this twisting.*

Now look at the end boundary at $z = L$. From (©.40) we have stress from outside

$$\mu k r^2 \mathbf{e}_\theta \otimes dr \wedge d\theta$$

acting in the $\mathbf{e}_\theta$ direction. This has to be supplied by **external tractions** since there is no part of the body past its ends. What is the **moment** of the traction? We have a disk, radius $a$, a force of magnitude $\mu k r^2\, dr\, d\theta$ acting in the $\mathbf{e}_\theta$ direction on an infinitesimal "rectangle" of "sides" $dr$ and $d\theta$. The **moment** about the $z$ axis is $r(\mu k r^2) dr\, d\theta$, and so the total moment is $\mu k \iint r^3 dr\, d\theta = \mu k (a^4/4) 2\pi = \pi \mu k a^4/2$. If the total twist at $z = L$ is an angle of twist $\alpha = kL$, then the total moment required is $\pi \mu a^4 \alpha / 2L$. An opposite moment is required at $z = 0$. An experiment could yield the value of $\mu$.

In the case of the floating body, treated near the beginning of our Section ©.p, our argument *really* showed the following. Take any blob of fluid $B''$ surrounded by fluid at rest under the surface $z = 0$. Then the hydrostatic stress (pressure) on $\partial B''$ due to the water surrounding $B''$ produced a "body force" that supported the weight of the water in $B''$. We now show that in the case of our twisted cylinder, to order $k$,

the Cauchy stresses produce no internal **body** forces inside the cylinder.

Look at an internal portion $B$ of the cylinder, with boundary $\partial B$. The Cauchy stress acting on $B$ from outside $B$ derives from the vector valued 2-form in (©.40) at points of $\partial B$. For *total* stress force on $\partial B$, we cannot just integrate this because it makes no sense to add vectors like $\mathbf{e}_\theta$ at different points. There is no problem with the $\mathbf{e}_z$ components because $\mathbf{e}_z$ is a constant vector field in $\mathbb{R}^3$. So let us express the unit vector $\mathbf{e}_\theta$ in terms of the constant basis $\mathbf{e}_x$ and $\mathbf{e}_y$. Again we leave the cylindrical coordinate 2-forms alone. Now

$$\boldsymbol{\partial}/\boldsymbol{\partial}\theta = (\partial x/\partial\theta)\boldsymbol{\partial}/\boldsymbol{\partial}x + (\partial y/\partial\theta)\boldsymbol{\partial}/\boldsymbol{\partial}y = (-r\sin\theta)\mathbf{e}_x + (r\cos\theta)\mathbf{e}_y$$

and $\mathbf{e}_\theta = r^{-1}(\boldsymbol{\partial}/\partial\theta) = -\mathbf{e}_x \sin\theta + \mathbf{e}_y \cos\theta$, and so (©.40) becomes

$$\mathfrak{t} = \mu k r^2(-\mathbf{e}_x \sin\theta + \mathbf{e}_y \cos\theta) \otimes dr \wedge d\theta + \mu k r \mathbf{e}_z \otimes dz \wedge dr$$

Then, with constant basis, $\iint_{\partial B} \mathbf{e}_x \mu k r^2 \sin\theta\, dr \wedge d\theta = \mathbf{e}_x \iint_{\partial B} \mu k r^2 \sin\theta\, dr \wedge d\theta$, etc., and so

$$\iint_{\partial B} \mathfrak{t} = -\mathbf{e}_x \iint_{\partial B} \mu k r^2 \sin\theta\, dr \wedge d\theta + \mathbf{e}_y \iint_{\partial B} \mu k r^2 \cos\theta\, dr \wedge d\theta$$
$$+ \mathbf{e}_z \iint_{\partial B} \mu k r\, dz \wedge dr$$

But each integral vanishes, e.g., $\mathbf{e}_x \iint_{\partial B} \mu k r^2 \sin\theta\, dr \wedge d\theta =$
$\mathbf{e}_x \iiint_B d[\mu k r^2 \sin\theta] \wedge dr \wedge d\theta = 0$, as desired.

It is a fact, alas, that this simple approach will not work to higher order, keeping terms of order $k^2$. One cannot realize such a simple twist; other deformations are required (see [Mu]).

I would like to emphasize one point brought out in the calculation above. When *integrating* vector valued exterior forms, such as Cauchy's $\boldsymbol{\partial}_i \otimes t^{ij} i(\boldsymbol{\partial}_j)\text{vol}^3$, we were forced to make a change to a constant basis for the vector part, $\boldsymbol{\partial}_i = \mathbf{e}_a A^a{}_i$, but kept the cylindrical exterior forms, yielding

$$\iint_{\partial B} \mathbf{e}_a \otimes A^a_i t^{ij} i(\boldsymbol{\partial}_j)\text{vol}^3 = \mathbf{e}_a \iint_{\partial B} A^a{}_i t^{ij} i(\boldsymbol{\partial}_j)\text{vol}^3 = \mathbf{e}_a \iiint_B d[A^a{}_i t^{ij} i(\boldsymbol{\partial}_j)\text{vol}^3]$$

and our exterior differential completely avoids Christoffel symbols and tensor divergence of $(t^{ij})$ in curvilinear coordinates, that appear in tensor treatments.

Finally, let us compute the work done by the traction acting on the face $Z = L$, moving each point $(R, \Theta)$ to the point $(R, \Theta + \alpha)$. Let $0 \leq \beta \leq \alpha$. The traction force on the small "rectangle" of sides $dR$, $d\Theta$ at $(R, \Theta + \beta)$ has, from $(\mathbb{O}.38')$, covariant component approximately $f_\Theta\, dR\, d\Theta = g_{\Theta\Theta}\mu k_\beta R\, dR\, d\Theta = \mu k_\beta R^3\, dR\, d\Theta$, where $k_\beta = \beta/L$. The work done in moving this rectangle from $\beta = 0$ to $\beta = \alpha$ is approximately $(dR\, d\Theta) \int_0^\alpha (\mu R^3 \beta/L)\, d\beta = (dR\, d\Theta)\mu R^3 \alpha^2/2L$. Thus the total work done in the twist of the face is $W = (\mu\alpha^2/2L) \iint R^3\, dR\, d\Theta = \pi\mu\, a^4\alpha^2/4L$. In most common materials (**hyperelastic**), in particular for our isotropic body, this work yields a **strain energy** of the same amount $W$, that is stored in the twisted body. Furthermore, for hyperelastic bodies, this can be computed from an integral over the undeformed body (see Sections A.d and D.a),

$$W = \tfrac{1}{2} \iiint S^{AB} E_{AB} \text{VOL}^3$$

and the reader can verify this in our example using $E$ and $S$ given before and after $(\mathbb{O}.38)$.

This is one reason for our choice, at the beginning of this section, of considering stress force as being contravariant, rather than covariant. Note that a **metric** $ds^2 = g_{ij}\, dx^i\, dx^j$ can be thought of as the **covector valued 1-form** $dx^i \otimes g_{ij}\, dx^j$ whose value on any vector $\mathbf{v}$ is the covariant version of $\mathbf{v}$, $dx^i \otimes g_{ij}\, dx^j(\mathbf{v}) = dx^i g_{ij} v^j = v_i\, dx^i$. Likewise, the Lagrange deformation tensor can be thought of as a covector valued 1-form

$$\mathcal{E} = dX^I \otimes E_{IJ}\, dX^J = dX^I \otimes \mathcal{E}^{(1)}{}_I$$

The stress tensor is a vector valued 2-form $\mathcal{S} = \boldsymbol{\partial}_A \otimes S^{AB} i(\boldsymbol{\partial}_B)\text{VOL}^3 = \boldsymbol{\partial}_A \otimes \mathcal{S}^{(2)A}$. It is natural then to construct a **scalar** valued 3-form by introducing a **new product** $\mathcal{S}(\wedge)\mathcal{E}$ by taking the wedge product of the forms in both and evaluating the 1-form $dX^I$ of $\mathcal{E}$ on the vector $\boldsymbol{\partial}_A$ of $\mathcal{S}$

$$\mathcal{S}(\wedge)\mathcal{E} := dX^I(\boldsymbol{\partial}_A)[S^{(2)A} \wedge \mathcal{E}^{(1)}{}_I] = \mathcal{S}^{(2)A} \wedge \mathcal{E}^{(1)}{}_A$$

