THE CONCEPTUAL ROOTS OF MATHEMATICS

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J. R. LUCAS

ROUTLEDGE

THE CONCEPTUAL ROOTS OF MATHEMATICS

The Conceptual Roots of Mathematics is a comprehensive study of the philosophy of mathematics, which is—as the book shows—central to epistemology and metaphysics.

J.R.Lucas argues for a 'chastened Logicism'. He claims that though mathematical arguments are a priori, they are not all always deductive. Our mathematical concepts are grounded in logic, although often developed to something essentially new.

Mathematical argument is viewed in a new way, as a two-person dialogue rather than a formal proof-sequence. Peano's Fifth Postulate and the Axiom of Choice can be justified if we construe them not as would-be truths, but as principles of argument between two rational seekers after truth. Geometers will be relieved to learn that geometry is neither a branch of physics nor a purely formal exercise, and will be intrigued by a 'consumers' guide' to geometry that recommends Euclid's as the Best Buy.

J.R.Lucas is a Fellow of the British Academy. He was Fellow and Tutor of Merton College, Oxford, and is the author of several books, including *A Treatise on Time and Space, Space, Time and Causality* and (jointly) *Spacetime and Electromagnetism*. He has written extensively on the implications of Gödel's theorem, and his article 'Minds, Machines and Gödel' has been much attacked and much defended.

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An essay on the philosophy of mathematics

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London and New York

First published 2000
by Routledge
11 New Fetter Lane, London EC4P 4EE

Simultaneously published in the USA and Canada
by Routledge
29 West 35th Street, New York, NY 10001

Routledge is an imprint of the Taylor & Francis Group

This edition published in the Taylor & Francis e-Library, 2002.

British Library Cataloguing in Publication Data
A catalogue record for this book is available from the British Library

Library of Congress Cataloging in Publication Data
Lucas, J.R. (John Randolph), 1929–
The conceptual roots of mathematics/J.R.Lucas.
p. cm.
Includes bibliographical references and index.
1. Mathematics—Philosophy. I. Title.
QA8.4.L833 1999 99–20193
510′.1–dc21 CIP

ISBN 0-203-02842-2 Master e-book ISBN

ISBN 0-203-20274-0 (Adobe eReader Format)
ISBN 0-415-20738-X (Print Edition)

Introduction

The *Roots?* For many years I lectured in Oxford on the Conceptual Foundations of Mathematics; but, as I did so, I became increasingly uncomfortable with Descartes' metaphor of an edifice for which foundations were the appropriate basis. Pedagogically it was bad: it meant spending a lot of time on recondite logic-chopping, the point of which could not be understood until much later in the course, when we were on to proper mathematical topics. Metaphysically it was dubious, too. If I had been as clever as Archimedes, I might have been able to move minds, given a firm starting point, but no firm starting point was, or, as I came to believe, could, be given. To start from the absolutely incontestable was to start from nowhere. Here was where we were, and there was where we might be, but any journey of exploration or justification had to start from some point or other, and any actual starting point could be put in question by a skilful sceptic.

I worked through these difficulties in my British Academy Lecture, "Philosophy and Philosophy Of", and there suggested that another of Descartes' metaphors was more appropriate: we should think of mathematics not as a building based on foundations, but as a tree grounded in the soil of our general conceptual structure, and growing both up and down.[1] This metaphor, too, is inadequate, but less so than the building one, and so I adopt it in the title for this work.

I start and end with Plato. He was right to distinguish mathematical arguments from empirical ones, but failed in his various attempts to characterize them positively. He was his own severest critic, and outlined, as alternatives to Platonism, approaches we can recognise as proto-formalist and proto-logicist.

The paradigm mathematical discipline was geometry, and in Chapter Two I argue that its axioms cannot be taken as being

[1] *Descartes: the Philosophical Works,* tr. E.S.Haldane, and G.T.R.Ross, i, 2nd edn., Cambridge, 1934, p.211; cited by Bernard Williams, *Descartes,* Penguin, 1978, p.34.

simply self-evident, and cannot be merely postulated either. Geometry is not just a formal discipline, but is subject to conceptual constraints arising from its place in our conceptual structure, and the accompanying operational needs. Various geometries are possible, but a *Which?* guide to geometry recommends Euclid's as the Best Buy.

Formalism is considered more fully in Chapter Three. It has its uses, not only for checking arguments, and revealing fallacies, but for revealing similarities between different branches of mathematics, and as a topic of study in itself. Nevertheless, Formalism offers an inadequate account of mathematical knowledge. It secures validity at the price of vacuity, and does not at all conform to the actual practice of mathematicians.

A logicist approach to the natural numbers is developed in Chapters Four, Five and Six. Frege's account of the cardinal numbers as answers to the question "How many?" reveals the similarity between the number nought and the negative existential "quotifier" (negative existential quantifier, as it is generally called). Dedekind's ordinal approach lights up the systematic structure of the natural numbers, which can be grounded uniquely if we take nought as the first number, the number than which there can be no fewer, so that counting down is always a finite procedure. Peano's abstract approach raises the converse problem of counting forwards. I suggest that Peano's Fifth Postulate (and the Axiom of Choice) should be construed as principles of argument between two rational seekers after truth, rather than assertions of arcane truth. Once the principle of recursive reasoning, as I call it, is accepted, we can think of all the natural numbers, of which, as the Schoolmen often observed, there are an infinite number. Different concepts of infinity, with their associated difficulties, are discussed in Chapter Seven. The Intuitionists, like Aristotle, reject actual infinites, allowing only the potential infinite. But their "selective scepticism" is unstable, being vulnerable to more extreme scepticism on the part of finitists and ultra finitists; and any arguments they can adduce against these can in turn be used by classical logicians against Intuitionist scruples.

Infinite sets can be mapped into proper subsets of themselves, and hence a formal system that includes the natural numbers can be so coded that statements about it can be represented within it. Gödel was thus able to construct a well-formed formula somewhat analogous to the Liar Paradox, and in Chapter Eight Gödel's theorem is explained, and its implications, which greatly alter our understanding of logic and logical argument, discussed.

Traditional Logicism has had too narrow a view of logic, and has sought to base the whole of mathematics on the natural numbers and set theory. In Chapter Nine I seek to show how much of mathematics is, instead, grounded in the logic of transitive relations. Equivalence relations give us group theory and category theory; ordering relations give rise to discrete, dense and continuous orderings, as well as to lattices and trees. Some interconnexions between equivalence and ordering relations are sketched, and two paradigms exhibited, set theory, seen as a structure rather than ontologically, and mereology, the logic of the part-whole relation, to which mathematicians have not given the attention it deserves.

Whitehead hoped to base geometry and topology on a mereological foundation, and though his project failed, it is illuminating to see how far we can go in developing "Prototopology", in order to have available the concept of extensive magnitude, which we need for a proper theory of measurement, discussed in Chapter Eleven.

Chapter Twelve is called, provocatively, "Down with Set Theory". Not that I have anything against sets—indeed some of them I have made use of myself. But sets have been accorded undue significance in the philosophy of mathematics, and I want to cut them down to size. In part it is merely a matter of nomenclature: what passes for set theory is really transfinite arithmetic, a fascinating subject of great interest, but beyond the horizon of most mathematicians. If we view it as a relational structure, we can compare the Axiom of Extensionality with the Identity of Indiscernibles, and see the Axiom of Foundation as a paradigmatic stipulation rather than a fundamental ontological fact. Not all the problems of set theory are thus easily dissolved, and much mystery remains. But less.

Philosophers of mathematics often pose their questions as between stark alternatives: Either Platonism or Intuitionism; Platonism is more than we can be expected to stomach; so Intuitionism is our only hope. It is a natural method of argument in metaphysics, but has the disadvantage of concentrating effort on pointing out the defects in rival doctrines, rather than accommodating their strong points within a more inclusive account. In the course of the first twelve chapters the starkness of the alternatives has been much softened. Logic is no longer confined to first-order logic, logical truths are no longer invariably analytic, logical argument is no longer confined to argument that must, on pain of self-contradiction, be conceded. Gödel's theorem shows that however fully we formalise a system, there will still be some arguments evidently valid, but not in virtue of conforming to the canons of formal validity laid down. Instead of taking formally valid deduction as the one and only paradigm, we should see it as an extreme case, where we are having to argue with a recalcitrant fool. In the standard case, however, the argument is with a like-minded seeker after truth, and much of mathematical argument is shaped by an underlying dialectical structure. Our Logicism is less sharp also in that our mathematical concepts are not held to be definable in purely logical terms, but only *grounded* in logic, often developed by extrapolation to something essentially new.

