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Exponential Families in Theory and Practice

Bradley Efron

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Exponential Families in Theory and Practice

During the past half-century, exponential families have attained a position at the center of parametric statistical inference. Theoretical advances have been matched, and more than matched, in the world of applications, where logistic regression by itself has become the go-to methodology in medical statistics, computer-based prediction algorithms, and the social sciences. This book is based on a one-semester graduate course for first year Ph.D. and advanced master's students. After presenting the basic structure of univariate and multivariate exponential families, their application to generalized linear models including logistic and Poisson regression is described in detail, emphasizing geometrical ideas, computational practice, and the analogy with ordinary linear regression. Connections are made with a variety of current statistical methodologies: missing data, survival analysis and proportional hazards, false discovery rates, bootstrapping, and empirical Bayes analysis. The book connects exponential family theory with its applications in a way that doesn't require advanced mathematical preparation.

BRADLEY EFRON is Professor Emeritus of Statistics and Biomedical Data Science at Stanford University. He is the inventor of the bootstrap method for assessing statistical accuracy. He has published extensively on statistical theory and its applications, with particular attention to exponential families. A MacArthur fellow, he is a member of the National Academy of Sciences. He received the National Medal of Science in 2007.

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Preface

Exponential Families in Theory and Practice is based on my notes for a graduate course designed for first-year Ph.D. and advanced master's degree students in the Statistics Department at Stanford. The course and the book focus on the elegant structure of exponential families, and how exponential family methods have transformed statistical applications in the age of high-speed computing.

Parts 1, 2, and 3 concern the basic ideas of univariate and multivariate exponential families, and their use in generalized linear models, particularly logistic and Poisson regression, the mainstays of modern applications in a variety of fields. The three parts can be covered in about twenty 50-minute lectures, leaving ten lectures for selections from Parts 4 and 5 in a one-quarter course, or fifteen in a semester. Applied topics touch on several statistical success stories: survival analysis and proportional hazards, empirical Bayes, missing data, and false discovery rates.

Homework problems, integrated into the text rather than gathered at the end, play an important role in getting the material across. For the most part the problems aren't very difficult, with the majority chosen to augment points raised in the lecture. Their main role is to help students incorporate the ideas rather than just hear them. Each week I usually assigned four or five homework problems to be turned in, and allowed students to work together on them.

Computational exercises utilize the programming language R, which is also used occasionally in the text to convey specific algorithmic details. Data sets appearing in the text are available from the author's website.

About the mathematical level: I have tried to keep this as low as possible consonant with the subject's needs. Asymptotic arguments are mostly absent, and there are almost no proofs except those that are vital to understanding the statistical points being made. A good background in multivariable calculus, linear algebra, and probability is sufficient mathematical background for the book. Exponential family theory has a strong geometri-

cal aspect, and, whenever possible, I have substituted geometry for algebra and figures for equations.

The physics profession has an honored cohort of practitioners called "phenomenologists" who work to connect theory with applications. In that spirit, the title *Exponential Families in Theory and Practice* could be better amended to *Exponential Families* Between *Theory and Practice*. My goal was to link the powerful theory of exponential families with the modern world of statistical applications, and I hope the book will be successful in that role for both teachers and students.

Acknowledgments

The material in this book accrued over fifty years of teaching, during which time the Stanford Statistics graduate students were almost always good sports and keen critics. My associate Cindy Kirby did heroic work as editor, compositor, and occasional artist in turning messy notes into the volume you are holding. My thanks also to my Cambridge University Press editors Lauren Cowles and Diana Gilooly for their kind support during the long process of publication.

Introduction

Some great ideas are born in a flash of inspiration, perhaps announced to the world by a pathbreaking paper. R. A. Fisher's 1925 article on maximum likelihood estimation is a classic example. Nothing at all like that happened with exponential families. The theory accrued slowly over a period extending roughly between 1932 and 1970. Applications lagged behind, a turning point being the advent of logistic regression and McCullagh and Nelder's 1983 book on generalized linear models.

