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Real and Complex Singularities

Edited By
M. Manoel, M. C. Romero Fuster
and C. T. C Wall

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10th International Workshop on Real and Complex Singularities
July 27 - August 01, 2008 ICMC-USP, São Carlos - SP, Brazil

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Real and Complex Singularities

Edited by

M. MANOEL

Universidade de São Paulo, Brazil

M. C. ROMERO FUSTER

Universitat de València, Spain

C. T. C. WALL

University of Liverpool, UK

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This volume is dedicated to
Maria Aparecida Ruas
and
Terence Gaffney
on their 60th birthdays

Maria Aparecida Ruas

Terence Gaffney

Contents

Preface

The *Workshops on Real and Complex Singularities* form a series of biennial meetings organized by the Singularities group at Instituto de Ciências Matemáticas e de Computação of São Paulo University (ICMC-USP), Brazil. Their main purpose is to bring together world experts and young researchers in singularity theory, applications and related fields to report recent achievements and exchange ideas, addressing trends of research in a stimulating environment.

These meetings started in 1990 following two pioneer symposia on Singularity Theory (in fact the very first talks on Singularities held in Brazil) organized respectively by G. Loibel and L. Favaro in 1982 and 1988 at ICMC-USP. Since then, with Maria Aparecida Ruas as a driving force, meetings have taken place every two years between singularists from around the world who find in São Carlos a centre to interact and develop new ideas.

The meeting held from the 27th of July to the 2nd of August 2008 was the tenth of these workshops. This was a special occasion, for it was also dedicated to Maria Aparecida Ruas (Cidinha) and Terence Gaffney on their 60th birthdays.

Cidinha and Terry started their scientific connection in 1976 when she was a Ph.D. student at Brown University in the U.S.A. At that time Terry held a position as instructor at that university. Their common interest in singularity theory brought them together and he became her (very young) thesis supervisor. Cidinha returned to Brazil in 1980 and joined the Singularities group created by G. Loibel at the ICMC-USP. Her great capacity and notable enthusiasm has brought the group to a leading position in the Brazilian mathematical community.

The mathematical interaction between Cidinha and Terry has had an important influence on the development of Singularity Theory at the ICMC-USP. From the beginning Terry has attended almost all workshops organized by the members of Cidinha's group, contributing to their research with stimulating discussions and seminars.

This workshop had a total of about 170 participants from about 15 different countries. The formal proceedings consisted of 27 plenary talks, 27 ordinary sessions and 3 poster sessions, with a total of 19 posters. The topics were divided into six categories: real singularities, classification of singularities, topology of singularities, global theory of singularities, singularities in geometry, and dynamical systems.

The Scientific Committee was composed of Lev Birbrair (Universidade Federal do Ceará, Brazil), Jean-Paul Brasselet (Institut de Mathématiques de Luminy, France), Goo Ishikawa (Hokkaido University, Japan), Shyuichi Izumiya (Hokkaido University, Japan), Steven Kleiman (Massachusetts Institute of Technology, USA), David Massey (Northeastern University, USA), David Mond (University of Warwick, UK), Maria del Carmen Romero Fuster (Universitat de València, Spain), Marcio Gomes Soares (Universidade Federal de Minas Gerais, Brazil), Marco Antonio Teixeira (Universidade Estadual de Campinas, Brazil), David Trotman (Université de Provence, France) and Terry Wall (University of Liverpool, UK).

Thanks are due to many people and institutions crucial in the realization of the workshop. We start by thanking the Organizing Committee: Roberta Wik Atique, Abramo Hefez, Isabel Labouriau, Miriam Manoel, Ana Claudia Nabarro, Regilene Oliveira and Marcelo José Saia. We also thank the members of the Scientific Committee for their support. The workshop was funded by FAPESP, CNPq, CAPES, USP and SBM, whose support we gratefully acknowledge. Finally, it is a pleasure to thank the speakers and the other participants whose presence was the real success of the tenth Workshop.

The editors

Introduction

This book is a selection of papers submitted for the proceedings of the *10th Workshop on Real and Complex Singularities*. They are grouped into three categories: singularity theory (7 papers), singular varieties (8 papers) and applications to dynamical systems, generic geometry, singular foliations, etc. (10 papers). Among them, four are survey papers: *Local Euler obstruction, old and new, II*, by N. G. Grulha Jr. and J.-P. Brasselet, *Global classifications and graphs*, by J. Martínez-Alfaro, C. Mendes de Jesus and M. C. Romero-Fuster, *Pairs of foliations on surfaces*, by F. Tari, and *Gaffney's work on equisingularity*, by C. T. C. Wall.

We thank the staff members of the London Mathematical Society involved with the preparation of this book. All papers presented here have been refereed.

1

On a conjecture by A. Durfee

E. ARTAL BARTOLO, J. CARMONA RUBER
AND A. MELLE-HERNÁNDEZ

Abstract

We show how *superisolated surface singularities* can be used to find a counterexample to the following conjecture by A. Durfee [8]: for a complex polynomial $f(x, y, z)$ in three variables vanishing at 0 with an isolated singularity there, "the local complex algebraic monodromy is of finite order if and only if a resolution of the germ $(\{f = 0\}, 0)$ has no cycles". A Zariski pair is given whose corresponding superisolated surface singularities, one has complex algebraic monodromy of finite order and the other not (answering a question by J. Stevens).

1. Introduction

In this paper we give an example of a *superisolated surface singularity* $(V, 0) \subset (\mathbb{C}^3, 0)$ such that a resolution of the germ $(V, 0)$ has no cycles and the local complex algebraic monodromy of the germ $(V, 0)$ is not of finite order, contradicting a conjecture proposed by Durfee [8].

For completeness in the second section we recall well known results about monodromy of the Milnor fibration, about normal surface singularities and state the question by Durfee.

In the third section we recall results on superisolated surface singularities and with them we study in detail the counterexample.

In the last section we show a Zariski pair (C_1, C_2) of curves of degree d given by homogeneous polynomials $f_1(x, y, z)$ and $f_2(x, y, z)$ whose

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corresponding superisolated surface singularities $(V_1, 0) = (\{f_1(x, y, z) + l^{d+1} = 0\}, 0) \subset (\mathbb{C}^3, 0)$ and $(V_2, 0) = (\{f_2(x, y, z) + l^{d+1} = 0\}, 0) \subset (\mathbb{C}^3, 0)$ (l is a generic hyperplane) satisfy: 1) $(V_1, 0)$ has complex algebraic monodromy of finite order and 2) $(V_2, 0)$ has complex algebraic monodromy of infinite order (answering a question proposed to us by J. Stevens).

2. Invariants of singularities

2.1. Monodromy of the Milnor fibration

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be an analytic function defining a germ $(V, 0) := (f^{-1}\{0\}, 0) \subset (\mathbb{C}^{n+1}, 0)$ of a hypersurface singularity. The *Milnor fibration* of the holomorphic function f at 0 is the C^∞ locally trivial fibration $f| : B_\varepsilon(0) \cap f^{-1}(\mathbb{D}^*_\eta) \to \mathbb{D}^*_\eta$, where $B_\varepsilon(0)$ is the open ball of radius ε centered at 0, $\mathbb{D}_\eta = \{z \in \mathbb{C} : |z| < \eta\}$ and $\mathbb{D}^*_\eta$ is the open punctured disk ($0 < \eta \ll \varepsilon$ and ε small enough). Milnor's classical result also shows that the topology of the germ $(V, 0)$ in $(\mathbb{C}^{n+1}, 0)$ is determined by the pair $(S^{2n+1}_\varepsilon, L^{2n-1}_V)$, where $S^{2n+1} = \partial B_\varepsilon(0)$ and $L^{2n-1}_V := S^{2n+1}_\varepsilon \cap V$ is the *link* of the singularity.

