# Companion Encyclopedia of the History and Philosophy of the Mathematical Sciences

VOLUME 2

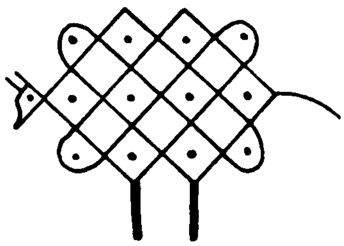
EDITED BY
I. GRATTAN-GUINNESS

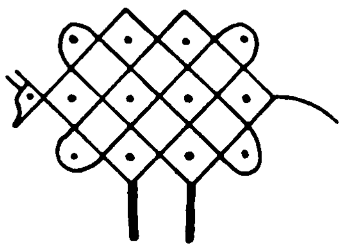
Routledge Reference

# *Companion Encyclopedia of the History and Philosophy of the Mathematical Sciences*

Volume 2

*Edited by*
I. GRATTAN-GUINNESS





ROUTLEDGE
LONDON AND NEW YORK

First published in 1994
Reprinted in 1994
by Routledge
2 Park Square, Milton Park, Abingdon, Oxon, OX14 4RN

Simultaneously published in the USA and Canada
by Routledge Inc.
270 Madison Ave, New York NY 10016

Transferred to Digital Printing 2006



Typeset in 10/12½pt times Compugraphic by
Mathematical Composition Setters Ltd, Salisbury, UK



*British Library Cataloguing in Publication Data*

A catalogue record for this book is available from the British Library

*Library of Congress Cataloging-in-Publication Data*

Companion encyclopedia of the history and philosophy of the mathematical sciences / edited by Ivor Grattan-Guinness.
p. cm.
Includes bibliographical references and index.
ISBN 0-415-03785-9
1. Mathematics—History. 2. Mathematics—Philosophy. I. Grattan-Guinness, I.
QA21.E57 1992
510'.9-dc20 92-13707
CIP

ISBN 0-415-03785-9 (set)
0-415-09238-8 (Vol. 1)
0-415-09239-6 (Vol. 2)

*Illustration on title page*:
Antelopes drawn by the (Ts)Chokwe people of Angola in their tradition of monolinear art; the main block of the pattern is composed of one continuous line, and is so designed that a regular lattice of dots can be inserted in the spaces (see P. Gerdes, *Lusona: Geometrical Recreations of Africa*, 1991, Maputo, Mozambique: Eduardo Mondlare University Press, 15). On this tradition see §1.8 by C. Zaslavsky, Section 3.

Printed and bound by CPI Antony Rowe, Eastbourne

# Contents

## VOLUME 1

# Part 7
# *Geometries and Topology*

# 7.0

# *Introduction*



As in the previous Part, the plural 'geometries' in the title is deliberately intended to refer to a variety of approaches and theories. §7.1–7.2 work strictly within the Euclidean realm, but the spread begins to emerge in §7.3 when non-Euclidean geometries make their bow. The next three articles (§7.4–7.6) are basically Euclidean in context but handle more general properties; §3.4 on differential geometry should also be noted. Corresponding in role to §6.9, the general philosophical situation as of the end of the nineteenth century is appraised in §7.7.

The next four articles are taken up with mathematical developments since around 1870. The most substantial addition is topology, of which the principal early features are described in §7.10–7.11. Finally, with finite vector spaces and graph theory, two aspects (rather than branches) of geometry are treated in §7.12–7.13; the overlaps with algebra are quite substantial.

One pleasing result of the development of geometry has been the construction of models of curves, surfaces and solids. Various books give instructions on their manufacture, and some departments of mathematics have collections. The history of this development is not treated here, but the pertinent theories are touched upon, and also in certain articles in Parts 1 and 12.

## BIBLIOGRAPHY

Coolidge, J. L. *1940*, *A History of Geometrical Methods*, Oxford: Oxford University Press. [Repr. 1963, New York: Dover.]

Simon, M. *1906*, *Über die Entwicklung der Elementar-Geometrie im XIX. Jahrhundert*, Leipzig: Teubner.

# 7.1

# *Algebraic and analytic geometry*

J. J. GRAY

## 1 COORDINATES IN THE PLANE

The present topic is also called 'Cartesian geometry', and few names are more appropriate in the history of mathematics, for the contribution of René Descartes was decisive in bringing together the methods of algebra and the study of geometry. Indeed, his *La Géométrie* (*1637*) can be read as the first modern mathematics book.

As the name 'algebraic geometry' suggests, the subject studied is geometry and the methods employed are algebraic. They are brought together by the idea of coordinates. Let us take the case of plane geometry. An arbitrary point is chosen, called the 'origin', and labelled O. Through it are drawn two lines at right angles called the $x$- and $y$-axes, respectively. Distances are measured along each axis according to a choice of scale. To give coordinates to a point P in the plane, one draws from P lines parallel to the $x$-axis and the $y$-axis, meeting the $x$-axis at A, say, and the $y$-axis at B. The first, or $x$-coordinate of the point P is the length OA; the second, or $y$-coordinate of the point P is the length OB.

Coordinates may be negative in the modern convention, for the scales along each axis measure positive to the right and up, negative to the left and down, as is usual in the representation of negative quantities. Amusingly enough, the sign convention, which still causes trouble to beginners, was also obscure to Descartes. The curve called the folium of Descartes, which has equation $x^3 + y^3 = 3xy$, actually looks as in Figure 1(a). By drawing it correctly in the first quadrant but incorrectly generalizing to the other quadrants, Descartes thought the curve looked as in Figure 1(b), whence the name 'folium', meaning a leaf-shaped curve.

Once points are given coordinates, it is possible to specify families of points whose coordinates satisfy certain equations, and to identify these families geometrically, thus turning geometry into algebra. For example,

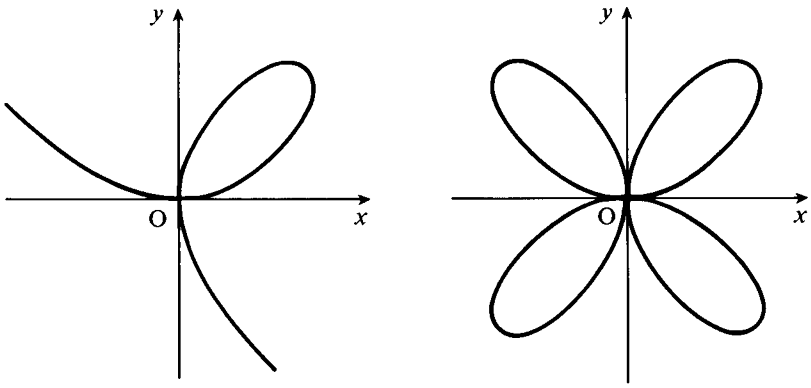



*Figure 1* The folium of Descartes, (a) correctly drawn and (b) as Descartes mistakenly drew it

the points P whose coordinates $(x, y)$ are related by the equation $x = 2y$ form a line through the origin; those satisfying the equation $x = y^2$ lie on a parabola (Figure 2). Conversely, given a curve in the plane and a choice of coordinate axes, the coordinates of points will be related by an equation: the sine curve, for example, has equation $y = \sin x$. It is usually to one's advantage to find coordinate axes with respect to which a given curve has a particularly simple equation.

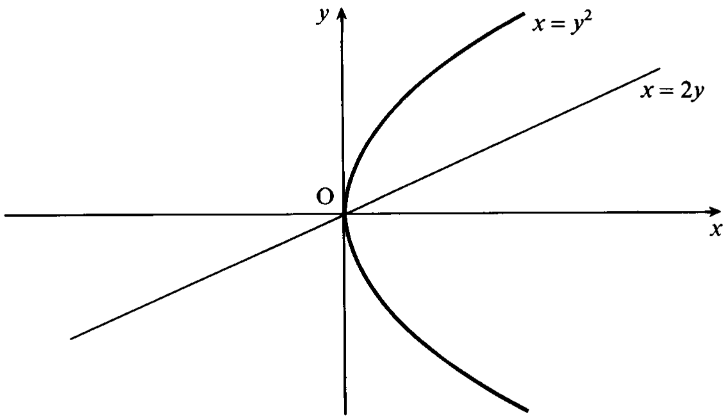



*Figure 2* A straight line and parabola plotted in Cartesian coordinates

Coordinate methods can be used to find where two curves meet. To find where the line with equation $y = 2x - 1$ meets the circle with equation

$x^2 + y^2 = 1$, one proceeds algebraically. Eliminating $y$ between the two equations, one obtains $5x^2 - 4x = 0$, whose solutions are $x = 0$ and $x = \frac{4}{5}$. The corresponding $y$-values are $y = -1$ and $y = \frac{3}{5}$, respectively. So the points of intersection are $(0, -1)$ and $(\frac{4}{5}, \frac{3}{5})$. Pursuing this argument as it applies to lines with equations $by = ax - b$, which all pass through the point $(0, -1)$ on the circle, one finds that the other point of intersection is

$$\left(\frac{2ab}{a^2 + b^2}, \frac{a^2 - b^2}{a^2 + b^2}\right).$$

This is an algebraic parametric representation of the circle; parametric representations are often a powerful method of studying curves.

Coordinate methods are also used to find the equation of curves satisfying given conditions. Let us find the equation of the circle with centre $(2, 3)$ and radius 5. We shall need to know the distance of a point A $(a, b)$ from another C $(c, d)$ (Figure 3); here $a = 2$ and $b = 3$, so we want the points $(x, y)$ such that $(x - 2)^2 + (y - 3)^2 = 5^2$. This simplifies to $x^2 - 4x + y^2 - 6y - 12 = 0$, which is the equation we seek.

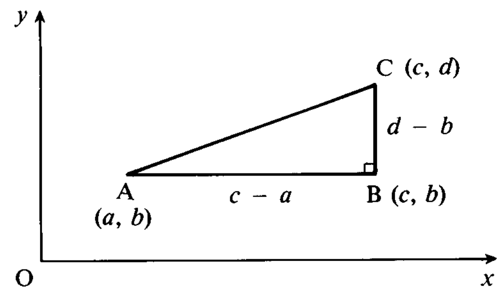



*Figure 3* Use of coordinates to establish an equation

Cartesian coordinates are by no means the only possible ones. Indeed, there are so many coordinate systems that it is not possible to describe them here; the reader should consult Coolidge's wide-ranging historical account *1940*.

## 2 THE CONTRIBUTION OF RENÉ DESCARTES

The motivating example for Descartes was a problem that had come down from the Hellenistic writer Pappus (§1.3). He had described a problem in which four lines and four angles are given, and points P are then sought which satisfy the following rather complicated condition. Lines are drawn from P to each given line $l_i$, meeting the given lines at the given angles $\alpha_i$;

four distances $d_i$ are thus obtained (Figure 4). The problem is to find the locus of points for which the product of the first two distances is proportional to the product of the remaining two. Pappus was able to show that the curve so defined was generally a conic section. However, the analogous problem can be formulated for larger numbers of lines and then there was, he said, no known solution. Descartes was able to show that the locus to four lines could be found rather easily by his methods, and that the locus to any number of lines was not in principle any harder to find. His success in this matter, of which he was very proud, was one of the reasons for his confidence in the general efficacy of the new approach.

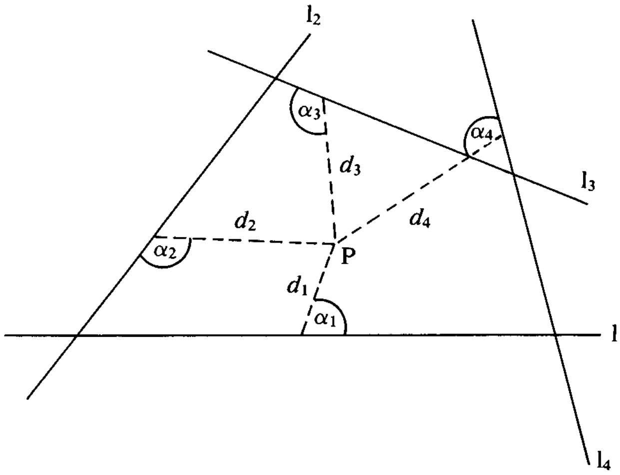



*Figure 4* Pappus's problem

However, all was not so simple. In Descartes's time, a solution to a problem in geometry had naturally to be given in geometric terms. As H. J. M. Bos shows in two important papers (*1981*, *1984*), this historical circumstance had a determining effect on how Descartes presented his ideas. For Pappus's problem, it was possible to describe the answer in geometrical terms: the solution curve is a conic section. But what could be said about a curve given only by an equation of higher degree than 2? One could hope to plot a finite number of points on it, but this fell well short of contemporary standards. To Descartes, the advocate of basing all reasoning on ideas clear and immediate to the mind, this was a problem. He invested considerable effort in pursuing the idea that for every equation there was an idealized machine that would draw its curve, just as a compass draws a circle. His hope was that simple curves would somehow generate more complicated ones, which in turn would generate still more complicated ones, and so on, complexity being measured by the degree of the equation. He never succeeded, but subsequent generations ceased to worry and accepted that a curve can be presented to the mind via an equation. Indeed, without admitting this it is hard to study curves at all: the ancient Greeks had bequeathed to posterity a mere dozen or so that could be analysed in any detail.

Descartes's aim was to take a geometrical problem, transcribe it into the language of algebra, solve it there, and then translate the solution back into geometrical terms. Even when the solution is the coordinates of a point, difficulties can arise, for it is by no means clear that *any* system of algebraic equations can be solved. This was always to remain a problem, and in practical situations where a numerical solution was required, approximate solutions were (and often still are) all that can be found. But there was a deeper problem, for by 'algebraic methods' Descartes meant those of finite polynomial algebra. When the solution sought is a curve, it is not true that the curve necessarily has an equation of this type. The sine curve does not; nor, as his critics pointed out, does the cycloid, the curve traced by a point on the rim of a wheel rolling along a straight line (Figure 5). Points on it have coordinates $(t - \sin t, 1 - \cos t)$, and there is no merely algebraic connection between the $x$- and $y$-coordinates. This mattered in its day, because Descartes regarded finding tangents to curves as the most important problem in mathematics. His critic Gilles Personne de Roberval had a simple argument by motion to find the tangent to a cycloid at a given point; but, as he pointed out, the cycloid necessarily eluded Descartes's method for finding tangents.

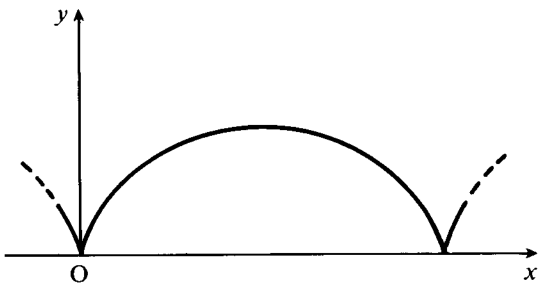



*Figure 5* The cycloid

Another critic of Descartes was Isaac Newton, who increasingly found fault with all the Frenchman's ideas, be they in mathematics or physics. In particular, he disputed Descartes's idea that a curve was simple if its equation was simple, and gave the example of the cycloid. Newton's ideas struck home, but the resolution of the matter would have satisfied neither man. Later generations took their cue from the masterly Leonhard Euler, the second volume of whose *Introductio in analysin infinitorum* (*1748*) made algebraic geometry seem so easy and natural that the need for a criterion for the simplicity of curves was no longer felt, and algebra reigned supreme.

In summary, Descartes associated curves and equations via the idea of a coordinate system, but he maintained the contemporary preference for geometrical answers to geometrical questions. In that way his algebraic geometry was more cumbersome than ours, but in another way it was more flexible. Our coordinate axes are usually at right angles to one another; his were not so restricted. This helps to simplify the equations that arise, but complicates their interpretation. The modern perpendicular system became standard with, and because of, Euler.

Was Descartes really the first to do this? Coordinate systems have been claimed for the Greeks (by Coolidge *1940*), and even for the Egyptians. There are fragmented pictures that suggest curves were discussed in a coordinate-based way, much as one might describe a sloping roof: 'this far along and this far up'. But such insights did not translate into an algebraic method, for the simple reason that algebra – reasoning with letters – did not exist in Egyptian mathematical culture (§1.2). Coolidge's case for the Greeks, which rests on Apollonius's profound study of conic sections, is much stronger. What is to be found in Apollonius's *Conics* is the idea that a conic section has axes, with respect to which the proportion (not, strictly, the equation) defining its shape can best be studied. This is a wonderful idea, but it falls well short of Descartes's realization that axes can be chosen in the plane with respect to which any curve can be profitably studied. And again, one could argue that the Greeks did not have algebra in anything like the modern sense.

A high point of algebraic development came with Francois Viète. He sought to purify algebra, extend it to solve all problems (many posed geometrically) and establish that this was how the Greeks had really proceeded (§6.9). One cannot credit Viète with the invention of coordinate geometry, for Viète's problems involve only one unknown, and there are no curves described by equations. Nor, indeed, does there seem to have been any influence of Viète on Descartes. However, Descartes's contemporary Pierre de Fermat did read Viète and did develop the coordinate idea in the language of Viète's algebra for the study of conic sections. He did this before Descartes, but he did not publish it, and he did not present the coordinate method in clear generality. The honour does go to Descartes.

Before turning to modern developments, the name 'analytic geometry' should also be explained. To analyse something in the mathematical terminology of the Greeks and the seventeenth century was to take something apart, much as one speaks of a chemical analysis today. In the Cartesian approach, figures are analysed in this sense: points are given coordinates, curves are successively given equations; above all, the unknown point or points are given coordinates and these are treated on a par with the known quantities until they can be found from certain equations that this process of analysis yields. In short, the Cartesian method is a discovery method. One starts knowing very little about the unknowns, and the method leads one to discover their values or some equations they satisfy. From its inception, this was thought to be a great advance over the synthetic methods of the Greeks, which sought to explain why an answer is correct. Compare chemistry: to synthesize a chemical is to check that one's prior analysis of its composition was correct, but synthesis is no way to discover the structure of a given substance.

## 3 THE DEVELOPMENT OF COORDINATE GEOMETRY

Coordinate geometry can be used to turn geometrical problems into algebraic ones and, conversely, to provide a geometrical interpretation of algebraic problems. The former case was extended by many writers to include most problems with a mechanical origin; the latter is helpful as an aid to thought and when approximate solutions are sought. Almost every aspect of eighteenth-century mathematics, both pure and applied, bears witness to the fertility of the union. But what really transformed mathematics and augmented its role in the growth of science was the yoking together of coordinate geometry and the calculus (§3.2 and §4.3). Invented in the 1660s and 1670s by Newton and Gottfried Wilhelm Leibniz independently, the calculus existed for over a hundred years in two superficially distinct formulations, each remarkably successful at reducing problems about tangents to curves and areas under curves to systematic calculation. Each was also well adapted to dealing with curves defined by equations, and so the calculus promoted just the vision of geometry that had most bothered Descartes and his contemporaries. In the hands of Euler, who wrote three great expository treatises on the subjects of curves, the differential and the integral calculus (*1748*, *1755*, *1768–70*), the result was a body of techniques seemingly easy to apply and almost certain of success.

The era of rational mechanics, the analysis of nature along the lines of Newton's *Principia* (*1687*), but conducted in the language of Cartesian geometry and the Leibnizian calculus, occupied the second half of the eighteenth century. Such are its breadth and its unity, encompassing everything from the geometry of curves to hydrodynamics, that it is difficult to select topics which belong to analytic geometry in the strict sense, but a few do stand out. In plane geometry, Newton was the first to take up the study of curves defined by equations of degree 3. Beginning in the 1660s, when he set himself this task in order to master the Cartesian methods, and culminating in his presentation as an appendix to his *Opticks* (*1704*), Newton completely classified curves of this type. His classification was cumbersome, for the problem is full of technicalities, and it was reworked by Euler in the second volume of his *Introductio* (*1748*). Euler, Gabriel Cramer *1750* and others also began the study of the singular points of curves. These are typically points where a curve either crosses itself or else has a cusp (Figure 6), although there are more complicated possibilities.

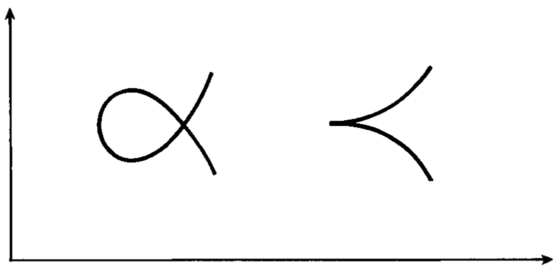


*Figure 6* Singular points: a crossing-point and a cusp

The natural extension of Cartesian methods to three dimensions, although hinted at by Descartes, had to wait until the eighteenth century, when it was set forth by the 18-year-old Alexis Clairaut *1731*. A single equation in three variables generally represents a surface; Clairaut considered the case of cubic surfaces (surfaces defined by an equation of degree 3) and their intersections by planes. The simpler case of equations of degree 2 yields the so-called 'quadric surfaces'. These include the ellipsoid, the hyperboloids of one and two sheets, and the cone.

Curves in space were (and are) much harder to represent. The most usual way was to realize a curve as the common points of two surfaces, and so as given by two equations. The alternative was to represent it parametrically, but this could be made to work in only a limited number of cases.

It is the fate of some vigorous branches of any subject to split into topics that between them exhaust the parent discipline. Such was the case with coordinate geometry. From it grew such subjects as differential geometry and algebraic projective geometry, but there is no such subject as 'advanced coordinate geometry'. Differential geometry has many roots (§3.4): one lies in the study of surfaces, specifically in the study of plane sections through each point; another lies in the study of curves in space, where it proved useful to look at the way a moving coordinate system varied as it travelled along a curve (Figure 7).

Algebraic projective geometry (§7.6) resumed at the start of the nineteenth century, when a reaction set in to the preference – then well established – for algebra and calculus over all other modes of mathematics. In France this move was led by Gaspard Monge, a highly influential founder-member of the Ecole Polytechnique (§11.1). In Germany, algebraic geometry was reinvigorated by August Ferdinand Möbius and, most notably, by Julius Plücker. Their work forms the natural generalization of the studies of Newton, Cramer and others, but the complexities of the subject inevitably pushed Cartesian geometry into projective geometry.

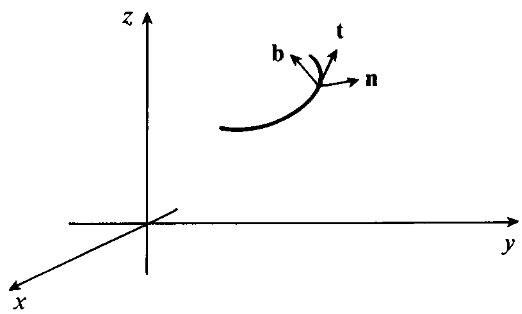



*Figure 7* A space curve

Typical of the richness of analytic geometry in the nineteenth century is the study of three surfaces: the cubic surface, Fresnel's wave surface and Kummer's quartic surface (for a good modern reference, full of photographs, see Fischer *1986*). The cubic surface was studied by many authors, but the English mathematician Arthur Cayley and his Irish colleague George Salmon made the most remarkable discovery: the general cubic surface contains 27 straight lines, all of which can be real. (Several of them are visible in Figure 8, which shows a late-nineteenth-century model.) It is surprising that a curved surface should carry straight lines, and their interrelation proved to be a fascinating object of study.

