A COURSE ON SET THEORY

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Set theory is the mathematics of infinity and part of the core curriculum for mathematics majors. This book blends theory and connections with other parts of mathematics so that readers can understand the place of set theory within the wider context. Beginning with the theoretical fundamentals, the author proceeds to illustrate applications to topology, analysis and combinatorics, as well as to pure set theory. Concepts such as Boolean algebras, trees, games, dense linear orderings, ideals, filters and club and stationary sets are also developed.

Pitched specifically at undergraduate students, the approach is neither esoteric nor encyclopedic. The author, an experienced instructor, includes motivating examples and over 100 exercises designed for homework assignments, reviews and exams. It is appropriate for undergraduates as a course textbook or for self-study. Graduate students and researchers will also find it useful as a refresher or to solidify their understanding of basic set theory.

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Note to the instructor

This book was written for an undergraduate set theory course, which is taught at Carnegie Mellon University every spring. It is aimed at serious students who have taken at least one proof-based mathematics course in any area. Most are mathematics or computer science majors, or both, but life and physical science, engineering, economics and philosophy students have also done well in the course. Other students have used this book to learn the material on their own or as a refresher. Mastering this book and learning a bit of mathematical logic, which is not included, would prepare the student for a first-year graduate level set theory course in the future. The book also contains the minimum amount of set theory that everyone planning to go on in math should know.

I have included slightly more than the maximum amount of material that I have covered in a fifteen-week semester. But I do not reach the maximum every time; in fact, only once. For a slower pace or shorter academic term, one of several options would be to skip Sections 5.6 and 7.2, which are more advanced.

There are over one hundred exercises, more than enough for eight homework assignments, two midterm exams, a final exam and review problems before each exam. Exercises are located at the ends of Chapters 1, 2, 3, 4 and 6. They are also dispersed throughout Chapters 5 and 7. This slight lack of uniformity is tied to the presentation and ultimately makes sense.

In roughly the first half of the book, through Chapter 4, I develop ordinal and cardinal arithmetic starting from the axioms of Zermelo–Fraenkel Set Theory with the Axiom of Choice (ZFC). In other words, this is not a book on what some call *naive set theory*. There is one minor way in which the presentation is not entirely rigorous. Namely, in listing the axioms of ZFC, I use the imprecise word *property* instead of the formal expression *first-order formula* because mathematical logic is not a prerequisite for the course.

Some other textbooks develop the theory of cardinality for as long as possible without using the Axiom of Choice (AC). I do not take this approach because it adds technicalities, which are not used later in the course, and gives students the misleading impression that AC is controversial. By assuming AC from the start, I am able to streamline the theory of cardinality. I may note how AC has been used in a proof but I do not belabor the point. Once, when an alternate proof without AC exists, it is outlined in an exercise.

The second half of the book is designed to give students a sense of the place of set theory within mathematics. Where I draw connections to other fields, I include all the necessary background material. Some of the other areas that come up in Chapter 5 are topology, metric spaces, trees, games and Ramsey theory. The real numbers are constructed using Dedekind cuts in Chapter 6. Chapter 7 introduces the student to filters and ideals, and takes up the combinatorics of uncountable sets. There is no section specifically on Boolean algebra but it is one of the recurring themes in the exercises throughout the book. For the reader's convenience, I have briefly summarized the results on Boolean algebra in the Appendix. All of this material is self-contained.

As I mentioned, before starting this book, students should have at least one semester's worth of experience reading and writing proofs in any area of mathematics; it does not matter which area. They should be comfortable with sets, relations and functions, having seen and used them at a basic level earlier. They should know the difference between integers, rational numbers and real numbers, even if they have not seen them explicitly constructed. And they should have experience with recursive definitions along the integers and proofs by induction on the integers. These notions come up again here but in more sophisticated ways than in a first theoretical mathematics course. There are no other prerequisites. However, because of the emphasis on connections to other fields, students who have taken courses on logic, analysis, algebra, or discrete mathematics will enjoy seeing how set theory and these other subjects fit together. The unifying perspective of set theory will give students significant advantages in their future mathematics courses.

Acknowledgements

As an undergraduate, I studied from *Elements of set theory* by Herbert Enderton and *Set theory: an introduction to independence proofs* by Kenneth Kunen. When I started teaching undergraduate set theory, I recommended *Introduction to set theory* by Karel Hrbacek and Thomas Jech to my students. The reader who knows these other textbooks will be aware of their positive influence.

This book began as a series of handouts for undergraduate students at Carnegie Mellon University. Over the years, they found typographical errors and indicated what needed more explanation, for which I am grateful. I also thank Michael Klipper for proofreading a draft of the book in Spring 2008, when he was a graduate student in the CMU Doctor of Philosophy program.