Any fiber $F_{f,0}$ of the Milnor fibration is called the *Milnor fiber* of f at 0. The *monodromy transformation* $h : F_{f,0} \to F_{f,0}$ is the well-defined (up to isotopy) diffeomorphism of $F_{f,0}$ induced by a small loop around $0 \in \mathbb{D}_\eta$. The *complex algebraic monodromy of f at* 0 is the corresponding linear transformation $h_* : H_*(F_{f,0}, \mathbb{C}) \to H_*(F_{f,0}, \mathbb{C})$ on the homology groups.

If $(V, 0)$ defines a germ of isolated hypersurface singularity then $\tilde{H}_j(F_{f,0}, \mathbb{C}) = 0$ but for $j = 0, n$. In particular the non-trivial complex algebraic monodromy will be denoted by $h : H_n(F_{f,0}, \mathbb{C}) \to H_n(F_{f,0}, \mathbb{C})$ and $\Delta_V(t)$ will denote its characteristic polynomial.

2.2. Monodromy Theorem and its supplements

The following are the **main properties of the monodromy operator**, see e.g. [11]:

(a) $\Delta_V(t)$ is a product of cyclotomic polynomials.
(b) Let N be the maximal size of the Jordan blocks of h, then $N \leq n + 1$.
(c) Let N_1 be the maximal size of the Jordan blocks of h for the eigenvalue 1, then $N_1 \leq n$.
(d) The monodromy h is called of *finite order* if there exists $N > 0$ such that $h^N = Id$. Lê D.T. [12] proved that the monodromy of an irreducible plane curve singularity is of finite order.
(e) This result was extended by van Doorn and Steenbrink [7] who showed that if h has a Jordan block of maximal size $n + 1$ then

$N_1 = n$, i.e. there exists a Jordan block of h of maximal size n for the eigenvalue 1.

Milnor proved that the link L_V^{2n-1} is $(n-2)$-connected. Thus the link is an *integer (resp. rational) homology* $(2n-1)$-*sphere* if $H_{n-1}(L_V^{2n-1}, \mathbb{Z}) = 0$ (resp. $H_{n-1}(L_V^{2n-1}, \mathbb{Q}) = 0$). These can be characterized considering the natural map $h - id : H_n(F_{f,0}, \mathbb{Z}) \to H_n(F_{f,0}, \mathbb{Z})$ and using Wang's exact sequence which reads as (see e.g. [19, 21]):

$$0 \to H_n\big(L_V^{2n-1}, \mathbb{Z}\big) \to H_n(F_{f,0}, \mathbb{Z}) \to H_n(F_{f,0}, \mathbb{Z}) \to H_{n-1}\big(L_V^{2n-1}, \mathbb{Z}\big) \to 0.$$

Thus rank $H_n(L_V^{2n-1}) = $ rank $H_{n-1}(L_V^{2n-1}) = \dim\ker(h - id)$ and:

- L_V^{2n-1} is a rational homology $(2n-1)$-sphere $\Longleftrightarrow \Delta_V(1) \neq 0$,
- L_V^{2n-1} is an integer homology $(2n-1)$-sphere $\Longleftrightarrow \Delta_V(1) = \pm 1$.

2.3. Normal surface singularities

Let $(V, 0) = (\{f_1 = \ldots = f_m = 0\}, 0) \subset (\mathbb{C}^N, 0)$ be a normal surface singularity with link $L_V := V \cap S_\varepsilon^{2N-1}$, L_V is a a connected compact oriented 3-manifold. Since $V \cap B_\varepsilon$ is a cone over the link L_V then L_V characterizes the topological type of $(V, 0)$. The link L_V is called a *rational homology sphere* ($\mathbb{Q}$HS) if $H_1(L_V, \mathbb{Q}) = 0$, and L_V is called an *integer homology sphere* ($\mathbb{Z}$HS) if $H_1(L_V, \mathbb{Z}) = 0$. One of the main problems in the study of normal surfaces is to determine which analytical properties of $(V, 0)$ can be read from the topology of the singularity, see the very nice survey paper by Nemethi [20].

The resolution graph $\Gamma(\pi)$ of a resolution $\pi : \tilde{V} \to V$ allows to relate analytical and topological properties of V. W. Neumann [22] proved that the information carried in any resolution graph is the same as the information carried by the link L_V. Let $\pi : \tilde{V} \to V$ be a *good* resolution of the singular point $0 \in V$. Good means that $E = \pi^{-1}\{0\}$ is a normal crossing divisor. Let $\Gamma(\pi)$ be the dual graph of the resolution (each vertex decorated with the genus $g(E_i)$ and the self-intersection E_i^2 of E_i in $\tilde{V}$). Mumford proved that the intersection matrix $I = (E_i \cdot E_j)$ is negative definite and Grauert proved the converse, i.e., any such graph comes from the resolution of a normal surface singularity.

Considering the exact sequence of the pair $(\tilde{V}, E)$ and using I is non-degenerated then

$$0 \longrightarrow \operatorname{coker} I \longrightarrow H_1(L_V, \mathbb{Z}) \longrightarrow H_1(E, \mathbb{Z}) \longrightarrow 0$$

and rank $H_1(E) = $ rank $H_1(L_V)$. In fact L_V is a $\mathbb{Q}$HS if and only if $\Gamma(\pi)$ is a tree and every E_i is a rational curve. If additionally I has determinant ± 1 then L_V is an $\mathbb{Z}$HS.

2.4. Number of cycles in the exceptional set E and Durfee's conjecture

In general one gets

$$\operatorname{rank} H_1(L_V) = \operatorname{rank} H_1(\Gamma(\pi)) + 2\sum_i g(E_i),$$

where $\operatorname{rank} H_1(\Gamma(\pi))$ is the number of independent cycles of the graph $\Gamma(\pi)$. Let $n : \tilde{E} \to E$ be the normalization of E. Durfee showed in [8] that the *number of cycles $c(E)$ in E*, i.e. $c(E) = \operatorname{rank} H_1(E) - \operatorname{rank} H_1(\tilde{E})$, does not depend on the resolution and in fact it is equal to $c(E) = \operatorname{rank} H_1(\Gamma(\pi))$. Therefore, E contains cycles only when the dual graph of the intersections of the components contains a cycle. Durfee in [8] proposed the following

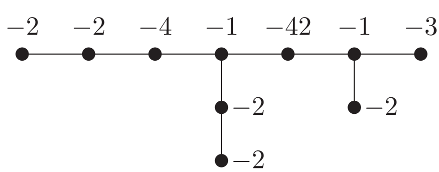
Conjecture. *For a complex polynomial $f(x, y, z)$ in three variables vanishing at* 0 *with an isolated singularity there, "the local complex algebraic monodromy h is of finite order if and only if a resolution of the germ* $(\{f = 0\}, 0)$ *has no cycles".*

He showed that the conjecture is true in the following two cases:

(1) if f is weighted homogeneuos (the resolution graph is star-shaped and therefore its monodromy is finite)
(2) if $f = g(x, y) + z^n$. Using Thom-Sebastiani [27], the monodromy of f is finite if and only if the monodromy of g is finite. Theorem 3 in [8] proves that the monodromy of f is of finite order if and only a resolution of f has no cycles.

2.5. Example (main result)

In this paper we show that the conjecture is not true in general, and for that we use superisolated surface singularities. Let $(V, 0) \subset (\mathbb{C}^3, 0)$ be the germ of normal surface singularity defined by $f := (xz - y^2)^3 - ((y - x)x^2)^2 + z^7 = 0$. Then the minimal good resolution graph Γ_V of (the superisolated singularity) $(V, 0)$ is

Γ_V:


where every dot denotes a rational non-singular curve with the corresponding self-intersection. Thus the link L_V is a rational homology sphere and in particular this graph is a tree, i.e. it has no cycles. But the complex algebraic monodromy of f at 0 does not have finite order because there exists a Jordan block of size 2×2 for an eigenvalue $\neq 1$.