Augustin Jean Fresnel was one of the architects of the classical wave theory of light (§9.1). One problem he set himself was to determine the image of a point seen from a fixed position through a biaxial crystal. As Buchwald *1989* has carefully described, this question led Fresnel to describe his wave surface, which is the locus, up to unit time, of all the plane waves initially emitted from the given point. The determination of the equation of

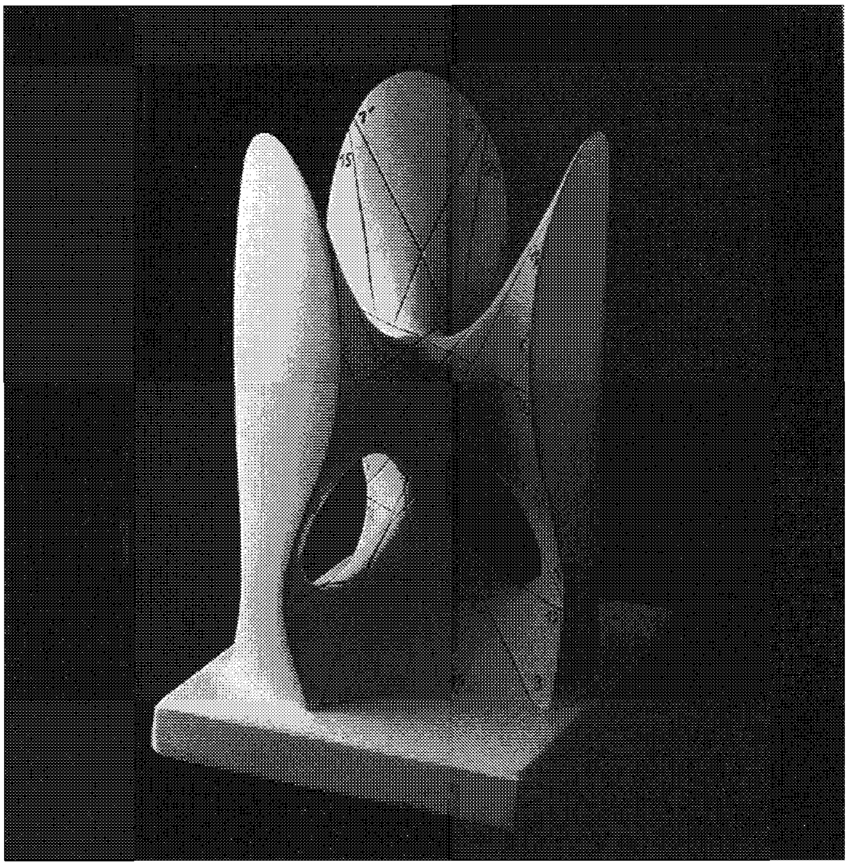


*Figure 8* Clebsch's model of a general cubic surface

this surface on optical grounds was not easy; it turned out to be

$$\frac{x^2}{r^2 - a^2} + \frac{y^2}{r^2 - b^2} + \frac{z^2}{r^2 - c^2} = 1, \tag{1}$$

where $a$, $b$ and $c$ are constants determined by the crystal, and $r^2 = x^2 + y^2 + z^2$. It is therefore a quartic surface (i.e. one of degree 4: see Figure 9). The differential geometry of curves on this surface interested mathematicians, but an even more remarkable fact soon caught their attention. Ernst Kummer, a leading light in the revival of mathematics in Berlin, investigated in the 1850s those quartic surfaces having the maximum possible number of singular points (16 double points, most of them visible in Figure 10). This purely geometrical line of enquiry turned out to yield a

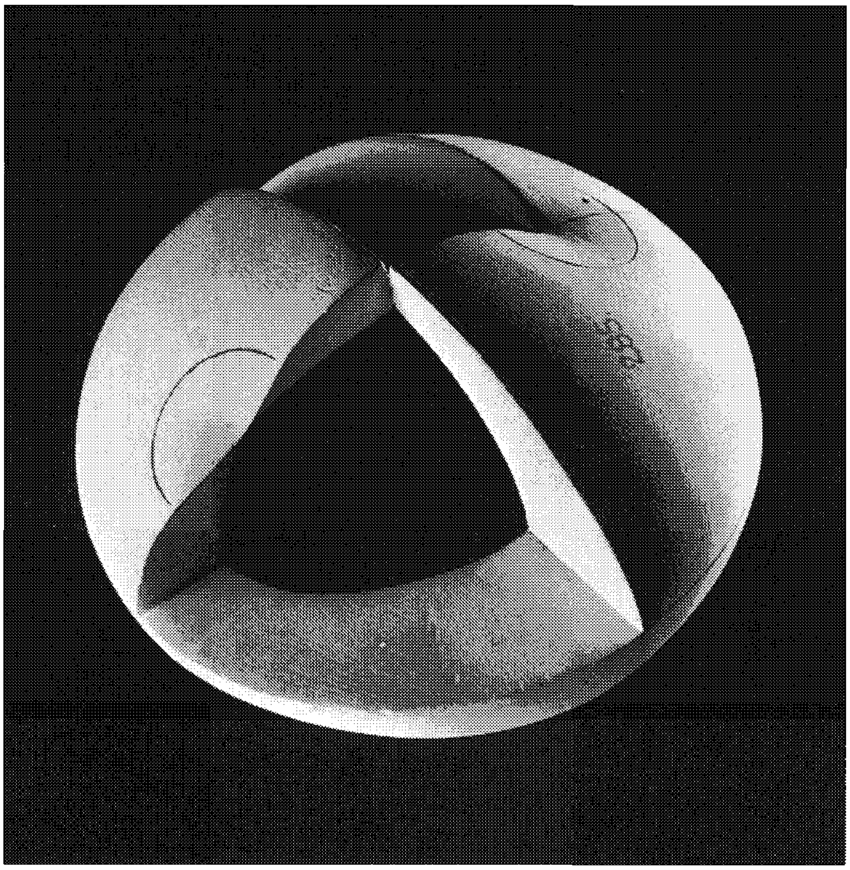


*Figure 9* Fresnel's wave surface

family of surfaces containing the wave surface as a special case, a surprising coincidence that stimulated much further work.

## 4 COORDINATE TRANSFORMATIONS

Although there is no such subject as advanced coordinate geometry, there is a mainstream development of Descartes's ideas that fills university courses and is a staple technique of both pure and applied mathematics: coordinate transformations. Coordinate geometry, whether in the plane, in space or in higher dimensions, lends itself naturally to a description by vectors (§7.12). Changing the origin of the coordinate system or moving the coordinate axes changes the equation of the curve, opening the prospect of finding a simplest possible equation for a curve. For this reason, in the

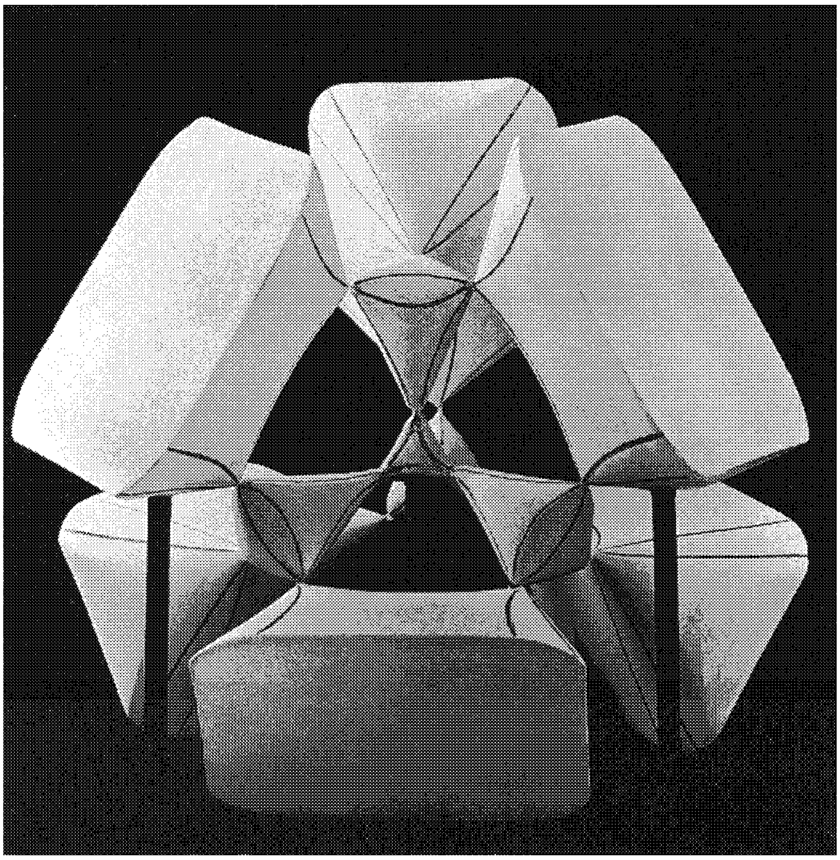


*Figure 10* Kummer's quartic surface with 16 double points

nineteenth century there was developed a powerful theory for describing coordinate transformations in the language of vector spaces, as Hawkins (*1975*, *1977*) has described. This theory, which has an independent interest, is discussed in the articles on matrices and invariants (§6.7–6.8).

## BIBLIOGRAPHY

Apollonius *1891–3*, *Apollonii Pergaei quae Graece existant cum commentariis antiquis*, 2 vols (ed. J. L. Heiberg), Leipzig: Teubner.

——*1896*, *Treatise on Conic Sections* (ed. T. L. Heath), Cambridge: Cambridge University Press.

Bos, H. J. M. *1981*, 'On the representation of curves in Descartes's *Géométrie*', *Archive for History of Exact Sciences*, **24**, 295–338.

——*1984*, 'Arguments on motivation in the rise and decline of a mathematical theory; the "construction of equations", 1637–*ca.* 1750', *Archive for History of Exact Sciences*, **30**, 331–80.

Boyer, C. B. *1956*, *History of Analytic Geometry*, New York: Scripta mathematica.

Buchwald, J. Z. 1989, *The Rise of the Wave Theory of Light*, Chicago, IL: University of Chicago Press.

Clairaut, A. C. *1731*, 'Sur les courbes que l'on forme en coupant une surface courbe quelconque par un plan donné de position', *Histoire de l'Académie Royale des Sciences*, 183–93.

Coolidge, J. L. *1940*, *A History of Geometrical Methods*, Oxford: Oxford University Press. [Repr. 1963, New York: Dover.]

Cramer, G. *1750*, *Introduction à l'analyse des lignes courbes algébriques*, Geneva: Cramer.

Descartes, R. *1637*, *La Géométrie* [one of the appendices to *Discours de la méthode*], Leiden: Maire. [Many subsequent editions. Repr. in D. E. Smith and M. L. Latham (transls), 1954, *The Geometry of René Descartes*, New York: Dover.]

Euler, L. *1748*, *Introductio in analysin infinitorum*, 2 vols, Lausanne: Bousquet. [Also *Opera omnia*, Series 1, Vols 8 and 9. English transl. by J. D. Blanton, 1988–90, *Introduction to the Analysis of the Infinite*, New York: Springer.]

——*1755*, *Institutiones calculi differentialis*, St Petersburg: Academy. [Also *Opera omnia*, Series 1, Vol. 10.]

——*1768–70*, *Institutiones calculi integralis*, 3 vols, St Petersburg: Academy. [Also *Opera omnia*, Series 1, Vols 11–13.]

Fischer, G. *1986*, *Mathematische Modelle/Mathematical Models*, 2 vols [Vol. 1 pictures, Vol. 2 commentary], Braunschweig: Vieweg.

Hawkins, T. *1975*, 'The theory of matrices in the 19th century', in *Proceedings of the International Congress of Mathematicians*, Vancouver, 1974, [no place]: Canadian Mathematical Congress, 561–70.

——*1977*, 'Another look at Cayley and the theory of matrices', *Archives Internationales d'Histoire des Sciences*, **26**, 82–112.

Newton, I. *1687*, *Philosophiae naturalis principia mathematica*, 1st edn, London: Streater. [English transl. by A. Motte, 1729; rev. F. Cajori, 1934, as *The Mathematical Principles of Natural Philosophy* [...], 2 vols, Berkeley, CA: University of California Press.]

——*1704*, *Opticks*, London: Smith and Walford. [See also 'Enumeration of the lines of the third order' (*circa* 1695), in *The Mathematical Papers of Isaac Newton* (ed. and transl. D. T. Whiteside), Cambridge: Cambridge University Press, Vol. 7, 1976, 589–645.]

# 7.2

# *Curves*

J. J. GRAY

## 1 CURVES IN GREEK MATHEMATICS

A good, naive, definition of a curve would be the path traced out by a moving point. While examples are legion, a curve can enter mathematics only when a mathematician has something specific to say about it, and the general concept arises only when there is something worthwhile and general to say. Yet it still comes as a surprise that the Greeks had no general concept, and few specific examples to study. Euclid's *Elements* discusses properties of the straight line and circle; following him Archimedes and Apollonius studied the conic sections (the ellipse, parabola and hyperbola), but only a few other curves were known, such as the spiral of Archimedes (§1.3). This paucity can be ascribed partly to the means available for defining and treating curves, partly to the range of problems discussed in Classical times. So, for example, the conic sections were defined as particular kinds of section of a cone by a plane. Their study was extremely difficult with the means available; that an extensive theory of the conics was obtained is eloquent testimony to the brilliance of Archimedes and Apollonius. No less a modern mathematician than B. L. van der Waerden has described Apollonius's *Conics* as a masterpiece whose author was 'a virtuoso in hiding his original line of thought [which] is what makes his work hard to understand' (*1961*: 248).

The conic sections and most of the other curves known to the Greeks were used by them to tackle significant problems, notably those known today under the collective title of 'the three classical problems'. These were: to trisect an angle (i.e. given any angle, to find an angle of one-third the size); to duplicate the cube (i.e. given a cube, to find one of twice the volume); and to square the circle (i.e. given a circle, to find a square of equal area). It seems that the Greeks despaired of finding solutions to any of these problems by straightedge and compass alone, and indeed it was shown rigorously in the nineteenth century that under those restrictions it cannot be done. So they set about solving these problems in other ways.

Trisecting an angle may be difficult, but trisecting a length is trivial, and the solution to the angle problem offered by Hippias of Elia in the fifth century reduced it to the trisection of an interval. He described a curve in this way (Figure 1). In a square ABCD of side $a$, one rod AR of length $a$ is attached at A, where it is free to pivot, while a second, horizontal rod $C'D'$ of the same length is free to move up and down. Initially the first rod is vertical and the second rod lies along CD. They are set in motion together, so as to move uniformly and at such speeds that when AR has rotated through a right angle, $C'D'$ has reached AB. Hippias's curve is traced by X, the intersection of the two rods; and AS = 2SB.

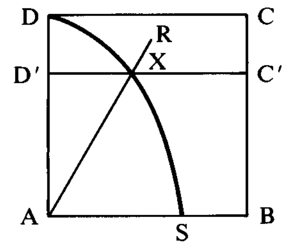


*Figure 1* The trisectrix

To use the curve to trisect a given angle of $q_0$, one has merely to allow the rod AR to rotate through that amount, to $AR_0$ say (Figure 2), and locate the corresponding position $C_0'D_0'$ of the horizontal rod. One then trisects the segment $DD_0'$, say at $D_1'$, and finds the corresponding position of the rod AR, say $AR_1$. The angle $DAR_1$ is one-third of the angle $DAR_0$. For this reason the curve is sometimes called the 'trisectrix'.

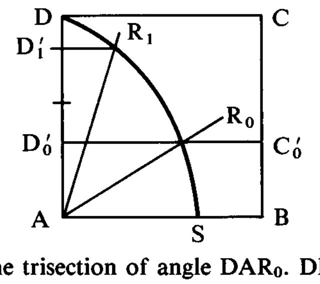


*Figure 2* The trisection of angle $DAR_0$. $DD_1' = DD_0'/3$

The trisectrix has another use: it helps to solve the problem of squaring the circle. It meets the side AB at the point S in Figure 2, for which

AS = $2a/\pi$. A circle of radius $r$ has an area of $\pi r^2$; for a circle of unit radius this reduces to $\pi$. With $a = 1/2$, the trisectrix gives a length of $1/\pi$, which can easily be made to yield a length of $\pi$, thus squaring the circle. For this reason the trisectrix is also called the 'quadratrix' (it has brought about the quadrature of the circle). At a time when it had to be shown rigorously that the concept of area, first introduced for rectilinear figures, applies to the circle, this was a considerable achievement.

Duplicating the cube recalls the problem of duplicating the square, but whereas the latter problem only requires $\sqrt{2}$ to be found, and was solved in Plato's dialogue the *Meno*, duplicating the cube requires $\sqrt[3]{2}$ to be found, which is much harder. Two solutions associated with the name of Menaechmus involve conic sections. Another, due to Nicomedes, introduced a curve called the conchoid. (It can also be used to trisect the angle.) He considered a line l, and a point O not on it, and looked at all the lines through O. Let the points $P_1$ and $P_2$ lie on such a line that meets l at Q. The conchoid is composed of the points $P_1$ and $P_2$ for which $P_1Q = QP_2 = a$ fixed distance $a$ (Figure 3). Yet another solution was proposed by Diocles. This curve, called the cissoid, was obtained as follows (Figure 4). Let AB and CD be perpendicular diameters of a circle, and let EB and BF be equal arcs. Draw FH perpendicular to CD and let it meet EC at P. As the point E moves round the circle, the point P traces out the cissoid. For a full account of these curves and their uses, see Heath *1921*: Vol. 1.

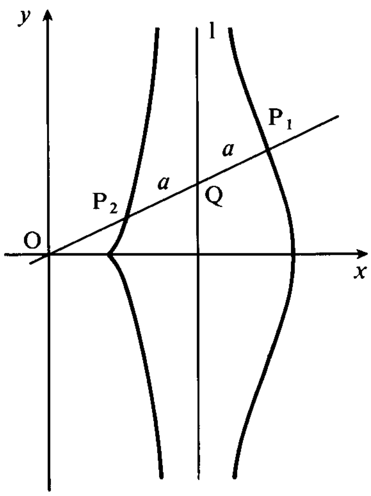



*Figure 3* The conchoid of Nicomedes

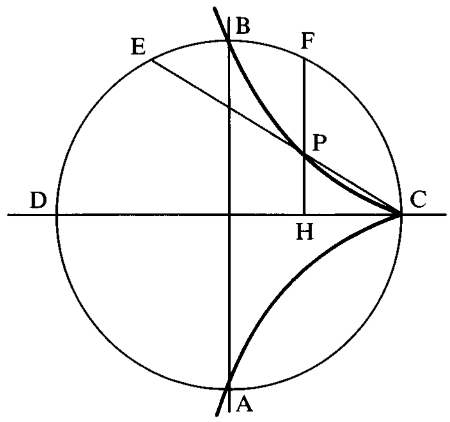



*Figure 4* The cissoid of Diocles

## 2 CURVES IN CARTESIAN GEOMETRY

Such elaborate constructions make it plain that very few curves can be introduced in this way. Much less did they elicit general concepts; the idea of a tangent to a curve does not seem to have been defined in general, although tangents to the circle, the conic sections, and the spiral were discussed. What is lacking is the concept of one quantity varying with another, so that the two between them define a moving point tracing out a curve. This idea was introduced into mathematics by René Descartes (§7.1). If, for each value of a variable $x$, one has one or more values of a variable $y$, then the points $(x, y)$ define a curve which can be represented in the Cartesian plane. At a stroke one has the capacity to define infinitely many curves, including all those for which the relationship between $x$ and $y$ is given by an algebraic equation, and all those for which $y$ is a known function of $x$. So in the seventeenth century it became possible to ask for the first time for a general method for finding the tangent to a given curve at a given point. Of course, such a method was soon provided by the rival versions of the calculus put forward by Isaac Newton and Gottfried Wilhelm Leibniz (§3.2).

The period from 1650 to 1850 was the heyday of the study of curves by Cartesian methods. It is usually easy to find the Cartesian equation of a curve defined in some other way. For example, the conchoid of Nicomedes has the equation $(x^2 + y^2)(x - b)^2 = a^2x^2$, and the cissoid of Diocles has the equation $y^2(a + x) = (a - x)^3$. When this cannot be done the curve can still usually be represented parametrically, that is, in the form $(x(t), y(t))$ as $t$ varies. For example, the cycloid has the parametric representation $(t - \sin t, 1 - \cos t)$. Two major books defined the theme: Euler *1748* and Cramer *1750* (see §7.1 for details).

## 3 UNEXPECTED PROPERTIES OF CURVES

With the rigorization of the calculus in the nineteenth century (§3.3), the question arose of what sort of behaviour a curve could have. For example, is the plausible claim that every curve has a tangent at every point always true, except where it is palpably false (where the curve crosses itself or suddenly reverses its direction at a cusp)? To everyone's initial surprise, it proved possible to construct continuous functions that never have a tangent at any point. After the first probable cases of such a phenomenon were brought to light by Bernhard Riemann in the 1850s, rigorous examples were found by Karl Weierstrass in the 1860s. It was then possible to define continuous curves that never have a tangent. At first these curves caused distress in some circles. Some regarded them as pathological, and wished to reject them; others, such as Henri Poincaré, eagerly accepted them. Rather surprisingly, it emerged from the work of Albert Einstein and others that the path of a particle undergoing Brownian motion was usually continuous but nowhere differentiable.

A further challenge to naive intuition about curves came when Georg Cantor showed in the 1880s that there is a 1–1 correspondence between the points of a curve and the points of a plane (§3.6). This seems to counter the notion of a curve as one-dimensional and a plane as two-dimensional; order was restored only when it was shown that there is no such 1–1 correspondence which is continuous in each direction. Nevertheless, there are continuous curves which pass through every point of a square or, for that matter, the whole plane. The first of these was defined in formulas derived by the Italian mathematician and logician Giuseppe Peano, and later illustrated by the American mathematician E. H. Moore and the German David Hilbert (§3.8). When the concept of dimension was later stretched by Felix Hausdorff to allow for spaces of non-integer dimension, it was soon found that these unexpected types of curve can have any dimension.

## 4 COMPLEX CURVES

At the same time, the concept of curve was being extended in another way. An algebraic relationship between two variables $x$ and $y$ can naturally be thought of as defining a curve. For example, the relationship $x^2 + y^2 = 1$ defines a circle. It was the profound idea of Riemann to extend this to the case where the variables $x$ and $y$ are complex (§7.9). Since each complex variable can be thought of as a pair of real ones, the relationship now defines a surface in a space of four real dimensions. This is not intuitively easy to grasp, but it proved possible to proceed as if an equation between two complex variables defined a curve (something of real dimension 2, but *complex* dimension 1), today called an algebraic curve or a Riemann surface. Riemann's approach allows the full resources of algebra to be exploited in studying curves defined by algebraic equations. Since then it has often proved worthwhile to think of an algebraic equation as defining a curve whatever field one is working in, a point of view that has abundantly proved its worth in number theory.