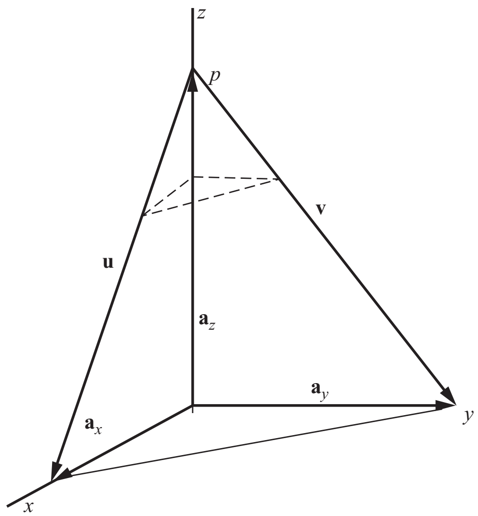
which is easily seen, since the two forms are of complementary dimension, to be the integrand of the strain energy $W$

$$\mathcal{S}(\wedge)\mathcal{E} = [S^{AB} i(\boldsymbol{\partial}_B)\text{VOL}^3] \wedge E_{AJ}\, dX^J = S^{AB} E_{AB} \text{VOL}^3$$

$$W = \tfrac{1}{2} \iiint \mathcal{S}(\wedge)\, \mathcal{E}$$

While work in particle mechanics pairs a force covector ($f_i$) with a contravariant tangent vector ($dx^i/dt$) to a curve, work done by traction in elasticity pairs the contravariant stress force 2-form $\mathfrak{S}$ with the covector valued deformation 1-form $\mathfrak{E}$, to yield a scalar valued 3-form. (Warning: the notation -($\wedge$)- does not appear in the literature.)

## 𝔒.q. Sketch of Cauchy's "First Theorem"




Consider a plane through a point $p$ on the $z$ axis of a cartesian coordinate system. This plane generically cuts the $x$ and $y$ axes at two points, yielding two vectors $\mathbf{u}$ and $\mathbf{v}$ that span the "roof" of a solid tetrahedron $T$, as in the figure above. The coordinate vectors $\mathbf{a}_x$, $\mathbf{a}_y$, $\mathbf{a}_z$ are *not* necessarily of the same length. The material outside $T$ exerts a stress force, call it $\frac{1}{2}\mathfrak{t}(\mathbf{u}, \mathbf{v})$ across the roof ($\frac{1}{2}$ because the roof is not a parallelogram). ($\mathbf{u}$, $\mathbf{v}$) tells us not only the roof, but also $\mathbf{u}$, $\mathbf{v}$, in that order is describing the normal pointing out of $T$. Likewise $\frac{1}{2}\mathfrak{t}(\mathbf{v}, \mathbf{u})$ describes a force that the material in $T$ exerts on material outside $T$. $\mathfrak{t}(\mathbf{v}, \mathbf{u}) = -\mathfrak{t}(\mathbf{u}, \mathbf{v})$ can be seen by considering the equilibrium of a small thin disk with faces parallel to the plane spanned by $\mathbf{u}$ and $\mathbf{v}$. This is the first part of Cauchy's first theorem.

Stress forces act also on the coordinate faces. We now let the tetrahedron $T$ shrink to the point $p$ by moving the $x$, $y$ plane up to the point $p$, the dashed triangle showing an intermediate position for the bottom face. At each stage the proportions of $T$ are preserved. As the vertical edge $\|\mathbf{a}_z\|$ shrinks to 0, the stress **forces** on the faces vanish as their areas, i.e., as $\|\mathbf{a}_z\|^2$ while the body forces, for example, gravity, if present, vanish as the volume, i.e., as $\|\mathbf{a}_z\|^3$. We will *neglect the body forces* for vanishingly small $T$.

For our small $T$ to be in equilibrium we must have, neglecting body forces

$$\begin{aligned} \mathfrak{t}(\mathbf{u}, \mathbf{v}) + \mathfrak{t}(\mathbf{a}_z, \mathbf{a}_y + \mathfrak{t}(\mathbf{a}_x, \mathbf{a}_z) + \mathfrak{t}(\mathbf{a}_y, \mathbf{a}_x) &\approx 0 \\ \mathfrak{t}(\mathbf{u}, \mathbf{v}) &\approx -\mathfrak{t}(\mathbf{a}_z, \mathbf{a}_y) - \mathfrak{t}(\mathbf{a}_x, \mathbf{a}_z) - \mathfrak{t}(\mathbf{a}_y, \mathbf{a}_x) \\ \mathfrak{t}(\mathbf{u}, \mathbf{v}) &\approx \mathfrak{t}(\mathbf{a}_y, \mathbf{a}_z) + \mathfrak{t}(\mathbf{a}_z, \mathbf{a}_x) + \mathfrak{t}(\mathbf{a}_x, \mathbf{a}_y) \end{aligned} \tag{𝔒.41}$$

Look at the first term $\mathfrak{t}(\mathbf{a}_y, \mathbf{a}_z)$. The normal to the pair $\mathbf{a}_y, \mathbf{a}_z$ is in the positive $x$ direction and so the area form for the $y, z$ face is $dy \wedge dz$. Let $\langle \mathfrak{t}_{yz} \rangle$ be the **area vector average** of the vector $\mathfrak{t}(\mathbf{a}_y, \mathbf{a}_z)$, so

$$\mathfrak{t}(\mathbf{a}_y, \mathbf{a}_z) = \langle \mathfrak{t}_{yz} \rangle dy \wedge dz(\mathbf{a}_y, \mathbf{a}_z)$$

Now note that for projected areas, $dy \wedge dz(\mathbf{u}, \mathbf{v}) = dy \wedge dz(\mathbf{a}_x - \mathbf{a}_z, -\mathbf{a}_z + \mathbf{a}_y) = dy \wedge dz(-\mathbf{a}_z, -\mathbf{a}_z) + dy \wedge dz(-\mathbf{a}_z, \mathbf{a}_y) = dy \wedge dz(-\mathbf{a}_z, \mathbf{a}_y) = -dy \wedge dz(\mathbf{a}_z, \mathbf{a}_y) = dy \wedge dz(\mathbf{a}_y, \mathbf{a}_z)$. Thus

$$dy \wedge dz\ (\mathbf{a}_y, \mathbf{a}_z) = dy \wedge dz(\mathbf{u}, \mathbf{v}) \quad \text{and so} \quad \mathfrak{t}(\mathbf{a}_y, \mathbf{a}_z) = \langle \mathfrak{t}_{yz} \rangle dy \wedge dz(\mathbf{u}, \mathbf{v})$$

and similarly for the other faces in (℘.41). We then have

$$\mathfrak{t}(\mathbf{u}, \mathbf{v}) \approx \langle \mathfrak{t}_{yz} \rangle dy \wedge dz(\mathbf{u}, \mathbf{v}) + \langle \mathfrak{t}_{zx} \rangle dz \wedge dx(\mathbf{u}, \mathbf{v}) + \langle \mathfrak{t}_{xy} \rangle dx \wedge dy(\mathbf{u}, \mathbf{v}) \qquad (℘.42)$$