Logicism thus watered down no longer excludes all elements of Platonism. Although epistemological platonism is, as Plato himself realised, too *simpliste* to be wholly satisfactory, there is an element of pattern-recognition in our achieving mathematical knowledge. Often the identification of the correct pattern is achieved only after a long and complicated dialectic, in which the "eye of the mind" is guided much more by argument than by any extra visual acuity. The visual metaphor, though sometimes natural, is apt to mislead.

Ontological platonism is often expressed in misleading terms too. The arguments we use are ineluctable, their conclusions objectively true, the patterns we discern are not mere figments of our imagining, but really there, out there. But however strongly we maintain that *?* and *p* are part of the furniture of the universe, there is nowhere out there that we can locate them, and unlocatable furniture is not an attractive bargain. But it is an effect of language rather than of thought. We can seek objectivity, without thinking it must be embodied in material objects. Chemists have no difficulty in believing in the Periodic Table, and would not change their mind if astronauts could not find it anywhere in space. A platonist ontology is objectionable, only if we take it for granted that to exist is to exist as a material thing. Many modern philosophers make that assumption, and much philosophy of mathematics is massaged so as to conform to a prevailing materialism. It is an important question, and one to which the philosophy of mathematics is relevant, but it is not the concern of this book, which is concerned only with mathematics, and how it should be correctly understood.

I have apologies to make. I idiosyncratically abandon the standard usage, and describe terms such as ‘all’, ‘some’ and ‘none’ as “quotifiers”. I hope thereby to bring home the great difference between the question “How many?”, in Latin *Quot?*, to which a natural number is a natural answer, and the question “How much?”, in Latin *Quantum?*, to which a numerical answer can be given only with considerable artifice, ultimately depending on the logic of transitive relations, the topic of Chapter Nine. Quantifying is what quantity surveyors should do, not logicians. In a similar spirit, I abandon the traditional symbols, $(\forall x)$ and $(\exists x)$. They are difficult to type. Worse, they obscure their relation to other Boolean symbols. By writing the existential quotifer as $(\mathrm{V}x)$, following the example of Geoffrey Hunter (*Metalogic,* London and Basingstoke, 1971, p.139), I bring out the analogies with $\vee$, and the set-theoretical $\cup$ and $\bigcup$. For the universal quotifier I desert the purity of an inverted V for the convenience of a simple (Ax), with a capital A the right way up because not only is it easier for typing, but it is easier for the beginner, since A suggests All. A does resemble an inverted V, and to that extent the analogies with $\wedge$, $\cap$, and $\bigcap$.

Perhaps I should apologize to rigorist readers for my lack of rigour. There is a trade-off between full formal rigour and intelligibility. I have tried to be intelligible, and have left out many trees in the hope of showing the outline of the wood. In part, no doubt, it betrays my natural sloppiness of mind. In part it is a reaction against the needless obfuscation of many articles in mathematics and logic. But I hope I shall succeed in conveying my argument to more readers than those put off by the informality of the exposition, and that those who are sharp enough to spot holes are clever enough also to see how they could be blocked.

I should also apologize to many North American thinkers for not discussing their work in the detail it deserves. I have learnt much from their writings, but very often have found that they are addressing a different question from the one I am seeking to answer. Very often they are starting with some wide-ranging metaphysical view—nominalism, physicalism, naturalism—and are trying to show how mathematics can be fitted into that scheme. It is a perfectly proper exercise—metaphysics on the grand scale—and one I hope some day to attempt myself. But my present objective is more limited. Although the philosophy of mathematics anyone adopts *is* influenced by his general metaphysical presuppositions, as also the philosophy of history, he adopts, or philosophy of art, I want, so far as possible, to concentrate on considerations arising within the discipline. It is a philosophy of mathematics arising from mathematics rather than a philosophy of mathematics stemming from elsewhere.

I have many debts to acknowledge: I have been fortunate in my position, my colleagues and my pupils. It was good for me that Oxford set up a joint school in Mathematics and Philosophy, thus bringing me in touch with colleagues who shared my interests, enabling me to try out ideas on pupils, and giving me the stimulus to focus my thoughts in communicable form. It was good for me also that in David Bostock I had a colleague who shared my interest in the philosophy of mathematics, and, while often disagreeing with what I said, was ready to think about it and respond to it; and who, by shouldering burdens which otherwise would have fallen on me, gave me the time to crystallize my thoughts in writing. It is for this reason that I dedicate the book to him, and to all my colleagues.

East Lambrook

Christmas 1998

Note on Logical Symbolism

Older text books express the universal quantifier by brackets surrounding the variable alone, *(x)*. Modern text books usually have an inverted A, $(\forall x)$. It is better to have $(\mathrm{A}x)$, with a capital A the right way up because not only is it easier for typing, but it is easier for the beginner, since A suggests All, and also resembles an inverted V, with useful analogies with $\wedge$, the sign for conjunction, with $\cap$, the cap of set theory, and $\bigcap$, the inverted U, for intersection. For the same reason it is desirable to symbolize the existential quotifier not by the usual $(\exists x)$, but by $(\mathrm{V}x)$, to bring out the analogies with $\vee$, and the set-theoretical $\cup$ and $\bigcup$. Likewise the null set (or empty set) is represented by Λ. (See fn. 44 in §9.11.)

Material implication, which is rendered, sometimes inaccurately, in English by 'if..., then...', is represented by $\rightarrow$. In older text books it was often rendered by a horseshoe $\supset$; $\supseteq$ would have been better, as it is an antisymmetric relation (see §9.1), but is inconveniently cumbersome.

For ease of reading, brackets are often omitted, the convention being that $\neg$ binds most tightly of all, and $\wedge$ binds more tightly than $\vee$, which in turn binds more tightly than $\leftrightarrow$, which in its turn binds more tightly than $\rightarrow$. Sometimes, however, brackets are retained, where they seem to make the outline of the formula easier to grasp. Many authors abbreviate 'if and only if' to 'iff', which is convenient, particularly in macaronic prose, where it is desirable to have mathematics and English all on one line.

Logical Symbols

English	Standard	Old	Here
and	$\wedge$	.	$\wedge$
and/or	$\vee$	$\vee$	$\vee$
not	$\neg$	-	$\neg$
if...then	$\rightarrow$	$\supset$	$\rightarrow$
if and only if	$\leftrightarrow$	$\equiv$	$\leftrightarrow$
all x	$(\forall x)$	(x)	$(\mathrm{A}x)$
some x	$(\exists x)$	$(\exists x)$	$(\mathrm{V}x)$
null set x	$\emptyset$	$\emptyset$	Λ

Contents

to

David Bostock

and

all my colleagues

Chapter 1

Plato's Philosophies of Mathematics

1.1 *Meno*

The philosophy of mathematics begins with Pythagoras, who believed that mathematics gave us the key to understanding reality, but it is Plato who first gave it articulate form. In the *Meno* he proves that mathematics is known *a priori*—that is, without appeal to sense experience.[1] He starts talking to a slave boy, and by a series of questions elicits from him a method of constructing a line $\sqrt{2}$ as long as a given one, using a special case of Pythagoras' theorem. The general proof of Pythagoras' theorem is difficult: for two thousand years it was the *Pons Asinorum* for school boys. In the Meno, however, Plato considers the special case of an isosceles right-angled triangle, where even someone who has never done geometry in his life can be brought to see how to construct a square with area twice that of a given square.

Readers often complain that Plato's argument is unfair, because Socrates uses leading questions. But, although he does use leading questions, he is careful not to tell the slave boy any particular empirical facts. This is borne out by the boy's making several mistakes along the way. He first suggests that, to get a square twice the area of the given one, we should take one with side twice as long. But, Socrates objects, that will not work, and draws a diagram on the sand, like Figure 1.1.1, and shows, very obviously,

[1] *Meno,* Stephanus pages 82–85. (All references to Plato's dialogues are to the pages in the Stephanus edition.)