## BIBLIOGRAPHY

Apollonius *1891–3*, *Apollonii Pergaei quae Graece existant cum commentariis antiquis*, 2 vols (ed. J. L. Heiberg), Leipzig: Teubner.

—— *1896*, *Treatise on Conic Sections* (ed. T. L. Heath), Cambridge: Cambridge University Press.

Brieskorn, E. and Knörrer, H. *1986*, *Plane Algebraic Curves*, Basel: Birkhäuser. [Part 1 is historical.]

Cramer, G. *1750*, *Introduction à l'analyse des lignes courbes algébriques*, Geneva: Cramer.

Descartes, R. *1637*, *La Géométrie* [one of the appendixes to *Discours de la méthode*], Leiden: Maire. [Repr. in D. E. Smith and M. L. Latham (transls), 1954, *The Geometry of René Descartes*, New York: Dover.]

Euler, L. *1748*, *Introductio in analysin infinitorum*, 2 vols, Lausanne: Bousquet. [Also *Opera omnia*, Series 1, Vols 8 and 9. English transl. by J. D. Blanton, 1988–90, *Introduction to the Analysis of the Infinite*, New York: Springer.]

Heath, T. L. *1921*, *A History of Greek Mathematics*, 2 vols, Oxford: Clarendon Press. [2nd edn 1981, New York: Dover.]

van der Waerden, B. L. *1961*, *Science Awakening* (transl. A. Dresden), New York: Oxford University Press.

# 7.3

# *Regular polyhedra*

BRANKO GRÜNBAUM

The idea of singling out certain polyhedra with unusual symmetry properties first occurred during the heyday of Greek geometry. Exact information about the origin of the choice of five specific polyhedra (shown in Figure 1) which we call 'regular' is not available, and neither is there any certainty about the first discovery of individual regular polyhedra. The regular polyhedra were discussed in Plato's Academy, where they were attributed to Theaetetus. The fanciful importance Plato bestowed upon them, as building blocks of the universe, led to the name 'Platonic solids'. The regular polyhedra are the crowning achievement of Euclid's *Elements*, Book 13, where it is proved that there are exactly five different regular polyhedra. In a certain sense, the specific determination of the five polyhedra may be considered less important than the generation of the concept of regularity; apparently this should be credited to Theaetetus (see Waterhouse *1972* for a stimulating discussion of this aspect). Illustrations and accounts of the properties of these polyhedra (and of most of the polyhedra mentioned in this article) can be found, for example, in Coxeter *1948* or Cundy and Rollett *1951*; a nice introduction to various aspects of polyhedra, their history and literature is given in the collection Senechal and Fleck *1988*.

## 1 SHORTCOMINGS OF THE EUCLIDEAN TREATMENT

From the point of view of logic, Euclid's account of the regular polyhedra presents a striking contrast between good intentions and actual execution. Although axiomatic geometry enjoyed through the ages the reputation of presenting logical thought at its best, this was undeserved, even in the case of the frequently discussed topic of regular polyhedra. In fact, the validity of the stated characterizations of regular polyhedra requires additional, unstated assumptions. The basic logical difficulty is that Euclid never defined (or informally explained) what is a 'solid' or 'figure' – although he was aiming to show that 'apart from the said five figures, there cannot be constructed any other figure which is contained by equilateral and

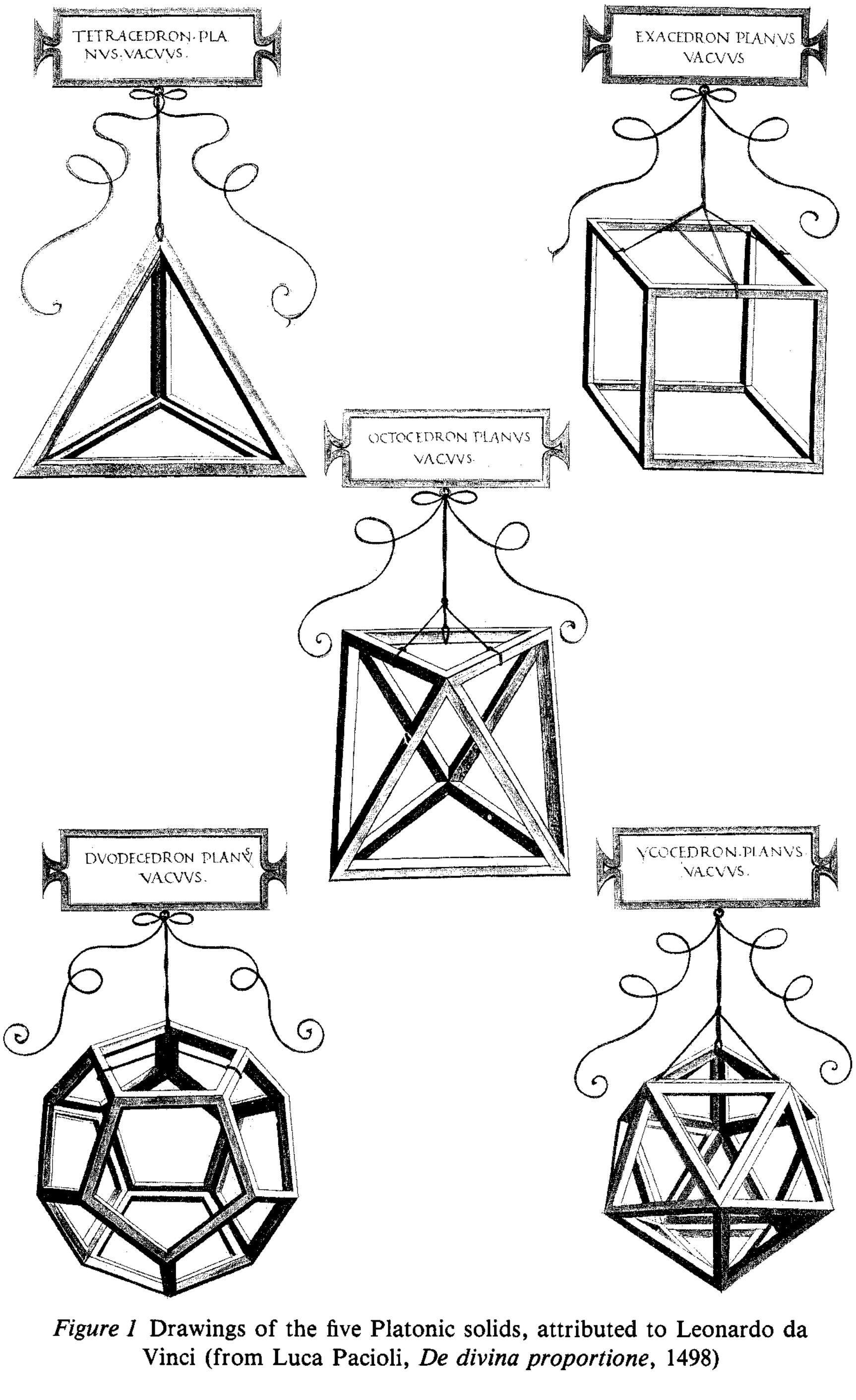



*Figure 1* Drawings of the five Platonic solids, attributed to Leonardo da Vinci (from Luca Pacioli, *De divina proportione*, 1498)

equiangular figures equal to one another'. Sympathetic commentators have explained that Euclid must have had only convex polyhedra in mind, but even then there are at least two deficiencies. First, convexity itself is not unambiguously defined: only in recent decades has there been a tendency to a very restrictive interpretation of this word – in fact, the regular polyhedra found by Johannes Kepler and (later) by Louis Poinsot, which are discussed next, were considered throughout the nineteenth century to be 'convex'. Second, even with interpretation of 'figure' as a convex polyhedron in the most restrictive sense, Euclid's statement is plainly wrong: there are five additional 'deltahedra', non-regular convex polyhedra bounded by congruent equilateral triangles.

To salvage Euclid's enumeration it is therefore necessary to impose additional restrictions, besides convexity in the narrow sense. Over the centuries, various conditions have been proposed, such as: all solid angles should be regular; all solid angles should be congruent; all dihedral angles should be equal; equal numbers of faces should meet at each vertex; there exist three concentric spheres, one through all vertices, one touching all edges, and one touching all faces. Although logically distinct, these and other conditions are mutually equivalent in the context under discussion, and indeed yield only the five Platonic solids among strictly convex polyhedra. More recently, it has become customary to define regularity in terms of equivalence under symmetries. The most frequent formulation, which in essence goes back to a paper by Augustin Louis Cauchy (published in 1810) and was apparently first formulated explicitly by Ernst Steinitz in his encyclopedic survey of polyhedra (1916), defines a polyhedron to be regular if its group of symmetries acts transitively on its 'flags', a 'flag' being a triplet consisting of a vertex, an edge and a face, all mutually incident.

## 2 POLYHEDRA OF KEPLER AND POINSOT

The lack of precision in Euclid's formulation has had the positive effect of admitting other interpretations that led to new investigations and to the designation of additional polyhedra as 'regular'; this helped to break the stranglehold of ossified traditions. The first such step was taken by Kepler in 1619. He started by observing that, if a polygon is interpreted as a circuit of edges, meeting in pairs at common end-points, then the pentagram (and other star polygons) are regular in the sense of having all sides equal as well as all angles equal. The fact that the sides cross each other has no relevance to the question of regularity. Since a similar reasoning applies to polyhedra, and since it is possible to combine pentagrams as faces to obtain polyhedra which satisfy the Euclidean conditions of regularity, Kepler claimed the status of regular polyhedra for two of his inventions (see Figure 2, taken

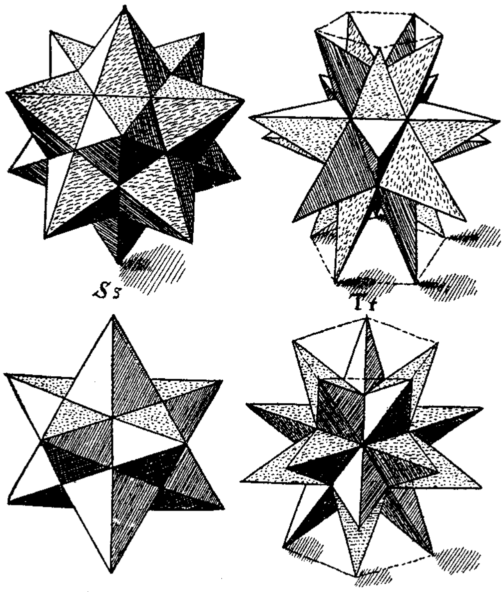



*Figure 2* Two views of each of the two regular polyhedra found by Kepler (from his *Harmonice mundi*, 1619)

from Kepler's work). In fact, they are regular even under the flag-transitivity criterion. It may be mentioned that star-shaped solids resembling one of Kepler's are found in the work of some earlier artists; but, as with Theaetetus, Kepler's principal merit was the insight that these polyhedra deserve to be considered regular, rather then just having an attractive shape.

Kepler's discovery lay largely forgotten for two centuries, and the next development is due to Poinsot in *1809*. Apparently unaware of Kepler's polyhedra, Poinsot followed a similar argument to describe them, as well as two additional regular polyhedra in which either triangles or pentagons (without self-intersections) are arranged around each vertex in a star-shaped way. These four polyhedra – usually known as the 'Kepler–Poinsot polyhedra' – are shown in Figure 3. Poinsot's discovery was followed in 1810 by Cauchy's proof of the completeness of that enumeration. However, Poinsot's work suffers from an internal inconsistency. In following Poinsot uncritically, as in other instances where he was insensitive to geometric subtleties, Cauchy made a mistake that has been noticed only recently; as a consequence of this error, additional restrictions are needed to make his result valid. (The discovery of the shortcoming was made during the preparation of this article and was first presented in a course that I was giving in spring 1990 at the University of Washington.)

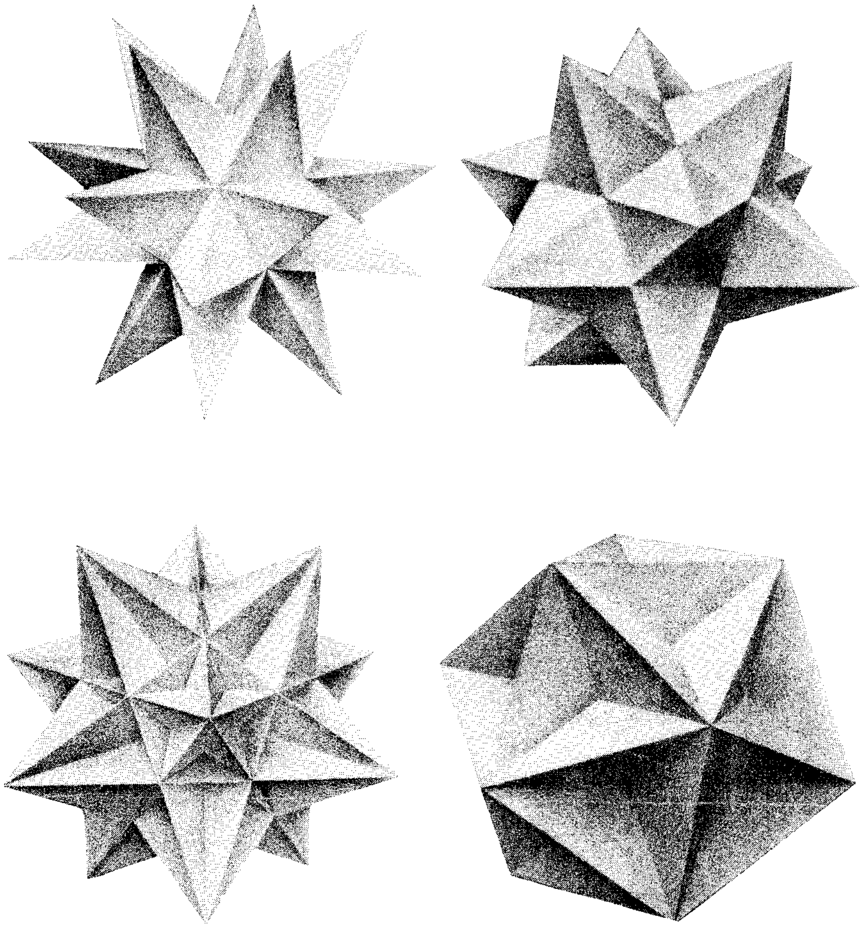


*Figure 3* The four Kepler–Poinsot polyhedra (from Wiener *1865*: Plate 3; this was the first published illustration of these four polyhedra)

## 3 ERRORS IN POINSOT'S WORK

The problem arises from Poinsot's disregard of his own definition of regular polygons, which – freely but faithfully translated – is as follows. To obtain a polygon, let $m$ points $a$, $b$, $c, \ldots$ be arbitrarily given in the plane, joined by $m$ segments $ab$, $bc$, $cd, \ldots$, so that the resulting figure is closed. A polygon is regular if its edges are equal and its angles are equal. (Poinsot gives a completely satisfactory definition of equality of angles in a polygon.) He goes on to say (correctly) that if $h$ is an integer relatively

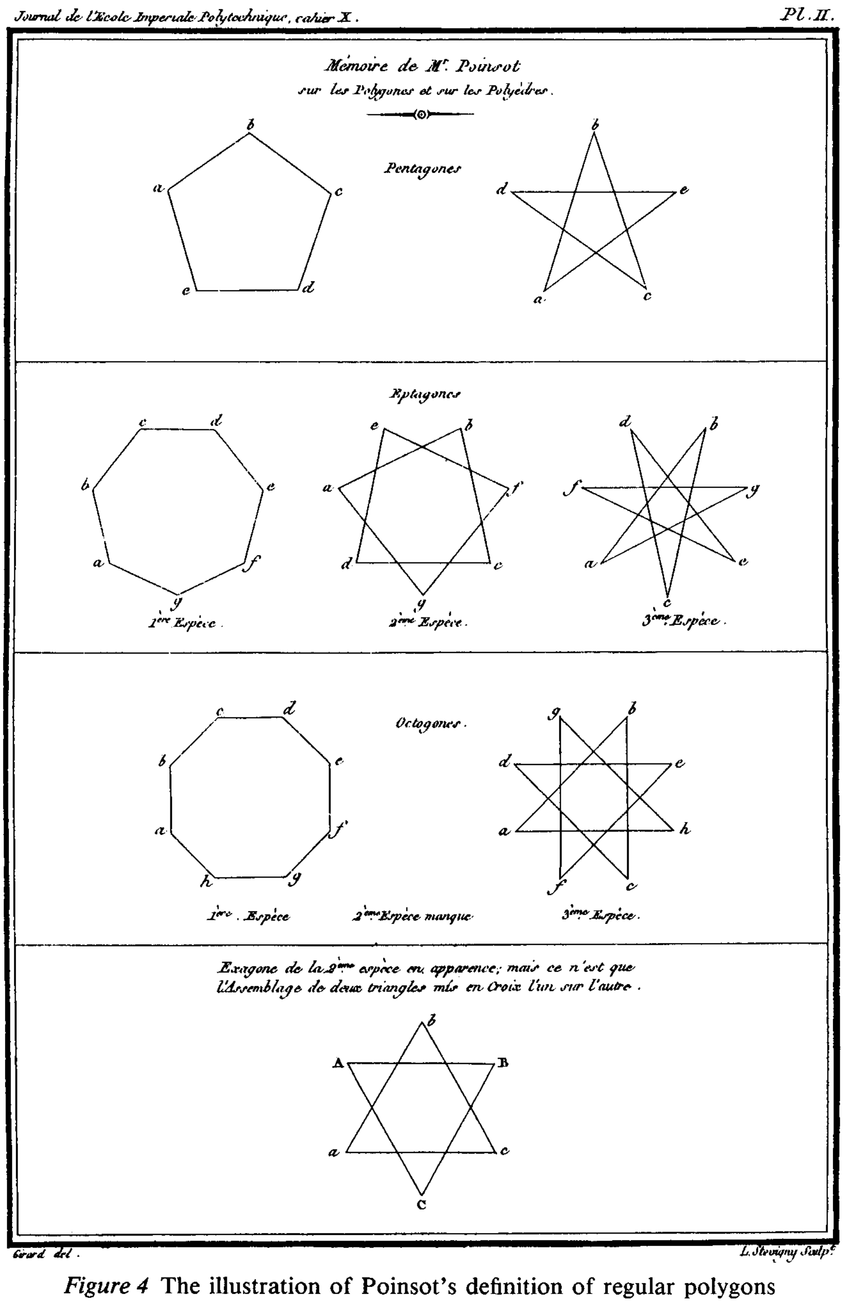



*Figure 4* The illustration of Poinsot's definition of regular polygons (from his *1809*)

prime to $m$, and if $m$ points, equidistributed on a circle, are connected by segments each of which spans $h$ of the arcs determined by the points, then a regular polygon is obtained. In modern notation, such a polygon would be designated $\{m/h\}$; convex polygons correspond to $h = 1$, and the pentagram $\{5/2\}$ to $m = 5$ and $h = 2$ (see Figure 4, taken from Poinsot's paper). The last row in Figure 4 illustrates (for $m = 6$ and $h = 2$) Poinsot's (correct) statement that this construction does not produce a regular polygon if $m$ and $h$ are not relatively prime; in such a case the resulting figure is composed of several regular polygons, and hence is not itself a regular polygon.

The logical error committed by Poinsot and uncritically accepted by all later writers is the assumption that this reasoning proves the non-existence of regular polygons (as defined) other than the ones corresponding to relatively prime $m$ and $h$. In fact, it is fully in accordance with Poinsot's definition (as well as with more modern ones) to consider as regular polygons obtained by starting from one point on a circle, connecting it to a second point by a segment spanning an arc which is $h/m$ times the length of the circle's perimeter, and continuing by such steps until, after $m$ steps, the original point is reached again. If $m$ and $h$ are relatively prime, the $m$th step is the first that leads to the starting-point; otherwise, there will have been

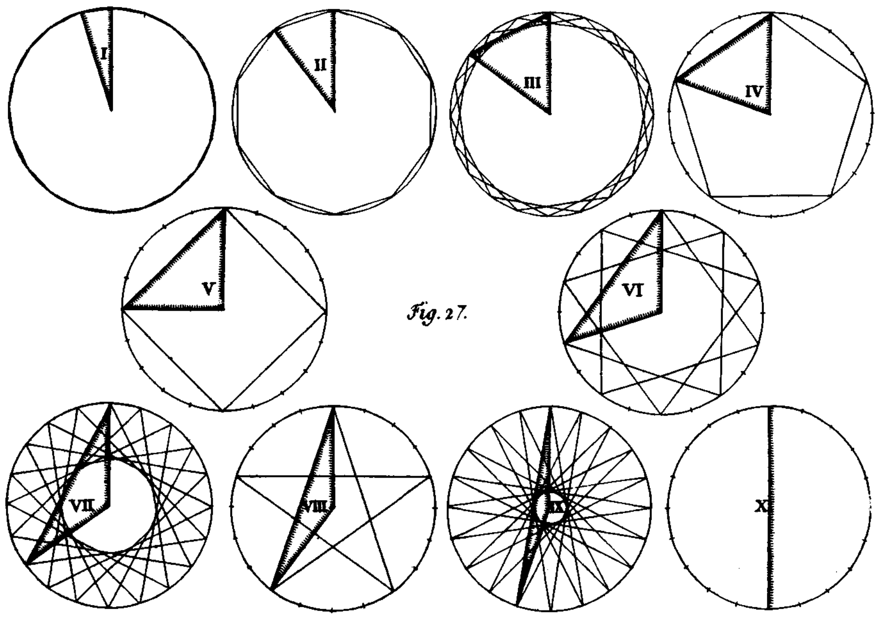
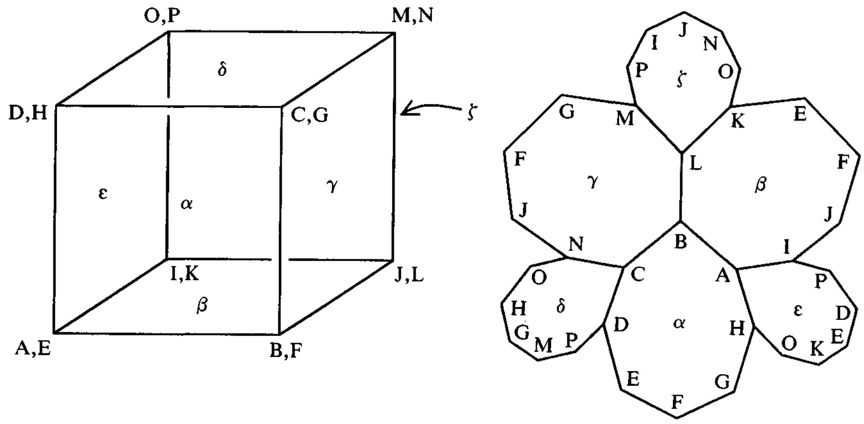



*Figure 5* An illustration of the definition of regular polygons, showing all the different polygons with 20 vertices (from Meister *1770*: Plate 6)

previous visits – but this is irrelevant to regularity. It is most remarkable and ironic that precisely this interpretation was originally advanced by A. F. L. Meister in *1770*; Figure 5 shows his illustration for $m = 20$. However, although Meister's article is often mentioned in accounts of the history of polygons, it is apparently rarely, if ever, read. The only discussion of some details of Meister's work, by the historian of mathematics Siegmund Günther (in 1876), misquotes Meister at the critical place, giving the false impression that Meister made the same error as Poinsot.