Now as $T$ shrinks to the point $p$ the average $\langle \mathfrak{t}_{yz} \rangle$ tends to a **vector** $\mathfrak{t}^x(p) = \mathfrak{t}^1(p)$ at $p$, etc. We can then approximate the stress in (℘.42), for a very small parallelogram at $p$ spanned by $\mathbf{u}$ and $\mathbf{v}$

$$\mathfrak{t}(\mathbf{u}, \mathbf{v}) \approx [\mathfrak{t}^x(p) \otimes dy \wedge dz + \mathfrak{t}^y(p) \otimes dz \wedge dx + \mathfrak{t}^z(p) \otimes dx \wedge dy](\mathbf{u}, \mathbf{v})$$

which suggests Cauchy's theorem, that for any surface $V^2$ with normal direction prescribed, the stress across $V$ is given by a vector valued integral of the form

$$\int_V \mathfrak{t}^x(x, y, z) \otimes dy \wedge dz + \mathfrak{t}^y(x, y, z) \otimes dz \wedge dx + \mathfrak{t}^z(x, y, z) \otimes dx \wedge dy$$

with Cauchy vector valued stress 2-form

$$\mathfrak{t} = \boldsymbol{\partial}_i \otimes t^{ij} i(\boldsymbol{\partial}_j)\mathrm{vol}^3 \qquad (℘.42)_{\text{Cauchy}}$$

but this is not the way it is written in engineering texts. Consider first just the surface integral of a 2-form $\beta^2 = i(\mathbf{b})\mathrm{vol}^3$ over a surface $V^2 \subset \mathbb{R}^3$ (using any coordinates $x^i$), with unit normal vector field $\mathbf{n}$ and covector version the 1-form $n_* = n_i\, dx^i$. Then, when applied to two vectors $\mathbf{v}$ and $\mathbf{w}$ tangent to $V$, "$dA$" $(\mathbf{v}, \mathbf{w}) := \mathrm{vol}(\mathbf{n}, \mathbf{v}, \mathbf{w}) = [i(\mathbf{n})\mathrm{vol}]$ $(\mathbf{v}, \mathbf{w})$ is the area spanned by $\mathbf{v}$ and $\mathbf{w}$. Then we can write, with $\mathbf{b}_{\tan}$ the tangential part of $\mathbf{b}$

$$\int_V \beta = \int_V i(\mathbf{b})\mathrm{vol}^3 = \int_V i[(\mathbf{b} \cdot \mathbf{n})\mathbf{n} + \mathbf{b}_{\tan}]\mathrm{vol} = \int_V (\mathbf{b} \cdot \mathbf{n})[i(\mathbf{n})\mathrm{vol}]$$

since $\mathrm{vol}(\mathbf{b}_{\tan}, \mathbf{v}, \mathbf{w}) = 0$ for three tangent vectors to $V^2$. Then

$$\int_V \beta = \int_V i(\mathbf{b})\mathrm{vol}^3 = \int_V (\mathbf{b} \cdot \mathbf{n})[i(\mathbf{n})\mathrm{vol}] = \int_V (\mathbf{b} \cdot \mathbf{n})dA = \int_V b^j n_j\, dA$$

Likewise, on a surface $V^2$, engineering texts write the stress

$$t^{ij} n_j\, dA \quad \text{instead of} \quad t^{ij} i(\boldsymbol{\partial}_j)\mathrm{vol}^3$$

## $\mathbb{0}$.r. Sketch of Cauchy's "Second Theorem," Moments as Generators of Rotations

For Cauchy's second theorem, the symmetry of the stress tensor $t^{ij} = t^{ji}$, we shall consider only the simplest case of a deformed body, at **rest** and in **equilibrium** with its external tractions on its boundary, and with no external body forces (like gravity) considered. We employ *cartesian* coordinates throughout. Then, since $g_{ij} = \delta_{ij}$, tensorial indices may be raised and lowered indiscriminately and we can use the summation convention for *all* repeated indices.

Let $B$ be any sub-body in the *interior* of the body, with boundary $\partial B$. Then the (assumed vanishing) total stress force *covector* on $B$ yields

$$0 = \int_{\partial B} \{dx^c\} \otimes t_c{}^b i(\boldsymbol{\partial} b)\mathrm{vol}^3 = \{dx^c\} \int_{\partial B} \mathfrak{t}_c = \{dx^c\} \int_B d\mathfrak{t}_c$$

where we use the braces { } just to remind us that the basis form to the left of $\otimes$ is a constant covector that plays no role in the integral. Since this holds for every interior $B$ we must have

$$d\mathfrak{t}_c = dt_c{}^b i(\boldsymbol{\partial}_b)\mathrm{vol}^3 = 0 \quad \text{for each } c \tag{$\mathbb{0}$.43}$$

which classically is written as a divergence $\partial t_c{}^b/\partial x^b = 0$.

For equilibrium we must also have that the total **moment** of stress forces on $\partial B$ must vanish. Now the moment about the origin, of a force **f** at position vector **r** is, in elementary point mechanics, $\mathbf{r} \times \mathbf{f}(\mathbf{r})$, but this expression makes no sense in more than 3 dimensions. But moments and torques surely make sense in any euclidean $\mathbb{R}^n$, indicating that we have not understood *mathematically* the notion of moment. Now in cartesian coordinates in $\mathbb{R}^n$, if we replace **r** and **f**(**r**) by 1-forms $\mathfrak{r} = x^a\, dx^a$ and $\mathfrak{f} = f_c(\mathbf{r})dx^c$, then $\mathfrak{r} \wedge \mathfrak{f}$ does make sense as a 2-form *at the origin* of $\mathbb{R}^n$ and its components, in the case of $\mathbb{R}^3$, coincide with those of $\mathbf{r} \times \mathbf{f}(\mathbf{r})$. There is a more important point. A moment about the origin 0 of $\mathbb{R}^n$ is *physically* a "generator" of a rotation about 0. Let us see why a 2-form at the origin of $\mathbb{R}^n$, with components forming a skew symmetric matrix, also is associated to a rotation there.

Let $g(t)$ be a 1-parameter group (i.e., $g(t)\, g(s) = g(t+s)$, and $g(0) = \mathrm{I}$) of **rotations** of $\mathbb{R}^n$ about the origin. Since each $g(t)$ is an "orthogonal" matrix, $g(t)\, g(t)^T = \mathrm{I}$, where $T$ is transpose. Differentiate with respect to $t$ (indicated by an overdot) and put $t = 0$. Then

$$0 = \dot{g}(0)g(0)^T + g(0)\dot{g}(0)^T = \dot{g}(0) + \dot{g}(0)^T$$

says that $A := \dot{g}(0)$ (the so-called "infinitesimal **generator**" of the 1-parameter group $g(t)$), is a skew symmetric $n \times n$ matrix, and so defines a 2-form $\mathfrak{A} = \Sigma_{j<k} A_{jk} dx^j \wedge dx^k$ at the origin. For example, a 1-parameter group of rotations about the $z$ axis of $\mathbb{R}^3$ is, with $\omega$ a constant,

$$g(t) = \begin{bmatrix} \cos(\omega t) & -\sin(\omega t) & 0 \\ \sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and has generator} \quad A = \dot{g}(0) = \begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with associated 2-form $\mathfrak{A} = -\omega\, dx \wedge dy$ at the origin. If **v** is a vector at the origin, then $A\mathbf{v}$ is the vector $(A\mathbf{v})_j = A_{jk}v^k = -v^k A_{kj}$, i.e., the covector version of $A\mathbf{v}$ is $-i(\mathbf{v})\mathfrak{A}$.

Conversely, if $A$ is a skew symmetric $n \times n$ matrix at the origin (a 2-form at the origin), then $A$ generates a 1-parameter group of rotations $g(t)$ by means of the exponential matrix

$$g(t) = e^{tA} = \exp tA := \Sigma_k t^k A^k/k!$$

(it is an orthogonal matrix since $g(t)^T = \exp tA^T = \exp(-tA) = g(-t) = g^{-1}(t)$). **A 2-form at the origin of $\mathbb{R}^n$ generates a 1 parameter group of rotations about the origin of $\mathbb{R}^n$**. (Linear algebra also shows that the generator of $e^{tA}$ is $[d/dte^{tA}]_{t=0} = Ae^0 = A$.)

Thus to each moment of a force **f** about the origin of $\mathbb{R}^n$ we may attach the generator of its rotations, i.e., a 2-form at the origin, which is simply a skew symmetric $n \times n$ matrix.