A salient fact is that no one person is credited with the development of exponential families, though it will be clear from these notes that Fisher's work was instrumental. The name "exponential familes" is comparatively recent. Until the late 1950s they were often referred to as "Koopman–Darmois–Pitman" families (crediting three prominent statisticians working separately in three different countries); the awkward nomenclature suggests only minor importance being attached to the ideas.

Figure 1 gives a rough schematic history of Twentieth Century statistics. The inner circle represents normal theory, the preferred venue of classical methodology. Exact inference – t tests, F tests, chi-squared statistics, ANOVA, multivariate analysis – was feasible inside the circle. Outside the circle was a general theory based on large-sample asymptotic approximations involving Taylor series, Edgeworth expansions, and the central limit theorem. A few special exact results lay outside the normal circle, relating to especially tractable distributions such as the binomial, Poisson, gamma and beta families. These are the figure's green stars. A happy surprise, though a slowly emerging one beginning in the 1930s, was that the special cases were all examples of a powerful general construction, *exponential families*, the intermediate circle in Figure 1. Within this circle, "almost exact" inferential calculations are possible, where any necessary approximations can be pictured in simple geometric diagrams. Such diagrams play a major role in what follows.

Two complementary types of mathematical development can be labeled

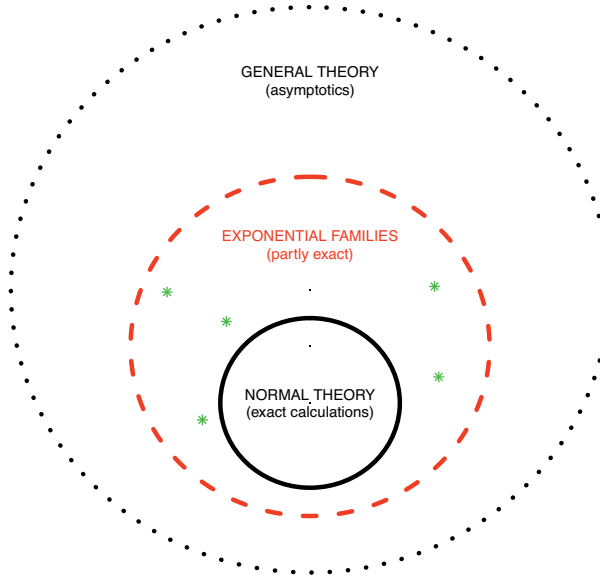


Figure 1 Three levels of statistical modeling.

as "theorem-proof" and "descriptive". The former has a worst-case aspect: "What I stated remains true even under the worst possibility of what I've allowed." Descriptive mathematics, of the kind encountered in introductions to calculus or linear algebra, is less pessimistic: disregarding pathologies, interest centers on the broad central run of useful results. This book pursues the theory of exponential families from a descriptive point of view, aiming at the parts of the theory most useful for applications.

Exponential families, used flexibly, can gracefully bridge the gap between statistical theory and its practice. These notes collect a large amount of material useful in statistical applications, but also of value to the theoretician trying to frame a new situation without immediate recourse to asymptotics. My own experience has been that when I can put a problem, applied or theoretical, into an exponential family framework, a solution is often imminent. There are almost no proofs in what follows, but hopefully enough motivation and heuristics to make the results believable if not obvious. References are given when this doesn't seem to be the case. Readers who desire a more thorough approach can look in Sundberg's excellent text, *Statistical Modelling by Exponential Families* (2019), or Brown's IMS monograph, *Fundamentals of Statistical Exponential Families* (1986).