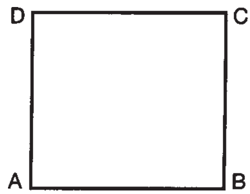

Figure 1.1.1 How can we draw a square with twice the area of the given square *ABCD*?

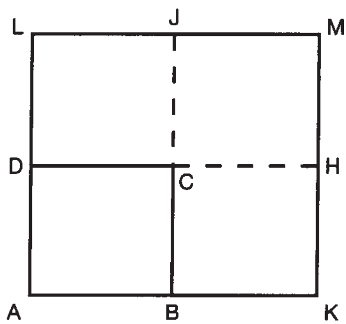

Figure 1.1.2 The Boy's First Suggestion: the square on *AK* is twice that on *AB*.

that if we have AK twice AB, then the square AKML will be not twice but four times the area of ABCD (Figure 1.1.2). So, says the boy, let us try with a side one-and-a-half times the length of the original side. That suggestion is gone over too, and we reach the conclusion that $(1.5)^2$ is not equal to 2 (Figure 1.1.3).

So, after these two false starts we arrive at the real solution, which is to construct a square on the diagonal of the original square, and Socrates shows why such a square must have an area that is half that of the big square, AKML, itself four times that of the original, ABCD. The slave boy finds this argument convincing, and—what is more important—so does everyone who attends to it. The argument is given in Figure 1.1.4.

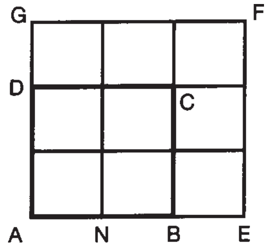

Figure 1.1.3 The Boy's Second Suggestion: $\sqrt{2} = 1.5$. But, argues Socrates, the square on *AE* is nine times that on *AN,* whereas the square which has twice the area of *ABCD* should be eight times that on *AN*.

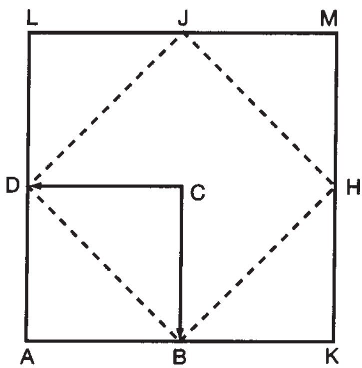

Figure 1.1.4 The Real Solution: the square on *DB* is twice that on *AB*. For the square *DBHJ* is the sum of the four triangles *DBC, CBH, HJC,* and *CJD,* and each of these is half the corresponding square, *ABCD, BKHC, CHMJ* and *JLDC,* and half four is two.

1.2 *A Priori*

Question

1. *How* do we know mathematics?

Plato's conclusion is that mathematical knowledge is *a priori,* that is, that it is not based on the evidence of the senses, either our own, or other people's transmitted to us by some form of communication. In this mathematics differs from most other subjects.

With the exception of logic, and possibly philosophy, most subjects depend one way or another on empirical evidence which is at some stage or other based on people seeing, hearing, or feeling things. In scientific laboratories there are balances and spectrometers, and complicated instruments to detect fundamental particles, such as muons. We could not detect a muon unless either we had seen something—a flash on a screen, a track on a photographic plate, a trail in a bubble chamber—or had heard something—a click on a Geiger counter, or had had some other sensory experience. Equally, those studying the humanities, although they do not go to laboratories, do go to libraries, where they can read reports of what men did in time past. We cannot know that the Battle of Hastings took place in 1066, unless we can find a report from someone who was there, or from someone who was told by someone who was there, or something that was done in consequence of the battle, or some other word or deed which we can take as evidence of there having been a battle. We may base our knowledge on the *Anglo-Saxon Chronicle,* or on the Bayeux Tapestry; but we reckon that they are reliable sources of information only because they are, immediately or ultimately, based on what men actually witnessed with their eyes, their ears, or their other sense organs. Even in literary criticism, which is much more a matter of imagination, insight and flair, if we want to propound a serious interpretation of, say *King Lear,* we must actually have read the text. Mathematics is different. Although later I shall have to qualify what I am now saying and admit that mathematics, too, is quite often known second-hand,[2] it is true, nevertheless, that one can do mathematics on one's own. It is potentially a solipsistic, or, as we might even say, autistic, discipline. Autistic children cannot relate to other people, and are characteristically bad at the humanities, but can be good at mathematics. One does not have to know, one does not have to like, other people, in order to be a mathematician, whereas it is very difficult to study the humanities without some liking, or at least some disliking, for people. So mathematics is something which can be done, although with difficulty, by someone who is blind, deaf and deprived of all tactile sensation, and who, moreover is in solitary confinement. If I had had the misfortune to fall into the hands of the Communists, and had been sent to Siberia for thirty years, I

[2] See below, §7.4, and §14.9.

should be unable to while away the time by studying chemistry or ancient history, but I could still in my cell do prime number theory. And that puts mathematics in a class apart.

If not Empiricism, then What?

1.3 Relevance

But there is a difficulty. What is the relation between mathematics, which is pure, and reality, which is relevant? If mathematical truths apply to empirical reality, as they surely do, they must be vulnerable to empirical refutation. Protagoras, one of the leading Greek sophists and a precursor of modern empiricism, argued that geometry was not true *a priori* but empirical in content and false. For geometry teaches that a tangent touches a circle in just one point, whereas casual observation reveals that they are coincident over a short length. If we observe a wheel on a road, or a hoop on a pavement, or a top lying on a table, we see that they do not touch at just one point, but are evidently touching over some small, but finite, distance.[3] This is a matter of simple observation. Look at any bicycle. The propositions of mathematics, Protagoras concluded, are not true *a priori,* but are simple synthetic propositions, and, in the case of those actually asserted by mathematicians, in fact false. Plato is worried by this argument: his answer is "So much the worse for wheels and hoops and tops, and all particular exemplifications of circles".[4] He distinguishes the ideal circles of our thought from their imperfect exemplifications in the world around us. If I turn a top on a lathe, although it is more or less a circle, it is not a perfect circle. Similarly, no bicycle wheel is a perfect circle. But that is not telling us anything about geometry, but only about bicycling. Geometry expresses *a priori* truths about ideal shapes which material objects only imperfectly approximate to. Protagoras has not produced a counter-example to geometrical truth, but simply an example of material imperfection.

[3] Aristotle, *Metaphysics,* II, 2, 997b34–998a4. (All references to Aristotle are to the pages in the Bekker edition.)

[4] Plato, *Seventh Letter,* 343a.

Two Questions

1. *How* do we know mathematics?
2. What is mathematics *about*?

The distinction between our concepts and the material objects to which they are applied is important and often lost sight of. Plato gives another example in the *Phaedo*.[5] We have the concept of two things being equal, for example two sticks being equal in length. If we were to examine them closely, it is a fair guess that we should find that they were not exactly equal. Indeed, until the advent of quantum mechanics, it was a fair guess that no two material objects were exactly equal in length. Nevertheless, we have the concept of being exactly equal in length. We know that if A is exactly equal to B and B is exactly equal to C, then A is exactly equal to C. We might be faced with a series of objects, each apparently exactly equal to the next, but the first visibly smaller than the last. We do not then suppose that we have refuted by experimental test the claim that exact equality is a transitive relation; we do not say that even if A is exactly equal to B and B is exactly to C, A may not be exactly equal to C. Instead, we blame our application of the concept, and say that although we could not see it, some of the objects cannot have been exactly equal to the others. Rather than amend our clear and distinct idea of exact equality, we cast doubt on our use of it in the individual case.

As a first move it is fair enough to try blaming the application rather than the concept itself for any discrepancy with empirical observation, but we cannot be sure that the attempt to shift the blame will always be successful. In the two cases cited the discrepancy is not great, and the explanation plausible—the hoop touches the road for only a short distance, the first and last members of the series are only a little different in length, and it is quite likely that the hoop should have deformed a little, as a rubber tyre visibly does, and that very small inequalities should have been invisible, although their sum was visible. In other cases, however, the discrepancy might be great and no plausible explanation available. If the angles of a triangle added

[5] *Phaedo,* 74–76.

up to much less or much more than two right angles, it would be implausible to impute gross experimental error, and a few checks could eliminate that possibility altogether. On any one occasion Agamemnon might have miscounted if he reckoned that there were seven heroes in one ship, five in another, and yet not twelve in the two together, but if repeated checking by a chartered accountant still failed to conform to our simple arithmetical equation

$$7+5=12$$

we should be at a loss to know what to say.