Then with our sub-body $B$ of an elastic body in $\mathbb{R}^3$, the Cauchy stress **covector** valued 2-form yields an "area covector force density" with "components" the 2-forms $\mathfrak{t}_c = t_c{}^b\, i(\boldsymbol{\partial}_b)\text{vol}^3$ at points of the boundary $\partial B$. The "moment about an origin (chosen inside $B$)" density, on $\partial B$, has *cartesian* "components" the matrix of 2-forms

$$\mathfrak{m}_{ac} = [x^a t_c{}^b - x^c t_a{}^b]i(\boldsymbol{\partial}_b)\text{vol}^3 = x^a \mathfrak{t}_c - x^c \mathfrak{t}_a$$

Thus the **total moment** about the **origin** due to these stress forces on $\partial B$ is the 2-form at the origin $\Sigma_{a<c} M_{ac}\, dx^a \wedge dx^c$ with components the matrix of numbers

$$M_{ac} = \int_{\partial B} [x^a \mathfrak{t}_c - x^c \mathfrak{t}_a] = \int_B d[x^a \mathfrak{t}_c - x^c \mathfrak{t}_a]$$

which, from (©.43) (i.e., assuming no external body forces), is

$$M_{ac} = \int_B dx^a \wedge \mathfrak{t}_c - dx^c \wedge \mathfrak{t}_a$$

In most common elastic materials, this must vanish if there are to be no "couple stresses" without applied internal torque sources. Since this holds for any portion $B$ we must have

$$dx^a \wedge \mathfrak{t}_c = dx^c \wedge \mathfrak{t}_a \qquad (©.44)$$

Since these are 3-forms in $\mathbb{R}^3$,

$$dx^a \wedge \mathfrak{t}_c = dx^a \wedge t_c{}^b\, i(\boldsymbol{\partial}_b)\text{vol}^3 = t_c{}^a \text{vol}^3 \qquad (©.44')$$

For example, in $\mathbb{R}^3$ with $a = 2$ and $c = 1$,

$$\begin{aligned} dx^2 \wedge t_1{}^b[i(\boldsymbol{\partial}_b)dx^1 \wedge dx^2 \wedge dx^3] &= dx^2 \wedge t_1{}^2[i(\boldsymbol{\partial}_2)dx^1 \wedge dx^2 \wedge dx^3] \\ &= -dx^2 \wedge t_1{}^2[i(\boldsymbol{\partial}_2)dx^2 \wedge dx^1 \wedge dx^3] \\ &= -dx^2 \wedge t_1{}^2\, dx^1 \wedge dx^3 \\ &= -t_1{}^2\, dx^2 \wedge dx^1 \wedge dx^3 = t_1{}^2\, dx^1 \wedge dx^2 \wedge dx^3 \\ &= t_1{}^2 \text{vol}^3 \end{aligned}$$

(☉.44) then yields $t_c{}^a \mathrm{vol}^3 = t_a{}^c \mathrm{vol}^3$, and since the coordinates are cartesian we have

$$t^{ca} = t^{ac} \tag{☉.45}$$

Since the Cauchy stress $t$ is a tensor, this symmetry holds in *any* coordinate system. This is Cauchy's second theorem.

**Warning:** In Section ☉.p we allowed and encouraged the use of different coordinates for the 2-form part and the value part of the stress vector valued 2-form

$$\boldsymbol{\partial}_i \otimes t^{ij} i(\boldsymbol{\partial}_j)\mathrm{vol}^3 = \mathbf{e}_a \otimes A^a_i t^{ij} i(\boldsymbol{\partial}_j)\mathrm{vol}^3 =: \mathbf{e}_a \otimes \tau^{aj} i(\boldsymbol{\partial}_j)\mathrm{vol}^3$$

The left index "$a$" on $\tau$ is associated with the **e** basis and the right index "$j$" is associated with the $\boldsymbol{\partial}$ basis. (Think, for example, of **e** as cartesian and $\boldsymbol{\partial}$ as cylindrical.) Does the fact that $t$ is symmetric, $t^T = t$, insure that $\tau = At$ is also ? No!

$$\tau^T = (At)^T = t^T A^T = t A^T = A^{-1}\tau A^T \neq \tau \quad \text{generically}$$

## ☉.s. A Remarkable Formula for Differentiating Line, Surface, and..., Integrals

Let $\mathbf{v}$ be a **time independent** vector field in a coordinate patch $U$ of $\mathbb{R}^n$ with any coordinates $x^i$. Roughly speaking, i.e., omitting some technicalities, by integrating the differential equations $dx^i/dt = v^i(x)$ we can move along the integral curves of $\mathbf{v}$ for $t$ seconds yielding a "flow" $\phi_t : U \to \mathbb{R}^n$. Since $\mathbf{v}$ is time independent, the $\phi_t$ form a 1 parameter commutative group of mappings, $\phi_t \phi_h = \phi_{t+h}$ and $\phi_0$ is the identity map. Let $V^r$ be an oriented $r$ dimensional "submanifold" of $U$. For examples, $V^1$ is an oriented curve , $V^2$ is an oriented 2 dimensional surface, .... $V^r$ is the kind of object over which one integrates an exterior $r$-form $\alpha = \alpha^r$ (a **scalar** valued, not vector valued form), yielding the number $\int_V \alpha^r$. As time changes, the flow moves $V$ from $V(0) = V$ to $V(t) = \phi_t(V)$. We consider only the simplest case where the r-form $\alpha$ is time independent. How does the integral change in time? The answer can be shown (see Section 4.3a) to be

$$d/dt|_{t=0} \int_{V(t)} \alpha^r = \int_V \mathcal{L}_{\mathbf{v}} \alpha^r \tag{☉.46}$$

where the $r$-form $\mathcal{L}_{\mathbf{v}}\alpha^r$, the **Lie derivative** of the form $\alpha$, is defined via the pull-backs

$$\begin{aligned} [\mathcal{L}_{\mathbf{v}}\alpha^r](\text{at } x) :&= [d/dt]_{t=0} \phi_t{}^*[\alpha^r(\text{at } \phi_t \mathrm{x})] \\ &= \lim_{t\to 0}\{\phi_t{}^*[\alpha^r(\text{at } \phi_t x)] - \alpha^r(\text{at } x)\}/t \end{aligned} \tag{☉.47}$$

Furthermore, there is a remarkable expression for computing the Lie derivative of any form, given by the **Henri Cartan** (son of Elie Cartan) **formula**

$$\mathcal{L}_{\mathbf{v}}\alpha^r = i_{\mathbf{v}}(d\alpha^r) + d(i_{\mathbf{v}}\alpha^r) \tag{☉.48}$$

Thus (☉.46) and Stokes say

$$d/dt|_{t=0} \int_{V(t)} \alpha^r = \int_V \mathcal{L}_{\mathbf{v}}\alpha^r = \int_V i_{\mathbf{v}}\, d\alpha + \int_{\partial V} i_{\mathbf{v}}\alpha \tag{☉.49}$$

Consider for example the case of a line integral in $\mathbb{R}^3$, which we also write in classical form in cartesian coordinates. $V^1$ is then a curve $C$ starting at point $P$ and ending at point $Q$. Symbolically $\partial C = Q - P$. Classically $\alpha = \mathbf{a} \cdot d\mathbf{x}$. Then $i_{\mathbf{v}}\alpha$ is the 0-form, i.e., function $v \cdot \mathbf{a}$, and $\int_{\partial C} \mathbf{v} \cdot \mathbf{a}$ is *by definition* simply $(\mathbf{v} \cdot \mathbf{a})(Q) - (\mathbf{v} \cdot \mathbf{a})(P)$. This is the second "integral" in ($\mathcal{O}$.49). Also, $d\alpha^1$ is the 2-form version of the vector curl $\mathbf{a}$, and so $i_{\mathbf{v}}\, d\alpha$, from ($\mathcal{O}$.34), is the 1-form version of $-\mathbf{v} \times \text{curl } \mathbf{a}$. We then have, in the classical version

$$d/dt]_{t=0} \int_{C(t)} \mathbf{a} \cdot d\mathbf{x} = -\int_C [\mathbf{v} \times \text{ curl } \mathbf{a}] \cdot d\mathbf{x} + (\mathbf{v} \cdot \mathbf{a})(Q) - (\mathbf{v} \cdot \mathbf{a})(P)$$