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1
Preliminaries

In one sense, set theory is the study of mathematics using the tools of mathematics. After millennia of doing mathematics, mathematicians started trying to write down the rules of the game. Since mathematics had already fanned out into many subareas, each with its own terminology and concerns, the first task was to find a reasonable common language. It turns out that everything mathematicians do can be reduced to statements about sets, equality and membership. These three concepts are so fundamental that we cannot define them; we can only describe them. About equality alone, there is little to say other than "two things are equal if and only if they are the same thing." Describing sets and membership has been trickier. After several decades and some false starts, mathematicians came up with a system of laws that reflected their intuition about sets, equality and membership, at least the intuition that they had built up so far. Most importantly, all of the theorems of mathematics that were known at the time could be derived from just these laws. In this context, it is common to refer to laws as *axioms*, and to this particular system as *Zermelo–Fraenkel Set Theory with the Axiom of Choice*, or *ZFC*. In the first unit of the course, through Chapter 4, we examine this system and get some practice using it to build up the theory of infinite numbers.

In another sense, set theory is a part of mathematics like any other, rich in ideas, techniques and connections to other areas. This perspective is emphasized more than the foundational aspects of set theory throughout the course but especially in the second half, Chapters 5–7. There, our choice of topics within set theory is designed to give the reader an impression of the depth and breadth of the subject and where it fits within the whole of mathematics.

To get started, we review some basic notation and terminology. We expect that the reader is familiar with the following notions but perhaps has not seen them expressed in exactly the same way.

Ordered pairs are used everywhere in mathematics, for example, to refer to points on the plane in geometry. The precise meaning of (x, y) is left to the imagination in most other courses but we need to be more specific.

Definition 1.1 $(x, y) = \{\{x\}, \{x, y\}\}$ is the *ordered pair with first coordinate x and second coordinate y.*

It is convenient that (x, y) is defined in terms of sets. After all, this is set theory, so everything should be a set! The main point of the definition is that from looking at $\{\{x\}, \{x, y\}\}$ we can tell which is the first coordinate and which is the second coordinate. Namely, if $\{\{x\}, \{x, y\}\}$ has exactly two elements, then the first coordinate is

$$x = \text{the unique } z \text{ such that } \{z\} \in \{\{x\}, \{x, y\}\}$$

and the second coordinate is

$$y = \text{the unique } z \neq x \text{ such that } \{x, z\} \in \{\{x\}, \{x, y\}\}.$$

And, if $\{\{x\}, \{x, y\}\}$ has just one element, which can only happen if $x = y$, then the first and second coordinates are both

$$x = \text{the unique } z \text{ such that } \{z\} \in \{\{x\}\}.$$

To understand this formula, keep in mind that

$$\{x, y\} = \{y, x\}$$

and

$$\{x, x\} = \{x\}.$$

In particular,

$$\{\{x\}, \{x, x\}\} = \{\{x\}, \{x\}\} = \{\{x\}\}$$

and $\{x\}$ is the only element of $\{\{x\}\}$.

Definition 1.2 $A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$ is the *Cartesian product of A and B.*

Definition 1.3 R is a *relation from A to B* iff R is a subset of $A \times B$, that is

$$R \subseteq A \times B.$$

Sometimes, if we know that R is a relation, then we write xRy instead of $(x, y) \in R$. For example, we write

$$\sqrt{2} < \pi$$

not

$$(\sqrt{2}, \pi) \in <$$

because the latter is confusing.

Definition 1.4 Let R be a relation from A to B and $S \subseteq A$.

1. The *domain of R* is

$$\mathrm{dom}(R) = \{x \in A \mid \text{there exists } y \text{ such that } xRy\}.$$

2. The *image of S under R* is

$$R[S] = \{y \in B \mid \text{there exists } x \in S \text{ such that } xRy\}.$$

3. The *range of R* is

$$\mathrm{ran}(R) = \{y \in B \mid \text{there exists } x \text{ such that } xRy\}.$$

Notice that $\mathrm{ran}(R) = R[\mathrm{dom}(R)]$.

Definition 1.5 f is a *function from A to B* iff f is a relation from A to B and, for every $x \in A$, there exists a unique y such that $(x, y) \in f$.

If we happen to know that f is a function, then we write

$$f(x) = y$$

instead of $(x, y) \in f$. When we write $f : A \to B$, it is implicit that f is a function from A to B. In certain situations, we refer to a function f by writing $x \mapsto f(x)$ or $\langle f(x) \mid x \in A \rangle$. There are times when we write f_x instead of $f(x)$; this is when we are thinking of elements x of A as *indices* and $\langle f_x \mid x \in A \rangle$ as an *indexed family.* If the domain of f consists of ordered pairs, then it is common to write $f(x, x')$ instead of $f((x, x'))$. Functions are also called *operations* and *maps*. Some people distinguish between a function $f : A \to B$ and its graph,

$$\text{graph}(f) = \{(x, f(x)) \mid x \in A\},$$

but we do not. To us they are the same, that is, $f = \text{graph}(f)$, as we see from Definition 1.5.