1

One-parameter Exponential Families

The basic unit of probability theory is a probability distribution. The basic unit of statistical inference is a *family* of probability distributions. Dating from the time of Laplace and Gauss, the one-dimensional normal family[1]

$$x \sim \mathcal{N}(\mu, \sigma^2), \tag{1.1}$$

[1] Equation (1.1) means that the real-valued random variable x has density $\exp\{-(x-\mu)^2/\sigma^2\} \cdot (2\pi\sigma^2)^{-1/2}$ on the real line.

with $\mu \in (-\infty, \infty)$ and σ^2 positive, has played a dominant role in both theory and practice. A strong desire to go beyond normal models fueled the development of exponential family theory. One-parameter exponential families are useful in their own right, and crucial to understanding the multiparameter exponential families of Parts 2 through 5. Here we will present the general one-parameter family theory, and show how it plays out in familiar contexts such as the Poisson, binomial, normal, and gamma distributions.

1.1 Definitions, Notation, and Terminology

This section reviews the basic definitions for exponential families. An exponential family is a set of probability densities $\mathcal{G}$, "density" here including the possibility of discrete atoms (as in the family of binomial densities). A *one-parameter exponential family* has densities $g_\eta(y)$ of the form

$$\mathcal{G} = \left\{ g_\eta(y) = e^{\eta y - \psi(\eta)} g_0(y) m(dy),\ \eta \in A,\ y \in \mathcal{Y} \right\}, \tag{1.2}$$

where A and $\mathcal{Y}$ are subsets of the real line $\mathcal{R}^1$.

There is a more-or-less standard terminology for the elements of (1.2):

- η is the *natural* or *canonical* parameter; in familiar families like the Poisson and binomial, it often isn't the parameter we are used to working with.
- y is the *sufficient* or *natural* statistic, a name that will be more meaningful when we discuss repeated sampling situations; in many cases (the more interesting ones) $y = y(x)$ is a function of an observed data set x (as in the binomial example below); y takes values in its sample space $\mathcal{Y}$.
- The densities in $\mathcal{G}$ are defined with respect to some *carrying measure* $m(dy)$, such as the uniform measure on $[-\infty, \infty]$ for the normal family, or the discrete measure putting weight 1 on the non-negative integers ("counting measure") for the Poisson family. Usually $m(dy)$ won't be indicated in our notation. We will call $g_0(y)$ the *carrying density*.
- $\psi(\eta)$ in (1.2) is the *normalizing function* or *cumulant generating function*; it scales the densities $g_\eta(y)$ to integrate to 1 over sample space $\mathcal{Y}$,

$$\int_{\mathcal{Y}} g_\eta(y) m(dy) = \int_{\mathcal{Y}} e^{\eta y} g_0(y) m(dy) / e^{\psi(\eta)} = 1. \tag{1.3}$$

- The *natural parameter space* A consists of all η for which the integral

on the right is finite,

$$A = \left\{ \eta : \int_{\mathcal{Y}} e^{\eta y} g_0(y) m(dy) < \infty \right\}. \tag{1.4}$$

Homework 1.1 Use convexity to prove that if η_1 and $\eta_2 \in A$ then so does any point in the interval $[\eta_1, \eta_2]$ (implying that A is a possibly infinite interval in $\mathcal{R}^1$).

Homework 1.2 We can reparameterize $\mathcal{G}$ in terms of $\tilde{\eta} = c\eta$ and $\tilde{y} = y/c$. Explicitly describe the reparameterized densities $\tilde{g}_{\tilde{\eta}}(\tilde{y})$.

Suppose $g_0(y)$ is any given positive function on a subset $\mathcal{Y}$ of the real line. We can construct an exponential family $\mathcal{G}$ through $g_0(y)$ by "tilting" it exponentially,

$$g_\eta(y) \propto e^{\eta y} g_0(y), \tag{1.5}$$

and then renormalizing $g_\eta(y)$ to integrate to 1,

$$g_\eta(y) = e^{\eta y - \psi(\eta)} g_0(y), \qquad \text{where } e^{\psi(\eta)} = \int_{\mathcal{Y}} e^{\eta y} g_0(y) m(dy). \tag{1.6}$$

The space A is all values of η such that the integral is finite. It seems like we might employ other tilting functions, say

$$g_\eta(y) \propto \frac{1}{1 + \eta|y|} g_0(y), \tag{1.7}$$

but only exponential tilting gives convenient properties under independent sampling.