Clearly, with consistent application of the definitions some points may correspond to several vertices of a regular polygon – a possibility which has not been excluded in any way (except possibly by arbitrary and unstated fiat). A variety of additional polyhedra can now be considered regular, with the requirement of flag transitivity under symmetries fulfilled. A simple example is shown in Figure 6(a), where each vertex of the cube represents two vertices of a regular polyhedron which has 16 vertices and 6 octagons




(a) (b)

*Figure 6* (a) A regular polyhedron $\{8/2, 3\}$ with 16 vertices and 6 octagonal faces:

$$\alpha = [A, B, C, D, E, F, G, H],$$
$$\beta = [A, I, J, F, E, K, L, B],$$
$$\gamma = [B, L, M, G, F, J, N, C],$$
$$\delta = [C, N, O, H, G, M, P, D],$$
$$\varepsilon = [A, H, O, K, E, D, P, I],$$
$$\zeta = [I, P, M, L, K, O, N, J]$$

(b) A topologically regular map of genus 2, which is isomorphic to the regular polyhedron $\{8/2, 3\}$; this map is denoted by $\{4 + 4, 3\}$ in Coxeter and Moser (*1957*: Section 8.8), where references to various papers discussing it may be found of type {8/2} as faces. This polyhedron can be interpreted as a polyhedral realization of the well-known, topologically regular map of genus 2, which is shown in Figure 6(b).

The complete enumeration of regular polyhedra is still outstanding.

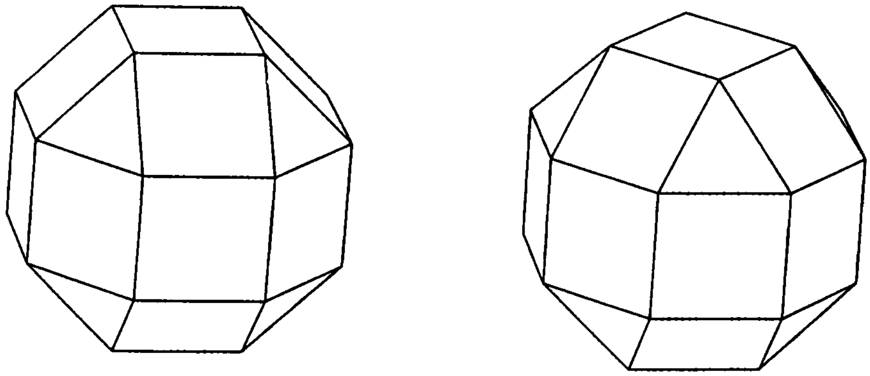
In the 1880s the concept of regular polyhedra was extended to higher-dimensional spaces. Related concepts were studied in hyperbolic spaces, and in other settings as well. Space here does not permit these and other developments to be presented; accounts of them can be found in Coxeter *1948*, *1974*, and Fejes Tóth *1964*.

## 4 OTHER SPECIAL POLYHEDRA

Since Antiquity, various classes of polyhedra that are slightly less special than the regular polyhedra have also been considered. In several cases logical errors as blatant as the one discussed above have been widely accepted; these errors are briefly mentioned in discussing some of these classes of polyhedra.

Archimedean polyhedra are usually defined as strictly convex polyhedra, all faces of which are regular polygons (not necessarily of the same kind), and all vertices of which are congruent (this means that the figure formed by the faces containing one vertex is congruent to the figure analogously formed at any other vertex). A related concept is that of a uniform polyhedron defined to be a regular-faced isogonal polyhedron; some writers confuse 'Archimedean' with 'uniform'. ('Isogonal' means that the vertices are all equivalent under symmetries of the polyhedron.) The enumeration of Archimedean polyhedra is believed to have been carried out by Archimedes, but, if so, no manuscript survived. In any case, from the Renaissance onwards there have been frequent claims that, besides the regular polyhedra, the prisms and the antiprisms, there are precisely 13 other Archimedean polyhedra. While this is correct if applied to uniform polyhedra, there is one additional Archimedean polyhedron which is not uniform. That exceptional polyhedron is the pseudorhombicuboctahedron (Figure 7), first constructed by J. C. P. Miller in the 1920s and described by H. S. M. Coxeter in 1930. It was previously missed by geometers because they assumed – with no logical basis – that if the faces of an Archimedean polyhedron are arranged around a vertex in a certain way, no other Archimedean polyhedron can be formed with this arrangement of faces.

During the last century, isogonal polyhedra have been studied without restriction to those with regular faces. The convex ones among them have the interesting property that each is isomorphic to one of the uniform polyhedra; this holds even for the more general class of those not necessarily convex isogonal polyhedra which are topologically equivalent to a solid



(a) (b)

*Figure 7* Two distinct Archimedean polyhedra with the same arrangement of faces around each vertex: (a) the rhombicuboctahedron and (b) the pseudorhombicuboctahedron; (a) is uniform, (b) is not

sphere. However, in most writings it is (explicitly or implicitly) assumed that all isogonal polyhedra (without self-intersections) are 'spherical' in that sense. Only recently, Grünbaum and Shephard *1984* observed that this assumption is not justified: there exist toroidal isogonal polyhedra, as well as isogonal polyhedra of genus 3, 5, 7, 11 or 19. These non-spherical isogonal polyhedra also show that the frequently made claims about 'natural duality' of isogonal polyhedra with isohedral polyhedra (i.e. those in which all faces are equivalent under symmetries) are mistaken: there exist no isohedral polyhedra (without self-intersection) which are dual to the toroidal isogonal polyhedra, or to those of higher genus. The logical error here is due to a widespread confusion between the projective duality which relates points and planes, and the topological duality which relates cells and vertices.

These (and other) facts about regular polyhedra and their generalizations show that, in the distant past as well as in modern times, geometers have been less adept at investigative and deductive geometry and critical thinking than at accepting traditional teachings, even if these are not supported by logic.

## BIBLIOGRAPHY

Coxeter, H. S. M. *1948*, *Regular Polytopes*, London: Methuen. [3rd edn repr. 1973, New York: Dover.]

——*1974*, *Regular Complex Polytopes*, Cambridge: Cambridge University Press. [2nd edn 1991.]

Coxeter, H. S. M. and Moser, W. O. J. *1957*, *Generators and Relations for Discrete Groups*, Berlin: Springer. [4th edn 1980.]

Cundy, H. M. and Rollett, A. P. *1951*, *Mathematical Models*, Oxford: Clarendon Press. [2nd edn 1961.]

Fejes Tóth, L. *1964*, *Regular Figures*, London: Pergamon.

Grünbaum, B. and Shephard, G. C. *1984*, 'Polyhedra with transitivity properties', *Société Royale de Canada. Académie des Sciences. Comptes Rendus Mathématiques*, **6**, 61–6.

Meister, A. F. L. *1770*, 'Generalia de genesi figuram planarum [...]', *Novi commentarii Societatis Regiae Scientarum Goettingensis*, **1**, 144–80.

Poinsot, L. *1809*, 'Mémoire sur les polygones et les polyèdres', *Journal de l'Ecole Polytechnique*, **4**, cahier 10, 16–49.

Senechal, M. and Fleck, G. (eds) *1988*, *Shaping Space: A Polyhedral Approach*, Boston, MA: Birkhäuser.

Waterhouse, W. C. *1972*, 'The discovery of the regular solids', *Archive for History of Exact Sciences*, **9**, 212–21.

Wiener, C. *1865*, *Über Vielecke und Vielfläche*, Leipzig: Teubner.

# 7.4

# *Euclidean and non-Euclidean geometry*

J. J. GRAY

## 1 EUCLIDEAN GEOMETRY

Euclidean geometry is named after the Greek mathematician Euclid (*circa* 300 BC), who wrote what became the definitive account of the elementary part of the subject in his *Elements* (§1.3). This work, in 13 books, begins by describing the properties of plane figures composed of straight lines and circles, starting from appropriate basic definitions, common notions and postulates. Common notions are undemonstrated truths of general applicability, such as 'the whole is greater than the part', while the postulates are specific mathematical statements their author saw fit to assume. Most of these are uncontroversial, for example 'all right angles are equal to one another', but one of them came to be increasingly discussed – the parallel postulate. This is the claim 'that, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than two right angles'. While the other assumptions of Book I alone enabled Euclid to prove some results, such as the isosceles-triangle theorem and the congruence theorems of plane geometry, the parallel postulate was needed to prove such results as Pythagoras's theorem, and that the angles of a triangle add up to two right angles. It follows that most of the rest of the *Elements* also required the postulate.

On the other hand, the postulate seemed open to objection on two opposing grounds: it was not as obviously true as the other assumptions made by Euclid, and it seemed more like a result that could be proved on the basis of the other assumptions alone. Indeed, a number of later Greek writers attempted proofs along these lines, for example the astronomer Ptolemy and the fifth-century commentator Proclus. The common result of these attempts and many that came to be made down the ages was either to lapse into a fallacy, or else to establish, explicitly or implicitly, the equivalence of the parallel postulate with some other assumption. These discoveries are interesting in themselves, and shed light on what Euclidean geometry was taken to be. For more detailed accounts, see Bonola *1912* and Gray *1989*.

## 2 ATTEMPTS ON THE PARALLEL POSTULATE

Most commonly, it was a property of distance that came to be invoked. Proclus's attempt rests on the claim, not discussed by him, that two parallel lines remain everywhere a bounded distance apart. Others, such as Ibn al-Haytham (*circa* AD 1000), endeavoured to prove that the locus of points equidistant from a straight line is itself straight, and on that basis to prove the postulate (§1.6). Weaker spirits merely assumed that Ibn al-Haytham's claim was true.

One of the most scrupulous attempts was by John Wallis in 1663. He showed that the parallel postulate was equivalent to the assumption that similar figures exist which are not congruent: that is, that a copy of a given figure can be made which has the same shape but not the same size. Wallis did not claim to have established the postulate as a theorem, merely to have reduced the task to that of defending the possibility of making scale copies of figures, for which he gave a different, philosophical argument. The assertions of Ibn al-Haytham and Wallis are, of course, theorems in the *Elements*; they can be proved once the parallel postulate is assumed. What is at stake is whether they are more elementary than the postulate, and so either a more natural initial assumption to make or easier to prove as theorems themselves.

Ibn al-Haytham's case is typical of many in resting on claims about the straight line, defined rather unhelpfully in the *Elements* as 'the curve that lies evenly upon itself'. It is probably no longer possible to determine what Euclid meant by this phrase, but it is likely that he was doing no more than hinting at a concept that was supposed to be grasped intuitively. Aristotle, writing rather earlier, made several comments about lines, including the remarkable one that 'if a line is what we recognize it to be from our physical intuition, then the angle sum of a triangle is two right angles'. It can be argued that to the Greeks, geometry was an abstraction from familiar, physical reality, and that its theorems were in some direct way true of the world. Or it can be argued that the statements on the *Elements* were intended to be taken more formally, the better to display the logical coherence of the work, and that the question of interpretation was to be kept at one remove. Certainly every attempt was made to indicate explicitly when extra assumptions had to be made, and no reliance in the work is placed on what we know from experience other than in the framing of the definitions.

But if this leaves open the question of the extent to which the *Elements* was taken to be true of the world by Greek writers, it is hard to deny that it was generally so interpreted by writers after AD 1500.

The most full-blooded attempt to prove the parallel postulate was made by Girolamo Saccheri and published in 1733, the year of his death. He distinguished between three kinds of geometry, in which the angle sum of triangles was greater than, equal to or less than two right angles, and he aimed to show that of these only the second kind was internally consistent. He succeeded in showing that the first kind 'destroyed itself', as he put it, but his account of the third kind was flawed. Significantly, it rested on establishing a result that for him was irreconcilable with the nature of the straight line. He was followed by Johann Heinrich Lambert, who in the 1760s also explored Saccheri's trichotomy. He did not discover any results that struck him as self-contradictory, finding only those which were contrary to his intuition, so he may be regarded as the first to describe theorems in a geometry other than Euclid's. But, perhaps because his investigations were inconclusive, he did not publish them, and they were first published posthumously by John III Bernoulli in 1786. They were reprinted in Engel and Stäckel *1895*.

Lambert accepted that two straight lines cannot enclose an area, which he regarded as an assumption also made by Euclid (although it is probably an Arab interpolation). This principle was taken by Franz Taurinus, writing in 1825 (also reprinted in Engel and Stäckel *1895*), to rule out of consideration the geometry on a sphere in which great circles are taken as straight lines, and in which indeed the angle sum of every triangle exceeds two right angles. The naturalness of this geometry strongly suggests that in investigating the parallel postulate mathematicians were trying not to devise formal systems differing in some way from Euclid's, but to describe the real world in mathematical terms. Spherical geometry differs from Euclidean geometry, but it is irrelevant if the aim is to describe the universe, because that cannot be like a sphere once it is agreed that two lines cannot enclose an area.

Lambert was a contemporary of Immanuel Kant, with whom he corresponded on the nature of geometry. Kant's view that statements of Euclidean geometry were synthetic *a priori* truths (synthetic because they could in principle be false, *a priori* because independent of any particular experience) seems not to have convinced him, perhaps because of his deep study of the parallel postulate. However, Kant's view that human intuition presented space to us as Euclidean did gain wide acceptance. The result was that, by the start of the nineteenth century, the widespread view was that Euclidean geometry correctly describes the world, and that it is built up logically from axioms which themselves are incontrovertible truths. All this was to change, with consequences not just for the *Elements* but for all of geometry and our very understanding of the nature of mathematics. For a discussion of the philosophical implications, see Torretti *1978*.

## 3 THE DISCOVERY OF NON-EUCLIDEAN GEOMETRY

The first to doubt the truth of Euclidean geometry was Carl Friedrich Gauss, who investigated the question from a variety of standpoints throughout his life but published almost nothing. Whether he lacked the final insight to clarify the question to his own satisfaction or merely wished to avoid public controversy is almost impossible to decide; both interpretations may be true. But his failure to resolve the matter means that the honour of being the first to describe a geometry other than Euclid's goes to the Russian Nikolai Lobachevsky and the Hungarian János Bolyai, writing independently in the 1830s. The degree of overlap in their descriptions is considerable. Both described a three-dimensional geometry in which coplanar lines $\lambda$ and $\nu$ meeting a line $\mu$ at angles of $\alpha$, which is less than a right angle, and $\gamma$, a right angle, may be asymptotic and therefore considered parallel (Figure 1). On the basis of this assumption, and the independent result that the theorems of spherical trigonometry do not depend on the parallel postulate, both men obtained trigonometric formulas relating the sides of a triangle. On the strength of these formulas, both proclaimed that their new geometries made sense and, being different from Euclid's, called into question the truthfulness of Euclidean geometry.

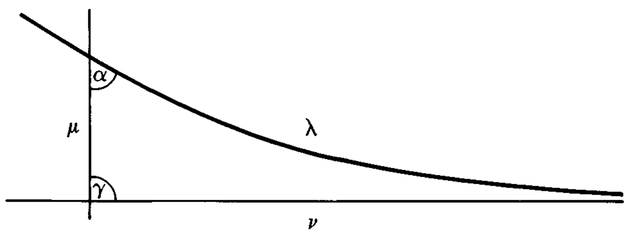



*Figure 1* Parallels in the non-Euclidean geometry of Bolyai and Lobachevsky

Reactions to the new geometry of Bolyai and Lobachevsky, called 'non-Euclidean geometry' because its assumptions differ from those of Euclidean geometry (although only in respect of the parallel postulate), were indifferent or even hostile in their lifetimes. Gauss accepted their findings, but did little in public to defend their cause. Lobachevsky's work was derided by some of his fellow Russians, and Bolyai's was left to languish in the obscurity of its original publication – an appendix to his father's treatise on geometry. Moreover, by the strictest canons, the work of Bolyai and Lobachevsky was flawed, for no proof of the consistency of their original assumption about parallel lines was given. The possibility remained that their theory was nothing more than some formulas in analysis, spuriously dressed up as geometry.

Matters changed decisively with the publication of Eugenio Beltrami's description of non-Euclidean geometry in 1868. He presented a description of Lobachevsky's non-Euclidean geometry in terms of the points inside a disc. By defining the distance between two points appropriately, Beltrami produced a map of non-Euclidean two-dimensional space in which straight lines in the space appeared as straight lines in the disc. This is a map in exactly the sense in which pages of an atlas are maps of the curved surface of the Earth, and so it resolved conclusively and affirmatively the question of whether Lobachevsky's definition of parallel lines was a tenable one. This flaw in the original presentation was undoubtedly one reason for the delay in accepting it, just as the posthumous publication of Gauss's views after 1855 helped to gain it acceptance. But the main reasons for the delay were the enormity of the consequences that accepting it entailed, and the difficulty in mastering the new approach to geometry that it required. The very existence of a new geometry seemed to challenge all one's intuitions about mathematics and space.

Given this challenge to the role of intuition in mathematical thought, it is interesting that both the discovery and the reception of non-Euclidean geometry illustrate a curious feature of mathematical discourse: its symbolism. By casting geometry into trigonometrical formulas, Bolyai and Lobachevsky both produced a language for analysing the behaviour of lines that evades the weight of tradition. Had non-Euclidean geometry turned out to be impossible after all, their formulas would still have been valid exercises in the hyperbolic trigonometric functions that they introduced (functions introduced originally by Lambert, curiously enough: see §4.2). Their formulas were ontologically obscure, if not actually ambiguous. By distancing mathematicians from spatial intuition, they made it possible for their successors to study the fundamental concepts of geometry in an original way, culminating in the reformulations by Beltrami and Bernhard Riemann. For exactly the opposite reason – a refusal to indulge in uninterpreted symbols – the new geometry was often hotly denounced by philosophers. The most remarkable of these denouncements was by Gottlob Frege, whose attitude to definitions and to the concept of number in particular was usually sharp (§5.2), but who could never accept non-Euclidean geometry because he felt it was precluded by the existence of a unique physical world.

## 4 THE MEANING OF NON-EUCLIDEAN GEOMETRY

The formulas of Bolyai and Lobachevsky did not remain obscure for long. Beltrami's interpretation fitted very well with the views of Riemann on the way geometry should be reformulated, namely as the study of sets of points forming a surface or higher-dimensional object upon which the notion of length along a curve makes sense. From this point of view there are three exceptionally interesting two-dimensional geometries: those on the plane, on the sphere, and on the surface of constant negative curvature that had been known since the time of Gauss but was best described by Beltrami. They are distinguished mathematically by being the geometries on surfaces of constant curvature. Curvature is a technical concept, but its intuitive meaning is straightforward. Applied to surfaces in space, the curvature of a plane is zero, that of a sphere of radius $R$ is $1/R^2$ and that of a saddle-shaped surface is negative.

From a Riemannian point of view, one starts with a surface and defines a sense of distance upon it; this yields a value for the curvature at each point. The surface with constant, zero curvature is a plane with the familiar Euclidean geometry. The geometry which has constant positive curvature is also familiar: it is that of the sphere, where a 'line' is defined as a great circle. Spherical geometry differs from Euclidean geometry in two fundamental ways: the parallel postulate is false because there are no parallel lines, and lines cannot be extended indefinitely. For this reason its existence does not invalidate Saccheri's and Lambert's proofs that a geometry in which the angle sum of a triangle exceeds two right angles cannot exist, although the angle sum of a spherical triangle always exceeds two right angles. Authors differ over whether spherical geometry should be called a non-Euclidean geometry or simply a geometry different from Euclid's. The simplest solution is to call only the geometry discovered by Bolyai and Lobachevsky non-Euclidean, and that has been adopted here. What Beltrami showed was that non-Euclidean geometry (in which there are many parallels to a given line through a given point not on the line, and in which the angle sum of a triangle is always less than two right angles) is identical to the geometry on a surface of constant negative curvature.

The three geometries on surfaces of constant curvature stand out because figures may be moved around freely on the surfaces without distortion, as we believe to be the case in (two-dimensional) physical space. This gives them all a degree of physical plausibility. Moreover, the crucial property of curvature is that it is intrinsic: it can be defined and measured entirely without reference to any three-dimensional space in which the surface may be placed. The idea of curvature can be generalized to higher dimensions, when there are many more interesting special cases. Again, the new geometries are intrinsically defined, so there is no need for an *a priori* Euclidean space for them to exist in. This makes it clear that, from the Riemannian point of view, Euclidean geometry is just one example of many. In this picture, which was widely adopted during the second half of the nineteenth century, Euclidean and non-Euclidean geometry become part of differential geometry (§3.4).

The implications of non-Euclidean geometry were soon taken up. Since it, like Euclid's, is a mathematically valid, physically plausible geometry (i.e. one which is consistent and in which basic terms like distance and angle make sense), people asked which, if either, was true. To what extent had geometry become an empirical science? Gauss, Friedrich Wilhelm Bessel, Lobachevsky and Bolyai were clear that measuring the angle sums of triangles would in principle serve to distinguish between Euclidean and non-Euclidean geometry, and that to this extent geometry was henceforth empirical. But once that is granted, what is one to make of two thousand years' worth of belief in the intuitive naturalness of the Euclidean terms? What, indeed, is a straight line? In the formulation proposed by Riemann that question had an answer for the first time: a straight line can be defined as a geodesic (curve of shortest length between its points) in a surface of zero curvature. What Beltrami showed, and Riemann himself hinted at, is that non-Euclidean geometry is the intrinsic geometry on a surface of constant negative curvature.

The empirical question was never to be resolved, and is today subsumed in theories of general relativity that interpret gravity as a variable curvature in a four-dimensional spacetime. But for mathematicians the problems posed by the discovery of non-Euclidean geometry were by no means over. By the 1880s projective geometry had come to assume a higher status than that of Euclidean geometry: its assumptions were fewer, and it contained Euclidean geometry as a special case. So although projective geometry was not physically plausible (it contains no concept of length or angle), it is logically prior to Euclid's. Moreover, there are no parallel lines in projective geometry. What, then, of the nature of a line? The intuitive concept of the Euclidean straight line has been refined (at the very least) or replaced (at worst) by that of the projective line. From a metrical point of view, should not the primitive intuition have been that of the non-Euclidean line? Mathematicians' intuition was no longer unproblematic; it had led them into an unwarranted confidence in Euclid. To anyone who based mathematics on some kind of intuitive apprehension of reality, this raised the question of what guarantee there could be that even mathematical truth could be discovered. The first to tackle this question was Moritz Pasch, who discovered a number of tacit assumptions made by Euclid that no one had remarked upon before because they rest on seemingly self-evident propositions (Nagel *1939*).