Protagoras would have known what to say. An empirical generalisation, though hitherto well confirmed, had at long last been falsified. It is an attractive answer. It was put forward in the last century by Mill, and in recent years by Kitcher and Gillies.[6] Mill's arguments, in spite of Frege's scathing criticisms,[7] are not bad. He argues, as does Hume, against all forms of *a priori* and necessary knowledge; it is by long experience that I learn that seven plus five equals twelve. It is a synthetic truth; not, as Kant had made out, synthetic *a priori,* but synthetic *a posteriori*. It is a position that has found favour with many modern philosophers, but it has never found favour with the majority of mathematicians.[8] Mathematicians, who have experienced the force of mathematical argument, do not believe that it is just the result of a Humean conditioning process, that they have so often discovered by experiment or been told by a teacher, that Pythagoras' theorem is true, that now they have got into the habit of believing it and cannot break themselves of the habit. They follow Plato, and maintain that mathematical truth is, indeed, *a priori,* and then seek to reconcile the Pythagorean intimation that it does give us some knowledge of reality with the objection put forward by Protagoras that any statements about reality are vulnerable to empirical falsification.

What Plato himself does is to disconnect mathematical truth from too close a contact with empirical reality. It is not something

[6] J.S.Mill, *A System of Logic,* London, 1843, Book II, chs. 5 & 6, Book III, ch.24; Philip Kitcher, *The Nature of Mathematical Knowledge,* New York, 1983; D.A.Gillies, *Frege, Dedekind, and Peano on the Foundations of Arithmetic,* Assen, 1982, chs. 3 & 4.

[7] G.Frege, *The Foundations of Arithmetic,* 1884, tr. J.L.Austin, Oxford, 1950, §7.

[8] But see below, §2.4.

Schools of Mathematical Philosophy

	Empiricism	Platonism
Leading Exponents	Protagoras Mill Gillies Kitcher	Early Plato Hardy Gödel
How Do We Know?	By Observation	By Thinking *A Priori*

Table 1.3.1

peculiar to Plato to insulate cherished truths against unfavourable counter-instances; it is the argument of experimental error, often used by scientists when their theories come into collision with the evidence. But Plato pushes it very much further than modern scientists do. Modern scientists are prepared in the end to abandon even their most cherished theories in the face of adverse empirical evidence, whereas Plato dismisses empirical evidence from the outset as completely irrelevant. He takes a very low view of empirical truths about material objects. Material objects, because of their imperfect and transitory natures, cannot be the subject of genuine knowledge. If there is a conflict between them of mere empirical observation, it is the latter that must be rejected. In Book VII of the *Republic* he ridicules astronomers who spend their nights looking up at the sky instead of doing mathematical calculations. "Don't be so silly", he says, "as to waste your time lying on your back, looking up at the night sky: just think."[9] Such a move is effective, but costly. It offers mathematical propositions absolute security against empirical refutation, but at the cost of complete disconnexion between mathematics and empirical reality. Plato means by *ἀστρονομία*, *astronomia*, not astronomy, but "rational mechanics", something in the same sort of line as the classical Newtonian mechanics taught in schools, in which one makes no observations but simply performs calculations and solves problems; or like the General Relativity studied by theoretical physicists or pure mathematicians,

[9] *Republic,* 529–530.

who are solely concerned with the solutions to differential equations, the determination of boundary conditions, and the discovery of elegant derivations, and hardly at all with empirical observation. But rational mechanics, although eminently rational and often highly elegant, may not be true of the external world. As we now know, Newtonian mechanics is not completely true under all conditions. It does not detract from the intellectual interest of Newtonian mechanics as a formal mathematical system, or its value as an educational exercise. But its claim to complete truth is compromised in our eyes by its failure to conform completely with empirical fact. Not so for Plato. In his eyes, if rational mechanics does not agree with the external world, so much the worse for the external world. Faced with the possibility of disagreement between mathematical rationality and empirical reality, he saves the relevance of mathematics by down-grading empirical reality so as to be less real than some other sort of reality, capable of being known *a priori*, of which mathematical truth is a paradigm example. Real reality for Plato is that possessed by the "forms", as the word *εἴδη (eide)* is usually translated, and any other sort of reality is only of lower degree. The relevance of mathematics is saved by denying the ultimate relevance of any other sort of reality that could impugn the truth of mathematical theorems. For Plato, therefore, mathematics is both true *a priori* and relevant, just because only the sort of reality that mathematical truth reveals is accounted fundamentally real and relevant.

1.4 What Are We Talking About?

Plato avoided refutation at the hands of Protagoras by claiming that the objects of mathematical discourse are not simple material objects, which manifestly fail to conform to mathematical expectations, but something else, more real and definitely different. But what exactly are the objects of mathematical discourse? Plato uses two words to describe them: *εἴδη (eide)*, which in Latin was translated as *species,* and *ἰδέαι (ideai)*, from which comes our word 'idea'; but in modern English the word 'idea' is too psychological to express what Plato meant. In technical philosophy the word 'universal' is the best equivalent, but fails to express the strong visual connotations of the Greek words, which was partly expressed in the standard translation 'forms', but might be better rendered by 'aspects' or 'features', or better, 'shapes' or 'patterns', since

Plato's whole theory is very much influenced by his thinking about geometry. He develops it in books VI and VII of the *Republic,* and says that what we are talking about in geometry and mathematics generally are abstract entities which are timeless, spaceless, and impersonal. When I talk about the diagonal of the square, or the nine-point circle, or the Euler line, I am not talking about the often rather sketchy and highly imperfect drawing on the blackboard, but about something which underlies all particular exemplifications of squares and diagonals, nine-point circles, or Euler lines, and is independent of each of them. The very fact that we use the definite article, and talk of *the* square, *the* nine-point circle, *etc.,* bears witness to this; and by the same token, it would be absurd to ask *where* the square was, or to ask *when* the nine-point centre came to be on the Euler line, or to suggest that Pythagoras' theorem might hold for you but not for me. So Plato's answer to the question "What is mathematics about?" is that it is about something timeless, spaceless and objective.

Plato has been much criticized, but also much imitated: chemists learn the Periodic Table; in the Periodic Table are arranged many abstract entities—Hydrogen, Deuterium, Lithium, Chlorine, and they have places in the Periodic Table, just as Plato said they should. These places in the Periodic Table are not spatial ones, they are not temporal ones, they are not a matter of personal predilection: they are relations of a quasi-spatial kind between entities that are spaceless, timeless and impersonal. Equally on a syllabus, there may be listed various topics to be covered—group theory, Lebesgue measure theory, geometry—which again are not located in space, nor in time, nor are a matter of personal predilection. Such entities can both be referred to and be meaningfully talked about. Unless we are prepared to make out that the whole of modern scientific discourse is meaningless, and to forswear all talk of the Periodic Table and of biological species, it is unreasonable to object to Plato's talk of mathematical objects on metaphysical grounds. It is difficult to deny that mathematics is in some sense objective, accessible to every ratiocinating thinker, and independent of time and space. And Plato's language expresses that.

But it also expresses more. Plato himself held that great consequences followed from the existence of forms, and many philosophers since have agreed, some holding with Plato that if the forms exist, then things are not the only things to exist, and that

> There are more things in heaven and earth, Horatio,
> than are dreamt of in your philosophy.[10]

and, in particular, that materialism is therefore proved false, and that souls can exist as independent entities and survive the dissolution of our mortal bodies. Others have concluded that since these consequences may well follow from the existence of abstract entities, abstract entities cannot really exist, and we must reconstrue our ways of talking about mathematics and natural science so as not to seem to presuppose the existence of anything with embarrassing metaphysical consequences. Since the time of Plato there have been strong and sustained objections to his account of a non-spatial, non-temporal, impersonal world of invisible, intangible and immaterial entities. It is partly a matter of taste, a dislike of ontological extravagance. But also there is an element of fear. If, besides all the ordinary things in the external world, there are a whole lot of other things that exist, the realm of being seems overcrowded; the over-population problem is particularly acute if we venture into Cantor's paradise of transfinite numbers, and find ourselves piling infinity upon infinity in endless, and incomprehensible profusion. They fear that if they once allow that there are more things than can be touched or seen, they are on a slippery path which may lead to their having to acknowledge the existence of values or even of God, than which nothing could, in their conception, be worse.