3. Superisolated surface singularities

Definition 3.1. A hypersurface surface singularity $(V, 0) \subset (\mathbb{C}^3, 0)$ defined as the zero locus of $f = f_d + f_{d+1} + \cdots \in \mathbb{C}\{x, y, z\}$ (where f_j is homogeneous of degree j) is *superisolated*, SIS for short, if the singular points of the complex projective plane curve $C := \{f_d = 0\} \subset \mathbb{P}^2$ are not situated on the projective curve $\{f_{d+1} = 0\}$, that is $\mathrm{Sing}(C) \cap \{f_{d+1} = 0\} = \emptyset$. Note that C must be reduced.

The SIS were introduced by I. Luengo in [17] to study the μ-constant stratum. The main idea is that for a SIS the embedded topological type (and the equisingular type) of $(V, 0)$ does not depend on the choice of f_j's (for $j > d$, as long as f_{d+1} satisfies the above requirement), e.g. one can take $f_j = 0$ for any $j > d + 1$ and $f_{d+1} = l^{d+1}$ where l is a linear form not vanishing at the singular points [18].

3.1. The minimal resolution of a SIS

Let $\pi : \tilde{V} \to V$ be the monoidal transformation centered at the maximal ideal $\mathfrak{m} \subset \mathcal{O}_V$ of the local ring of V at 0. Then $(V, 0)$ is a SIS if and only if $\tilde{V}$ is a non-singular surface. Thus π is the *minimal resolution* of $(V, 0)$. To construct the resolution graph $\Gamma(\pi)$ consider $C = C_1 + \cdots + C_r$ the decomposition in irreducible components of the reduced curve C in $\mathbb{P}^2$. Let d_i (resp. g_i) be the degree (resp. genus) of the curve C_i in $\mathbb{P}^2$. Then $\pi^{-1}\{0\} \cong C = C_1 + \cdots + C_r$ and the self-intersection of C_i in $\tilde{V}$ is $C_i \cdot C_i = -d_i(d - d_i + 1)$, [17, Lemma 3]. Since the link L_V can be identified with the boundary of a regular neighbourhood of $\pi^{-1}\{0\}$ in $\tilde{V}$ then the topology of the tangent cone determines the topology of the abstract link L_V [17].

3.2. The minimal good resolution of a SIS

The minimal good resolution of a SIS $(V, 0)$ is obtained after π by doing the minimal embedded resolution of each plane curve singularity $(C, P) \subset (\mathbb{P}^2, P)$, $P \in \mathrm{Sing}(C)$. This means that the support of the minimal good resolution graph

Γ_V is the same as the minimal embedded resolution graph Γ_C of the projective plane curve C in $\mathbb{P}^2$. The decorations of the minimal good resolution graph Γ_V are as follows:

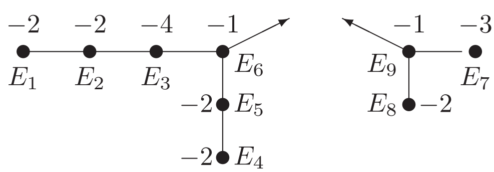
1) the genus of (the strict transform of) each irreducible component C_i of C is a birational invariant and then one can compute it as an embedded curve in $\mathbb{P}^2$. All the other curves are non-singular rational curves.

2) Let C_j be an irreducible component of C such that $P \in C_j$ and with multiplicity $n \geq 1$ at P. After blowing-up at P, the new self-intersection of the (strict transform of the) curve C_j in the (strict transform of the) surface $\tilde{V}$ is $C_j^2 - n^2$. In this way one constructs the minimal good resolution graph Γ of $(V, 0)$.

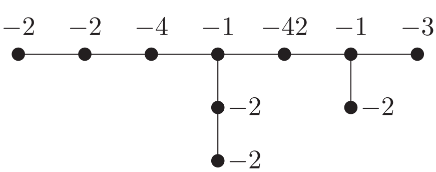
In particular the theory of hypersurface superisolated surface singularities "contains" in a canonical way the theory of complex projective plane curves.

Example 3.2. If $(V, 0) \subset (\mathbb{C}^3, 0)$ is a SIS with an irreducible tangent cone $C \subset \mathbb{P}^2$ then L_V is a rational homology sphere if and only if C is a rational curve and each of its singularities (C, p) is locally irreducible, i.e a cusp.

Example 3.3. For instance, if $f = f_6 + z^7$ is given by the equation $f_6 = (xz - y^2)^3 - ((y - x)x^2)^2$. The plane projective curve C defined by $f_6 = 0$ is irreducible with two singular points: $P_1 = [0 : 0 : 1]$ (with a singularity of local singularity type $u^3 - v^{10}$) and $P_2 = [1 : 1 : 1]$ (with a singularity of local singularity type $\mathbb{A}_2$) which are locally irreducible. Let $\pi : X \to \mathbb{P}^2$ be the minimal embedded resolution of C at its singular points $P_1,\ P_2$. Let $E_i, i \in I$, be the irreducible components of the divisor $\pi^{-1}(f^{-1}\{0\})$.



The minimal good resolution graph Γ_V of the superisolated singularity $(V, 0)$ is given by

Γ_V:


3.3. The embedded resolution of a SIS

In [2], the first author has studied, for SIS, the Mixed Hodge Structure of the cohomology of the Milnor fibre introduced by Steenbrink and Varchenko, [28], [29]. For that he constructed in an effective way an embedded resolution of a SIS and described the MHS in geometric terms depending on invariants of the pair $(\mathbb{P}^2, C)$.

The first author determined the Jordan form of the complex monodromy on $H_2(F_{f,0}, \mathbb{C})$ of a SIS. Let $\Delta_V(t)$ be the corresponding characteristic polynomial of the complex monodromy on $H_2(F_{f,0}, \mathbb{C})$. Denote by $\mu(V, 0) = \deg(\Delta_V(t))$ the Milnor number of $(V, 0) \subset (\mathbb{C}^3, 0)$.

Let $\Delta^P(t)$ be the characteristic polynomial (or Alexander polynomial) of the action of the complex monodromy of the germ (C, P) on $H_1(F_{g^P}, \mathbb{C})$, (where g^P is a local equation of C at P and F_{g^P} denotes the corresponding Milnor fiber). Let μ^P be the Milnor number of C at P. Recall that if $n^P : \tilde{C}^P \to (C, P)$ is the normalization map then $\mu^P = 2\delta^P - (r^P - 1)$, where $\delta^P := \dim_{\mathbb{C}} n^P_*(\mathcal{O}_{\tilde{C}^P})/\mathcal{O}_{C,P}$ and r^P is the number of local irreducible components of C at P.

Let H be a $\mathbb{C}$-vector space and $\varphi : H \to H$ an endomorphism of H. The i-th Jordan polynomial of φ, denoted by $\Delta_i(t)$, is the monic polynomial such that for each $\zeta \in \mathbb{C}$, the multiplicity of ζ as a root of $\Delta_i(t)$ is equal to the number of Jordan blocks of size $i + 1$ with eigenvalue equal to ζ.

Let Δ_1 and Δ_2 be the first and the second Jordan polynomials of the complex monodromy on $H_2(F_{f,0}, \mathbb{C})$ of V and let Δ_1^P be the first Jordan polynomial of the complex monodromy of the local plane singularity (C, P). After the Monodromy Theorem these polynomials joint with $\Delta_V(t)$ and Δ^P, $P \in \mathrm{Sing}(C)$, determine the corresponding Jordan form of the complex monodromy. Let us denote the Alexander polynomial of the plane curve C in $\mathbb{P}^2$ by $\Delta_C(t)$, it was introduced by A. Libgober [13, 14] and F. Loeser and Vaquié [16].

Theorem 3.4 [2]. *Let $(V, 0)$ be a SIS whose tangent cone $C = C_1 \cup \ldots \cup C_r$ has r irreducible components and degree d. Then the Jordan form of the complex monodromy on $H_2(F_{f,0}, \mathbb{C})$ is determined by the following polynomials*

(i) *The characteristic polynomial $\Delta_V(t)$ is equal to*

$$\Delta_V(t) = \frac{(t^d - 1)^{\chi(\mathbb{P}^2 \setminus C)}}{(t - 1)} \prod_{P \in \mathrm{Sing}(C)} \Delta^P(t^{d+1}).$$

(ii) *The first Jordan polynomial is equal to*

$$\Delta_1(t) = \frac{1}{\Delta_C(t)(t-1)^{r-1}} \prod_{P\in\mathrm{Sing}(C)} \frac{\Delta_1^P(t^{d+1})\Delta_{(d)}^P(t)}{\Delta_{1,(d)}^P(t)^3},$$

where $\Delta_{(d)}^P(t) := \gcd(\Delta^P(t), (t^d-1)^{\mu^P})$ *and* $\Delta_{1,(d)}^P(t) := \gcd(\Delta_1^P(t), (t^d-1)^{\mu^P})$.