If η_0 is any point in A we can write

$$g_\eta(y) = \frac{g_\eta(y)}{g_{\eta_0}(y)} g_{\eta_0}(y) = e^{(\eta - \eta_0)y - (\psi(\eta) - \psi(\eta_0))} g_{\eta_0}(y). \tag{1.8}$$

This is the same exponential family, now represented with

$$\eta \longrightarrow \eta - \eta_0, \quad \psi \longrightarrow \psi(\eta) - \psi(\eta_0), \quad \text{and} \quad g_0 \longrightarrow g_{\eta_0}. \tag{1.9}$$

Any member $g_{\eta_0}(y)$ of $\mathcal{G}$ can be chosen as the carrier density, with all the other members as exponential tilts of g_{η_0}. *Important*: the sample space $\mathcal{Y}$ is the *same* for all members of $\mathcal{G}$, and all put positive probability on every point in $\mathcal{Y}$. The members of $\mathcal{G}$ are absolutely continuous with respect to each other, which greatly reduces the opportunities for pathologies in exponential families.

The Poisson Family

As an important first example we consider the Poisson family. A Poisson random variable Y having expectation $\mu > 0$ takes values on the nonnegative integers $\mathcal{Z}_+ = \{0, 1, \ldots\}$,

$$\Pr_\mu\{Y = y\} = e^{-\mu}\mu^y/y!, \qquad \text{for } y \in \mathcal{Z}_+. \tag{1.10}$$

The densities $e^{-\mu}\mu^y/y!$, taken with respect to counting measure on $\mathcal{Y} = \mathcal{Z}_+$, can be written in exponential family form as

$$g_\eta(y) = e^{\eta y - \psi(\eta)} g_0(y) \begin{cases} \eta = \log\mu & (\mu = e^\eta) \\ \psi(\eta) = e^\eta & (= \mu) \\ g_0(y) = 1/y!. \end{cases} \tag{1.11}$$

(Here $g_0(y)$ is not a member of $\mathcal{G}$, and is not even a proper density.)

Homework 1.3 (a) Rewrite $\mathcal{G}$ so that $g_0(y)$ corresponds to the Poisson distribution with $\mu = 1$.

(b) Carry out the numerical calculations that tilt Poi(12), seen in Figure 1.1, into Poi(6).

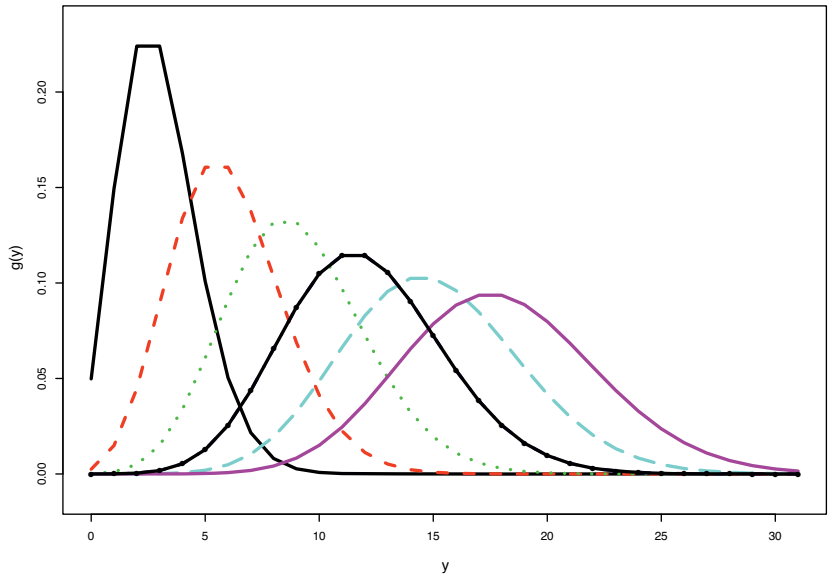


Figure 1.1 Poisson densities for $\mu = 3, 6, 9, 12, 15, 18$; heavy curve with dots for $\mu = 12$.