These endeavours are not perverse: any serious thinker will try to form as good a picture as he can of what the world is really like, and will need to consider the evidence of our mathematical experience and our mathematical knowledge, and whether they can be accommodated in the view he is being led to adopt. But these extraneous concerns make it difficult to formulate a philosophy of mathematics *per se,* and difficult to do justice to the accounts being offered. Since a full resolution of these issues lies outside the philosophy of mathematics, it will not be attempted in this book. So far as we can, we shall concern ourselves only with those considerations that arise within mathematics, and only in the final chapter take account of extraneous considerations.

Within mathematics, the existence of abstract mathematical objects has been generally reckoned to commit us to more than

[10] *Hamlet,* I, v, 166.

the bare legitimacy of using ordinary mathematical discourse. Platonic Realism, as it may be termed, lays claim to there being objective truth. We discover mathematics rather than invent it. Mathematical truth does not depend on our say-so. If Beethoven had been aborted, there would have been no Eroica: but if Pythagoras had never been born, Pythagoras' theorem would still have been true. It follows from the objectivity of mathematical truth that mathematical entities have properties independently of our knowing what they are, and so are subject to the Law of the Excluded Middle and the Principle of Bivalence. Platonic Realism is on this score opposed to Intuitionism.[11] The third claim is similar, and is to do with reference. If numbers and other mathematical entities exist independently of us, they can be referred to successfully without precise specification, whereas if they were only artefacts of our own devising, it would be incumbent on us to say exactly what it was we had in mind before we could expect someone else to know what it was we were talking about. This has become important with regard to the "vicious circle principle". In analysis and elsewhere we sometimes have occasion to define a number as a Dedekind cut of numbers satisfying some particular condition. There is no problem about this if numbers exist independently of us. In that case all we are doing is to give a pointer towards the number we are talking about in terms of a readily understood general specification. But if numbers do not exist independently of us, and are simply called into being as mental constructs of our own devising, then a critic can complain that in constructing a particular number by reference to all the numbers satisfying a certain condition, we have constructed it in terms of itself. On this score Platonic Realism is opposed to Constructivism. The real existence of mathematical entities entitles us to refer to them without completely characterizing them, and therefore can quantify (or as we shall later term it, "quotify") over them "impredicatively" in process of defining them.[12]

Platonic Realism thus has many attractions. It not only legitimises mathematical discourse generally, but licenses certain forms of

[11] See further below, §§7.4, 7.5.

[12] See Kurt Gödel, "Russell's Mathematical Logic", reprinted in Paul Benacerraf and Hilary Putnam, eds., *Philosophy of Mathematics,* 2nd ed., Cambridge, 1983, p.456. See further below, §16.3.

1. Mathematical entities are talkable about
2. Mathematical truth is objective
3. Principle of Bivalence holds (and LEM and DN)
4. Impredicative sets are admissible

Table 1.4.1

inference which mathematicians often have occasion to use. Although philosophers have qualms about platonism, most working mathematicians are platonists, in so far as they articulate any philosophy of mathematics. Platonism is not to be dismissed simply on the score that many philosophers are alarmed at the ontological extravagance of Platonic Realism. But the objections are not to be dismissed either. No less telling than the ontological doubts about the existence of mathematical entities are epistemological problems, which turn on the question "How do we come to know mathematical truth?". "Platonism seems obvious when you are thinking about mathematical truth", said W.D.Hart, "but impossible when you are thinking about mathematical knowledge."[13]

1.5 How Do We Know?

Philosophers find *a priori* knowledge difficult. Many are empiricists, and hold that all knowledge must be founded on sense-experience, even though it is evident that much of our knowledge is not founded on sense-experience—I know where I shall have lunch tomorrow not because I have made a prediction founded on sense-experience, but because I have made a decision. But even non-empiricists find it hard to free themselves from perceptual metaphors with dangerously misleading connotations. Plato, in his first thinking about mathematics, says we come to know mathematical truth with the "eye of the mind". It is an intelligible answer in view of the *Meno* experience. Later Whewell, the Master of Trinity,

[13] Reviewing Mark Steiner, *Mathematical Knowledge,* in *Journal of Philosophy,* **74,** 1977, pp.118–119.

Schools of Mathematical Philosophy

	Empiricism	Platonism
Leading Exponents	Protagoras Mill Gillies Kitcher	Early Plato Hardy Gödel
How Do We Know?	By Observation	By Thinking *A Priori*
What Are We Talking About?	Empirical Phenomena	εἴδη Forms (or Patterns)

Table 1.4.2

Cambridge, spoke of "imaginary looking".[14] It accords with our experience. When one is trying to master a proof, one concentrates very much, and one's whole body becomes tense, much as when one is trying to see something at a distance. One feels one is trying to focus one's mind in much the same way as one focuses one's gaze, even though one may have one's eyes shut. The perceptual metaphor is a very natural one. Admittedly, it is a metaphor; but it is one that commends itself to mathematicians, as characterizing how it seems when they are trying to do mathematics. To give a more modern example, G.A.Hardy, in an article in *Mind* wrote:

> I have myself always thought of a mathematician as in the first instance an *observer,* a man who gazes at a distant range of mountains and notes down his observations. His object is simply to distinguish clearly and notify to others as many different peaks as he can. There are some peaks

[14] W.Whewell, *Philosophy of the Inductive Sciences,* London, 1840, vol.i, Book II, ch.8, §5, p.130, (in 1847 edn. vol.i, ch.9, §5, p.135), *History of Scientific Ideas,* London, 1858, i, 140.

> which he can distinguish easily, while others are less clear. He sees A sharply, while of B he can obtain only transitory glimpses. At last he makes out a ridge which leads from A, and following it to its end he discovers that it culminates in B. B is now fixed in his vision, and from this point he can proceed to further discoveries. In other cases perhaps he can distinguish a ridge which vanishes in the distance, and conjectures that it leads to a peak in the clouds or below the horizon. But when he sees a peak he believes that it is there simply because he sees it. If he wishes someone else to see it, *he points to it,* either directly or through the chain of summits which led him to recognize it himself. When his pupil also sees it, the research, the argument, the *proof,* is finished. The analogy is a rough one, but I am sure that it is not altogether misleading. If we were to push it to its extreme we should be led to a rather paradoxical conclusion; that there is, strictly, no such thing as mathematical proof; that we can, in the last analysis, do nothing but *point;* that proofs are what Littlewood and I call *gas,* rhetorical flourishes designed to affect psychology, pictures on the board in the lecture, devices to stimulate the imagination of pupils. This is plainly not the whole truth, but there is a good deal in it.[15]

Hardy is one of the few mathematicians to express for the benefit of philosophers his own thinking about the nature of mathematical truth and mathematical discovery. He is an uncompromising Platonist. To the question *What are we talking about?,* he answers that we are talking about things that are there; they are real objects, although not visible objects. And to the question *How do we know?,* he answers that we know by seeing with the eye of the mind, by actually looking. It is a form of realism in which independently existing objects are seen with the mind's eye, and objective truths discovered about them.

[15] *Mind,* 1929, p.18; but see S.F.Barker, "Realism as a Philosophy of Mathematics", in J.J.Buloff, ed., *Foundations of Mathematics,* Berlin, 1969, pp.1–9, and P.Bernays, "Sur la Platonisme dans les mathématiques", *L'Enseignment Mathématique,* **34,** 1935, pp.52–69; tr. in Paul Benacerraf and Hilary Putnam, eds., *The Philosophy of Mathematics,* 2nd ed., Cambridge, 1983, pp.258–271.