(iii) *The second Jordan polynomial is equal to*

$$\Delta_2(t) = \prod_{P\in\mathrm{Sing}(C)} \Delta_{1,(d)}^P(t).$$

Corollary 3.5 [2, Corollaire 5.5.4]. *The number of Jordan blocks of size 2 for the eigenvalue* 1 *of the complex monodromy h is equal to*

$$\sum_{P\in\mathrm{Sing}(C)} (r^P-1)-(r-1). \tag{3.1}$$

Let $\tilde{D}_i$ be the normalization of D_i and $\tilde{C}$ the disjoint union of the $\tilde{D}_i$ and $n:\tilde{C}\to C$ be the projection map. Thus the first Betti number of $\tilde{C}$ is $2g := 2\sum_i g(D_i)$ and the first Betti number of C is $2g+\sum_{P\in\mathrm{Sing}(C)}(r^P-1)-r+1$. Then $\sum_{P\in\mathrm{Sing}(C)}(r^P-1)-(r-1)$ is exactly the difference between the first Betti numbers of C and $\tilde{C}$. In fact this non-negative integer is equal to the first Betti number of the minimal embedded resolution graph Γ_C of the projective plane curve C in $\mathbb{P}^2$, which is nothing but rank $H_1(\Gamma_V)$.

Corollary 3.6. *Let $(V,0)$ be a SIS whose tangent cone $C = C_1\cup\ldots\cup C_r$ has r irreducible components. Then the number of independent cycles $c(E) =$* rank $H_1(\Gamma_V) = \sum_{P\in\mathrm{Sing}(C)}(r^P-1)-(r-1)$.

In particular E has no cycles if and only if $\sum_{P\in\mathrm{Sing}(C)}(r^P-1) = (r-1)$ if and only if the complex monodromy h has no Jordan blocks of size 2 for the eigenvalue 1.

Corollary 3.7 [2, Corollaire 4.3.2]. *If for every $P\in\mathrm{Sing}(C)$, the local monodromy of the local plane curve equation g^p at P acting on the homology $H_1(F_{g^P},\mathbb{C})$ of the Milnor fibre F_{g^P} has no Jordan blocks of maximal size* 2 *then the corresponding SIS has no Jordan blocks of size 3.*

Corollary 3.8. *Let $(V,0)\subset(\mathbb{C}^3,0)$ be a SIS with a rational irreducible tangent cone $C\subset\mathbb{P}^2$ of degree d whose singularities are locally irreducible. Then:*

(i) *the link L_V is a $\mathbb{Q}$HS link and E has no cycles,*
(ii) *the complex monodromy on $H_2(F_{f,0},\mathbb{C})$ has no Jordan blocks of size 2 for the eigenvalue* 1,

(iii) *the complex monodromy on* $H_2(F_{f,0}, \mathbb{C})$ *has no Jordan blocks of size 3.*
(iv) *The first Jordan polynomial is equal to*

$$\Delta_1(t) = \frac{1}{\Delta_C(t)} \prod_{P \in \mathrm{Sing}(C)} \gcd(\Delta^P(t), (t^d - 1)^{\mu^P}).$$

The proof follows from the previous description and the fact that if every $P \in \mathrm{Sing}(C)$ is locally irreducible then by Lê D.T. result (see 2.2) the plane curve singularity has finite order and $\Delta_1^P(t) = 1$.

Corollary 3.9. *Let* $(V, 0) \subset (\mathbb{C}^3, 0)$ *be a SIS whose tangent cone* $C = C_1 \cup \ldots \cup C_r$ *has* r *irreducible components. Assume that* $\sum_{P \in \mathrm{Sing}(C)} (r^P - 1) = (r-1)$*, then:*

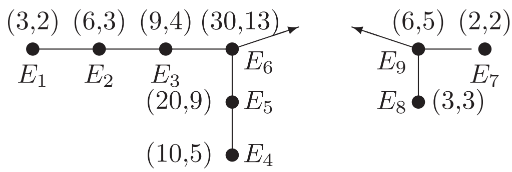
(i) E *has no cycles,*
(ii) *the complex monodromy on* $H_2(F_{f,0}, \mathbb{C})$ *has no Jordan blocks of size 2 for the eigenvalue* 1,
(iii) *the complex monodromy on* $H_2(F_{f,0}, \mathbb{C})$ *has no Jordan blocks of size 3.*
(iv) *The first Jordan polynomial is equal to*

$$\Delta_1(t) = \frac{1}{\Delta_C(t)(t-1)^{r-1}} \prod_{P \in \mathrm{Sing}(C)} \gcd(\Delta^P(t), (t^d - 1)^{\mu^P}).$$

The proof follows from Corollary 3.6 and the part (e) Monodromy Theorem 2.2.

3.4. The first Jordan polynomial in Example 3.3

As we described above, the plane projective curve C defined by $f_6 = (xz - y^2)^3 - ((y-x)x^2)^3 = 0$ is irreducible, rational and with two singular points: $P_1 = [0:0:1]$ (with a singularity of local singularity type $u^3 - v^{10}$) and $P_2 = [1:1:1]$ (with a singularity of local singularity type $\mathbb{A}_2$) which are unibranched. Let $\pi : X \to \mathbb{P}^2$ be the minimal embedded resolution of C at its singular points P_1, P_2. Let E_i, $i \in I$, be the irreducible components of the divisor $\pi^{-1}(f^{-1}\{0\})$. For each $j \in I$, we denote by N_j the multiplicity of E_j in the divisor of the function $f \circ \pi$ and we denote by $\nu_j - 1$ the multiplicity of E_j in the divisor of $\pi^*(\omega)$ where ω is the non-vanishing holomorpic 2-form $dx \wedge dy$ in $\mathbb{C}^2 = \mathbb{P}^2 \setminus L_\infty$. Then the divisor $\pi^*(C)$ is a normal crossing divisor. We attach to each exceptional divisor E_i its numerical data (N_i, ν_i).



Thus $\Delta^{P_1}(t) = \frac{(t-1)(t^{30}-1)}{(t^3-1)(t^{10}-1)} = \phi_{30}\phi_{15}\phi_6$ and $\Delta^{P_2}(t) = \frac{(t-1)(t^6-1)}{(t^3-1)(t^2-1)} = \phi_6$, where ϕ_k is the k-th cyclotomic polynomial. Thus, by Corollary 3.8, the only possible eigenvalues of with Jordan blocks of size 2 are the roots of the polynomial $\Delta_1(t) = \frac{\phi_6^2}{\Delta_C(t)}$.

The proof of our main result will be finished if we show that the Alexander polynomial $\Delta_C(t) = \phi_6$. The Alexander polynomial, in particular of sextics, has been investigated in detail by Artal [1], Artal and Carmona [3], Degtyarev [6], Oka [24], Pho [25], Zariski [30] among others. In [23] Corollary 18, I.2, it is proved that $\Delta_C(t) = \phi_6$.

Consider a generic line L_∞ in $\mathbb{P}^2$, in our example the line $z = 0$ is generic, and define $f(x, y) = f_6(x, y, 1)$. Consider the (global) Milnor fibration given by the homogeneous polynomial $f_6 : \mathbb{C}^3 \to \mathbb{C}$ with Milnor fibre F. Randell [26] proved that $\Delta_C(t)(t-1)^{r-1}$ is the characteristic polynomial of the algebraic monodromy acting on $F : T_1 : H_1(F, \mathbb{C}) \to H_1(F, \mathbb{C})$.