Even though the mathematics in (1.11) is straightforward, it is still a little surprising to see that any Poisson density is a simple exponential tilt of any other.

1.2 Moment Relationships

The name *cumulant generating function* for the normalizer $\psi(\eta)$ reflects an older methodology for finding expectations, variances, and higher-order moments. The methodology is particularly useful and easy to apply within exponential families.

Expectation and Variance

Differentiating $\exp\{\psi(\eta)\} = \int_{\mathcal{Y}} e^{\eta y} g_0(y) m(dy)$ with respect to η, and indicating differentiation by dots, gives

$$\dot{\psi}(\eta) e^{\psi(\eta)} = \int_{\mathcal{Y}} y e^{\eta y} g_0(y) m(dy) \tag{1.12}$$

and

$$\left(\ddot{\psi}(\eta) + \dot{\psi}(\eta)^2\right) e^{\psi(\eta)} = \int_{\mathcal{Y}} y^2 e^{\eta y} g_0(y) m(dy). \tag{1.13}$$

(The dominated convergence conditions for differentiating inside the integral are always satisfied inside exponential families; see Theorem 2.2 of Brown, 1986.) Multiplying by $\exp\{-\psi(\eta)\}$ gives expressions for the expectation μ_η and variance V_η of Y,

$$\dot{\psi}(\eta) = \mu_\eta = E_\eta\{Y\}, \tag{1.14}$$

$$\ddot{\psi}(\eta) = V_\eta = \mathrm{Var}_\eta\{Y\}, \tag{1.15}$$

where E_η and Var_η indicate expectation and variance under density g_η. V_η is greater than 0, implying that $\psi(\eta)$ has a positive second derivative everywhere, in other words, that $\psi(\eta)$ is convex. Except in trivial cases, the variance V_η is positive for all $\eta \in A$.

Notice that

$$\dot{\mu} = \frac{d\mu}{d\eta} = V_\eta > 0.$$

The mapping from η to μ is 1:1 increasing and infinitely differentiable. We can index the family $\mathcal{G}$ just as well with μ, the *expectation parameter*, as with η. Functions like $\psi(\eta)$, E_η, and V_η can just as well be thought of as

functions of μ. We will sometimes write ψ, V, etc. when it's not necessary to specify the argument. Notations such as V_μ formally mean $V_{\eta(\mu)}$.

Note Suppose that ζ is a parameter that can be defined as a function of either η or μ,

$$\zeta = h(\eta) = H(\mu).$$

Let $\dot{h} = dh/d\eta$ and $H' = dH/d\mu$. Then

$$H' = \dot{h}\frac{d\eta}{d\mu} = \frac{\dot{h}}{V}. \tag{1.16}$$

Skewness and Kurtosis

The first two moments of a random variable Y describe its expectation and variance. The third and fourth moments give its *skewness* and *kurtosis*, valuable for higher-order asymptotic approximations. For instance, a first-order Edgeworth expansion says that

$$\Pr\{Y \le \text{ median } (Y)\} \doteq 0.5 + \frac{1}{6\sqrt{2\pi}} \text{ SKEWNESS } (Y),$$

while the second-order approximation also involves Y's kurtosis.

A pre-computer technology, *cumulants*[2] are certain linear combinations of moments that are easy to deal with in repeated sampling situations (Section 1.3). $\psi(\eta)$ is the *cumulant generating function* for g_0 and $\psi(\eta) - \psi(\eta_0)$ is the CGF for $g_{\eta_0}(y)$, that is,

$$e^{\psi(\eta)-\psi(\eta_0)} = \int_{\mathcal{Y}} e^{(\eta-\eta_0)y} g_{\eta_0}(y) m(dy).$$