Besides the virtues already noted of making mathematics independent of empirical evidence, not limited by space or time, and of securing the objectivity of mathematical truth, Hardy's account has the further virtue of expressing its accessibility to every mind. Although the abstract entities talked about by mathematicians— call them forms, or universals, or sets, or shapes, or patterns, or concepts, or what you will—are a bit hard to swallow, once swallowed they give a simple and satisfying answer to the question of what mathematicians talk about and also the question of what makes mathematical propositions true. Mathematicians talk about numbers in much the same way as artists talk about colours and to say that six comes between five and seven is to ascribe a certain relation to five, six and seven, in much the same way as to say that yellow comes between green and orange or purple between red and blue ascribes a similar relation to colours. In each case the proposition is reporting facts, not superficial contingent facts but deep necessary facts, and it is in virtue of those facts that what is said is true. In order to ascertain them it is necessary to focus attention on them, but not to be in any particular place at any particular time, or to use any particular sense organ. With history or astronomy, I can tell you that you were not at the right place at the right time to make the relevant observation, whereas it is clean contrary to the *ethos* of mathematics that any aspiring mathematician should be disqualified on such grounds. Mathematics is, in principle, accessible to all who wish to study it. It may be difficult in practice, but in its ideology it is totally non-esoteric.

Although few mathematicians have been as articulate as Hardy, most have had Platonist leanings. Of course, we need to be careful. To ascribe explicit views to someone who has not explicitly espoused them is always hazardous. And there is much ambiguity in exactly what Platonism in mathematics is. In due course we shall have to refine the term. But it is useful to give general guidance, even though it may need to be modified in detail. Besides Hardy, Frege (in a manner of speaking), Gödel, and Bernays, are mathematicians who have expressed doctrines which are generally in line with those I have ascribed to Plato. And most working mathematicians, although reluctant to express any very definite views on the philosophy of mathematics, when they do say what they think they are doing, incline towards Plato's views rather than to any of his rivals'. Gödel draws a specific parallel between the apprehension of mathematical truth and sense perception:

> But, despite their remoteness from sense experience, we do have something like a perception also of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true. I don't see any reason why we should have less confidence in this kind of perception, *i.e.,* in mathematical intuition, than in sense perception, which induces us to build up physical theories and to expect that future sense perceptions will agree with them and, moreover, to believe that a question not decidable now has meaning and may be decided in the future. The set-theoretical paradoxes are hardly any more troublesome for mathematics than deceptions of the senses are for physics.[16]

Many other mathematicians have spoken similarly.[17] Their testimony, like that of Hardy and Gödel, should be taken seriously, but is not conclusive. There are objections. Whereas the objections to Plato's ontology were on account of its being too lush, the objections to his mathematical epistemology are on account of its being too *simpliste*. The perceptual analogy does not carry any weight. No coherent account is offered of how we come to know mathematical truths. What relation can a finite and time-bound mortal like myself have with the timeless and immaterial entities of Plato's imagining? Particularly in recent years, when causal theories of knowledge and reference have been in vogue, the epistemological consequences of Plato's theory of forms have been taken as a powerful argument against it.

[16] Kurt Gödel, "What Is Cantor's Continuum Problem?" (Supplement to Second Edition), reprinted in Paul Benacerraf and Hilary Putnam, eds., *Philosophy of Mathematics,* 2nd ed., Cambridge, 1983, pp.483–484; and in *The Collected works of Kurt Gödel,* ii, ed.S.Feferman, Oxford, 1990, p.268. For further discussion, see below, §14.9.

[17] Mark Steiner, *Mathematical Knowledge,* Ithaca, NY, 1975, ch.4, §VI, pp.134–137; C.Parsons, "Frege's Theory of Number", in Max Black, ed., *Philosophy in America,* London, 1965, pp.180–203; L.H.Tharp, "Ontological Reduction", *Journal of Philosophy,* 68, 1971, p.162; P.Cohen, *Set Theory and the Continuum Hypothesis,* New York, 1966, pp.150–151; Penelope Maddy, "Perception and Intuition", *The Philosophical Review,,* LXXXIX, 1980, pp.163–196.

Both the ontological and the epistemological objections have been put forward in a very different context by J.L.Mackie. In his book, *Ethics: Inventing Right and Wrong,* he argues for moral subjectivism on the grounds that the only alternative is Platonism, and that values, if they existed, would be extremely queer sorts of things, and that there is no intelligible account available of how we could come to know them.[18] Exactly the same points can be made against mathematical Platonism. And the same rebuttals can be used to counter them. Values may be queer sorts of object, as also imaginary numbers, the nine-point centre, and the alternating group of order twelve: but so are quarks, electrons, alpha-particles and pions. The world is full of strange objects. Queerness is no bar to existing. Unless we adopt a know-nothing policy, and refuse to acknowledge that anything could exist unless it be of a familiar sort, we must allow that many unusual objects have been shown to exist, and that we should not lay down antecedent stipulations on what we may be led to posit. A hard-line common-sense philosopher can maintain that neither mathematics nor ethics nor metaphysics nor science can compel us to acknowledge the existence of anything we were not familiar with at the age of ten, but he does so at the cost of cutting himself off from most modern science and many traditional realms of discourse. Although we may legitimately be wary of multiplying entities needlessly, we cannot rule out Plato's abstract entities on those grounds alone.

A more specific difficulty with mathematical entities assails those modern philosophers who believe in a causal theory of knowledge: they do not see how it is possible to account for knowledge of immaterial objects within the confines of the causal theory of knowledge.[19] But the causal theory of knowledge is barely a theory at all. In part it is an attempt to block a loophole in the account of knowledge as justified true belief: it is not

[18] J.L.Mackie, *Ethics: Inventing Right and Wrong,* Penguin, 1977, ch.1, §9, pp.38–42.

[19] Paul Benacerraf, "Mathematical Truth", *Journal of Philosophy,* 70, 1973, pp.661–680; reprinted in Paul Benacerraf and Hilary Putnam, eds., *The Philosophy of Mathematics,* 2nd ed., Cambridge, 1983, pp.403–420. See also W.Hart, "The Epistemology of Abstract Objects", *Proceedings of the Aristotelian Society, Supplementary Volume,* **53,** 1979, pp.153–164; and P.Kitcher, "Introduction", *Revue Internationale de Philosophie,* **42,** issue 4, 1988, p.397.

enough that a true belief is justified—it must be justified in the right way, and the causal theory claims that the right way is a causal way. But a justified true belief could be caused in the wrong way: a brain in a vat may be induced by suitable stimulation to hold a belief which is true, and for which it has adequate justification. The causal theory does not succeed in blocking the hole in traditional accounts of knowledge. Only if we take 'cause' in a wide, and elastic, sense can the causal theory serve its purpose, and then there is no difficulty in having abstract entities explaining observable phenomena. I can explain the distribution of bagatelle balls rolling down a pinboard by the Gaussian curve $y = e^{-x^2}$, the distribution of spots on a leopard's skin by the solution to a differential equation, and the shape of crystals, and the properties of fundamental particles, by the theory of groups. The reason why we all agree that 257 is a prime number is not that we have mysterious commerce with an abstract entity, but that, being rational truth-seekers, we believe what is true, and it is true that 257 is a prime number. Only if we are wedded to an extremely *simpliste* account of knowledge should knowledge of abstract entities seem problematic, and then the trouble lies in the account of knowledge, not in mathematics. A modern philosopher who does not see how he can enter into cognitive causal relations with *p*, should have comparable difficulty in getting into cognitive contact with the halogen group in the periodic table, the colour aquamarine, or the Alleluia Chorus, for these too are abstract entities, not located in time or space.

Objections based on the causal theory of reference are equally unpersuasive. The theory is based on a paradigm of reference to material objects, and has been over-extrapolated from too narrow a base. It may be that Fido obtains his name by baptism: it does not follow that π and and e $\sqrt{-1}$ needed similarly to be immersed in terrestrial water in order to acquire their names, or to be intelligibly talked about. We should be cautious in accepting the deliverances of modern theories. Modern theories may be wrong. It is their function to account for what we actually know and can talk about, not to be a strait-jacket preventing us from knowing things we do know, or talking about things we do talk about. Philosophers may be unable to understand how we can know mathematical truths, but they have a bad track record at being able to understand anything. In the middle of the twentieth century, philosophers aired their inability to see how we could know anything about the past, but it did not lead historians to go out of business; nor did the inability to solve the problem of "other minds" lead to actual autism. The contemporary inability to see how mathematical knowledge is possible should be seen in context of the mid-century inability to have knowledge of the past, or to see how one person could know whether another person had feelings. It is a difficulty to which anyone studying the philosophy of mathematics should address himself, but not a doubt that should debar him from adopting that account which seems to do most justice to the phenomena.