## 5 AXIOMATIC GEOMETRY

Pasch may well have hoped to refine our intuition until it became reliable. The far more radical step of denying mathematical terms any meaning and relying completely on formal rules of inference was taken by Guiseppe Peano and, independently, by David Hilbert. (Hilbert's route to this discovery has recently been described in Toepell *1986*.) Hilbert proposed that the geometric terms 'point', 'line', 'plane', and so forth be controlled by a system of axioms that determine what one may say about them, but which makes no attempt to say what they are. A self-consistent set of axioms would be taken to guarantee the existence of the objects it provided for, but all intepretations of them would be equally valid. Self-consistency was assured by any model exemplifying them. So, plane Cartesian geometry is an algebraic model assuring the consistency of the Euclidean axioms, and Beltrami's disc can be seen as a model of the consistency of the non-Euclidean axioms. It also establishes that Euclidean and non-Euclidean geometry are relatively consistent, since it can be interpreted as a picture in Euclidean geometry. This means that attempts to refute non-Euclidean geometry would, if successful, have shown Euclidean geometry to be impossible also – a result no one can have wanted.

Hilbert's axioms for geometry began by proposing three kinds of object (points, lines and planes) with properties to be described by words such as 'lie', 'between' and 'congruent'. These were controlled by five families of axioms: those of incidence, order, congruence, parallelism and continuity. For example, the first axiom is 'For every two points A and B, there exists a line *a* that contains each of the points A and B.' Pasch's axiom appeared among the axioms of order, and Euclid's parallel postulate was given in the equivalent form: 'Let *a* be any line, and A any point not on it. Then there is at most one line in the plane, determined by *a* and A, that passes through A and does not intersect *a*.' Hilbert then showed how the familiar theorems of Euclidean geometry could be derived within his formal system by chains of reasoning that followed the axioms he had laid down.

Hilbert's proposal neatly side-stepped the question of how intuition had led people astray by avoiding all questions of meaning. His intentions, however, were not to render mathematics meaningless, but to make it clear; he continued to assert the importance of intuition in guiding and developing one's thought. Indeed, he might well not have put his abstract axiomatics forward had he not discovered they were of use in generating new theorems. Examples were already known of geometries where the coordinates are taken from finite fields; Hilbert's work led to the discovery of geometries that cannot be described in terms of coordinates at all. His approach to geometry therefore greatly enlarged its scope and took it beyond the domain of simple, continuous manifolds where Riemann and Felix Klein had placed it.

## 6 AXIOMATIC MATHEMATICS

Hilbert's presentation came to have a decisive effect on many branches of mathematics. It was as if the pure mathematician's task was to provide axiom systems and check that they were self-consistent, which applied mathematicians, physicists and others could then use as they saw fit. This neatly defined a new relationship between pure and applied mathematics. Within pure mathematics, what was done for Euclidean geometry was done for other geometries. Hilbert showed the next year (1900) how non-Euclidean geometry can be obtained by changing just his version of the parallel postulate. Other mathematicians joined in, describing geometries which differed more and more in their nature from Euclidean geometry. Then other systems of mathematical ideas were given axiomatic treatments, starting with the theories of groups and fields. In time, there set in a reaction against a baroque profusion of axiom systems, serving less and less purpose; the present generation of mathematicians regards axiom systems as a good way to define or create mathematical objects, but prefers (with Hilbert) to concentrate on their intuitive meanings for the purposes of carrying out research. However, Hilbert's ideas had by then grown to embrace the hope that all of mathematics could be not only axiomatized, but given axioms that guaranteed every truth a proof expressible in a finite number of statements. This hope, partly forged in the battle against intuitionism, was shown to be futile by Kurt Gödel (§5.5).

## 7 THE CURRENT SITUATION

A further step away than non-Euclidean geometry lies the domain of special relativity (§9.12). With the rapid acceptance of Albert Einstein's ideas, space and time, to quote Hermann Minkowski, faded away to be replaced by spacetime. The geometry of spacetime, although linear (unlike that of non-Euclidean geometry), differs markedly from Euclid's in other ways, and so the dominant geometrical paradigm among physicists is no longer Euclidean. What, then, remains of Euclid's creation? It remains the naive formulation of space taught in schools. It is perfectly adequate for classical physics. The *Elements* proposed an ideal of rigorous deductive reasoning from clearly stated premisses that mathematicians continue to uphold. It is unlikely that anything like Kantian intuitionism will survive as an account of why we perceive space the way we do, but it is quite likely that we will learn that small, nearby regions of space are like patches of Euclidean space. Meanwhile, non-Euclidean geometry continues to arouse the greatest interest among mathematicians. Outstanding, but widely believed, conjectures by William Thurston imply, for example, that in some sense most three-dimensional manifolds are made up of patches of three-dimensional non-Euclidean space.

## BIBLIOGRAPHY

Bonola, R. *1912*, *Non-Euclidean Geometry. A Critical and Historical Study of its Development*, Chicago, IL: Open Court. [English transl., with additional appendices, by H. S. Carslaw, repr. 1955, New York: Dover. Contains J. Bolyai, *Science of Absolute Space*, and N. I. Lobachevsky, *Geometrical Researches in the Theory of Parallels*, both transl. G. B. Halsted.]

Engel, F. and Stäckel, P. *1895*, *Die Theorie der Parallellinien von Euklid bis auf Gauss*, Leipzig: Teubner.

Euclid *1918*, *Elements* (ed. T. L. Heath), 3 vols, Cambridge: Cambridge University Press. [Repr. 1956, New York: Dover.]

Gray, J. J. *1989*, *Ideas of Space: Euclidean, non-Euclidean, and Relativistic*, 2nd edn, Oxford: Oxford University Press.

Hilbert, D. *1899*, *Grundlagen der Geometrie*, Leipzig: Teubner. [2nd edn transl. by L. Unger as *Foundations of Geometry*, 1902, La Salle, IL: Open Court; repr. 1971.]

Nagel, E. *1939*, 'The formation of modern conceptions of formal logic in the development of geometry', *Osiris*, 7, 142–224.

Toepell, M.-M. *1986*, *Über die Entstehung von David Hilberts 'Grundlagen der Geometrie'*, Göttingen: Vandenhoeck & Ruprecht.

Torretti, R. *1978*, *Philosophy of Geometry from Riemann to Poincaré*, Dordrecht: Reidel.

# 7.5

# *Descriptive geometry*

KIRSTI ANDERSEN AND

I. GRATTAN-GUINNESS

Today, 'descriptive geometry' denotes a discipline which deals with how to represent three-dimensional objects on a plane. The term was introduced by Gaspard Monge, and used by him in one sense more narrowly and in another more broadly. He required that knowledge about the shape and measures of an object could be obtained from its image, and restricted his considerations to a representation fulfilling this. However, not all his theory was directly connected with two-dimensional representations: some of it dealt more generally with synthetic geometrical methods; his book on descriptive geometry covers in particular topics which later became part of differential geometry. In treating the prehistory of the subject and its development after Monge, we leave out the history of differential geometry (on which see §3.4); see §12.6 for more details on the particular two-dimensional representation called 'perspective'.

## 1 PREHISTORY

What is now known as Monge's descriptive geometry developed out of a technique of representing a three-dimensional object on a plane by projecting it onto two suitable planes, perpendicular to each other, projections known as the plan and the elevation. The origin of this technique is unknown: plans have been used since very early days; elevations were also used in Antiquity, but apparently not together with plans. It is unclear when the two representations were first used simultaneously. But Renaissance architects knew the technique, and in the second part of the fifteenth century a perspective method relating directly to plans and elevations was described by the Italian painter Piero della Francesca in the book *De prospettiva pingendi* ('On Perspective in Painting') which, however, remained unpublished for more than four centuries. The German artist

Albrecht Dürer published a method similar to Piero's in his *Underweysung der Messung* in 1525 (Figure 1). Moreover, Dürer used the technique of plan-and-elevation for describing skew curves (Figure 2).

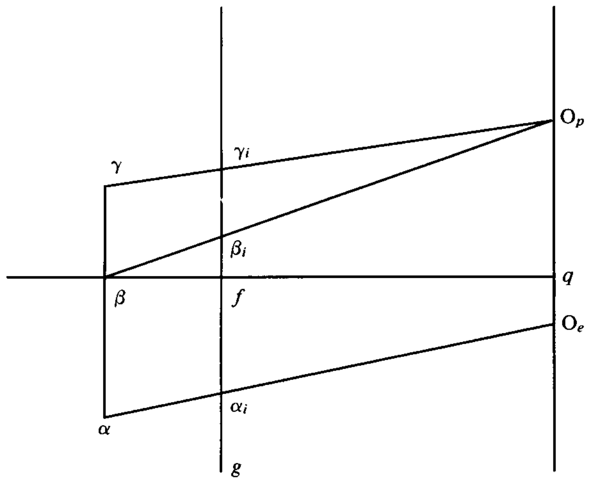



*Figure 1* Dürer's perspective method based on plan and elevation; a redrawing of part of Dürer's figure 56 in *Underweysung der Messung* (1525) (the letters are not original). Below the line $fq$ is the plan, and above it the elevation; the line $fg$ represents the picture plane. The eye-point has been projected orthogonally onto the points $O_p$ and $O_e$, the points $\alpha$ and $\beta$ are the projections of a point that lies in the plan, and the points $\alpha$ and $\gamma$ are the projections of a point lying a distance $\beta\gamma$ from the plan above $\alpha$. By joining the projections of the eye and the projections of a point, and finding their intersections with $fg$, Dürer obtained the plan and elevation of the perspective image of the point; thus $f\alpha_i$ determines the horizontal position and $f\beta_i$ the vertical position of the image of the point that is projected onto $(\alpha, \beta)$

Plans and elevations were handy tools for architects and engineers in planning buildings and fortifications, and they were required in the majority of perspective constructions. The method for producing them was therefore taught in many textbooks on civil and military architecture, and in textbooks on perspective; it was also included in some works on projections. In general the method was described rather briefly, and no new theoretical insights were added. A more comprehensive study of orthogonal projections of curved surfaces was published by the French engineer Amédé-François Frézier in the 1730s; like Monge later, Frézier combined his investigation with a kinematic description of curves and surfaces.

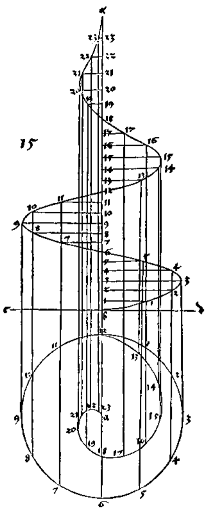


*Figure 2* Dürer's plan and elevation of the cylindrical spiral and the helix (from Dürer, *Underweysung der Messung* (1525), figure 15)

## 2 MONGE'S NEW TRADITION OF DESCRIPTIVE GEOMETRY

Monge began to develop descriptive geometry during the late 1760s, when he was in his twenties and teaching at the military school at Mézières; he continued to practise and advocate its use throughout his life (Taton *1951*). Because it was applied to warlike needs, it was classified mathematics; for example, he used it to prepare the plates for his book *Description de l'art de fabriquer les canons* (1793), but he was not permitted to describe the manner of their preparation.

When the Parisian system of higher education was renovated in the 1790s, Monge found opportunities to publicize the details. Courses on descriptive geometry were given at the Ecole Normale from 1795, and soon after also at the Ecole Polytechnique; the lecture notes from the former were edited in a stenographic version in a special journal. In 1799 Monge's lectures on descriptive geometry were collected and published as the book *Géométrie descriptive* by J. N. P. Hachette, who helped Monge with the teaching. This book went through many editions, from 1820 enlarged with sections on shadows and perspective edited by Monge's student, B. Brisson.

The first chapter of *Géométrie descriptive* deals with projections of points, lines and planes. Among Monge's innovations to traditional plan-and-elevation methods were a systematization of the description of the images and an investigation of how some basic problems about determining lines and angles are solved when the constructions concern the images rather than the original objects in Euclidean space. In Figure 3, LMNO is the plan and LMPQ the elevation, and they intersect in the ground line LM; onto these two planes objects are projected by orthogonal projection. To obtain a plane configuration, the elevation LMPQ is rotated about LM into the plane of LMNO. It is handy to have a word for the two half-planes containing, respectively, plans and elevations of objects; hence the term 'descriptive plane' is used. The image of a point like A is the pair of points *a* and *a* ″lying on a line perpendicular to LM intersecting it in C; the position of the original point A is determined by the distances C*a* and C*a*″. The image of a line segment which cuts both the plan and the elevation is a pair of line segments determined by the images of the points where the line segment meets these two planes (Figure 4). Rather untraditionally, Monge also

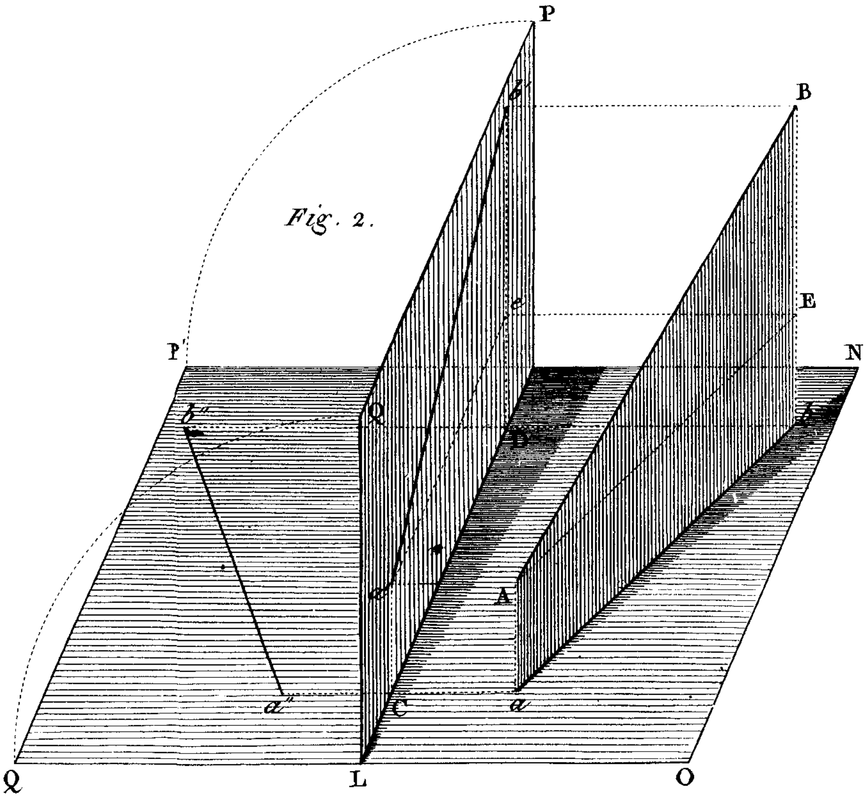



*Figure 3* Monge's projections of points and lines onto the plan and the elevation (from Monge, *Géométrie descriptive* (1820), figure 2)

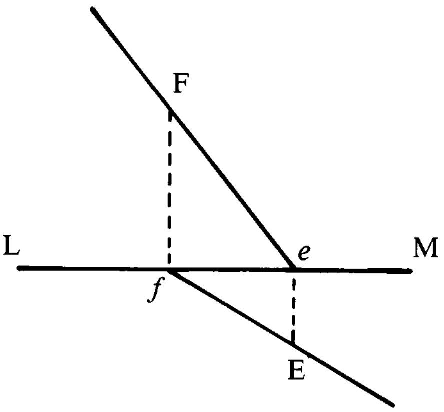



*Figure 4* A line segment intersecting the plan in E and the elevation in F is represented by the lines (E*f*, *e*F); the points of intersection are represented as (E, *e*) and (F, *f*)

considered the images of planes; the image of a plane that meets the plan as well as the elevation he defined as the two lines of intersection. Thus, in general, a plane is represented in the descriptive plane by two half-lines intersecting each other on the ground line (Figure 5).

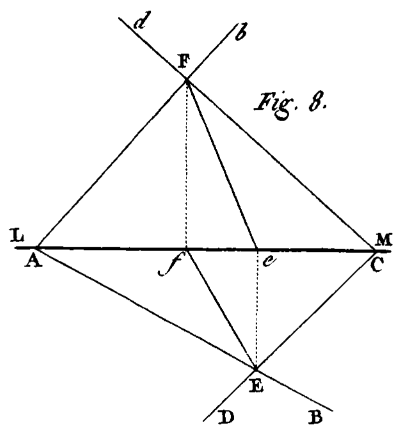



*Figure 5* The two pairs of lines (BA, A*b*) and (DC, C*d*) represent two planes, their trace is represented by the lines (E*f*, *e*F), where (E, *e*) and (F, *f*) are the representations of the points that are common for both planes and, respectively, the plan and elevation (from Monge, *Géométrie descriptive* (1820), figure 8)

Incidences between points and lines are obviously preserved when they are mapped into the descriptive plane. Since a plane is represented by a pair of lines in the descriptive plane, the incidence relations involving planes are a little more complicated. It is, however, quite straightforward to construct, for instance, the image of a line which is the trace between two planes whose images in the descriptive plane are given (Figure 5). Monge described this construction; he also showed how the lengths of original line segments can be determined from their images, and similarly how the original angles between lines and planes can be determined from the images of these objects. Moreover, he demonstrated how a number of basic constructions can be performed in the descriptive plane. For instance, for a given point and a line (plane) not containing the point in the descriptive plane, he constructed the line (plane) through the given point parallel to the given line (plane), and the plane orthogonal (the normal) to the given line (plane) through the given point. This idea of investigating the geometry in the descriptive plane is analogous to an examination of direct constructions in the perspective plane carried out through the seventeenth century and the first part of the eighteenth.

Although some of the direct constructions in a descriptive plane are simple, the procedures are in general most interesting from a theoretical point of view, because the practical constructions involve so many steps that it becomes complicated to keep track of them. Figure 6 shows an example of a rather intricate construction. In the applications, the scheme was to work out various things on the descriptive representations to determine properties in space, and maybe even to modify the object by manipulating the projections and then projecting back again.

The largest part of Monge's *Géométrie descriptive* is devoted to curved surfaces and skew curves; in treating these Monge turned to kinematics. He considered in particular surfaces that can be generated by two moving curves, and through an intuitive interpretation of motion he found several results, for example a method for determining the tangent plane through a given point on a surface. His approach was a continuation of the line which had been followed by Gilles Personne de Roberval and Isaac Newton, among others. Monge referred to the former and generalized his method of tangents, or rather an incorrect application of it (§3.1), to three dimensions, the consequence being that Monge obtained a wrong result.

Besides tangents and tangent planes, Monge dealt with the problem of determining the curve of intersection of two surfaces, and with curvature; his treatment of the latter subject was not much related to descriptive representations.

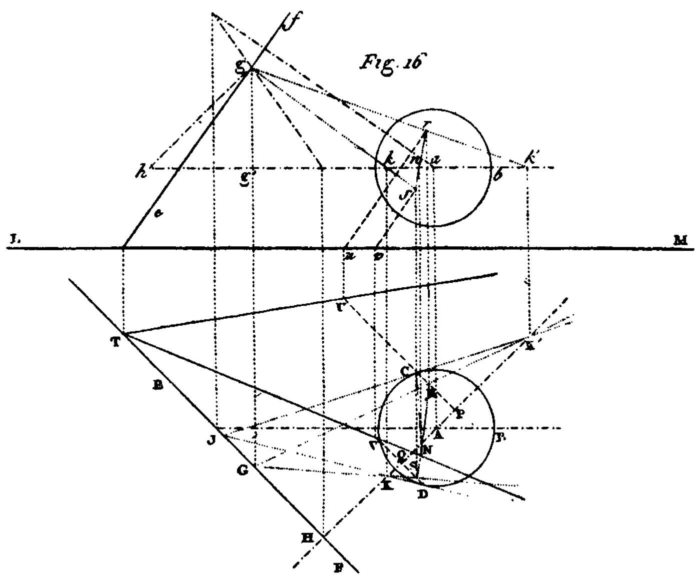


*Figure 6* Illustration to a problem of determining a tangent plane through the line LM to a sphere of which the two projections (with centres A and *a*) are shown (from Monge, *Géométrie descriptive* (1820), figure 16)

## 3 MONGE'S FOLLOWERS

Monge's dream was that descriptive geometry should become an important part of all mathematical curricula. He constantly claimed that the discipline was very useful, and could be applied almost everywhere, but in illustrating this point he chose rather academic applications. Thus he seems to have been more interested in elevating the discipline into a recognized branch of mathematics than developing it into a valuable tool for engineering mathematicians. In the event, it never became a major topic in pure mathematics, but it was relevant for the creation of projective geometry (§7.6).

During the early years of the Ecole Polytechnique, Monge was very influential in setting the policy, and he saw to it that descriptive geometry was given much time on the syllabus. However, the critics (especially Laplace) gradually gained the upper hand in the new century, and the time devoted to the topic was substantially reduced (Paul *1980*).

The industrious textbook writer Sylvestre Lacroix who, like Hachette, assisted Monge, included elements of descriptive geometry in editions of his *Essais de géométrie* from as early as 1795; Lacroix's treatment was presumably inspired by, but not completely dependent on, Monge's work.

In 1822 Hachette wrote his own textbook on the discipline. Monge's other followers were mainly from his circle at the Ecole polytechnique. The most vocal was Théodore Olivier, who taught the topic to industrial and commercial engineers at the Ecole Centrale des Arts et Manufactures, which he helped to found in 1829, and published textbooks from the 1840s. However, as at the Ecole Polytechnique, descriptive geometry lost ground there and also at other French educational institutions.

The most interesting practitioner among Monge's followers was Charles Dupin, who applied descriptive geometry to the stability of non-symmetric floating bodies, and used it together with differential geometry to find many nice geometrical properties of surfaces. Thus, when descriptive geometry was allied to differential geometry valuable results could be found, but even then the mathematical scope was limited.

## 4 LATER DEVELOPMENTS IN DESCRIPTIVE GEOMETRY

Although descriptive geometry did not become as central in mathematical education as Monge had wished, it did spread and enjoyed a long life. Throughout the nineteenth century and in the first decades of the twentieth, descriptive geometry was taught in many countries at the various kinds of new institutions of engineering that were being created (Wiener *1884*, Papperitz *1910*, Loria *1921*, Booker *1963*). The preparation of scientific and technological patents was an important stimulus to making machine drawing an important discipline. Otherwise, the normal range of applications covered the construction of arches, stone-cutting, carpentry and topography; it was also used for determining shadows and perspective pictures, and later in photogrammetry.

The various textbooks on descriptive geometry reflect the ambiguity in Monge's programme; some focused on the theoretical aspects and others on the practical; moreover, each category contains rather varied approaches. Thus some authors derived their main results by analytical methods, whereas others built upon projective geometry. In the practical books, different traditions developed in various countries and caused much difficulty for international projects where drawing was important. For example, in Britain, a particular fashion developed (Figure 7). In the USA (where the subject became popular only from the mid-century onwards, with the growth of industrialization), some preference emerged for projection onto plan, section and elevation.

Theoretical books on descriptive geometry have become rarer, whereas there still is a market for textbooks on technical drawing.