The reader might enjoy computing the rates of change of surface and volume integrals

$$\int_S \mathbf{b} \cdot \mathbf{n}\, dA \quad \text{and} \quad \int_M \text{vol}^3$$

**A final remark about time dependent flows and forms**. In the real world, vector fields and forms are frequently time dependent. Consider, for example, $\mathbb{R}^n$ with local coordinates $\mathbf{x} = (x^i)$, and let $\alpha^r$ be an $r$-form (with components that may be time $t$ dependent) and $\mathbf{v} = \boldsymbol{\partial}_i v^i(t, \mathbf{x})$. We may again solve the differential equations $d\mathbf{x}/dt = \mathbf{v}(t, \mathbf{x})$ to get maps $\phi_t$ but (as discussed in Section 4.3b) generically they will not satisfy the crucial $\phi_a \circ \phi_b = \phi_{a+b}$. To circumvent this we introduce the space $\mathbb{R} \times \mathbb{R}^n$ with $n + 1$ local coordinates $(x^0 = t, x^i)$, $1 \leq i \leq n$, that is, we enlarge the space $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$ by introducing time as another dimension. We then augment the original vector field $\mathbf{v}$ on $\mathbb{R}^n$ to the new field $\nu\,(t, \mathbf{x}) = \boldsymbol{\partial}_t + \mathbf{v}(t, \mathbf{x})$ on $\mathbb{R}^1 \times \mathbb{R}^n$. Then it is shown in Theorem (4.42) that we get new maps $\phi_t : \mathbb{R}^1 \times \mathbb{R}^n \to \mathbb{R}^1 \times \mathbb{R}^n$ that do form a flow, and if $V = V_0$ is an $r$ dimensional submanifold of the $\mathbb{R}^n$ slice $t = 0$, then $V(a) = \phi_a V$ is in slice $t = a$, and ($\mathcal{O}$.49) is replaced by

$$\begin{aligned} d/dt|_{t=0} \int_{V(t)} \alpha &= \int_V \mathcal{L}_\nu \alpha = \int_V i(\nu) d\alpha + \int_V d[i(\nu)\alpha] \\ &= \int_V (\partial\alpha/\partial t) + i_{\mathbf{v}} \mathbf{d}\alpha + \mathbf{d} i_{\mathbf{v}} \alpha \end{aligned} \qquad (\mathcal{O}.50)$$

(note $i_{\mathbf{v}} = i(\mathbf{v})$ uses the original vector field $\mathbf{v}$, not the augmented $\nu = \mathbf{v} + \boldsymbol{\partial}_t$). The bold $\mathbf{d}$ is the "spatial" exterior differential of $\mathbb{R}^n$ (keeping $t$ constant) and $\partial\alpha/\partial t$ is the $r$-form (with no $dt$ term) where each term of $\alpha$

$$a_{i \ldots j}(x, t) dx^i \wedge \ldots \wedge dx^j$$

is replaced by

$$[\partial a_{i \ldots j}(x, t)/\partial t]_{t=0}\, dx^i \wedge \ldots \wedge dx^j$$

For example, ($\mathcal{O}$.50) tells us that Faraday's law of section $\mathcal{O}$.1 says that for a *moving* surface $V^2(t)$

$$d/dt \iint_{V(t)} \mathcal{B}^2 = -\oint_{\partial V} (\mathcal{E} - i_{\mathbf{v}}\mathcal{B}) = -\oint_{\partial V} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot d\mathbf{x}$$

is the line integral of the *electromotive force* along the boundary curve.

Applications to fluid flows, vorticity, and magnetohydrodynamics can be seen in Section 4.3c.

PART ONE

# Manifolds, Tensors, and Exterior Forms

CHAPTER 1

# Manifolds and Vector Fields

Better is the end of a thing than the beginning thereof.

Ecclesiastes 7:8

As students we learn differential and integral calculus in the context of euclidean space $\mathbb{R}^n$, but it is necessary to apply calculus to problems involving "curved" spaces. Geodesy and cartography, for example, are devoted to the study of the most familiar curved surface of all, the surface of planet Earth. In discussing maps of the Earth, latitude and longitude serve as "coordinates," allowing us to use calculus by considering functions on the Earth's surface (temperature, height above sea level, etc.) as being functions of latitude and longitude. The familiar Mercator's projection, with its stretching of the polar regions, vividly informs us that these coordinates are badly behaved at the poles: that is, that they are not defined everywhere; they are not "global." (We shall refer to such coordinates as being "local," even though they might cover a huge portion of the surface. Precise definitions will be given in Section 1.2.) Of course we may use two sets of "polar" projections to study the Arctic and Antarctic regions. With these three maps we can study the entire surface, provided we know how to relate the Mercator to the polar maps.

We shall soon define a "manifold" to be a space that, like the surface of the Earth, can be covered by a family of local coordinate systems. *A manifold will turn out to be the most general space in which one can use differential and integral calculus with roughly the same facility as in euclidean space*. It should be recalled, though, that calculus in $\mathbb{R}^3$ demands special care when curvilinear coordinates are required.

The most familiar manifold is $N$-dimensional euclidean space $\mathbb{R}^N$, that is, the space of ordered $N$ tuples $(x^1, \ldots, x^N)$ of real numbers. Before discussing manifolds in general we shall talk about the more familiar (and less abstract) concept of a submanifold of $\mathbb{R}^N$, generalizing the notions of curve and surface in $\mathbb{R}^3$.

## 1.1. Submanifolds of Euclidean Space

What is the configuration space of a rigid body fixed at one point of $\mathbb{R}^n$?

### 1.1a. Submanifolds of $\mathbb{R}^N$

Euclidean space, $\mathbb{R}^N$, is endowed with a global coordinate system $(x^1, \ldots, x^N)$ and is the most important example of a manifold.

In our familiar $\mathbb{R}^3$, with coordinates $(x, y, z)$, a locus $z = F(x, y)$ describes a (2-dimensional) surface, whereas a locus of the form $y = G(x)$, $z = H(x)$, describes a (1-dimensional) curve. We shall need to consider higher-dimensional versions of these important notions.

A subset $M = M^n \subset \mathbb{R}^{n+r}$ is said to be an $n$-dimensional **submanifold** of $\mathbb{R}^{n+r}$, if *locally* $M$ can be described by giving $r$ of the coordinates differentiably in terms of the $n$ remaining ones. This means that given $p \in M$, a neighborhood of $p$ on $M$ can be described in *some* coordinate system $(x, y) = (x^1, \ldots, x^n, y^1, \ldots, y^r)$ of $\mathbb{R}^{n+r}$ by $r$ differentiable functions

$$y^\alpha = f^\alpha(x^1, \ldots, x^n), \qquad \alpha = 1, \ldots r$$

We abbreviate this by $y = f(x)$, or even $y = y(x)$. We say that $x^1, \ldots, x^n$ are **local (curvilinear) coordinates** for $M$ near $p$.

**Examples:**

**(i)** $y^1 = f(x^1, \ldots, x^n)$ describes an $n$-dimensional submanifold of $\mathbb{R}^{n+1}$.

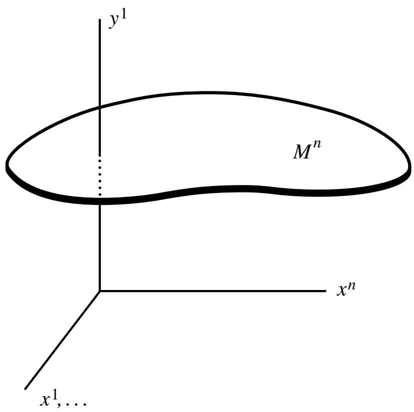



**Figure 1.1**

In Figure 1.1 we have drawn a portion of the submanifold $M$. This $M$ is the **graph** of a function $f : \mathbb{R}^n \to \mathbb{R}$, that is, $M = \{(\mathbf{x}, y) \in \mathbb{R}^{n+1} \mid y = f(\mathbf{x})\}$. When $n = 1$, $M$ is a curve; while if $n = 2$, it is a surface.