1.6 Modality

The severest critic of platonism was Plato. Although he never abandoned the Theory of Forms completely, he was acutely aware of the difficulties it gave rise to, and kept on criticizing it and revising his thought not only about the Forms but about the nature of mathematical thinking generally. Mathematical thinking is coercive. Mathematical truth not only can be known *a priori,* but is necessary. There is something compelling about it. We see it in the *Meno* proof. It is not only valid, but irresistible. It leaps out of the page at us. Once we have seen it, we are compelled to accept it. That feature guided Plato, and has guided the majority of other mathematicians and mathematical philosophers, in their thinking about the nature of mathematics. Mathematical truth has some sort of necessity about it, which contrasts with the merely contingent beliefs we have about the world of sense experience. There is a hardness about mathematical truth which makes it not only ineluctably true, but profoundly true, because it is immune to the changes and chances of this fleeting world of transient phenomena, and tells us about not what just is, but what must be, the case. It is difficult to give a visual exegesis of necessity. As Hume pointed out, we cannot see necessity. The logical geography of the world of forms may show how different forms are connected, but cannot show that they have to be the way they are. The metaphors we use—cogent, coercive, compelling—are muscular, rather than visual, and call for a different sort of exegesis. Typically we come to feel its force by trial and failure.[20] The boy tried $(1.5)^2$, and found that it did not equal 2. If mathematical proofs are alleged to be

[20] See J.R.Lucas, *Space, Time and Causality,* Oxford, 1985, ch.3, pp.35–36.

irresistible, we naturally ask what will happen if we try to resist them. We ask not simply "How do we come to know mathematics?" and "What is mathematics about?", but the further question "What happens if we refuse to concede the conclusion of a proof?".

Three Questions

1. *How* do we know mathematics?
2. What is mathematics *about*?
3. *What happens* if we do not accept a mathematical proof?

According to the Theory of Forms, if we do not see, it simply shows that we lack mathematical vision. It is a little like being colour-blind. But this is implausible. If someone does not concede the truth of Gödel's theorem, we *argue* with him, not take him to an oculist. More importantly, although to be colour-blind is to lack a certain perceptual capacity, which is a disqualification for a career in the navy or driving trains, it is relatively disconnected from other intellectual powers. Mathematics is more intimately connected with them. To assimilate mathematical incompetence to colour-blindness is like assimilating moral insensitivity to colour-blindness. In each case the failure is wrongly characterized as mere perceptual failure; too much else is involved. Although Plato would have been happy to acknowledge that great mathematical ability is a natural gift, which a few people have and most people lack, he cannot continue to hold that it is *just* a form of seeing without losing a sense of its integral connexion with the rest of thought and of its necessity.[21]

Plato rejects "platonism", because it gives no account of modal force of *Meno* argument.

One cannot "see" necessity.

Once we grasp the proof of a mathematical theorem, we feel the force, the irresistible force, of the argument. We are compelled

[21] See further, §16.6.

to acknowledge that it must be so. The cogency of mathematical argument is quite different from the cogency of empirical fact. With the latter we are compelled to acknowledge that it is so; with the former we are compelled to acknowledge that it *must* be so. Mathematics has modal subtleties that geography lacks. Plato's analogy, between the world of mathematical objects and the world of everyday experience, is to that extent defective. And Plato himself began to develop a theory of mathematical argument to take account of its compelling force and the necessity of its conclusions.

If not Platonic realism, then What?

1.7 Cogency

Plato sought to give an account of why mathematical truth was necessarily true. The necessity arose from the nature of argument. As he developed his theory of argument, he was impelled to view good arguments as inherently incontrovertible, and to accept as valid only those that could not be coherently gainsaid.

Most arguments are not like that. We often argue with the intent of getting those we argue with to accept our views, and sometimes we succeed—else we should have abandoned the activity altogether. But although arguments can carry conviction, in most cases they are not *maximally* coercive. The force of reason is, to a greater or lesser extent, resistible. Often we do not resist it: we may not want to, or we may be unwilling to pay the price of resistance: but we could, and whether we do or not depends on us, our attitudes and purposes. For argument in general is two-sided. We can distinguish a *proponent* and a *respondent,* the former putting forward a claim, the latter responding, perhaps accepting it, perhaps disputing it, sometimes asking for justification, sometimes putting forward objections and counter-claims or counter-arguments. Arguments are variable and complex, and it is not at all easy to schematize them. But for our present purpose it is enough to remark that typically the development of an argument depends on both the proponent and the respondent. Arguments are arguments *between*.

Arguments are for the most part holistic and cumulative. In history, in literature, in the law courts, in morals, the proponent is trying to convey a complete over-all judgement as to what happened, what the interpretation of a play really is, whether the accused committed the crime, what ought in the circumstances to be done. Sometimes he succeeds in conveying his way of seeing them, and we come to see things as he sees them, much as a mathematician does on Hardy's account.[22] Sometimes, indeed, his account may be compelling. But it does not have to be. Some small detail may entirely alter the aspect of the case, and we may be unconvinced that no such detail will emerge. A holistic cumulative case may convince, but there is always logical room for some further consideration, some further 'but', and so the argument, even if weighty, even if decisive, is not conclusive, not *maximally* coercive.

If mathematical arguments are to be maximally coercive, there must be no room for a further 'but', and so they cannot be effectively two-sided. We need to know with whom the mathematician is arguing, and be sure that he does not let any weak contention pass. Often the mathematician is arguing with other mathematicians, who are very ready to follow the wave of his hand, and see whatever it is that his proofs are intended to indicate. But Plato had occasion to argue with sophists, who would not concede anything willingly, and he had to use arguments that would have to be accepted by the most stubborn of them, even by Thrasymachus, who appears in the first book of the *Republic* and is not going to give an inch unless he has to. Valid arguments, Plato came to think, must be ones that would compel assent from anyone, even the most recalcitrant, even the fool with whom Anselm argued. Ironically, mathematics, one of the most difficult of disciplines, aims to address its arguments to the most moronic, and to articulate them so that even a computer is compelled to concede. Many difficulties have ensued for our proper understanding of the nature of mathematics. But the connexion that Plato sensed between universal validity and maximun coerciveness none the less remains.

Equally, if mathematical arguments are to be maximally coercive, they cannot be cumulative and holistic: a cumulative case could always be improved by some further consideration which finally clinched the argument; and with a holistic argument, someone may fail to appreciate the whole picture, or point out some detail, which, he maintains, flaws the whole case. Plato rejected

[22] See above, §1.5, pp.11–12.

the holistic approach of poetry, drama and rhetoric, and laid it down as a mark of rationality that a serious argument could stand up to the most searching and detailed scrutiny by question and answer. Although he himself, in his myths and elsewhere, was well able to mount a concerted artistic appeal, and although in the *Protagoras* he showed himself aware of the fact that a face could not be regarded simply as a sum of its parts,[23] the main thrust of his thought was against the holistic approach, and in favour of the analytic, because only thus could he force assent from the Thrasymachusly disinclined. If we had merely a platonist vision of mathematical truth, it would be indistinguishable from our apprehension of insights in, say, literary criticism. I might see a pattern, and point it out to a friend, but he might see it quite differently, and press his differing interpretation on me. One of us might be right, and might be successful in convincing the other of the rightness of his views; but wise men do not always think alike, and their judgements, though weighty, are not maximally coercive.

In order to be coercive, a proof must not rest simply on a wide-ranging ability to discern patterns. If you fail to see it when I point it out to you, we must be able to locate the area of disagreement, and pin it down to something definite. A mathematical proof is not an assessment of the case as a whole, where the whole is wide open to different specifications, and contrary interpretations, but a finite list of definite items which can be checked over for the elimination of all dispute. If I put forward what claims to be a mathematical argument, and you do not accept the conclusion, I am entitled to ask you where it breaks down, and you are obliged to point out what you regard as the flaw in my reasoning. We can then concentrate on that, and you must show that this is a weak link in the argument, and my purported proof is invalid, or I must show that it is a valid step, and your objection, on this point at least, fails. We reduce the disagreement from a general debate about the argument as a whole to a particular dispute about the individual steps.