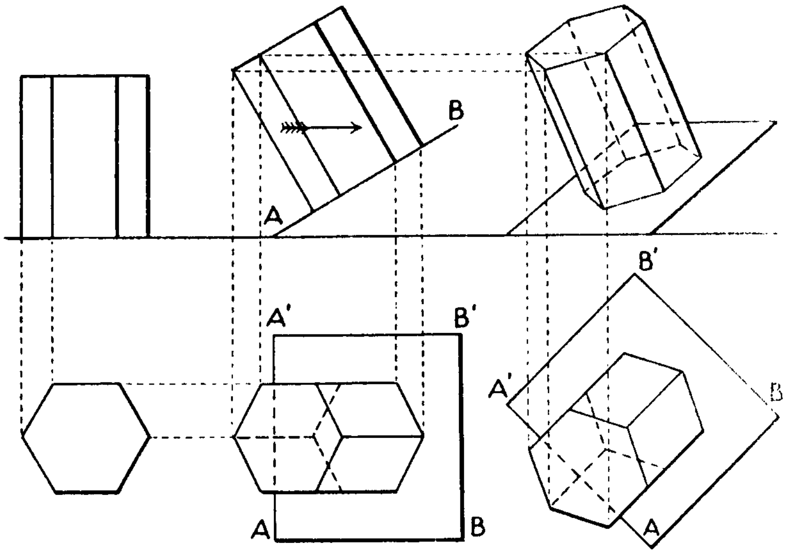



*Figure 7* An example of British descriptive geometry. The image of an obliquely lying hexagonal prism is produced in the following way. First, the plan and elevation of the prism are drawn, then the ground line of the elevation is rotated into AB. At the same time, the plan is drawn twice corresponding to the upper and lower faces of the prism. In the third step the plan is rotated through the same angle as AB, and the image is constructed by determining the points of intersection of vertical lines through the vertices of the rotated plan, and horizontal lines through the vertices of the rotated elevation. (From Booker *1963*: 138)

## BIBLIOGRAPHY

Booker, P. J. *1963*, *A History of Engineering Drawing*, London: Chatto & Windus. [Repr. 1979, London: Northgate.]

Loria, G. *1921*, *Storia della geometria descrittiva dalle origini sino ai giorni nostri*, Milan: Hoepli. [Good coverage of Europe. Worth learning Italian to appreciate his ironic remarks!]

Papperitz, E. *1910*, 'Darstellende Geometrie', in *Encyclopädie der mathematischen Wissenschaften*, Vol. 3, Part 1, 520–95 (article III AB 6).

Paul, M. *1980*, *Gaspard Monges 'Géométrie descriptive' und die Ecole Polytechnique* [. . .], Bielefeld: University.

Rowe, C. E. and McFarland, J. D. *1939*, *Engineering Descriptive Geometry* [. . .], 1st edn, Princeton, NJ: Princeton University Press. [2nd edn 1953. Not historical, but gives an excellent presentation of the different forms and uses of descriptive geometry.]

Schelling, F. *1904*, *Über die Anwendungen der darstellenden Geometrie* [...], Leipzig and Berlin: Teubner. [Includes art, photogrammetry and map-making.]

Taton, R. *1951*, *L'Oeuvre scientifique de Monge*, Paris: Presses Universitaires de France.

—— *1954*, *L'Histoire de la géométrie descriptive*, Paris: Palais de la Découverte.

Wiener, C. *1884*, 'Geschichte der darstellende Geometrie', in his *Lehrbuch der darstellenden Geometrie*, Vol. 1, Leipzig: Teubner, 1–61.

# 7.6

# *Projective geometry*

J. J. GRAY

## 1 INTRODUCTION

Projective geometry is the study of geometric figures and their transformations under projection. For example, the shadow of a figure in a plane cast onto a screen by a point source of light may be thought of as a projective transformation of the figure, called a 'perspectivity' (Figure 1(a)). The original figure and its image are said to be in perspective, or to be perspective images of each other; older books speak of 'transformation by projection and section'. A sequence of perspectivities is called a projective transformation. If the direction of the light is reversed, and the source of light replaced by an (idealized or simplified) eye, it is apparent that a figure and its perspective image can look exactly alike (Figure 1(b)). The precise appearance is dependent on the correct positioning of the eye, which compensates for some distortions but not others. For an account of the relationships between perspective theory, art and descriptive geometry, see §12.6 and §7.5.

A connection between optics and geometry had been appreciated by Greek mathematicians (§1.5). In Euclid's *Optics* the apparent sizes of objects is discussed; properties of reflection were discussed in his *Catoptrics*. Several theorems in the *Collection* of the later commentator and mathematician, Pappus, were devoted to reconstructing a lost book by Apollonius called *Determinate Section*. Pappus also discovered this result, later to become central in the theory of projective geometry (Figure 2): three points, A, B and C, lie on one line, and three more points, A′, B′ and C′, lie on another. Let the lines AB′ and A′B meet at R, the lines BC′ and B′C at P, and the lines CA′ and C′A at Q; then the points P, Q and R lie on a line. Modern commentators from Michel Chasles *1837* to B. L. van der Waerden *1961* have indicated how profitably these theorems can be seen as part of projective geometry; but it is unlikely that they were so regarded in Classical times.

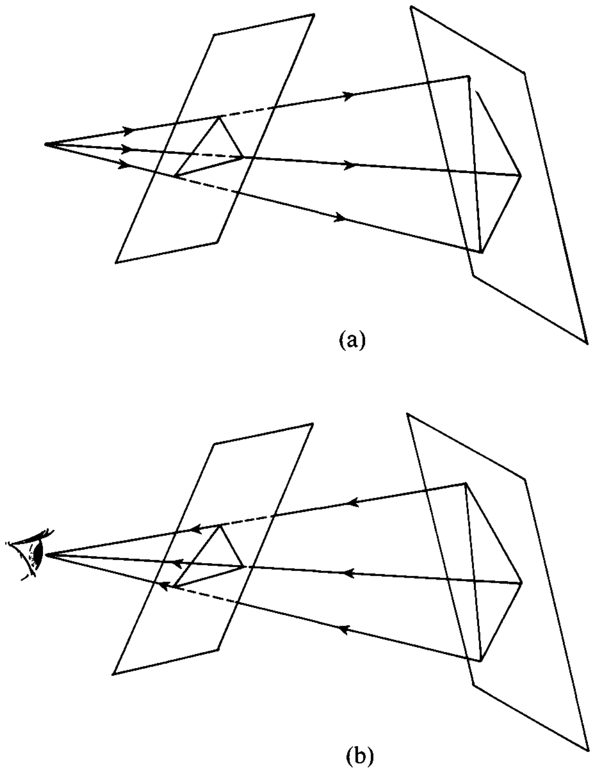



*Figure 1* Projective geometry: (a) a 'perspectivity'; (b) similarity to the eye of figure and projection

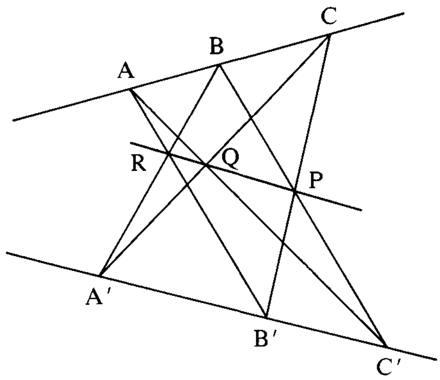



*Figure 2* Pappus's theorem

## 2 DESARGUES

A mathematical theory of perspective and projective transformations requires two things: the discovery of interesting properties of figures that remain true under such transformations, and satisfactory proofs of the main theorems. Rigorous proofs of the known constructions in the theory of perspective were given from the late sixteenth century on. For the most part they were ingenious or subtle exercises in Euclidean geometry, aimed at showing that, although a perspectivity changes lengths and angles, one can rigorously show how to draw a perspective image correctly. The first to appreciate that it was worth while looking for properties of figures that are not altered was the architect and mathematician Girard Desargues, who wrote about it in a short pamphlet of 1639 called the 'Brouillon project', or 'Rough draft on conics'. This difficult work circulated for a time in Paris before almost disappearing; for a recent discussion of the origin of Desargues's ideas and their impact, see Field and Gray *1987*.

It is trivially true that the projective transformation of a straight line is another straight line. Desargues found that a certain ratio of six points on a line was equal to the same ratio calculated for the image points. A special case of this observation, when two pairs of points coincide, was also studied by him because it arises naturally in geometrical problems; it is called 'four points in involution'.

Four points, A, B, C and D, on a line are said to be in involution if the ratios AC/CB and AD/DB are equal and opposite: $AC/CB = -AD/DB$. The minus sign arises because one of the segments is measured in the opposite direction to the others (DB in Figure 3). Desargues showed that if a perspectivity from a point O casts the points A, B, C and D onto the four points A′, B′, C′ and D′, respectively, then the image points are also in involution. Later mathematicians expressed this by saying that the 'cross-ratio' $(AC/CB)/(-AD/DB)$ is unaltered by the perspectivity:

$$(AC/CB)/(-AD/DB) = (A'C'/C'B')/(-A'D'/D'B'). \qquad (1)$$

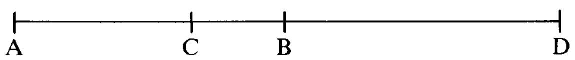



*Figure 3* Four points in involution on a straight line

The configuration of four points in involution is important because it occurs frequently in the theory of conics. For example, given an ellipse and a point P outside it, one may draw the two tangents from P to the ellipse, touching it at the points R and S, say (Figure 4). Then any line through P

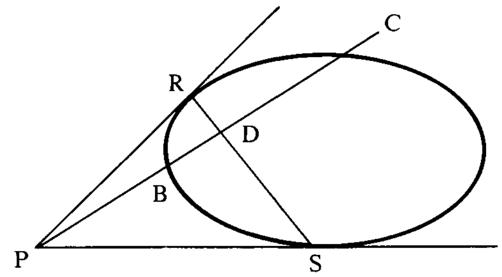



*Figure 4* Four points in involution on a conic

cutting the ellipse in two points, B and C say, and the line RS at D provides four points in involution: PBDC. Since every conic arises as a plane section of a cone on a circular base, Desargues could regard Figure 4 as a perspective image of a figure involving tangents and chords in a circle. As a result, many theorems about conics, whether ellipses, hyperbolas or parabolas, could be proved by formulating them in terms of points in involution and reducing to the simple case of the circle. In this way, Desargues's 'Brouillon project' gave the first unified treatment of the conic sections, surpassing the classical theory due to Apollonius. Some time later he contributed some theorems to his friend Abraham Bosse's work on perspective, including the lovely result that bears his name: if two triangles ABC and A′B′C′ are in perspective from a point O, and the lines AB and A′B′, BC and B′C′, and CA and C′A′ meet at the points R, P and Q respectively, then the points R, P and Q lie on a straight line (Figure 5).

As Desargues clearly saw, any theory of the projective properties of figures requires radical revision of some of the most elementary concepts of geometry, not least that of the straight line. For, by suitably positioning the image plane it is possible to cast the image of two intersecting lines as two parallel lines, and vice versa. Projective geometry cannot therefore distinguish between intersecting and parallel lines. In order to make sense of the point of intersection of two parallel lines, Desargues spoke of their intersection at infinity. This unhappy locution satisfies and repels readers in equal numbers, and was only made rigorous in the nineteenth century. Similarly, the hyperbola and parabola are given points at infinity, which is how they can be projectively equivalent to the ellipse.

Points at infinity were not the only source of difficulty in Desargues's work. His original manuscript, which circulated in fifty copies, seems to have been written with a view to the eventual production of a more polished version. Indeed, a later book may have been written, but only its title, *Leçons sur ténèbres*, survives. The original was very poorly organized,

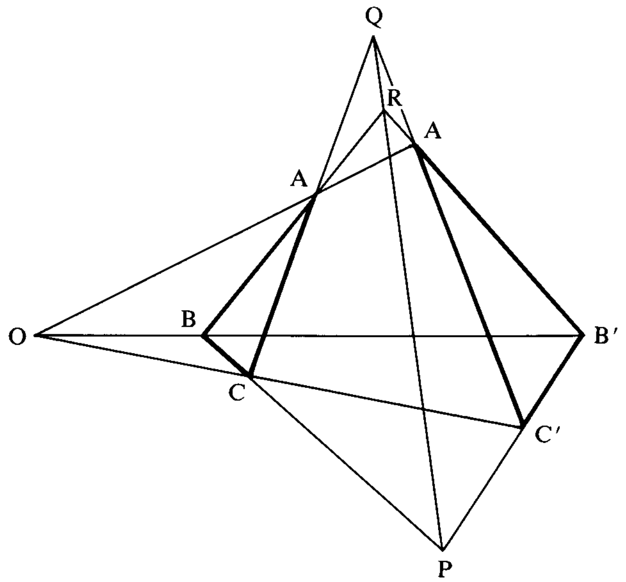



*Figure 5* Desargues's theorem

written in an opaque style and marred by misprints. Even René Descartes, a friend of Desargues, found it hard to read, and its impact was probably less than its author hoped for. But impact it had. The gifted young Blaise Pascal read it when only 16, and produced his wonderful theorem about a hexagon inscribed in a conic: if A, B, C, D, E and F are six points on a conic, and the lines AB and DE, BC and EF, and CD and FA meet at the points P, Q and R, respectively, then the points P, Q and R lie on a straight line (Figure 6). In Pascal's original presentation the special case of a circle

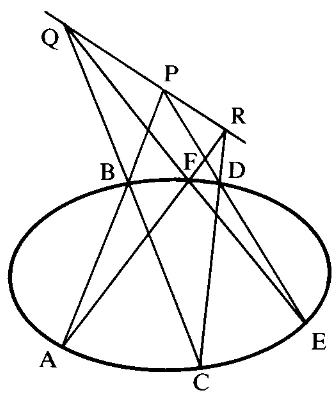



*Figure 6* Pascal's theorem

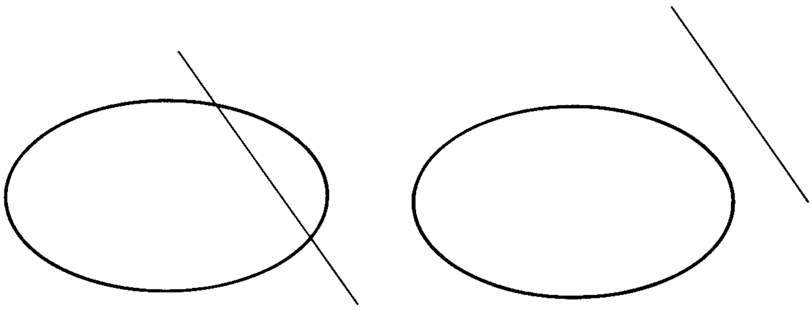
immediately precedes the general case of a conic; presumably the one was reduced to the other by a projection, but unhappily his proof for the circular case has not come down to us. Another, later author who was inspired by Desargues was Phillipe De La Hire, who had come to some of the same ideas independently. La Hire's later treatment *1685* emphasized the role of four points in involution, and is much clearer and more systematic than Desargues's.

Another who came to projective geometry was Isaac Newton, whether by reading La Hire or independently it is not possible to determine. He described a projective transformation in Book I of the *Principia* (1687), and used it to solve all the problems of the form 'given $k$ points and $5 - k$ lines, find the conic through those points and having those lines as tangents'. The most dramatic use of the idea of projective transformations, however, he reserved for his *Opticks* (1704), in which he summarized his analysis of the different curves, more than seventy of them, that can be represented by cubic equations in $x$ and $y$ by saying that the shadows of such curves were of only five different types. This cryptic remark towards the end of a difficult and lengthy analysis eventually prompted book-length explanations from J. P. de Gua de Malves *1740*, and from Patrick Murdoch *1746*.

At this juncture, projective geometry might have blossomed. A projective account of the conic sections written in Cartesian terms might have been produced, and the Newtonian projective theory of cubic curves acted as a spur to further research. But this did not happen. The influential treatises on geometry in the second half of the eighteenth century, those by Gabriel Cramer and Leonhard Euler, turned their backs on projective methods and the subject went into a fifty-year decline. When its study was resumed in the 1820s it was as an independent discovery, although French scholars did their best to make amends to Desargues, whose work had by then almost completely disappeared.

## 3 THE REDISCOVERY OF PROJECTIVE GEOMETRY

The stimulus for the rediscovery was the influential French geometer Gaspard Monge, a founder and initial organizer of the Ecole Polytechnique in Paris. Through his lectures he created a new school of geometers in France (§11.1). One of the most gifted of these, Jean Victor Poncelet, took the occasion of his capture by the Russians during the Napoleonic invasion of 1812 to rethink Monge's use of geometrical transformations. He generalized Monge's projection along parallel rays to projection along a pencil of intersecting rays, and thus rediscovered projective transformations. He also saw that in this way he could unify the study of the conics, and so obtain for geometry a level of generality that he felt had hitherto been the province of algebra (geometry being tied too closely to specific figures). His book *1822* gave a remarkably thorough treatment of the results of these insights. A problem for Poncelet, as it had been earlier for Desargues, was how to treat simultaneously those situations where a line does, and where it does not, cut a conic (Figure 7). Poncelet's solution was ingenious, but not easy to accept. Later geometers split into two camps: those who, like Poncelet, wanted to reason exclusively in geometrical terms (line, curve, intersection, involution), and those who were happier to pass into algebra. The former group, the 'synthetic geometers', following the example of C. G. C. von Staudt, came to unify Figures 7(a) and 7(b) around the concept of involution. The latter group, the 'algebraic geometers', invoked complex numbers and spoke instead of real and complex points of intersection; they were led by August Ferdinand Möbius and Julius Plücker. In particular, Plücker showed what a projective theory of cubic and quartic curves could accomplish. In a remarkable extension of these ideas towards the end of his life, he showed how one could study the geometry of all lines in space; this was a four-dimensional geometry which he and others, notably Ferdinand Lindemann, showed to be useful in the study of mechanics (§8.2).



(a) (b)

*Figure 7* The line and the conic: in contact or not?

Synthetic projective geometry produced a distinguished line of adherents attracted by its methodological purity: Jacob Steiner, a Swiss who taught for most of his life in Berlin, von Staudt in Germany, Chasles in France, Arthur Cayley in England and Luigi Cremona in Italy. To those gifted with the ability to see projective transformations (in their mind's eye), it was a suggestive field full of rich results. But the problem of points of intersection bedevilled the theory, making it unappealing to read and difficult to master. For these reasons J. L. Coolidge, the American mathematician and historian of mathematics, lamented in *1940* that it had died in the late nineteenth century. Algebraic projective geometry, on the other hand, had only to win acceptance for points with complex coordinates and the way would be open for a dramatic extension of Cartesian methods for the simultaneous study of algebra and geometry. The generation of Plücker and Otto Hesse equivocated on the issue, but complex points were firmly accepted by Bernhard Riemann throughout the 1850s. Consequently, those who wished to understand his ideas and to apply them – as he had occasionally done – to the study of curves, could bring the rigour of algebra to their study of geometry. Prominent in this endeavour were the members of the school that formed around Alfred Clebsch: above all Paul Gordan, Alexander Brill, Max Noether and Felix Klein.

Of these, all but Klein sought to recast as much as possible of Riemann's theory of complex functions in terms of the projective geometry of curves, and in this they were successful. Klein stood alone for a while in feeling that Riemann's ideas had in fact struck deeper than that, and his view was satisfactorily vindicated somewhat later by Hermann Weyl. But Klein's own original contribution lay in another direction. As a young man he was struck by the profusion of geometries that had been created in the nineteenth century: not just Euclidean and projective, but also non-Euclidean geometry and affine geometries now existed. In his famous *Erlanger Programm* of 1872, Klein showed how all these geometries could be thought of as variants of projective geometry, thus reunifying a subject he felt had become regrettably broken up. He went on to show how other geometries, such as Plücker's line geometry, could be considered, and how they too fitted his scheme of things. Klein's unification proceeded by regarding any geometry as consisting of a space of points and a group of transformations that moved the figures in the space around while preserving the properties appropriate to that geometry. So plane Euclidean geometry consists of a plane of points and has for its group all the transformations that preserve length; plane projective geometry consists of a plane of points suitably augmented by a line at infinity, and has for its group all the continuous transformations that preserve the cross-ratio of four points along a line. By the 1890s, when the idea of groups acting on spaces had become central (because of the work of Henri Poincaré and Sophus Lie), Klein's presentation, which had originally aroused little interest, came to seem prescient, and was widely seen as a forceful statement of the view (held also for example by Cayley) that projective geometry was the most basic kind of geometry. So, from being an unorthodox way to transform figures in a

Euclidean setting, projective methods had become fundamental and Euclidean ones but special cases.

In the latter half of the nineteenth century, a start was also made on the study of surfaces using projective transformations (§7.9). That topic led Giuseppe Veronese in 1880 to describe *n*-dimensional projective geometry. He was able to show that a number of difficulties in the study of surfaces had arisen because the surfaces in question were confusing projections onto three-dimensional space of simple surfaces in some higher-dimensional space. In this way he inspired mathematicians to look for simplification by passing to higher dimensions, and so promoted the study of geometry beyond the familiar space of physical intuition.

## 4 THE FOUNDATIONS OF GEOMETRY

Although geometry had become extensively reformulated by the end of the nineteenth century, with the advent of non-Euclidean as well as *n*-dimensional geometries, it retained traces of its original Euclidean state. It was still possible to draw figures and to imagine them being moved around. But even this feature changed with the publication of David Hilbert's *Grundlagen der Geometrie* in *1899*. (Hilbert's route to his discoveries is well described in Toepell *1986*.) Hilbert had been led to his ideas by a book by Moritz Pasch. Pasch had sought to give a rigorous treatment of geometry, one that would be worthy of the esteem that had been heaped upon Euclid's *Elements* (so naively, it was now thought). He discovered rules of inference used – but never made explicit – in elementary geometry, for example the one now known as Pasch's axiom: if a line enters a triangle ABC through the side AB, and does not pass through the vertex C, then it must leave the triangle either between B and C or between C and A. He sought to free the deductive side of geometry from hidden appeals to intuition, not because such appeals were flawed, but because until they were made explicit one could not be sure of the validity of a geometric argument. But he equivocated about the role of intuition in the formation of geometrical concepts such as that of the straight line, arguing that in these respects geometry was akin to the natural sciences.

Hilbert took up Pasch's ideas and found, to his surprise, that they led him not just to a philosophical critique of geometry but also to the possibility of doing geometry in a new way. By emphasizing the rules of inference to the exclusion of any consideration of the objects of geometry, Hilbert saw that one could start with certain rules, which he called axioms, and deduce their consequences. Different initial axioms would give different sets of consequences; in particular, there were sets of axioms from which particular theorems could be derived, and others from which they could not. Ingenuity in proving theorems in this new setting appealed to Hilbert, as did the production of counter-examples to conjectures suggesting that one set of axioms implied another.