By definition, the Taylor series for $\psi(\eta) - \psi(\eta_0)$ has the cumulants k_j of $g_{\eta_0}(y)$ as its coefficients,

$$\psi(\eta) - \psi(\eta_0) = k_1(\eta - \eta_0) + \frac{k_2}{2}(\eta - \eta_0)^2 + \frac{k_3}{6}(\eta - \eta_0)^3 + \cdots .$$

[2] Cumulants add correctly under independent sampling: if X and Y are independent then the jth cumulant of $X + Y$ is the sum of their jth cumulants, this holding for all j. This isn't true for central jth moments $E_0\{Y - \mu_0\}^j$ for $j > 3$. Cumulants are an algebraic computational tool for simplifying higher-order moment relationships, but here we will never go beyond $j = 4$. Older texts, such as Kendall and Stuart (1958), tabulate the relations of cumulants and moments up to $j = 10$.

Equivalently, letting dots indicate derivatives,

$$\dot{\psi}(\eta_0) = k_1 \quad (= \mu_0), \qquad \ddot{\psi}(\eta_0) = k_2 \quad (= V_0),$$
$$\dddot{\psi}(\eta_0) = k_3 \quad \left(= E_0\{Y - \mu_0\}^3\right),$$
$$\ddddot{\psi}(\eta_0) = k_4 \quad \left(= E_0\{Y - \mu_0\}^4 - 3V_0^2\right),$$

etc., where $k_1, k_2, k_3, k_4, \ldots$ are the cumulants of g_{η_0}.

A real-valued random variable Y has skewness and kurtosis defined by

$$\text{SKEWNESS}(Y) = \frac{E(Y - EY)^3}{(\text{Var}(Y))^{3/2}} \equiv \text{“}\gamma\text{”} = \frac{k_3}{k_2^{3/2}}$$

and

$$\text{KURTOSIS}(Y) = \frac{E(Y - EY)^4}{(\text{Var}(Y))^2} - 3 \equiv \text{“}\delta\text{”} = \frac{k_4}{k_2^2}.$$

Putting this together, if $Y \sim g_\eta(\cdot)$ is an exponential family, then

$$Y \sim \Big[\underset{\substack{\uparrow \\ \text{expectation}}}{\dot{\psi}}, \quad \underset{\substack{\uparrow \\ \text{standard} \\ \text{deviation}}}{\ddot{\psi}^{1/2}}, \quad \underset{\substack{\uparrow \\ \text{skewness}}}{\dddot{\psi}/\ddot{\psi}^{3/2}}, \quad \underset{\substack{\uparrow \\ \text{kurtosis}}}{\ddddot{\psi}/\ddot{\psi}^2} \Big], \tag{1.17}$$

where the derivatives are taken at η.

For the Poisson family

$$\psi = e^\eta = \mu,$$

so all the cumulants equal μ

$$\dot{\psi} = \ddot{\psi} = \dddot{\psi} = \ddddot{\psi} = \mu,$$

giving

$$Y \sim \Big[\underset{\substack{\uparrow \\ \text{exp}}}{\mu}, \quad \underset{\substack{\uparrow \\ \text{st dev}}}{\sqrt{\mu}}, \quad \underset{\substack{\uparrow \\ \text{skew}}}{1/\sqrt{\mu}}, \quad \underset{\substack{\uparrow \\ \text{kurt}}}{1/\mu} \Big]. \tag{1.18}$$

A Useful Result

Continuing to use dots for derivatives with respect to η and primes for derivatives with μ, notice that

$$\gamma = \frac{\dddot{\psi}}{\ddot{\psi}^{3/2}} = \frac{\dot{V}}{V^{3/2}} = \frac{V'}{V^{1/2}} \tag{1.19}$$

(using $H' = \dot{h}/V$). Therefore

$$\gamma = 2\left(\sqrt{V}\right)' = 2\frac{d}{d\mu}\,\mathrm{sd}_\mu, \tag{1.20}$$

where $\mathrm{sd}_\mu = V_\mu^{1/2}$ is the standard deviation of y. In other words, $\gamma/2$ is the rate of change of sd_μ with respect to μ; this plays a role in the theory of bootstrap confidence intervals (Part 5).