Definition 1.6 If $f : A \to B$ is a function and $S \subseteq A$, then the *restriction of f to S* is

$$f \restriction S = \{(x, f(x)) \mid x \in S\}.$$

Definition 1.7 Let $f : A \to B$ be a function.

1. f is an *injection* iff for all $x, x' \in A$, if $x \neq x'$, then $f(x) \neq f(x')$.
2. f is a *surjection* iff for every $y \in B$, there exists $x \in A$ such that $f(x) = y$.
3. f is a *bijection* iff f is both an injection and a surjection.

Injections are also called *one-to-one* functions. Surjections from A to B are also called functions from A *onto* B. Bijections are also called *one-to-one correspondences.*

Definition 1.8 If f is an injection from A to B, then we write f^{-1} for the unique injection $g : f[A] \to A$ with the property that $g(f(x)) = x$ for every $x \in A$. In other words,

$$f^{-1} = \{(f(x), x) \mid x \in A\}.$$

Finally, we assume that the reader has good intuition about the set of integers,

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\},$$

the set of rational numbers,

$$\mathbb{Q} = \{m/n \mid m, n \in \mathbb{Z} \text{ and } n \neq 0\}$$

and the set of real numbers, $\mathbb{R}$. One thing we will do in this course is define all these kinds of numbers, starting from the natural numbers 0, 1, 2, 3, 4, etc. Each natural number will be the set of natural numbers that precedes it. Thus $0 = \emptyset$, where $\emptyset$ is the set with no members. After that, $1 = \{0\}$, $2 = \{0, 1\}$, $3 = \{0, 1, 2\}$,

$4 = \{0, 1, 2, 3\}$, etc. This happens to be very convenient because then

$$m < n \iff m \in n.$$

In other words, the usual ordering on the natural numbers coincides with membership.

We use natural numbers to denote cardinality, for example, when we say, "Lance Armstrong won the Tour de France seven times." And we use natural numbers to denote order, for example, when we say, "the attorney general is seventh in the presidential line of succession." Another thing we will do in this course is extend the notions of cardinality and order into the infinite. Finite cardinal and ordinal numbers are basically the same thing; one could say that the difference between "seven" and "seventh" is just grammatical. However, the difference between infinite cardinal and ordinal numbers is more profound, as we will explain in Chapters 3 and 4.

Exercises

Exercise 1.1 If R is a relation, then we define

$$R^{-1} = \{(y, x) \mid xRy\}.$$

Give an example where R is a function but R^{-1} is not.

Exercise 1.2 How many functions whose domain is the empty set are there? In other words, given a set B, how many functions $f : \emptyset \to B$ are there?

Exercise 1.3 Explain why $(x, y, z) = (x, (y, z))$ is a reasonable definition of an *ordered triple.*

Exercise 1.4 Equivalence relations play an important role in this book. We assume that the reader has studied them before but this exercise reviews all the necessary definitions and facts. Let A be a set and R be a *relation on* A, that is, $R \subseteq A \times A$. Then:

- R is a *reflexive relation on* A iff for every $x \in A$, xRx.
- R is a *symmetric relation on* A iff for all $x, y \in A$, if xRy, then yRx.

- R is a *transitive relation on A* iff for all $x, y, z \in A$, if xRy and yRz, then xRz.[1]
- R is an *equivalence relation on A* iff R is a reflexive, symmetric and transitive relation on A.

Assuming that R is an equivalence relation on A, for every $x \in A$, we define the *equivalence class of x* to be

$$[x]_R = \{y \in A \mid xRy\}.$$

It is also standard to write

$$A/R = \{[x]_R \mid x \in A\}.$$

A *partition of A* is a family $\mathcal{F}$ of non-empty subsets of A such that

- A is the union of $\mathcal{F}$, that is,

  $$A = \bigcup \mathcal{F} = \{x \mid \text{there exists } X \in \mathcal{F} \text{ such that } x \in X\}$$

  and
- the elements of $\mathcal{F}$ are pairwise disjoint, that is, for all $X, Y \in \mathcal{F}$, if $X \neq Y$, then $X \cap Y = \emptyset$.

Now here are the exercises:

1. Let R be an equivalence relation on A. Prove that A/R is a partition of A.
2. Let $\mathcal{F}$ be a partition of A. Prove that there exists a unique equivalence relation R such that $\mathcal{F} = A/R$.

[1] Later in the book we will define *transitive set*, which is different from *transitive relation*. Unfortunately, it will be important to pay attention to this subtle difference in terminology.