Lemma 3.10 (Divisibility properties) [13]. *The Alexander polinomial $\Delta_C(t)(t-1)^{r-1}$ divides $\prod_{P\in\mathrm{Sing}(C)} \Delta^P(t)$ and the Alexander polynomial at infinity $(t^d-1)^{d-2}(t-1)$. In particular the roots of the Alexander polynomial are d-roots of unity.*

To compute the Alexander polynomial $\Delta_C(t)$ we combined the method described in [1] with the methods given in [13], [16] and [9].

Consider for $k = 1, \ldots, d-1$ the ideal sheaf $\mathcal{I}^k$ on $\mathbb{P}^2$ defined as follows:

- If $Q \in \mathbb{P}^2 \setminus \mathrm{Sing}(C)$ then $\mathcal{I}^k_Q = \mathcal{O}_{\mathbb{P}^2,Q}$.
- If $P \in \mathrm{Sing}(C)$ then $\mathcal{I}^k_P$ is the following ideal of $\mathcal{O}_{\mathbb{P}^2,P}$: if $h \in \mathcal{O}_{\mathbb{P}^2,P}$ then $h \in \mathcal{I}^k_P$ if and only if the vanishing order of $\pi^*(h)$ along each E_i is, at least, $-(\nu_i - 1) + [\frac{kN_i}{d}]$ (where [.] stands for the integer part of a real number).

For $k \geq 0$ the following map

$$\sigma_k : H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k-3)) \to \bigoplus_{P\in\mathrm{Sing}(C)} \mathcal{O}_{\mathbb{P}^2,P}/\mathcal{I}^k_P : h \mapsto (h_P + \mathcal{I}^k_P)_{P\in\mathrm{Sing}(C)}$$

is well defined (up to scalars) and the result of [13] and [16] reinterpreted in this language as [1] and [9] reads as follows:

Theorem 3.11 (Libgober, Loeser-Vaquié).

$$\Delta_C(t) = \prod_{k=1}^{d-1} (\Delta^k(t))^{l_k}, \tag{3.2}$$

where $\Delta^k(t) := (t - \exp(\frac{2k\pi i}{d})(t - \exp(\frac{-2k\pi i}{d})$ *and* $l_k = \dim \operatorname{coker} \sigma_k$

In our case, by the Divisibility properties (Lemma 3.10), $\Delta_C(t)$ divides $\Delta^{P_1}(t)\Delta^{P_2}(t) = \phi_{30}\phi_{15}\phi_6^2$. Thus, by Theorem 3.11, we are only interested in the case $k = 1$ and 5, $\Delta^1(t) = \Delta^5(t) = \phi_6 = (t^2 - t - 1)$. In case $k = 1$, we have $l_1 = 0$.

In case $k = 5$, the ideal $\mathcal{I}^5_{P_1}$ is the following ideal of $\mathcal{O}_{\mathbb{P}^2, P_1}$:

$$\mathcal{I}^5_{P_1} = \{h \in \mathcal{O}_{\mathbb{P}^2, P_1} : (\pi^* h) \geq E_1 + 3E_2 + 4E_3 + 4E_4 + 8E_5 + 13E_6\}$$

and with the local change of coordinates $u = x - y^2$, $w = y$, the generators of the ideal are $\mathcal{I}^5_{P_1} =< uw, u^2, w^5 >$ and the dimension of the quotient vector space $\mathcal{O}_{\mathbb{P}^2, P_1}/\mathcal{I}^5_{P_1}$ is 6. A basis is given by $1, u, w, w^2, w^3, w^4$. The ideal

$$\mathcal{I}^5_{P_2} = \{h \in \mathcal{O}_{\mathbb{P}^2, P_2} : (\pi^* h) \geq 0E_7 + 0E_8 + E_9\} = \mathfrak{m}_{\mathbb{P}^2, P_2}$$

and the dimension of the quotient vector space $\mathcal{O}_{\mathbb{P}^2, P_2}/\mathcal{I}^5_{P_2}$ is 1. A basis is given by 1.

If we take as a basis for the space of conics $1, x, y, x^2, y^2, xy$, the map σ_5

$$\begin{aligned} \sigma_5 : H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)) &\to \mathcal{O}_{\mathbb{P}^2, P_1}/\mathcal{I}^5_{P_1} \times \mathcal{O}_{\mathbb{P}^2, P_2}/\mathcal{I}^5_{P_2} \\ &= \mathbb{C}^6 \times \mathbb{C} : h \mapsto \left(h + \mathcal{I}^5_{P_1}, h + \mathcal{I}^5_{P_2}\right) \end{aligned}$$

is given in such coordinates by (using $u = x - y^2$): $\sigma_5(1) = (1, 0, 0, 0, 0, 0, 1)$, $\sigma_5(x) = (0, 1, 0, 1, 0, 0, 1)$, $\sigma_5(y) = (0, 0, 1, 0, 0, 0, 1)$, $\sigma_5(x^2) = (0, 0, 0, 0, 0, 1, 1)$, $\sigma_5(y^2) = (0, 0, 0, 1, 0, 0, 1)$ and $\sigma_5(xy) = (0, 0, 0, 0, 1, 0, 1)$.

Therefore σ_5 is injective and $\dim \operatorname{coker} \sigma_5 = 7 - 6 + 0 = 1$. The key point is that $u \notin \mathcal{I}^5_{P_1}$.

4. Zariski pairs

Let us consider $C \subset \mathbb{P}^2$ a reduced projective curve of degree d defined by an equation $f_d(x, y, z) = 0$. If $(V, 0) \subset (\mathbb{C}^3, 0)$ is a SIS with tangent cone C, then the link L_V of the singularity is completely determined by C. Let us recall, that L_V is a Waldhausen manifold and its plumbing graph is the dual graph of the good minimal resolution. In order to determine L_V we do not need the embedding of C in $\mathbb{P}^2$, but only its embedding in a regular neighborhood. The needed data can be encoded in a combinatorial way.

Definition 4.1. Let $\mathrm{Irr}(C)$ be the set of irreducible components of C. For $P \in \mathrm{Sing}(C)$, let $B(P)$ be the set of local irreducible components of C. The *combinatorial type* of C is given by:

- A mapping $\deg : \mathrm{Irr}(C) \to \mathbb{Z}$, given by the degrees of the irreducible components of C.
- A mapping $\mathrm{top} : \mathrm{Sing}(C) \to \mathrm{Top}$, where Top is the set of topological types of singular points. The image of a singular point is its topological type.
- For each $P \in \mathrm{Sing}(C)$, a mapping $\beta_P : T(P) \to \mathrm{Irr}(C)$ such that if γ is a branch of C at P, then $\beta_)(\gamma)$ is the global irreducible component containing γ.

Remark 1. There is a natural notion of isomorphism of combinatorial types. It is easily seen that combinatorial type determines and is determined by any of the following graphs (with vertices decorated with self-intersections):

- The dual graph of the preimage of C by the minimal resolution of $\mathrm{Sing}'(C)$. The set $\mathrm{Sing}'(C)$ is obtained from $\mathrm{Sing}(C)$ by forgetting ordinary double points whose branches belong to distinct global irreducible components. We need to mark in the graph the r vertices corresponding to $\mathrm{Irr}(C)$.
- The dual graph of the minimal good resolution of V. Since the minimal resolution is unique, it is not necessary to mark vertices.

Note also that the combinatorial type determine the characteristic polynomial $\Delta_V(t)$ of V (see Theorem 3.4).

Definition 4.2. A *Zariski pair* is a set of two curves $C_1, C_2 \subset \mathbb{P}^2$ with the same combinatorial type but such that $(\mathbb{P}^2, C_1)$ is not homeomorphic to $(\mathbb{P}^2, C_2)$. An *Alexander-Zariski pair* $\{C_1, C_2\}$ is a Zariski pair such that the Alexander polynomials of C_1 and C_2 do not coincide.