**(ii)** The *unit sphere* $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$. Points in the northern hemisphere can be described by $z = F(x, y) = (1 - x^2 - y^2)^{1/2}$ and this function is differentiable everywhere except at the equator $x^2 + y^2 = 1$. Thus $x$ and $y$ are local coordinates for the northern hemisphere except at the equator. For points on the equator one can solve for $x$ or $y$ in terms of the others. If we have solved for $x$ then $y$ and $z$ are the two local coordinates. For points in the southern hemisphere one can use the negative square root for $z$. The unit sphere in $\mathbb{R}^3$ is a 2-dimensional submanifold of $\mathbb{R}^3$. We note that we have *not* been able to describe the *entire* sphere by expressing one of the coordinates, say $z$, in terms of the two remaining ones, $z = F(x, y)$. We settle for local coordinates.

More generally, given $r$ functions $F^\alpha(x_1, \ldots, x_n, y_1, \ldots, y_r)$ of $n + r$ variables, we may consider the locus $M^n \subset \mathbb{R}^{n+r}$ defined by the equations

$$F^\alpha(x, y) = c^\alpha, \qquad (c^1, \ldots, c^r) \text{ constants}$$

If the **Jacobian** determinant

$$\left[\frac{\partial(F^1, \ldots, F^r)}{\partial(y^1, \ldots, y^r)}\right](x_0, y_0)$$

at $(x_0, y_0) \in M$ of the locus is not 0, the **implicit function theorem** assures us that locally, near $(x_0, y_0)$, we may solve $F^\alpha(x, y) = c^\alpha$, $\alpha = 1, \ldots, r$, for the $y$'s in terms of the $x$'s

$$y^\alpha = f^\alpha(x^1, \ldots, x^n)$$

We may say that "a portion of $M^n$ near $(x_0, y_0)$ is a submanifold of $\mathbb{R}^{n+r}$." If the Jacobian $\neq 0$ *at all points of the locus*, then the entire $M^n$ is a submanifold.

Recall that the Jacobian condition arises as follows. If $F^\alpha(x, y) = c^\alpha$ can be solved for the $y$'s differentiably in terms of the $x$'s, $y^\beta = y^\beta(x)$, then if, for fixed $i$, we differentiate the identity $F^\alpha(x, y(x)) = c^\alpha$ with respect to $x^i$, we get

$$\frac{\partial F^\alpha}{\partial x^i} + \sum_\beta \left[\frac{\partial F^\alpha}{\partial y^\beta}\right] \frac{\partial y^\beta}{\partial x^i} = 0$$

and

$$\frac{\partial y^\beta}{\partial x^i} = -\sum_\alpha \left(\left[\frac{\partial F}{\partial y}\right]^{-1}\right)^\beta_{\ \alpha} \left[\frac{\partial F^\alpha}{\partial x^i}\right]$$

provided the subdeterminant $\partial(F^1, \ldots, F^r)/\partial(y^1, \ldots, y^r)$ is not zero. (Here $([\partial F/\partial y]^{-1})^\beta{}_\alpha$ is the $\beta\alpha$ entry of the inverse to the matrix $\partial F/\partial y$; we shall use the convention that for matrix indices, the index to the *left* always is the *row* index, whether it is up or down.) This *suggests* that if the indicated Jacobian is nonzero then we might indeed be able to solve for the $y$'s in terms of the $x$'s, and the implicit function theorem confirms this. The (nontrivial) *proof* of the implicit function theorem can be found in most books on real analysis.

Still more generally, suppose that we have $r$ functions of $n+r$ variables, $F^\alpha(x^1, \ldots, x^{n+r})$. Consider the locus $F^\alpha(x) = c^\alpha$. Suppose that at each point $x_0$ of the locus the Jacobian *matrix*

$$\left(\frac{\partial F^\alpha}{\partial x^i}\right) \qquad \alpha = 1, \ldots, r \qquad i = 1, \ldots, n + r$$

has rank $r$. Then the equations $F^\alpha = c^\alpha$ define an $n$-dimensional submanifold of $\mathbb{R}^{n+r}$, since we may locally solve for $r$ of the coordinates in terms of the remaining $n$.

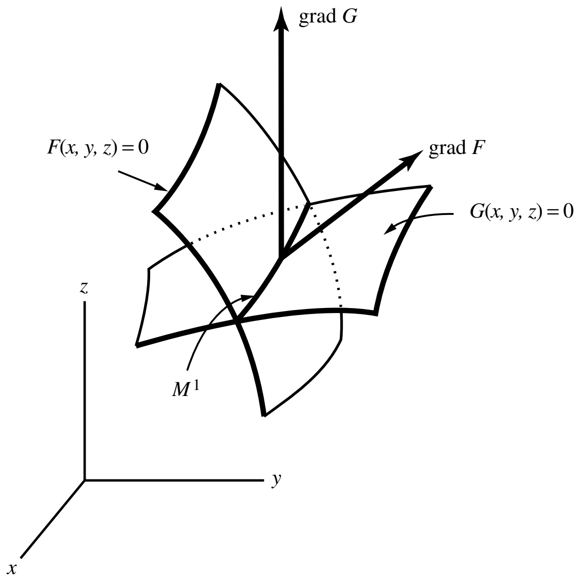



**Figure 1.2**

In Figure 1.2, two surfaces $F = 0$ and $G = 0$ in $\mathbb{R}^3$ intersect to yield a curve $M$.

The simplest case is *one* function $F$ of $N$ variables $(x^1, \ldots, x^N)$. If *at each point of the locus* $F = c$ there is always *at least one partial derivative that does not vanish*, then the Jacobian (row) matrix $[\partial F/\partial x^1, \partial F/\partial x^2, \ldots, \partial F/\partial x^N]$ has rank 1 and we may conclude that *this locus is indeed an* $(N-1)$*-dimensional submanifold of* $\mathbb{R}^N$. This criterion is easily verified, for example, in the case of the 2-sphere $F(x, y, z) = x^2 + y^2 + z^2 = 1$ of Example (ii). The column version of this row matrix is called in calculus the gradient vector of $F$. In $\mathbb{R}^3$ this vector

$$\begin{bmatrix} \frac{\partial F}{\partial x} \\ \frac{\partial F}{\partial y} \\ \frac{\partial F}{\partial z} \end{bmatrix}$$

is orthogonal to the locus $F = 0$, and we may conclude, for example, that if this gradient vector has a nontrivial component in the $z$ direction at a point of $F = 0$, then locally we can solve for $z = z(x, y)$.

A submanifold of dimension $(N-1)$ in $\mathbb{R}^N$, that is, of "**codimension**" 1, is called a **hypersurface**.

**(iii)** The $x$ axis of the $xy$ plane $\mathbb{R}^2$ can be described (perversely) as the locus of the quadratic $F(x, y) := y^2 = 0$. Both partial derivatives vanish on the locus, the $x$ axis, and our criteria would not allow us to say that the $x$ axis is a 1-dimensional submanifold of $\mathbb{R}^2$. Of course the $x$ axis *is* a submanifold; we should have used the usual description $G(x, y) := y = 0$. Our Jacobian criteria are *sufficient* conditions, not necessary ones.

**(iv)** The locus $F(x, y) := xy = 0$ in $\mathbb{R}^2$, consisting of the union of the $x$ and $y$ axes, is not a 1-dimensional submanifold of $\mathbb{R}^2$. It seems "clear" (and can be proved) that in a neighborhood of the intersection of the two lines we are not going to be able to describe the locus in the form of $y = f(x)$ or $x = g(y)$, where $f$, $g$, are differentiable functions. The best we can say is that this locus *with the origin removed* is a 1-dimensional submanifold.

### 1.1b. The Geometry of Jacobian Matrices: The "Differential"

The **tangent space** to $\mathbb{R}^n$ at the point $x$, written here as $\mathbb{R}^n_x$, is by definition the vector space of all vectors in $\mathbb{R}^n$ based at $x$ (i.e., it is a copy of $\mathbb{R}^n$ with origin shifted to $x$).