In his book *1899*, Hilbert gave a set of axioms for Euclidean geometry, grouped into axioms governing incidence, order, congruence, parallelism and continuity. He then investigated the connection between geometry as he presented it and coordinate geometry. He showed that his axioms, and even suitable subsets of them, yielded an arithmetic of segments. That is, it was possible to define the length of a segment, and to add, subtract, multiply and divide these lengths. Depending on the axioms chosen, such an arithmetic need not faithfully resemble ordinary arithmetic, but Hilbert showed that if Desargues's theorem is true in a geometry then the arithmetic of segments is a (possibly non-commutative) field, and if Pappus's theorem holds then the associated arithmetic is commutative. Moreover, if in a geometry Pappus's theorem holds, then so does Desargues's, but not conversely.

Hilbert soon incorporated non-Euclidean geometry into this new setting (§7.4), and the book by Veblen and Young *1910–13* showed how all known geometries, most notably projective geometry, could be described in this spirit. The result, as Hilbert was not slow to emphasize, was that for the first time geometrical concepts could be handled rigorously without the need for any definitions. 'Line', 'point' and 'plane' could be undefined terms in a set of axioms, the axioms being such statements as (to quote from the start of the *Grundlagen*) 'for every two points there exists a line that contains each of the points', and 'for every two points there exists no more than one line that contains each of the points'. The axioms implied certain consequences, and if those consequences were those of, say, Euclidean geometry, then specific undefined terms could be interpreted as corresponding to specific Euclidean concepts. But the interpretation was not necessary, and in particular it did not intervene in the process of deduction. It was as if applied mathematicians could take a ready-made system of axioms and interpret it as they wished, secure in the knowledge that the deductions were always valid. This radical approach struck at the heart of any view that the truth of mathematics rested on the nature of its basic definitions. Other mathematicians were quick to praise this achievement, and to see that it could be extended. A. Hurwitz, for example, wrote to Hilbert to say that he had created a new field of study, the mathematics of axioms, going far beyond the domain of geometry (see §5.5, on metamathematics).

The mathematics of axioms did indeed flourish for a while, only to abate when it became clear that some systems of axioms were of very little interest. It is now usual in most areas of pure mathematics to give an axiomatic description of the basic entities. As for projective geometry, the techniques of modern algebraic geometry have given it a new lease of life. There is much interest in the study of such objects as the space of all curves of a given genus, or of all surfaces of a given kind. Such collections often form subspaces of a projective space, and so invite one to parametrize them algebraically. While the techniques are formidable, the problems addressed are once again in the mainstream of nineteenth-century geometry, and many questions once treated only intuitively have recently yielded to modern investigations.

## BIBLIOGRAPHY

Apollonius *1891–3*, *Apollonii Pergaei quae Graece existant cum commentariis antiquis*, 2 vols (ed. J. L. Heiberg), Leipzig: Teubner.

—— *1896*, *Treatise on Conic Sections* (ed. T. L. Heath), Cambridge: Cambridge University Press.

Chasles, M. *1837*, *Aperçu historique sur l'origine et le développement des méthodes en géométrie*, Brussels: Académie des Sciences.

Coolidge, J. L. *1940*, *A History of Geometrical Methods*, Oxford: Oxford University Press. [Repr. 1963, New York: Dover.]

Field, J. V. and Gray, J. J. *1987*, *The Geometrical Work of Girard Desargues*, New York: Springer.

Gua de Malves, J. P. de *1740*, *Usages d'analyse* [...], Paris: Briasson.

Hilbert, D. *1899*, *Grundlagen der Geometrie*, Leipzig: Teubner [2nd edn, transl. by L. Unger as *Foundations of Geometry*, 1902, La Salle: IL: Open Court, repr. 1971.]

La Hire, P. de *1685*, *Sectiones conicae*, Paris: Michallet.

Murdoch, P. *1746*, *Neutoni genesis curvarum per umbras*, London: Miller.

Poncelet, J. V. *1822*, *Traité des propriétés projectives des figures*, Paris: Bachelier.

Toepell, M.-M. *1986*, *Über die Entstehung von David Hilberts 'Grundlagen der Geometrie'*, Göttingen: Vandenhoeck & Ruprecht.

van der Waerden, B. L. *1961*, *Science Awakening* (transl. A. Dresden), New York: Oxford University Press.

Veblen, O. and Young, J. W. *1910–18*, *Projective Geometry*, 2 vols, Boston, MA: Ginn.

# 7.7

# *Line geometry*

DAVID E. ROWE

## 1 ORIGINS AND SOURCES

The origins of line geometry can be traced back to two sources: early nineteenth-century investigations at the crossroads of projective geometry and mechanics, and optics. Its actual inception as an independent field of research, however, came about only in 1865, the year in which Julius Plücker, returning from a twenty-year hiatus from geometry, introduced the fundamental idea of regarding the four-parameter collection of lines in space as the ground elements in a new system of geometry. Three years later, he laid the foundations for the subject of line geometry in the first volume of his *Neue Geometrie des Raumes gegründet auf die Betrachtung der geraden Linie als Raumelement* (Plücker *1868–9*). The appearance of the second volume was interrupted by Plücker's sudden death, but his young assistant, Felix Klein, completed the work and went on to undertake ground-breaking research in the field (Rowe *1989*).

Line geometry is concerned with the properties of line complexes, which are three-parameter families of lines in space satisfying an algebraic condition in the system of coordinates for the lines. Although Gaspard Monge had already found a coordinatization for the lines of space as early as 1770, and Arthur Cayley had applied these in 1860, it was Plücker who took the decisive step by investigating the complex geometrical configurations associated with line complexes. Nearly all the more fundamental notions are due to him, although much of his terminology was supplanted by others, mostly German synthetic geometers, who came after him. Plücker's fame as one of the most important contributors to modern analytic geometry and his rivalry with Jacob Steiner, for many years the leading synthetic geometer in Germany, have sometimes obscured the fact that his approach actually represented a mixture of the analytic and synthetic styles. His characteristic *modus operandi* was to strive to simplify the analytic apparatus whenever possible, typically by introducing elegant parameters that lead to a straightforward geometric interpretation.

## 2 ORIGINS IN PHYSICS

Intimately related to Plücker's line complexes are so-called ray systems, which are two-parameter collections of lines. These arise, for example, by taking the lines common to two line complexes, a type of ray system Plücker called a 'congruence'. The theory of ray systems arose in a natural way from problems in optics involving the convergence of light rays: the determination of focal points, caustic curves and caustic surfaces (in German: *Brenn-punkte, -kurven, -flächen*). Important contributions to this field came from such figures as E. Malus, William Rowan Hamilton and Ernst Kummer. It was in connection with his researches on ray systems (*Strahlensysteme*) that Kummer found that famous surface which bears his name and which contains the Fresnel wave surface as a special case (§7.1).

Just as the study of ray systems had its origins in problems that arose in optics (§2.9), so can one find a rich prehistory for the theory of line complexes in the field of mechanics. Here one should distinguish the tradition of analytical mechanics (Leonhard Euler, Joseph Louis Lagrange, Pierre Simon Laplace and others) from a geometrical tradition that included Louis Poinsot, J. P. M. Binet, Michel Chasles and August Ferdinand Möbius (Ziegler *1985*). Their collective contributions were largely concerned with the geometry of motion for rigid bodies independent of considerations of the time variable, the kinematical properties alone. In considering the motion of a rigid body not subject to net forces or torque, one can easily show that the centre of mass either remains at rest or moves uniformly. Thus, without loss of generality, one can confine the analysis to motions in which the centre of mass stays fixed. Euler gave a classical analytic treatment of this problem, deriving the equations of motion (Euler's equations) and proving that at each instant in time the body's motion consists of a rotation about a certain axis. With the exception of three special (inertial) axes, the axis of rotation will vary with time.

A similar situation arises if one holds some arbitrary point fixed: for each point P one obtains three inertial axes about which the body will rotate without wobbling. In 1811 Binet found that by varying the point P the collection of all such inertial axes forms a three-parameter family of lines – i.e. a line complex – consisting of the normals to a one-parameter system of second-degree confocal surfaces. By Dupin's theorem, exactly three mutually orthogonal confocal surfaces pass through an arbitrary point P, and these intersect one another along lines of curvature, which means that the three inertial axes through P are tangents to these lines of curvature. Considerations such as these led A. M. Ampère and others to further investigations of this kind.

## 3 POINSOT, CHASLES AND MÖBIUS

Although he appears to have made no direct references to line complexes as such, Poinsot, the veritable father of geometrical mechanics, deserves some passing mention. In his *Eléments de statique* (1803), Poinsot first showed how a system of forces acting on a rigid body in space could be uniquely resolved into a single force together with a couple whose axis is parallel to the force. His analysis revealed a close connection between systems of forces and lines in space. Later, in studying the torque-free motion of a rigid body, Poinsot held the angular momentum vector **L** fixed and noted that the motion associated with the inertial ellipsoid traces out two curves, the 'polhode' and 'herpolhode' (meaning 'snakelike'), that give a clear, intuitive picture of the motion. In effect, the polhode rolls without slipping along the herpolhode, which lies in a certain invariant plane. For a symmetrical body these curves reduce to two circles generated by the motion of the 'body cone' and 'space cone'. In the same context, Binet considered what would happen in a reference system in which **L** were no longer held fixed. This led to his discovery of the so-called 'Binet ellipsoid', which essentially describes the possible positions that **L** may assume. Its three axes are related to the stability of the body's motion (minor deviations from the largest and smallest of the axes are stable, but not those near the intermediate axis).

Systems of lines played a central role in Chasles's kinematical investigations. In 1861 he studied a direct connection between rigid-body mechanics and line geometry by noting that the collection of lines connecting corresponding points of a body in two distinct positions forms a line complex of the second degree. Another key theorem of his states that, by an appropriate choice of reference system, the general motion of a rigid body can be reduced to a rotation together with a translation along the axis of the rotation, a so-called 'screw' motion. The elaborate theory of screw motions and their intimate connection with linear line complexes was developed by Robert S. Ball, whose work was championed in Germany by Felix Klein (Ball *1900*, Klein *1892–3*). Chasles's theorem, coupled with Poinsot's characterization of force systems acting on a rigid body, forged an important connection between the analysis of such systems and infinitesimal motions. The latter were investigated by Sophus Lie in connection with non-Euclidean geometries and the Riemann–Helmholtz–Lie space problem, which deals with the geometrical properties of spaces that permit the free movement of rigid bodies.

The most probing of the numerous studies connecting systems of lines and mechanics was undertaken by August Ferdinand Möbius, whose name is largely remembered today in connection with the single-sided surface known as the Möbius strip or band (§7.10). Although he was a pioneering figure in geometry and mechanics, he spent most of his career working as an astronomer in Leipzig. In 1828 Möbius made the following key discovery: if one considers any system of forces and fixes a plane M, then the collection of lines in M with moment equal to zero all pass through a definite point, the null point of M. This led him to investigate so-called 'null systems' of lines, and these turned out to be identical to the first-degree line complexes of Plücker's theory.

## 4 THE APPROACH OF PLÜCKER AND KLEIN

Although Plücker was well aware of the numerous connections between line geometry and mechanics, he was principally concerned with the elaboration of the geometrical theory rather than its potential applications. There are several different ways to coordinatize the lines in space, but the most elegant approach represents the four-parameter family of lines in $P^3(\mathbb{C})$ by six homogeneous coordinates, $p_{ij}$. Following Hermann Grassmann, these $p_{ij}$ can be obtained by taking any two points, $(x_1, x_2, x_3, x_4)$ and $(y_1, y_2, y_3, y_4)$, on a given line $l$ and forming the six possible $2 \times 2$ determinants $\rho(p_{ij}) = x_i y_j - x_j y_i$. The $p_{ij}$ thereby satisfy the fundamental relation

$$P = p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0. \qquad (1)$$

One also has the possibility of a dual representation by taking any two planes, $(u_1, u_2, u_3, u_4)$ and $(v_1, v_2, v_3, v_4)$ that contain $l$ and forming $\sigma(q_{ij}) = u_i v_j - u_j v_i$, where the $q_{ij}$ satisfy the same identity:

$$Q = q_{12}q_{34} + q_{13}q_{42} + q_{14}q_{23} = 0. \qquad (2)$$

The coordinates $p_{ij}$ and $q_{ij}$ represent the same line if

$$\rho p_{ij} = \frac{\partial Q}{\partial q_{ij}} \quad \text{or} \quad \sigma q_{ij} = \frac{\partial P}{\partial p_{ij}}. \qquad (3)$$

An $n$th-degree line complex is determined by a homogeneous $n$th-degree equation $F(p_{ij}) = 0$. Its local behaviour can be studied by fixing a point $x = (x_1, x_2, x_3, x_4)$, so that $F(p_{ij}) = F(x_i y_j - x_j y_i) = 0$ results in an $n$th-degree equation in $(y_1, y_2, y_3, y_4)$; the lines through $x$ then determine a cone of the $n$th order. As mentioned above, the lines shared by two line complexes, $\Omega_m = 0$ and $\Omega_n = 0$, determine a 'congruence' with order and class equal to $mn$. This means that $mn$ lines pass through a typical point, and $mn$ lines lie in a typical plane. The lines of a congruence envelope a caustic surface, which generally has two components. If one adjoins a third line complex $\Omega_l = 0$, then the lines common to all three determine a ruled surface of order $2lmn$, and the intersection with a fourth complex $\Omega_o$ results in $2lmno$ lines in space.

The theory of first-degree or linear line complexes clearly illustrates the central role played by invariant theory in this field. Such complexes have the form $\Omega_1(p_{ij}) = \Sigma a_{ij}p_{ij} = 0$, which leads to a null system of lines. Following the approach taken by Plücker and Klein, the expression

$$A := a_{12}a_{34} + a_{13}a_{42} + a_{14}a_{23} \tag{4}$$

is called the invariant of the complex. Its geometric significance can be interpreted as follows: if $A = 0$, then the $a_{ij}$ satisfy the identity $P = 0$, and therefore can themselves be viewed as line coordinates. By setting $a_{ij} = q'_{ij}$, one has $\Sigma a_{ij}p_{ij} = \Sigma q'_{ij}p_{ij} = 0$, and the complex consists of all lines $p_{ij}$ that intersect a fixed line $q'_{ij}$, a so-called special linear complex.

After Plücker's death, Klein promoted both the theoretical development of line geometry and its applications to mechanics. In his doctoral dissertation of 1868, he introduced a classification scheme for second-degree line complexes based on G. B. Battaglini's earlier work, and utilizing Karl Weierstrass's new theory of elementary divisors (§6.6). Klein left it to his Erlangen student A. Weiler to work out the detailed analysis, which leads to 57 different types. Klein also established the connection between non-Euclidean line geometry, based on the Cayley metric, and non-Euclidean kinematics and statics. This became the dissertation topic of Ferdinand Lindemann, who also studied under Klein at Erlangen. Klein's own publications on line geometry had already culminated in the broad vision of geometry he set forth in his *Erlanger Programm* (Klein *1872*: see §7.6).

## BIBLIOGRAPHY

Ball, R. S. *1900*, *Theory of Screws*, Cambridge: Cambridge University Press.

Klein, F. *1872*, *Vergleichende Betrachtungen über neuere geometrische Forschungen. Programm zum Eintritt in die philosophische Facultät und den Senat der Friedrich-Alexanders-Universität zu Erlangen*, Erlangen: Deichert. [Also in *Gesammelte mathematische Abhandlungen*, Vol. 1, 460–97.]

—— *1892–3*, *Vorlesungen über höhere Geometrie*, lithograph. [3rd edn (revised and ed. W. Blaschke), 1949, New York: Chelsea.]

Plücker, J. *1868–9*, *Neue Geometrie des Raumes gegründet auf die Betrachtung der geraden Linie als Raumelement*, Vol. 1 (ed. A. Clebsch), Vol. 2 (ed. F. Klein), Leipzig: Teubner.

Rowe, D. *1989*, 'The early geometrical works of Sophus Lie and Felix Klein', in D. Rowe and J. McCleary (eds), *The History of Modern Mathematics*, Vol. 1, Boston, MA: Academic Press, 209–73.

Ziegler, R. *1985*, *Die Geschichte der geometrischen Mechanik im 19. Jahrhundert*, Stuttgart: Steiner.

# 7.8

# *The philosophy of geometry to 1900*

JOAN L. RICHARDS

## 1 EUCLIDEAN GEOMETRY

Throughout the nineteenth century, long after other classical works were rendered obsolete by the developments of the scientific revolution, Euclid's *Elements* remained the text from which many schoolchildren learned the subject. Even today, Euclidean theorems and proofs are readily accessible. In writing this remarkable text, Euclid seems to have been acting primarily as the compiler of a system of geometry which had been developing for years before he appeared on the scene. His major innovation was also the most controversial part of his book, the fifth or parallel postulate:

> That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles. (Euclid *1926*: Vol. 1, 202)

Euclid accepted this postulate without proof, despite its being so crabbed and complex in comparison to the others (e.g. 'that all right angles are equal to one another'). Much of his posterity was unwilling to do so, and, according to the fifth-century commentator Proclus, unsuccessful attempts to prove it directly date back to the appearance of his system; all these attempts ended in failure, however (§7.4).

Thus Euclidean geometry, complete with its disputed but unrivalled theory of parallels, remained constant over several millennia. However, there was always a wide variety of interpretations of its meaning and significance. From the ancients come two radically different interpretations: that which saw geometry as the most essential subject, figuring the ultimate reality underlying all superficial impressions; and that which saw geometry as a highly abstract subject, far removed from the real world of experience. These two perspectives, which can be loosely labelled as Platonic and

Aristotelian, respectively, formed the bases upon which were developed a whole range of views of the nature of geometry in the Arab and medieval worlds.

## 2 NEWTON, KANT AND SPACE

The beginning of the modern discussion of the philosophy of geometry can be tied to the work of Isaac Newton. His formulation of dynamics entailed identifying the mathematical space so well described by Euclid with the physical space in which the Sun, Earth, planets and stars move. In the Scholium to the Definitions of the *Principia* (1687), he explicitly indicated the relationship between mathematical and physical space:

> Absolute space, in its own nature, without relation to anything external, remains always similar and immovable. Relative space is some movable dimension or measure of the absolute spaces; which our senses determine by its position [with respect] to bodies; and which is commonly taken for immovable space .... Absolute and relative space are the same in figure and magnitude; but they do not remain always numerically the same.

The essential linkage which Newton here maintained between relative, physical space and absolute, mathematical space allowed him to cloak his physics with the mantle of mathematical certainty. This meant that his system was remarkably solid but, at the same time, it raised the epistemological question of how such certain knowledge of a physical reality was possible. For many philosophers, from John Locke onward, the power of Newton's physics was indubitable, and the knowledge of space it rested upon exemplified the kind of certainty to which human knowledge could aspire.

The most notable example of the philosophical power of Newtonian space is found in the eighteenth-century philosophy of Immanuel Kant. For Kant, space was a form of our pure intuition, ontologically prior to the objects we perceive as existing in it. The unique combination of the absolute certainty of geometry and the substantial reality of physics made Newtonian spatial knowledge the quintessential exemplar of Kant's category of synthetic, *a priori* knowledge.

## 3 NON-EUCLIDEAN GEOMETRIES

From a geometrical standpoint, when he spoke of space Kant meant the space of Euclid's *Elements*. Even as he and others were incorporating the certainties of this geometry into their philosophies, however, mathematicians were re-examining the geometrical bases for such claims. The attempt to clarify geometry's certainty led them again to consider the status of Euclid's fifth postulate.

Recognizing the futility of trying to prove the postulate outright, the Italian Jesuit Girolamo Saccheri spearheaded the eighteenth-century movement to prove it indirectly by indicating the depths of contradiction entailed in assuming it to be false. Focusing on a quadrilateral whose base angles were assumed to be equal and right, Saccheri divided the various alternatives to Euclid's geometry into clearly demarcated categories. He was able to prove quite easily that the two remaining angles were equal to each other. From there he went on to show that if they were both right angles, Euclid's theory of parallels would hold; if, on the other hand, the remaining angles were either acute or obtuse, then the theory would not hold. Thus, in order to prove the necessity of Euclidean geometry, Saccheri set out to demonstrate the impossibility of both the acute-angle and the obtuse-angle hypothesis (Bonola *1912*: 22–44; Gray *1979a*: 51–62).

Saccheri apparently believed he had succeeded in this endeavour; other mathematicians who followed him, like Johann Heinrich Lambert, seemed more ambivalent about the implications of their results. Whatever their conclusions, however, efforts to generate contradictions from alternative postulates led to a number of geometrical theorems developed from systems with non-Euclidean parallels.

In the early decades of the nineteenth century these scattered results began to bear fruit; Ferdinand Schweikart and Carl Friedrich Gauss privately, and Nikolai Lobachevsky and János Bolyai more publicly, developed self-consistent 'non-Euclidean' geometries (§7.4). These men recognized that their success in creating alternatives to Euclid's geometrical system raised significant questions about the exact certainty which had been ascribed to Newtonian space; whether physical space actually was Euclidean became for them an empirical issue as yet undetermined (Bonola *1912*: 84–121; Gray *1979a*: 96–116).

## 4 RIEMANN AND HELMHOLTZ

Despite their importance, these mathematical developments had remarkably little immediate impact. They were not widely discussed, either philosophically or mathematically, until the 1860s. In 1863 Gauss's correspondence about non-Euclidean geometry was published for the first time; the rest of the decade saw the publication of a French edition of Lobachevsky's work, and Bernhard Riemann's 'Über die Hypothesen, welche der Geometrie zu Grunde liegen' (*1867*).

This last was Riemann's *Habilitationsvortrag*, initially delivered in Göttingen in 1854. Whereas Lobachevsky, Bolyai and their contemporaries were 'synthetic geometers' who constructed alternative spaces based on non-Euclidean assumptions, in this work Riemann worked analytically. In his view the concept of space was a particular instance of a more general concept, that of multiply extended magnitude. The identifying mark of space within this broader category was that a metric is defined on it:

> Hence [it] follows as a necessary consequence that the propositions of geometry cannot be derived from general notions of magnitude, but that the properties which distinguish space from other conceivable triply extended magnitudes are only to be deduced from experience. (Riemann *1867*: 272–3).

He went on to establish that Euclidean space was characterized by the particular distance function, $ds = (dx^2 + dy^2)^{1/2}$, but that other functions would generate alternative spaces. He further speculated that space might not even have a single, constant distance function, but that the measure of distance might be different for the infinitely large or small.

Riemann did not try to specify the actual experiences which might ground the Euclidean spatial concept; he saw this as 'a problem which from the nature of the case is not completely determinate, since there may be several systems of matters of fact which suffice to determine the measure-relations of space' (*Ibid.*: p. 273). The issue seemed more determinate to the German physicist Hermann von Helmholtz, who was initially drawn into geometrical discussion by studies of vision. In an attempt to justify his conviction that visual space had its roots in childhood experiences, Helmholtz set out to identify them. He settled on the motions of rigid bodies, which all infants have experienced by the end of their first year, and showed how the basic structures of Euclidean space could be generated from them.