In [2], (see here Theorem 3.4) it is shown that the Jordan form of complex monodromy of a SIS is determined by the combinatorial type and the Alexander polynomial of its tangent cone. The first example of Zariski pair was given by Zariski, [30, 31]; there exist sextic curves with six ordinary cusps. If these cusps are (resp. not) in a conic then the Alexander polynomial equals $t^2 - t + 1$ (resp. 1). Then, it gives an Alexander-Zariski pair. Many other examples of Alexander-Zariski pairs have been constructed (Artal [1], Degtyarev [6]).

We state the main result in [2].

Theorem 4.3. *Let V_1, V_2 be two SIS such that their tangent cones form an Alexander-Zariski pair. Then V_1 and V_2 have the same abstract topology and characteristic polynomial of the monodromy but not the same embedded topology.*

Recall that the Jordan form of the monodromy is an invariant of the embedded topology of a SIS (see Theorem 3.4); since it depends on the Alexander polynomial $\Delta_C(t)$ of the tangent cone.

4.1. Zariski pair of reduced sextics with only one singular point of type $\mathbb{A}_{17}$

Our next Zariski-pair example (C_1, C_2) can be found in [1, Théorème 4.4]. The curves C_i, $i = 1, 2$ are reduced sextics with only one singular point P of type $\mathbb{A}_{17}$, locally given by $u^2 - v^{18}$.

(I) the irreducible componentes of C_1 are two non-singular cubics. These cubics meet at only one point P which moreover is an inflection point of each of the cubics, i.e. the tangent line to the singular point P goes through the infinitely near points P, P_1 and P_2 of C_1. The equations of C_1 are given for instance by $\{f_1(x, y, z) := (zx^2 - y^3 - ayz^2 - bz^3)(zx^2 - y^3 - ayz^2 - cz^3) = 0\}$, with $a, b, c \in \mathbb{C}$ generic.

(II) the irreducible componentes of C_2 are two non-singular cubics. These cubics meet at only one point P which is not an inflection point of any of the cubics, i.e. the tangent line to the singular point P goes through the infinitely near points P, P_1 of C_1 but it is not going through P_2. The equations of C_2 are given for instance by $\{f_2(x, y, z) := (zx^2 - y^2x - yz^2 - a_1(z^3 - y(xz - y^2)))(zx^2 - y^2x - yz^2 - a_2(z^3 - y(xz - y^2))) = 0\}$ with $a_1, a_2 \in \mathbb{C}$ generic.

Consider the superisolated surface singularities $(V_1, 0) = (\{f_1(x, y, z) + l^7 = 0\}, 0) \subset (\mathbb{C}^3, 0)$ and $(V_2, 0) = (\{f_2(x, y, z) + l^7 = 0\}, 0) \subset (\mathbb{C}^3, 0)$ (l is a generic hyperplane). In both cases the tangent cone has two irreducible components and it has only one singular point P of local type $u^2 - v^{18}$ and therefore $\Delta^P(t) = (t^{18} - 1)(t - 1)/(t^2 - 1) = \phi_{18}\phi_9\phi_6\phi_3\phi_1$, where ϕ_k is the k-th cyclotomic polynomial. Thus the number of local branches is 2 and $\sum_{P\in \mathrm{Sing}(C)}(r^P - 1) = (r - 1)$. By Corollary 3.9, for $(V_i, 0)$, $i = 1, 2$, the complex monodromy has no Jordan blocks of size 2 for the eigenvalue 1, and it has no Jordan blocks of size 3. Moreover the first Jordan polynomial is equal to

$$\Delta_1(t) = \frac{\gcd(\Delta^P(t), (t^6 - 1)^{\mu^P})}{\Delta_{C_i}(t)(t - 1)} = \frac{\phi_6\phi_3}{\Delta_{C_i}(t)}. \tag{4.1}$$

To compute $\Delta_{C_i}(t)$ we use the same ideas as in Theorem 3.11.

Lemma 4.4. *For the point P at the curve C_1 the ideals $\mathcal{I}_P^k = \mathcal{O}_{\mathbb{P}^2,P}$ if $k \leq 3$, $\mathcal{I}_P^4 =< y^3, z > \mathcal{O}_{\mathbb{P}^2,P}$ and $\mathcal{I}_P^5 =< y^6, z - y^3 - ay^4 - by^5 > \mathcal{O}_{\mathbb{P}^2,P}$.*

Lemma 4.5. *For the point P at the curve C_2 the ideals $\mathcal{I}_P^k = \mathcal{O}_{\mathbb{P}^2,P}$ if $k \leq 3$, $\mathcal{I}_P^4 =< y^3, z - y^2 > \mathcal{O}_{\mathbb{P}^2,P}$ and $\mathcal{I}_P^5 =< y^6, z - y^2 - y^5 > \mathcal{O}_{\mathbb{P}^2,P}$.*

Thus $\Delta_{C_i}(t) = \phi_6^{\dim \operatorname{coker} \sigma_5} \phi_3^{\dim \operatorname{coker} \sigma_4}$.

Therefore the map σ_4 is

$$\sigma_4 : H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \simeq \mathbb{C}^3 \to \mathcal{O}_{\mathbb{P}^2,P}/\mathcal{I}_P^4 \simeq \mathbb{C}^3,$$

and if we choose as basis of the first space $\{1, y, z\}$ and of the second $\{1, y, y^2\}$ then

(1) by Lemma 4.4, for C_1 the dimension $\dim \operatorname{coker} \sigma_4 = \dim \ker \sigma_4 = 1$.
(2) by Lemma 4.5, for C_2 the dimension $\dim \operatorname{coker} \sigma_4 = \dim \ker \sigma_4 = 0$.

On the other hand for the map σ_5

$$\sigma_5 : H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)) \simeq \mathbb{C}^6 \to \mathcal{O}_{\mathbb{P}^2,P}/\mathcal{I}_P^5 \simeq \mathbb{C}^6,$$

if we choose as basis of the first space $\{1, y, z, y^2, yz, z^2\}$ and of the second $\{1, y, y^2, y^3, y^4, y^5\}$ then we can compute

(3) by Lemma 4.4, for C_1 the dimension $\dim \operatorname{coker} \sigma_5 = \dim \ker \sigma_5 = 1$.
(4) by Lemma 4.5, for C_2 the dimension $\dim \operatorname{coker} \sigma_5 = \dim \ker \sigma_5 = 0$.

Therefore, $\Delta_{C_1}(t) = \phi_6\phi_3$ and $\Delta_{C_2}(t) = 1$ and by (4.1) we have proved that the pair (C_1, C_2) is a Alexander-Zariski pair.

Example 4.6. Consider the superisolated surface singularities $(V_1, 0) = (\{f_1(x, y, z) + l^7 = 0\}, 0) \subset (\mathbb{C}^3, 0)$ and $(V_2, 0) = (\{f_2(x, y, z) + l^7 = 0\}, 0) \subset (\mathbb{C}^3, 0)$ (l is a generic hyperplane). Then the complex algebraic monodromy of $(V_1, 0) \subset (\mathbb{C}^3, 0)$ has finite order and the complex algebraic monodromy of $(V_2, 0) \subset (\mathbb{C}^3, 0)$ has not finite order

This answers a question proposed to us by J. Stevens: find a Zariski pair C_1, C_2 such that for the corresponding SIS surface singularities $(V_1, 0) \subset (\mathbb{C}^3, 0)$ and $(V_2, 0) \subset (\mathbb{C}^3, 0)$ one has a finite order monodromy and the other it does not.

There are also examples of Zariski pairs which are not Alexander-Zariski pairs (see [23], [3], [4]). Some of them are distinguished by the so-called characteristic varieties introduced by Libgober [15]. These are subtori of $(\mathbb{C}^*)^r$, $r := \# \operatorname{Irr}(C)$, which measure the excess of Betti numbers of finite Abelian coverings of the plane ramified on the curve (as Alexander polynomial does it for cyclic coverings).

Problem 1. How can one translate characteristic varieties of a projective curve in terms of invariants of the SIS associated to it?