Let $x^1, \ldots, x^n$ and $y^1, \ldots, y^r$ be coordinates for $\mathbb{R}^n$ and $\mathbb{R}^r$ respectively. Let $F : \mathbb{R}^n \to \mathbb{R}^r$ be a **smooth** map. ("Smooth" ordinarily means infinitely differentiable. For our purposes, however, it will mean differentiable at least as many times as is necessary in the present context. For example, if $F$ is once continuously differentiable, we may use the chain rule in the argument to follow.) In coordinates, $F$ is described by giving $r$ functions of $n$ variables

$$y^\alpha = F^\alpha(x) \qquad \alpha = 1, \ldots, r$$

or simply $y = F(x)$. We will frequently use the more dangerous notation $y = y(x)$.

Let $y_0 = F(x_0)$; the Jacobian *matrix* $(\partial y^\alpha/\partial x^i)(x_0)$ has the following significance.

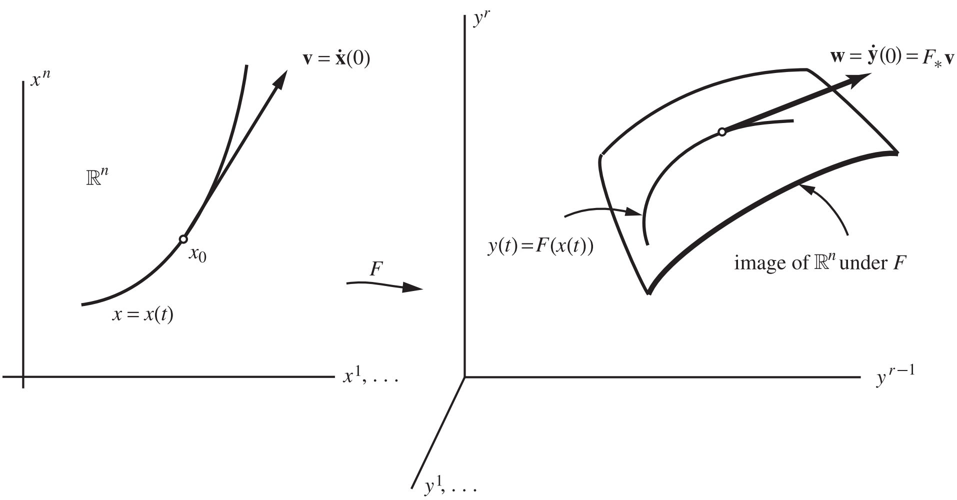



**Figure 1.3**

Let $\mathbf{v}$ be a tangent vector to $\mathbb{R}^n$ at $x_0$. Take *any* smooth curve $x(t)$ such that $x(0) = x_0$ and $\dot{x}(0) := (dx/dt)(0) = \mathbf{v}$, for example, the straight line $x(t) = x_0 + t\mathbf{v}$. The image of this curve

$$y(t) = F(x(t))$$

has a tangent vector $\mathbf{w}$ at $y_0$ given by the chain rule

$$w^\alpha = \dot{y}^\alpha(0) = \sum_{i=1}^{n} \left(\frac{\partial y^\alpha}{\partial x^i}\right)(x_0)\dot{x}^i(0) = \sum_{i=1}^{n} \left(\frac{\partial y^\alpha}{\partial x^i}\right)(x_0)v^i$$

The assignment $\mathbf{v} \mapsto \mathbf{w}$ is, from this expression, independent of the curve $x(t)$ chosen, and defines a *linear transformation*, the **differential** of $F$ at $x_0$

$$F_* : \mathbb{R}^n_{x_0} \to \mathbb{R}^r_{y_0} \qquad F_*(\mathbf{v}) = \mathbf{w} \tag{1.1}$$

whose matrix is simply the Jacobian matrix $(\partial y^\alpha/\partial x^i)(x_0)$. *This interpretation of the Jacobian matrix, as a linear transformation sending tangents to curves into tangents to the image curves under $F$, can sometimes be used to replace the direct computation of matrices*. This philosophy will be illustrated in Section 1.1d.

### 1.1c. The Main Theorem on Submanifolds of $\mathbb{R}^N$

The main theorem is a geometric interpretation of what we have discussed. Note that the statement "$F$ has rank $r$ at $x_0$," that is, $[\partial y^\alpha/\partial x^i](x_0)$ has rank $r$, is geometrically the statement that the differential

$$F_* : \mathbb{R}^n_{x_0} \to \mathbb{R}^r_{y_0=F(x_0)}$$

is **onto** or "surjective"; that is, given any vector $\mathbf{w}$ at $y_0$ there is at least one vector $\mathbf{v}$ at $x_0$ such that $F_*(\mathbf{v}) = \mathbf{w}$. We then have

**Theorem (1.2):** *Let $F : \mathbb{R}^{r+n} \to \mathbb{R}^r$ and suppose that the locus*

$$F^{-1}(y_0) := \{x \in \mathbb{R}^{r+n} \mid F(x) = y_0\}$$

*is not empty. Suppose further that for all $x_0 \in F^{-1}(y_0)$*

$$F_* : \mathbb{R}^{n+r}_{x_0} \to \mathbb{R}^r_{y_0}$$

*is onto. Then $F^{-1}(y_0)$ is an $n$-dimensional submanifold of $\mathbb{R}^{n+r}$.*

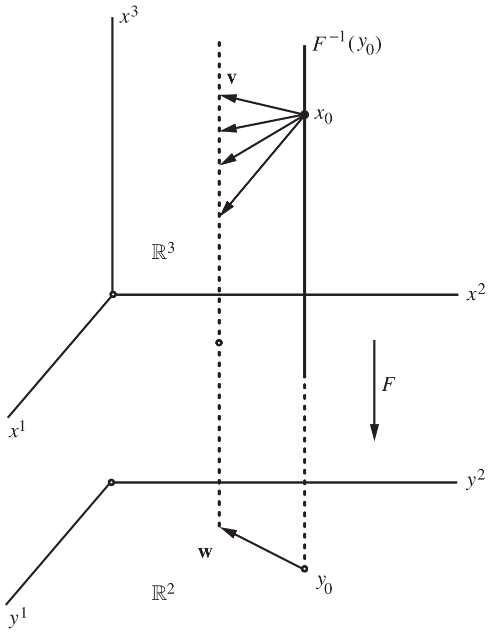



**Figure 1.4**

The best example to keep in mind is the linear "projection" $F : \mathbb{R}^3 \to \mathbb{R}^2$, $F(x^1, x^2, x^3) = (x^1, x^2)$, that is, $y^1 = x^1$ and $y^2 = x^2$. In this case, $x^3$ serves as global coordinate for the submanifold $x^1 = y_0^1, x^2 = y_0^2$, that is, the vertical line.

### 1.1d. A Nontrivial Example: The Configuration Space of a Rigid Body

Assume a rigid body has one point, the origin of $\mathbb{R}^3$, fixed. By comparing a cartesian right-handed system fixed in the body with that of $\mathbb{R}^3$ we see that the configuration of the body at any time is described by the rotation matrix taking us from the basis of $\mathbb{R}^3$ to the basis fixed in the body. The configuration space of the body is then the **rotation group** SO(3), that is, the $3 \times 3$ real matrices $x = (x_{ij})$ such that

$$x^T = x^{-1} \quad \text{and} \quad \det x = 1$$

where $T$ denotes transpose. (If we omit the determinant condition, the group is the full **orthogonal** group, O(3).) By assigning (in some fixed order) the nine coordinates $x_{11}, x_{12}, \dots, x_{33}$ to *any* matrix $x$, we see that the space of all $3 \times 3$ real matrices, $M(3 \times 3)$, is the euclidean space $\mathbb{R}^9$. The group O(3) is then the locus in this $\mathbb{R}^9$ defined by the equations $x^T x = I$, that is, by the system of nine quadratic equations $(i, k)$

$$(i, k) \qquad \sum_{j=1}^{3} x_{ji} x_{jk} = \delta_{ik}$$

We then have the following situation. The configuration of the body at time $t$ can be represented by a point $x(t)$ in $\mathbb{R}^9$, but in fact the point $x(t)$ lies on the locus O(3) in $\mathbb{R}^9$. We shall see shortly that *this locus is in fact a 3-dimensional submanifold* of $\mathbb{R}^9$. As time $t$ evolves, the point $x(t)$ traces out a curve on this 3-dimensional locus. Since O(3) *is* a submanifold, we shall see, in Section 10.2c from the principle of least action, that this path is a very special one, a "geodesic" on the submanifold O(3), and this in turn will yield important information on the existence of periodic motions of the body even when the body is subject to an unusual potential field. All this depends on the fact that O(3) is a submanifold, and we turn now to the proof of this crucial result.