Before he had quite finished his first paper on this subject, however, Helmholtz read Riemann's newly published work. This was his first introduction to non-Euclidean geometries, and Helmholtz quickly realized that they could also be constructed from his rigid-body motions. In subsequent papers Helmholtz offered these motions as the 'matters of fact' on which Riemannian measure relations were based. At the same time, he used the possibility of non-Euclidean geometries as the basis for a frontal attack on the so-called 'nativist' physiologists who maintained that Euclidean space is a physiologically determined concept. The fact that we could have constructed several different geometries, and yet see only one, indicated to Helmholtz that we have a choice which we make based on experience. This argument was easily extended to challenge philosophies which, like Kant's, asserted that space is known intuitively and absolutely without contingent empirical input (Richards *1977*; Torretti *1978*: 155–71).

Helmholtz's argument and analysis were clear and persuasive; his writings served to introduce many mathematicians as well as others to non-Euclidean ideas. His empirical conclusions were not universally embraced, however. Philosophers who were committed to the necessity of geometrical truth, like the Dutchman J. P. N. Land, countered his arguments by focusing on the nature of conceivability. The ability to describe a space mathematically, they argued, was not enough to render it truly conceivable; in their view, despite the mathematical alternatives Helmholtz had constructed, Euclidean space remained uniquely certain because of its conceptual status. This line of argument led away from mathematics into discussions of the nature of conception.

## 5 PROJECTIVE GEOMETRY

A related development grew out of yet another mathematical approach to non-Euclidean geometries, through projective geometry (§7.6). Initially rooted in techniques of technical drawing developed after the French Revolution by Gaspard Monge, Jean Victor Poncelet, and others, projective geometry focused on those aspects of figures which remain constant throughout the transformations of projection and section (Daston *1986*). Metric relations are not invariant in this way and all lines intersect at infinity, so, initially, there was no relation between projective and non-Euclidean ideas. However, in his 'Sixth Memoir on Quantics' (1859), Arthur Cayley developed from projective properties a function which displayed the defining characteristics of a Euclidean distance function. Having shown in this way how metric Euclidean space may be generated from the less structured projective space, Cayley concluded: 'Metrical geometry is thus a part of [projective] geometry, and [projective] geometry is *all* geometry, and reciprocally' (Richards *1988*: 130). Cayley completed this work before he knew of non-Euclidean geometry; but in the wake of the non-Euclidean publications of the 1860s the German mathematician Felix Klein showed that non-Euclidean metrics could also be generated within projective spaces. With this insight, he opened a powerful new projective approach to non-Euclidean geometry.

Projective geometry held a great deal of mathematical interest in the final decades of the nineteenth century, and for some mathematicians who were committed to the necessity of geometrical truth it held out a solution to Helmholtz's empirical challenge. By emphasizing the conceptual primacy of projective space over any metric spaces, they were able to preserve the necessity of geometrical knowledge, albeit now of projective rather than Euclidean space; in this view the choice among metric geometries might be contingent, but knowledge of the projective space in which they were embedded was not. Bertrand Russell's book of 1897, *An Essay on the*

*Foundations of Geometry*, marks the culmination of this tradition (Richards *1988*: Chap. 5).

## 6 CONVENTIONALISM AND AXIOMATICS

In the last decade of the nineteenth century, the French mathematician Henri Poincaré developed yet a different approach to geometry. He focused on the structural equivalences among the various geometries, and claimed that choices among them could not be physically determined; rather, he emphasized, they were the result of decisions about which was the most convenient (*'la plus commode'*) (Torretti *1978*: 320–56).

In Poincaré's conventionalist view, the category of necessary truth is basically irrelevant to geometry; in this he was feeding into a movement towards an axiomatic or formal view of geometry which had been growing since the middle of the century. The 1844 *Ausdehnungslehre* of Hermann Grassmann was an early attempt to avoid the epistemological quagmire surrounding geometry by developing a geometrical structure which did not rely at all on intuitive content (§6.2). In 1882, Moritz Pasch published a series of 'Lectures on Modern Geometry' with a similar intent. Starting from a carefully circumscribed set of undefined concepts drawn from experience, Pasch tried to erect a purely deductive, non-conceptual geometrical structure. In his words:

> If geometry is to be truly deductive, the processes of inference must be independent in all its parts from the meaning of the geometrical concepts, just as it must be independent from the diagrams. All that need to be considered are the relations between the geometrical concepts, recorded in the propositions and definitions. (Torretti *1978*: 211)

The Italian Giuseppe Peano went a step further by not only proceeding from a strictly limited number of geometrical terms, but also creating an artificial language as a way of controlling the logical content of theory. It was under the influence of his ideas that Russell abandoned the kind of work found in his early *Essay*, and moved towards the logicism of *Principia mathematica* (1910–13; see §5.2).

Thus, by the end of the nineteenth century the mathematical understanding of geometry had changed considerably. With these developments came challenges to the claims of special, necessary truth which had underlain the subject for two centuries. The philosophical interest of the subject changed accordingly, and the twentieth century has opened a whole new chapter in its development.

## BIBLIOGRAPHY


Bonola, R. *1912*, *Non-Euclidean Geometry. A Critical and Historical Study of its Development*, Chicago, IL: Open Court. [English transl., with additional appendices, by H. S. Carslaw, repr. 1955, New York: Dover.]

Daston, L. J. *1986*, 'The physicalist tradition in early nineteenth century French geometry', *Studies in the History and Philosophy of Science*, **17**, 269–95.

Euclid *1926*, *The Thirteen Books of Euclid's Elements* (introduction and commentary by T. L. Heath), 3 vols, Cambridge: Cambridge University Press. [Repr. 1956, New York: Dover.]

Gray, J. J. *1979a*, *Ideas of Space: Euclidean, Non-Euclidean, and Relativistic*, 1st edn, Oxford: Oxford University Press. [2nd edn 1989.]

—— *1979b*, 'Non-Euclidean geometry – A re-interpretation', *Historia mathematica*, **6**, 236–58.

Gray, J. J. and Tilling, L. *1978*, 'Johann Heinrich Lambert, mathematician and scientist, 1728–1777', *Historia mathematica*, **5**, 13–41.

Helmholtz, H. *1977*, *Epistemological writings* (transl. by M. F. Lowe, ed. with an introduction and bibliography by R. S. Cohen and Y. Elkana), Dordrecht: Reidel.

Nowak, G. *1989*, 'Riemann's *Habilitationsvortrag* and the synthetic *a priori* status of geometry', in D. Rowe and J. McCleary (eds), *The History of Modern Mathematics*, Boston, MA: Academic Press, Vol. 1, 17–46.

Richards, J. L. *1977*, 'The evolution of empiricism: Hermann von Helmholtz and the foundations of geometry', *British Journal for the Philosophy of Science*, **28**, 235–53.

—— *1988*, *Mathematical Visions: The Pursuit of Geometry in Victorian England*, Boston, MA: and San Diego, CA: Academic Press.

Riemann, G. F. B. *1867*, 'Über die Hypothesen, welche der Geometrie zu Grunde liegen', in *Gesammelte mathematische Werke*, 2nd edn, 1892, 272–87.

Rougier, L. *1920*, *La Philosophie géométrique de Henri Poincaré*, Paris: Alcan.

Scholz, E. *1982*, 'Herbart's influence on Bernhard Riemann', *Historia mathematica*, **9**, 413–40.

Torretti, R. *1978*, *Philosophy of Geometry from Riemann to Poincaré*, Dordrecht: Reidel.

# 7.9

# *Early modern algebraic geometry*

J. J. GRAY

Algebraic geometry is the application of algebra to the study of geometry. As such, its origins lie in the invention of coordinate or analytic geometry by René Descartes in the 1630s (§7.1). His work began a tradition whereby the algebra of polynomial expressions was applied to the study of curves defined by equations relating their coordinates. Over the subsequent centuries both the algebraic and the geometrical sides developed in sophistication, but a crucial aspect of the modern period has been the development of newer kinds of algebra to solve geometrical problems, as is described with his customary briskness and mathematical clarity by Dieudonné *1974*. These new problems often involve three or more variables, and often arise in contexts which are not exclusively geometric, such as the study of complex functions. Most influentially, this was the case with the work of Bernhard Riemann.

## 1 RIEMANN

The history of algebraic geometry from the work of Riemann onwards can be regarded as a dialogue between two approaches, one emphasizing the geometry and the other the algebra. In two major papers of 1851 and 1857 (see his *1990*), Riemann showed how the ideas of complex-function theory could be extended and applied to study integrals on an algebraic curve (later also to be called a 'Riemann surface'). The central theorem in his presentation of the subject is Niels Abel's theorem concerning integrals on such a curve. This asserts that there is a number, $q$, determined by the curve, such that the sum of any number of integrals (with the same integrand), having a fixed initial point but variable end-points, is the sum of some rational functions and some logarithmic terms minus exactly $q$ integrals having the same integrand. Moreover, the end-points of these integrals depend algebraically on the given variable end-points.

Riemann simplified this theorem by considering the integrands. If they have simple poles (if, that is, they look like $dz/z$ near the origin), then logarithmic terms will arise if the path of integration winds around the pole. If there are poles of higher order (terms like $dz/z^n$, $n > 1$) then a rational function can be expected. But if the integrand is 'holomorphic' (i.e. has no poles) then neither of these terms can arise, and Abel's theorem becomes the statement that the sum of any number of integrals having a fixed initial point but variable end-points, is equal to minus exactly $q$ integrals having specified integrands and whose end-points depend algebraically on the given variable end-points. The number $q$ was given a novel interpretation by Riemann (it had been left unexplained by Abel): he showed that the original algebraic curve could be regarded as a real surface upon which a system of $2q$ (but not $2q + 1$) cuts can be made, leaving the dissected surface still in one piece. He then showed that, as a result, there are exactly $q$ linearly independent holomorphic integrands.

Riemann's approach was a difficult mixture of the profound and the naive, and to make sense of it the next generation of German mathematicians sought to rewrite it in a language they better understood – that of algebra (Gray *1989*). Alfred Clebsch (the leader of this movement) and his colleague Paul Gordan found Riemann's topological arguments obscure, and the deduction about the number of integrands, resting as it did on Dirichlet's principle (§3.17), they found unacceptable. So they based their book of *1866* on the idea that a complex curve is a locus defined by a polynomial equation in two complex variables, $x$ and $y$ (such a curve is called an algebraic curve), and went back to Abel's explicit presentation of the holomorphic integrands. Abel had written them in the form $\int_0^z g\,dx/F_y$, where $F(x, y) = 0$ is the equation of the curve, $F_y$ is the partial derivative of $F$ with respect to $y$, and $g = g(x, y)$ is a polynomial of low enough degree for the quotient $g\,dx/F_y$ to be holomorphic. He had then obtained the number $q$ by counting the number of possible terms in $g$. If the curve $F = 0$ has no singular points and is of degree $n$, then this number is $q = (n - 1)(n - 2)/2$.

Clebsch and Gordan used geometry to understand the way in which the end-points enter Abel's theorem. They considered families of algebraic curves depending linearly on some parameters, and cutting the curve in the end-points of the given integrals. It is then clear that the remaining points of intersection of the original curve and these variable ones are determined by algebraic conditions. Problems arose with this approach when the curve could have singular points, self-intersections or cusps. It was possible to show that each of $d$ double points and $r$ cusps lowers this number $q$ by 1.

This meant that Plücker's formulas from projective geometry could be used to show that $\frac{1}{2}(n-1)(n-2)-d-r$ was invariant under any projective transformation of the curve, and could therefore be taken as the definition of $q$. But for curves with more complicated singularities the definition of $q$ was not clear. Alexander Brill and Max Noether, following the work of Clebsch, then sought to show that birational transformations could always be found to reduce a curve to one having singularities which gave only a straightforward contribution to the number $q$. In this way they gave a definition of $q$ which was indeed birationally invariant. This they did by insisting that the variable curves through the singular points have the right degree and passed through the singular points the right number of times ($n-1$ times through a singularity of order $n$). Riemann's topological approach was replaced with a geometrical definition of $q$ (called the genus by Clebsch) which was, however, valid for both singular and non-singular curves. Then in 1880 Richard Dedekind and Heinrich Weber put forward a remarkable theory of Riemann surfaces which showed how a complex curve could be studied entirely in terms of the families of meromorphic functions it supported.

## 2 HIGHER DIMENSIONS

For a long time the path to studying geometrical objects other than curves (for example, surfaces) seemed blocked, until an approach was developed by Leopold Kronecker and extended by David Hilbert which provided a rich, uniform treatment and offered the securities of algebra in exchange for the insights of geometry. Kronecker's work was but a part of what he hoped would be a theory embracing algebraic number theory as well as geometry. In the event, he put forward the important definition of an algebraic variety as the set of points in $\mathbb{C}^n$ satisfying a family of polynomial equations, and showed that each such variety could be decomposed into a number of irreducible pieces, each having a definite dimension. He observed that the variety depends only on the ideal in the polynomial ring $C[x_1, x_2, \ldots, x_n]$ that the given polynomials define, and showed that if the ideal is prime then the variety it generates is irreducible. (An ideal $a$ in a ring $R$ is a set such that $u, v \in a$ implies that $u+v \in a$, and $u \in a$ and $r \in R$ imply that $ru \in a$; it is prime if $uv \in a$ implies either $u \in a$ or $v \in a$; see §6.4, Section 2.2.)

Hilbert's profound contributions were part of his reformulation of invariant theory in the late 1880s (§6.8). Above all, he investigated the polynomials that vanished at every point of a variety, and showed that if $f$ has this property then some power of $f$ belongs to the ideal defining the variety. This is Hilbert's *Nullstellensatz* ('theorem of the zeros'). Powers of $f$ need to be taken in case the variety has multiple points. It means that one may study either the variety or the so-called radical of the ideal defining it, where the radical of an ideal $a$ is the set of all $u \in \mathbb{R}$ such that $u^k \in \mathbb{R}$ for some $k$.

## 3 MODERN ALGEBRA

Kronecker's work opened the way for the transcription of geometrical properties of varieties into a new algebraic language, that of commutative algebra. In this spirit, the type of ideal that an irreducible variety generates was first elucidated in 1905 by Emanuel Lasker (who later became the world chess champion). The important concept of a field was axiomatized by Ernst Steinitz, who in 1910 isolated the concepts that underlay a profusion of examples previously discussed by Heinrich Weber in his influential book on algebra (§6.4, Section 2). He also defined the concept of the transcendence degree of one field over a field that it contains, which counts the number of linearly independent elements of the larger field, and connected it to Kronecker's definition of the dimension of a variety. The connection is provided by the idea of the coordinate ring of a variety (loosely, the polynomials defined on the variety). It turns out that every geometric property of a variety is expressible as an algebraic property of its coordinate ring. In particular, the construction of the coordinate ring yields a field of transcendence degree $d$ over $\mathbb{C}$, where $d$ is the dimension of the variety.

## 4 GEOMETRICAL RESPONSES

A programme that aimed to solve problems in geometry by translating them so completely into the language of commutative algebra naturally did not appeal to geometers, nor did everyone welcome the drift away from the questions that had so profitably animated Abel and Riemann. It seemed to geometers of the flourishing Italian school in particular that at least algebraic surfaces could be tackled more geometrically, and concentrated effort was devoted to this problem from 1893, when two decisive papers were published. One, by E. Bertini, summarized the Brill–Noether approach to curves. The other, by Corrado Segre, reformulated that theory in a way that invited generalization to higher dimensions. What Bertini described via families of curves through certain points on a given curve, Segre sought to describe in terms of certain sets of points on the curve. In particular, Segre moved away from Bertini's description of a curve as a locus in the plane and towards Riemann's hazy appreciation of a curve as an object to be studied intrinsically, without reference to any particular embedding in an ambient space.

Segre's paper opened the way to a series of papers by Guido Castelnuovo and Federigo Enriques over the next twenty years that culminated in a classification of algebraic surfaces analogous to Riemann's classification of complex curves according to their genus. They also gave complete descriptions of several families of curves, analogous to the rich picture one has of curves of low genus. Meanwhile, Emile Picard had been working methodically on a generalization of Riemann's function-theoretic ideas to complex surfaces. The result was an analysis of surfaces according to the kinds of function they can support (as described by their singularities and their integrals) that exactly complemented the Italian theory of surfaces and the families of curves they can have. When Picard wrote up his findings in the definitive form of his book with F. Simart (*1897*, *1906*) this overlap was often alluded to, and in a lengthy appendix Castelnuovo and Enriques set out the main features of their approach. The roles were reversed in their article on the algebraic geometry of surfaces in the *Encyklopädie der mathematischen Wissenschaften* (*1908*), in which the Italians showed how Picard's discoveries fitted into their approach.

## 5 THE NEED FOR RIGOUR

For good reasons, the Italian theory was soon felt to lack rigour. It was criticized from within most decisively by Francesco Severi for relying too heavily on intuition, and even Castelnuovo came to feel that progress required other methods. For this reason he encouraged his student O. Zariski in Rome, saying to him, however, 'You are here with us, but you are not of us'. Zariski's book *Algebraic Surfaces* (1935) was a turning point for him. He went as far as he felt he could to present the ideas underlying the Italian approach, but at the price, he later wrote, of feeling compelled to leave the geometric paradise for the greater rigour of algebra. The problem was the singular points and curves of an algebraic surface, which can be very complicated. He therefore turned to the ideas that E. Krull, and Emmy Noether and her student B. L. van der Waerden, were developing in the theory of rings.

Meanwhile, complex algebraic geometry developed under the forceful guidance of Solomon Lefschetz in the direction of algebraic topology. Picard and Simart's book had been a great source of inspiration for Lefschetz, but they had not been able to draw on Henri Poincaré's visionary ideas about topology (§7.10). It was Lefschetz's contribution to do just that, in forging an algebraic topology of algebraic varieties of higher dimension. This theory was consummated by W. V. D. Hodge's theory of harmonic forms, which represents the subtle generalization to higher dimensions of Riemann's insight into the intimate connection between holomorphic and harmonic functions.

Throughout the period, other questions also motivated algebraic geometers. It must suffice here to describe just two of these. A remarkable calculus (the enumerative calculus) had been developed for counting the number of curves touching a given set of curves. Proposed by Michel Chasles, and developed by H. Schubert and others, notably H. G. Zeuthen, in the later nineteenth century, the move towards rigour in geometry found this theory almost incomprehensible. Only recently has it been rescued from oblivion and several of Schubert's remarkable counts confirmed. The study of particular surfaces was a continuing source of discovery. The 27 lines on a cubic surface, discovered by Arthur Cayley and counted by George Salmon, have yielded many gems, including a connection with the Weyl groups of exceptional Lie groups, while Ernst Kummer's quartic surface with its 16 nodal points, which has its origins in Augustin Jean Fresnel's theory of optics (§9.1), has recently yielded the discovery of non-standard differentiable surfaces on $\mathbb{R}^4$.

## 6 A GLIMPSE OF THE MODERN SITUATION

Some of the problems in algebraic geometry today curiously resemble those that inspired Kronecker. In the hands of A. Grothendieck, a language has been developed specifically to move algebraic geometry beyond its reliance on field-theoretic concepts, to give rings equal weight. This amounts to passing from the study of the rational numbers to the integers, almost always a considerable step, and it has brought to the problems of algebraic number theory the resources of modern geometry, often with considerable success. Grothendieck shares with Zariski and others an unwillingness to accord the complex numbers any special preference, but some problems remain that have finally been solved only over the complex numbers. For example, in 1966 the Japanese mathematician H. Hironaka showed how singularities of any kind may be resolved, but only over the complex numbers and certain related fields (those of characteristic zero). This theory permits one to pass in a controllable way from a singular variety to a non-singular one related to it. Sometimes even this theory does not suffice; the recent successful classification of algebraic varieties of (complex) dimension 3 requires an analysis of certain mildly singular cases, unlike the Enriques classification in dimension 2.

Since the 1930s there has been a fruitful tension between the half of algebraic geometry that sought to stay near the intuitively accessible complex varieties and the half that, attracted by the generality of algebra, sought to deal with objects defined over any field (or ring). This tension is perhaps at its most productive in its analysis of the so-called problem of moduli. Riemann had observed that curves of a given genus $g > 1$ seemed to form a family of complex dimension $3g - 3$. In other words, a curve of genus $g$ could be specified uniquely by choosing $3g - 3$ parameters or moduli. Despite some early attempts to describe the space of all parameters (for a given value of $g$), no progress was made on this problem until the work of K. Kodaira and D. C. Spencer in the 1950s. What they began for the case of complex curves and certain types of surface was extended by David Mumford for other examples and broadened to include fields of arbitrary characteristic. Most unexpectedly, the hope today is that these theories will play an essential part in rigorizing the theory of the Feynman integral in quantum-field theory, which has already begun in its turn to produce remarkable insights in the difficult territory of low-dimensional topology.

## BIBLIOGRAPHY

Clebsch, A. and Gordan, P. *1866*, *Theorie der Abelschen Functionen*, Leipzig: Teubner. [Repr. 1967, Würzburg: Physica.]

Dieudonné, J. *1974*, *Cours de géométrie algébrique*, Paris: Presses Universitaires de France.

Enriques, F. and Castelnuovo, G. *1908*, 'Grundeigenschaften der algebraischen Flächen', in *Encyklopädie der mathematischen Wissenschaften*, Vol. 3, Part 2, 635–73 (article III C 6a).

Gray, J. J. *1989*, 'Algebraic geometry in the late nineteenth century', in D. Rowe and J. McCleary (eds), *The History of Modern Mathematics*, Boston, MA: Academic Press, Vol. 1, 361–85.

Picard, E. and Simart, F. *1897*, *1906*, *Théorie des fonctions algébriques de deux variables indépendantes*, 2 vols, Paris: Gauthier-Villars. [Repr. 1971, New York: Chelsea.]

Riemann, B. *1990*, *Gesammelte mathematische Werke*, 3rd edn, New York: Springer.

Schubert, H. *1979*, *Kalkül der abzählenden Geometrie* (reprint of original 1879 edition, with an introductory essay by S. L. Kleiman), New York: Springer.

# 7.10

# *Topology: Geometric, algebraic*

E. SCHOLZ

The move from classical to modern geometry was linked with a reorganization of geometrical thought around new concepts which were much more general than the traditional concept of space. This change opened new reference fields for geometry within mathematics, and created a whole network of methods which gave rise to new geometrical subdisciplines. Most important in this respect, certainly for the nineteenth century and with its strong impact in the twentieth, was the new concept of manifold, introduced by Bernhard Riemann in his inaugural lecture of 1854 (Scholz *1980*: Chap. 2). One of the striking new features of Riemann's approach was the investigation of global properties of manifolds: *analysis situs*, in his terminology, now better known as topology.

## 1 RIEMANN'S APPROACH TO GEOMETRY

### 1.1 Manifolds

Riemann started his inaugural lecture with a semi-philosophical introduction of the concept of a multiply extended magnitude, or manifold. By this he meant a (somehow given) general concept, the individual instances of which allow continuous transitions between one another and a locally bijective correspondence to a finite (or in some cases even infinite) number of real quantities $x_1, x_2, \ldots, x_n$ which may serve as coordinates for the description of the points in the manifold (of dimension $n$).

Riemann initiated investigations of manifolds on different levels. He distinguished metric-free investigations from those using differential geometrical metrical concepts as the two main research areas for manifolds. Of the investigations which are free from metrical considerations, he emphasized particularly those of *analysis situs* (topology), although in his function-theoretic work he also studied the analytical and/or algebraic