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E. Artal Bartolo
Departamento de Matemáticas, IUMA
Universidad de Zaragoza
Campus Pza. San Francisco s/n
E-50009 Zaragoza SPAIN
artal@unizar.es

A. Melle-Hernández
Departamento de Álgebra,
Universidad Complutense,
Plaza de las Ciencias s/n,
Ciudad Universitaria,
28040 Madrid, SPAIN
amelle@mat.ucm.es

J. Carmona Ruber
Departamento de Sistemas
Informáticos y Computación,
Universidad Complutense,
Plaza de las Ciencias s/n,
Ciudad Universitaria,
28040 Madrid, SPAIN
jcarmona@sip.ucm.es

2

On normal embedding of complex algebraic surfaces

L. BIRBRAIR, A. FERNANDES AND W. D. NEUMANN

Abstract

We construct examples of complex algebraic surfaces not admitting normal embeddings (in the sense of semialgebraic or subanalytic sets) whose image is a complex algebraic surface.

1. Introduction

Given a closed and connected subanalytic subset $X \subset \mathbb{R}^m$ the *inner metric* $d_X(x_1, x_2)$ on X is defined as the infimum of the lengths of rectificable paths on X connecting x_1 to x_2. This metric defines the same topology on X as the Euclidean metric on $\mathbb{R}^m$ restricted to X (also called *"outer metric"*). This follows from the famous Lojasiewicz inequality and the subanalytic approximation of the inner metric [6]. But the inner metric is not necessarily bi-Lipschitz equivalent to the Euclidean metric on X. To see this it is enough to consider a simple real cusp $x^2 = y^3$. A subanalytic set is called *normally embedded* if these two metrics (inner and Euclidean) are bi-Lipschitz equivalent.

Theorem 1.1 [4]. *Let $X \subset \mathbb{R}^m$ be a connected and globally subanalytic set. Then there exist a normally embedded globally subanalytic set $\tilde{X} \subset \mathbb{R}^q$, for some q, and a global subanalytic homeomorphism $p\colon \tilde{X} \to X$ bi-Lipschitz with respect to the inner metric. The pair $(\tilde{X}, p)$ is called a normal embedding of X.*

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The original version of this theorem (see [4]) was formulated in a semialgebraic language, but it is easy to see that this result remains true for a global subanalytic structure. The proof remains the same as in [4].

Complex algebraic sets and real algebraic sets are globally subanalytic sets. By the above theorem these sets admit globally subanalytic normal embeddings. T. Mostowski asked if there exists a complex algebraic normal embedding when X is a complex algebraic set, i.e., a normal embedding for which the image set $\tilde{X} \subset \mathbb{C}^n$ is a complex algebraic set. In this note we give a negative answer for the question of Mostowski. Namely, we prove that a Brieskorn surface $x^b + y^b + z^a = 0$ does not admit a complex algebraic normal embedding if $b > a$ and a is not a divisor of b. For the proof of this theorem we use the ideas of the remarkable paper of A. Bernig and A. Lytchak [3] on metric tangent cones and the paper of the authors on the (b, b, a) Brieskorn surfaces [2]. We also briefly describe other examples based on taut singularities.

2. Proof

Recall that a subanalytic set $X \subset \mathbb{R}^n$ is called *metrically conical* at a point x_0 if there exists an Euclidean ball $B \subset \mathbb{R}^n$ centered at x_0 such that $X \cap B$ is bi-Lipschitz homeomorphic, with respect to the inner metric, to the straight cone over its link at x_0. When such a bi-Lipschitz homeomorphism is subanalytic we say that X is *subanalytically metrically conical* at x_0.

Example 2.1. The Brieskorn surfaces in $\mathbb{C}^3$

$$\{(x, y, z) \mid x^b + y^b + z^a = 0\}$$

$(b > a)$ are subanalytically metrically conical at $0 \in \mathbb{C}^3$ (see [2]).

We say that a complex algebraic set admits a *complex algebraic normal embedding* if the image of a subanalytic normal embedding of this set can be chosen complex algebraic.

Example 2.2. Any complex algebraic curve admits a complex algebraic normal embedding. This follows from the fact that the germ of an irreducible complex algebraic curve is bi-Lipschitz homeomorphic with respect to the inner metric to the germ of $\mathbb{C}$ at the origin (e.g., [8], [5]).

Theorem 2.3. *If* $1 < a < b$ *and* a *is not a divisor of* b*, then no neighborhood of* 0 *in the Brieskorn surface in* $\mathbb{C}^3$

$$\{(x, y, z) \in \mathbb{C}^3 \mid x^b + y^b + z^a = 0\}$$

admits a complex algebraic normal embedding.

We need the following result on tangent cones:

Theorem 2.4. *If* (X_1, x_1) *and* (X_2, x_2) *are germs of subanalytic sets which are subanalytically bi-Lipschitz homeomorphic with respect to the induced Euclidean metric, then their tangent cones* $T_{x_1}X_1$ *and* $T_{x_2}X_2$ *are subanalytically bi-Lipschitz homeomorphic.*

This result is a weaker version of the results of Bernig-Lytchak ([3], Remark 2.2 and Theorem 1.2). We present here an independent proof.

Proof of Theorem 2.4. Let us denote

$$S_xX = \{v \in T_xX : |v| = 1\}.$$

Since T_xX is a cone over S_xX, in order to prove that $T_{x_1}X_1$ and $T_{x_2}X_2$ are subanalytically bi-Lipschitz homeomorphic, it is enough to prove that $S_{x_1}X_1$ and $S_{x_2}X_2$ are subanalytically bi-Lipschitz homeomorphic.

By Corollary 0.2 in [9], there exists a subanalytic bi-Lipschitz homeomorphism with respect to the induced Euclidean metric

$$h\colon (X_1, x_1) \to (X_2, x_2)\,,$$

such that $|h(x) - x_2| = |x - x_1|$ for all x. Let us define

$$dh\colon S_{x_1}X_1 \to S_{x_2}X_2$$

as follows: given $v \in S_{x_1}X_1$, let $\gamma : [0, \epsilon) \to X_1$ be a subanalytic arc such that

$$|\gamma(t) - x_1| = t \ \forall\, t \in [0, \epsilon) \quad \text{and} \quad \lim_{t\to 0^+} \frac{\gamma(t) - x_1}{t} = v\,;$$

we define

$$dh(v) = \lim_{t\to 0^+} \frac{h \circ \gamma(t) - x_2}{t}.$$

Clearly, dh is a subanalytic map. Define $d(h^{-1})\colon S_{x_2}X_2 \to S_{x_1}X_1$ in the same way. Let $k > 0$ be a Lipschitz constant of h. Let us prove that k is a Lipschitz constant of dh. In fact, given $v_1, v_2 \in S_{x_1}X_1$, let $\gamma_1, \gamma_2\colon [0, \epsilon) \to X_1$ be subanalytic arcs such that

$$|\gamma_i(t) - x_1| = t \ \forall\, t \in [0, \epsilon) \quad \text{and} \quad \lim_{t\to 0^+} \frac{\gamma_i(t) - x_1}{t} = v_i \quad \text{for } i = 1, 2.$$

Then

$$\begin{aligned}|dh(v_1) - dh(v_2)| &= \left| \lim_{t\to 0^+} \frac{h\circ\gamma_1(t) - x_2}{t} - \lim_{t\to 0^+} \frac{h\circ\gamma_2(t) - x_2}{t} \right| \\ &= \lim_{t\to 0^+} \frac{1}{t}|h\circ\gamma_1(t) - h\circ\gamma_2(t)| \\ &\le k \lim_{t\to 0^+} \frac{1}{t}|\gamma_1(t) - \gamma_2(t)| \\ &= k|v_1 - v_2|.\end{aligned}$$

Since $d(h^{-1})$ is Lipschitz (by the same argument) and dh and $d(h^{-1})$ are mutual inverses, we have proved the theorem. □

Corollary 2.5. *Let $X \subset \mathbb{R}^n$ be a normally embedded subanalytic set. If X is subanalytically metrically conical at a point $x \in X$, then the germ (X, x) is subanalytically bi-Lipschitz homeomorphic to the germ $(T_x X, 0)$.*