Note first that since $x^T x$ is a symmetric matrix, equation $(i, k)$ is the same as equation $(k, i)$; there are, then, only 6 independent equations. This suggests the following. Let

$$\text{Sym}^6 := \{x \in M(3 \times 3) \mid x^T = x\}$$

be the space of all *symmetric* $3 \times 3$ matrices. Since this is defined by the three *linear* equations $x_{ik} - x_{ki} = 0$, $i \neq k$, we see that $\text{Sym}^6$ is a 6-dimensional linear subspace of $\mathbb{R}^9$; that is, it can be considered as a copy of $\mathbb{R}^6$. To exhibit O(3) as a locus in $\mathbb{R}^9$, we consider the map

$$F : \mathbb{R}^9 \to \mathbb{R}^6 = \text{Sym}^6 \quad \text{defined by } F(x) = x^T x - I$$

O(3) is then the locus $F^{-1}(0)$. Let $x_0 \in F^{-1}(0) = \text{O}(3)$. We shall show that $F_* : \mathbb{R}^9_{x_0} \to \text{Sym}^6$ is onto.

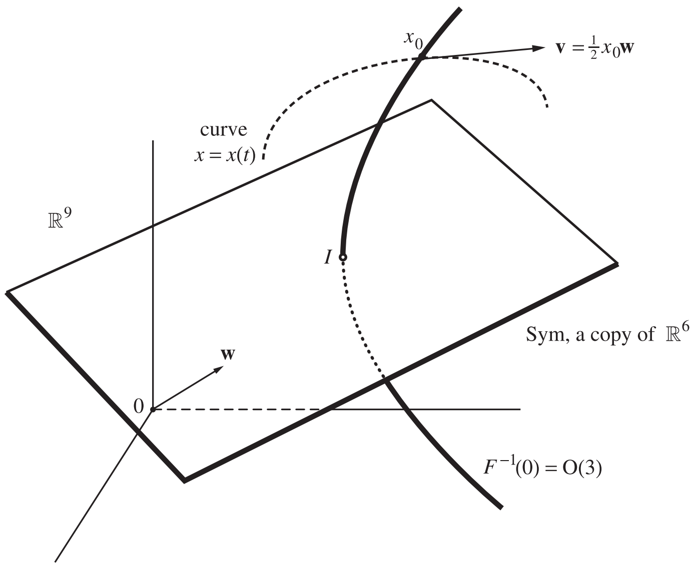



**Figure 1.5**

Let $\mathbf{w}$ be tangent to Sym$^6$ at the zero matrix. As usual, we identify a vector at the origin of $\mathbb{R}^n$ with its endpoint. *Then* $\mathbf{w}$ *is itself a symmetric matrix.* We must find $\mathbf{v}$, a tangent vector to $\mathbb{R}^9$ at $x_0$, such that $F_*\mathbf{v} = \mathbf{w}$. Consider a general curve $x = x(t)$ of matrices such that $x(0) = x_0$; its tangent vector at $x_0$ is $\dot{x}(0)$. The image curve

$$F(x(t)) = x(t)^T x(t) - I$$

has tangent at $t = 0$ given by

$$\frac{d}{dt}[F(x(t))]_{t=0} = \dot{x}(0)^T x_0 + x_0^T \dot{x}(0)$$

We wish this quantity to be $\mathbf{w}$. You should verify that it is sufficient to satisfy the matrix equation $x_0^T \dot{x}(0) = \mathbf{w}/2$. Since $x_0 \in$ O(3), $x_0^T = x_0^{-1}$ and we have as solution the matrix product $\mathbf{v} = \dot{x} = x_0\mathbf{w}/2$. Thus $F_*$ *is* onto at $x_0$ and by our main theorem O(3)$= F^{-1}(0)$ is a $(9 - 6) = 3$-dimensional submanifold of $\mathbb{R}^9$.

What about the subset SO(3) of O(3)? Recall that each orthogonal matrix has determinant $\pm 1$, whereas SO(3) consists of those orthogonal matrices with determinant $+1$. The mapping

$$\det : \mathbb{R}^9 \to \mathbb{R}$$

that sends each matrix $x$ into its determinant is continuous (it is a cubic polynomial function of the coordinates $x_{ik}$) and consequently the two subsets of O(3) where det is $+1$ and where det is $-1$ must be separated. This means that SO(3) itself must have the property that it is locally described by giving 6 of the coordinates in terms of the remaining 3, that is, SO(3) is a 3-dimensional submanifold of $\mathbb{R}^9$.

*Thus the configuration space of a rigid body with one point fixed is the group* SO(3). *This is a 3-dimensional submanifold of* $\mathbb{R}^9$. *Each point of this configuration space lies in some local curvilinear coordinate system.*

In physics books the coordinates in an $n$-dimensional configuration space are usually labeled $q^1, \ldots, q^n$. For SO(3) physicists usually use the three "Euler angles" as coordinates. These coordinates do not cover all of SO(3) in the sense that they become singular at certain points, just as polar coordinates in the plane are singular at the origin.

### Problems

**1.1(1)** Investigate the locus $x^2 + y^2 - z^2 = c$ in $\mathbb{R}^3$, for $c > 0$, $c = 0$, and $c < 0$. Are they submanifolds? What if the origin is omitted? Draw all three loci, for $c = 1$, 0, $-1$, in one picture.

**1.1(2)** SO($n$) is defined to be the set of all *orthogonal* $n \times n$ matrices $x$ with $\det x = 1$. The preceding discussion of SO(3) extends immediately to SO($n$). What is the dimension of SO($n$) and in what euclidean space is it a submanifold?

**1.1(3)** Is the **special linear group**

$$\mathrm{Sl}\,(n) := \{n \times n \text{ real matrices } x \mid \det x = 1\}$$

a submanifold of some $\mathbb{R}^N$ ? Hint: You will need to know something about $\partial/\partial x_{ij}$ (det $x$); expand the determinant by the $j^{\text{th}}$ column.This is an example where it might be easier to deal directly with the Jacobian matrix rather than the differential.

**1.1(4)** Show, in $\mathbb{R}^3$, that if the cross product of the gradients of $F$ and $G$ has a nontrivial component in the $x$ direction at a point of the intersection of $F = 0$ and $G = 0$, then $x$ can be used as local coordinate for this curve.

## 1.2. Manifolds

In learning the sciences examples are of more use than precepts.

Newton, *Arithmetica Universalis* (1707)

The notion of a "topology" will allow us to talk about "continuous" functions and points "neighboring" a given point, in spaces where the notion of distance and metric might be lacking.

The cultivation of an intuitive "feeling" for manifolds is of more importance, at this stage, than concern for topological details, but some basic notions from point set topology are helpful. The reader for whom these notions are new should approach them as one approaches a new language, with some measure of fluency, it is hoped, coming later.

In Section 1.2c we shall give a technical (i.e., complete) definition of a manifold.

### 1.2a. Some Notions from Point Set Topology

The **open ball** in $\mathbb{R}^n$, of radius $\epsilon$, centered at $\mathbf{a} \in \mathbb{R}^n$ is

$$B_{\mathbf{a}}(\epsilon) = \{\mathbf{x} \in \mathbb{R}^n \mid \| \mathbf{x} - \mathbf{a} \| < \epsilon\}$$