1.8 Deduction

A proof breaks an argument down into a finite number of separate steps, and will be valid as a whole, provided all the steps are.

[23] *Protagoras* 329d.

And these, Plato saw, would be incontrovertible if they were deductive inferences.

Plato discovered deduction. In his disputes with sophists he found that there was almost no position so unreasonable that some would not take it. The only way to oust them was to show that their position was completely self-contradictory. In the first book of the *Republic* we see him manoeuvring Thrasymachus into a formal self-contradiction,[24] and this is exactly in accord with the prescription he gives in the *Phaedo*. There he gives as one test of the tenability of a philosophical thesis "to see whether its consequences agree or disagree among themselves".[25] It is clear that this must be a negative test. If the consequences of a thesis disagree among themselves, that is, if they are, when taken together, self-contradictory, or, as logicians say, mutually inconsistent, then the thesis must be rejected: but if the consequences are consistent, that does not show that the thesis is true—many consistent positions are nonetheless false. Plato also proposes a positive test.[26] A thesis is to be accepted if it can be derived from another which is itself acceptable to both the parties. And this must be true, even for Thrasymachus, if the derivations are deductive, that is, if it would be self-contradictory to concede the premises and to deny the conclusion. We often define deductive arguments by saying that a deductively valid argument is one where the conjunction of the premises with the negation of the conclusion is inconsistent; just as we can define an analytic proposition as one whose negation is itself inconsistent.

One might ask why Thrasymachus should be all that worried about avoiding inconsistency: many people get away with inconsistency, and Thrasymachus and his friends were only concerned with what they could get away with. Athens, like Britain, was a free country, and there was no law against contradicting oneself. If I am minded to stand up at Hyde Park Corner, and proclaim that the square on the hypotenuse of a right-angled triangle is not equal to the sum of the squares on the other two sides, nothing very terrible will happen to me. So what is the sanction? Where is the necessity? Plato found an answer. He discovered the Law of Non-Contradiction. He formulates it in the fourth book of the *Republic,* and makes

[24] *Republic* I, 339.

[25] *Phaedo* 101 de.

[26] *Phaedo* 100a.

considerable use of it in his own arguing. Even the most recalcitrant sophist, even Thrasymachus, cannot afford to be caught out in self-contradiction—because those who contradict themselves thereby render themselves unintelligible, and cease to be talking comprehensible Greek, and are merely exercising their vocal chords in meaningless babble. There is thus some constraint on what can be meaningfully said, and some sanction even against the most determined sceptic. From this necessity of avoiding inconsistency there follows a canon of coercive argument.

Schools of Mathematical Philosophy

	Empiricism	Platonism	Formal Logicism
Leading Exponents	Protagoras Mill Gillies Kitcher	Early Plato Hardy Gödel	Late Plato Frege Russell
How Do We Know?	By Observation	By Thinking *A Priori*	Deducing
What Are We Talking About?	Empirical Phenomena	εἴδη Forms (or Patterns)	Propositions
What Happens If You Do Not See?	Abandon Thesis	Change Your Subject	You Can't Be Understood

Table 1.8.1

If I cannot maintain, with any hope of being understood, an inconsistent set of propositions, then if I allow all except one of that set of propositions to be true, I cannot myself deny the negation of the remaining proposition in my own mouth or gainsay its negation in anyone else's. That is, I cannot, on pain of inconsistency, refuse to concede the negation of that proposition, having acknowledged the others to be true. And this is what it is to be a *deductive* argument. In symbols, if

$$P, Q, R \vdash$$

then

$$P, Q \vdash \neg R$$

that is,

if *P, Q, R* together are inconsistent,

then *P, Q* together entail $\neg R$).

Plato was immensely taken with deductive argument. Many simple mathematical arguments can be put in deductive form, and then possess the sense of necessity that simple seeing is unable to convey. Later we shall argue that inconsistency is not the only sanction a reasonable man is sensitive to, and that Plato has in consequence construed mathematical argument too narrowly.[27] But almost all philosophers have followed Plato in taking deductive argument as the paradigm of valid argument, and seeking to explicate all mathematical reasoning in terms of it alone.

1.9 Whence the Premises?

In the sixth and seventh books of the *Republic*[28] Plato continued the theory of argument he began in the *Phaedo*. Although it is easy to formulate and apply a negative criterion for rejecting a thesis—a thesis is to be rejected if it is inconsistent—it is more difficult to apply the positive criterion. The positive criterion—that a thesis is to be accepted if it deductively follows from premises acknowledged to be true—is applicable *provided* some premises are acknowledged to be true. But the same question then arises about them. Whenever we try to justify mathematics in terms of deductive logic alone, we are faced with the problem of where we are going to get our initial premises from.

There are three possibilities: the premises may be self-evident, and should be granted without demur; they may be established as following from other already established premises; or they some may be simply postulated—granted for the sake of argument perhaps—but with no justification offered for their truth. Each of

[27] §6.6, §7.5, and §14.6.

[28] *Republic,* 509–535.

Status of Axioms

The Axiomatic approach can be developed in three ways:

1. The axioms are self-evident; epistemological Platonism and traditional geometry
2. The axioms are truths of logic, and can be proved; Logicism
3. The axioms are neither true nor false; Formalism

these possibilities has been adopted by some thinkers, and gives rise to a different philosophy of mathematics.

That some propositions should be self-evident is not evidently absurd—the American Declaration of Independence is not to be laughed out of court. Although it is not clear how we come to know self-evident truths, many truths have seemed self-evident to many thinkers down the ages. In particular, Euclid's postulates were generally taken to express self-evident truths. In the next chapter we shall explore that approach in seeking to understand geometry in the light of principles we can reasonably regard as self-evident.

If not self-evident, then What?
perhaps justified?

But Plato is uneasy. He feels impelled to try and give an account *λόγον διδόναι (logon didonai),* of the suppositions, *ὑποθέσεις (hupotheseis),* to free them from their hypothetical status and show that they are really to be believed, not just supposed. These are axioms, *ἀξιώματα*, things worthy to be believed. For Plato they are truths yet to be established, but in due course to be vindicated. That can be done. We can justify the axioms of one system by showing them to be theorems of another, seemingly less open to question. Plato here shows himself to be proto-Logicist. The Logicists hope to derive the whole of mathematics from pure logic, which can plausibly be regarded as a starting point that does not need to be supposed, but can be taken for granted without further question. The Logicist programme, which will be further discussed in Chapters Four, Five and Six, has been very influential, even if not completely successful on the terms originally set. But Plato is still uneasy. However far we go back, there will always be some fundamental system whose axioms are in need of justification without there being any more fundamental system within which they can be established as theorems. At the end of the sixth book of the *Republic,* he embarks on the search for the ἀρχὴ ἀνυπόθετος *(arche anupothetos),* the unpostulated starting point for all mathematical argument, but does not really find it,[29] although he thinks it has something to do with the Form of the Good, ἡ ἰδέα τοῦ ἀγαθοῦ *(he idea tou agathou)*. There is an impression of his trying to perform a logical version of the Indian rope trick. The ἀρχὴ ἀνυπόθετος *(arche anupothetos)* will always elude us, if the only method of establishing truth is a deductive proof from premises.

If the premises are not self-evident and cannot be justified, the only recourse is to postulate them. It is indisputable, once postulates have been granted, that they have been granted, and since the whole argument depends on their being granted, no further argument can arise on that score. The Formalist, by lowering his sights from what is alleged to be true to what has actually been conceded, achieves a correspondingly greater degree of certainty. Plato himself felt this, and the geometers who carried through his programme of axiomatization bear witness to the same tendency in not attempting to justify the hypotheses, but simply demanding that certain assumptions be granted, and going on from there. The Greek word αἴτημα, *aitema,* like the English word 'postulate' carries the sense of the imperative (or optative) rather than the indicative mood, much better than the word ἀξιώμα, *axioma,* does. The axiomatic approach is, as far as the axioms are concerned, "*fiat*ory" rather than justificatory. And it is reasonable then to enquire whether in view of the difficulties in finding an ultimate justification for first principles, we should not adopt a Formalist philosophy of mathematics.

If not justified, then What?
perhaps postulated?

The Greeks were not driven far along the *fiat*ory track, and never sought to formalise mathematics completely, but the fact that Plato's

[29] But see further below, §3.7.