Proof. The tangent cone of the straight cone at the vertex is the cone itself. So the result is a direct application of Theorem 2.4. □

Proof of Theorem 2.3. Let $X \subset \mathbb{C}^3$ be the complex algebraic surface defined by

$$X = \{(x, y, z) \mid x^b + y^b + z^a = 0\}.$$

We are going to prove that the germ $(X, 0)$ does not have a normal embedding in $\mathbb{C}^N$ which is a complex algebraic surface. In fact, if $(\tilde{X}, 0) \subset (\mathbb{C}^N, 0)$ is a complex algebraic normal embedding of $(X, 0)$ and $p\colon (\tilde{X}, 0) \to (X, 0)$ is a subanalytic bi-Lipschitz homeomorphism, since $(X, 0)$ is subanalytically metrically conical [2], then $(\tilde{X}, 0)$ is subanalytically metrically conical and, by Corollary 2.5, $(\tilde{X}, 0)$ is subanalytically bi-Lipschitz homeomorphic to $(T_0\tilde{X}, 0)$. Now, the tangent cone $T_0\tilde{X}$ is a complex algebraic cone, thus its link is an S^1-bundle. On the other hand, the link of X at 0 is a Seifert fibered manifold with b singular fibers of degree $\frac{a}{\gcd(a,b)}$. This is a contradiction because the Seifert fibration of a Seifert fibered manifold (other than a lens space) is unique up to diffeomorphism. □

The following result relates the metric tangent cone of X at x and the usual tangent cone of the normally embedded sets. See [3] for a definition of a metric tangent cone.

Theorem 2.6 [3], Section 5. *Let $X \subset \mathbb{R}^m$ be a closed and connected subanalytic set and $x \in X$. If $(\tilde{X}, p)$ is a normal embedding of X, then $T_{p^{-1}(x)}\tilde{X}$ is bi-Lipschitz homeomorphic to the metric tangent cone.*

Remark 1. We showed that the metric tangent cones of the above Brieskorn surface singularities are not homeomorphic to any complex cone.

2.1. Other examples

We sketch how taut surface singularities give other examples of complex surface germs without any complex analytic normal embeddings.

Both the inner metric and the outer (Euclidean) metric on a complex analytic germ (V, p) are determined up to bi-Lipschitz equivalence by the complex analytic structure (independent of a complex embedding). This is because $(f_1, \dots, f_N)\colon (V, p) \hookrightarrow (\mathbb{C}^N, 0)$ is a complex analytic embedding if and only if the f_i's generate the maximal ideal of $\mathcal{O}_{(V,p)}$, and adding to the set of generators gives an embedding which induces the same metrics up to bi-Lipschitz equivalence.

A *taut* complex surface germ is an algebraically normal germ (V, p) (to avoid confusion we say "algebraically normal" for the algebro-geometric concept of normality) whose complex analytic structure is determined up to isomorphism by its topology. So if its inner and outer metrics are not bi-Lipschitz equivalent then it has no complex analytic normal embedding with algebraically normal image. Taut complex surface singularities were classified by Laufer [7] and include, for example, the simple singularities. A simple singularity (V, p) of type B_n, D_n, or E_n has non-reduced tangent cone, from which follows easily that it has non-equivalent inner and outer metrics. Thus (V, p) admits no complex algebraic normal embedding as an algebraically normal germ.

If we drop the requirement that the image be algebraically normal, (V, p) still has no complex analytic normal embedding. Indeed, suppose we have a subanalytic embedding $(V, p) \to (Y, 0) \subset (\mathbb{C}^n, 0)$ whose image Y is complex analytic but not necessarily algebraically normal (see also [1]). By tautness, the normalization of Y is isomorphic to V, which has non-reduced tangent cone. Hence Y also has non-reduced tangent cone, so it is not normally embedded.

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L. Birbrair and A. Fernandes
Dep. de Matemática
Universidade Federal do Ceará (UFC)
Campos do Pici, Bloco 914
60455-760 Fortaleza-CE
Brazil
birb@ufc.br
alexandre.fernandes@ufc.br

W. D. Neumann
Dep. of Mathematics,
Barnard College
Columbia University
New York, NY 10027
USA
neumann@math.columbia.edu

3

Local Euler obstruction, old and new, II

JEAN-PAUL BRASSELET AND NIVALDO G. GRULHA JR.

Abstract

This paper is a continuation of the first author's survey *Local Euler Obstruction, Old and New* (1998). It takes into account recent results obtained by various authors, in particular concerning extensions of the local Euler obstruction for frames, functions and maps and for differential forms and collections of them.

1. Introduction

The local Euler obstruction was first introduced by R. MacPherson in [34] as a key ingredient for his construction of characteristic classes of singular complex algebraic varieties. Then, an equivalent definition was given by J.-P. Brasselet and M.-H. Schwartz in [7] using vector fields. This new viewpoint brought the local Euler obstruction into the framework of "indices of vector fields on singular varieties", though the definition only considers radial vector fields. There are various other definitions and interpretations in particular due to Gonzalez-Sprinberg, Verdier, Lê-Teissier and others, and there is a very ample literature on this topic, see for instance [3] and also [1, 7, 9, 12, 13, 17, 23, 33, 34].

A survey was written by the first author [2]. Then, the notion of local Euler obstruction developed mainly in two directions: the first one comes back to MacPherson's definition and concerns differential forms. That is

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developed by W. Ebeling and S. Gusein-Zade in a series of papers. The second one relates local Euler obstruction with functions defined on the variety [5, 6] and with maps [23]. That approach is useful to relate local Euler obstruction with other indices. The aim of this paper is to update the previous survey in order to present together the new features on the subject. We use (and abuse) the monographies [3, 9] for the most classical features. Let us provide a brief "history" of the subject.

The local Euler obstruction at a point p of an algebraic variety V, denoted by $Eu_V(p)$, was defined by MacPherson. It is one of the main ingredients in his proof of Deligne-Grothendieck conjecture concerning existence of characteristic classes for complex algebraic varieties [34]. An equivalent definition was given in [7] by J.-P. Brasselet and M.-H. Schwartz, using stratified vector fields.

Independently of MacPherson, M. Kashiwara [29] introduced a local invariant of singular complex spaces in relation to his famous local index theorem for holonomic $\mathcal{D}$-modules. It was later observed by Dubson to be the same as MacPherson's local Euler obstruction [11, 12, 13].

The computation of local Euler obstruction is not so easy by using the definition. Various authors propose formulae which make the computation easier. G. Gonzalez-Sprinberg and J.L. Verdier give a formula in terms of Chern classes of the Nash bundle [17]. Lê D.T. and B. Teissier provide a formula in terms of polar multiplicities [33]. V. H. Jorge-Perez, D. Levcovitz and M. J. Saia [27], use the Lê-Teissier result to emphasize interest of local Euler obstruction in the context of maps.

In the paper [5], J.-P. Brasselet, D. T. Lê and J. Seade give a Lefschetz type formula for the local Euler obstruction. The formula shows that the local Euler obstruction, as a constructible function, satisfies the Euler condition relatively to generic linear forms. A natural continuation of the result is the paper by J.-P. Brasselet, D. Massey, A. J. Parameswaran and J. Seade [6], whose aim is to understand what is the obstacle for the local Euler obstruction to satisfy the Euler condition relatively to analytic functions with isolated singularity at the considered point. That is the role of the so-called local Euler obstruction of f, denoted by $Eu_{f,V}(0)$.

The relation between local Euler obstruction of f and the number of Morse points of a Morsification of f is described, for particular germs of singular varieties, in [40] by J. Seade, M. Tibar and A. Verjovsky. They compare $Eu_{f,V}(0)$ with two different generalizations of the Milnor number for functions with isolated singularities on singular varieties: one is the notion of the Milnor number given by Lê D. T. [30], the other is the one given by D. Mond, and D. van Straten [37] and by V. Goryunov [22] for curves, and considered by