Homework 1.4 Show that

$$\text{(a)}\ \ \delta = V'' + \gamma^2 \quad \text{and} \quad \text{(b)}\ \ \gamma' = \frac{\delta - {}^3\!/_2\gamma^2}{\mathrm{sd}}.$$

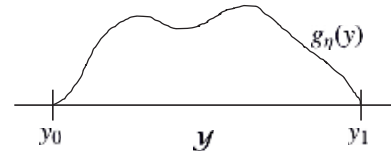
Note The classical exponential families – binomial, Poisson, normal, etc. – are those with closed-form CGFs ψ, yielding neat expressions for means, variances, skewnesses, and kurtoses.

Modern computing power lets us work with general exponential families where results like (1.17) can be exploited numerically, no matter what the form of $\psi(\eta)$.

Unbiased Estimate of η

By definition y is an unbiased estimate of μ (and, in fact, by completeness the only unbiased estimate of form $t(y)$). What about η?

- Let $l_0(y) = \log g_0(y)$ and $l_0'(y) = dl_0(y)/dy$.
- Suppose $\mathcal{Y} = [y_0, y_1]$ (a possibly infinite interval) and that $m(y) = 1$ for all $y \in \mathcal{Y}$.



Lemma 1.1

$$E_\eta\{-l_0'(y)\} = \eta - \left(g_\eta(y_1) - g_\eta(y_0)\right).$$

Homework 1.5 Prove Lemma 1.1. (*Hint*: Integration by parts.)

So, if $g_\eta(y) = 0$ (or $\to 0$) at the extremes of $\mathcal{Y}$, then $-l_0'(y)$ is a unbiased estimate of η.

Homework 1.6 Numerically calculate values of $-l_0'(y)$ to estimate η using Lemma 1.1 for $y \sim \mathrm{Poi}(\mu)$. Does it work?

1.3 Repeated Sampling

One-parameter exponential families have a crucial property that makes them simple to deal with, both in theory and practice: in repeated sampling situations, they retain one-parameter exponential family structure.[3]

Suppose $y_1, \ldots, y_n$ is an independent and identically distributed (i.i.d.) sample from an exponential family $\mathcal{G}$:

$$y_1, \ldots, y_n \overset{\text{iid}}{\sim} g_\eta(\cdot), \tag{1.21}$$

for an unknown value of the parameter $\eta \in A$. The density of $\boldsymbol{y} = (y_1, \ldots, y_n)$ is

$$\begin{aligned}\prod_{i=1}^{n} g_\eta(y_i) &= e^{\sum_1^n (\eta y_i - \psi)} \prod_{i=1}^{n} g_0(y_i) \\ &= e^{n(\eta \bar{y} - \psi)} \prod_{i=1}^{n} g_0(y_i),\end{aligned}$$

where $\bar{y} = \sum_{i=1}^n y_i/n$. Letting $g_\eta^{(n)}(\boldsymbol{y})$ indicate the density of $\boldsymbol{y}$ with respect to $\prod_{i=1}^n m(dy_i)$,

$$g_\eta^{(n)}(\boldsymbol{y}) = e^{n(\eta \bar{y} - \psi(\eta))} \prod_{i=1}^{n} g_0(y_i). \tag{1.22}$$

This is a one-parameter exponential family, with:

- natural parameter $\eta^{(n)} = n\eta$ (so $\eta = \eta^{(n)}/n$);
- sufficient statistic $\bar{y} = \sum_1^n y_i/n$ ($\bar{\mu} = E_{\eta(n)}\{\bar{y}\} = \mu$);
- normalizing function $\psi^{(n)}(\eta^{(n)}) = n\psi(\eta^{(n)}/n)$;
- carrier density $\prod_{i=1}^n g_0(y_i)$ (with respect to $\prod m(dy_i)$).

Homework 1.7 Show that, in the bracket notation of (1.17),

$$\bar{y} \sim \left[\mu, \sqrt{\frac{V}{n}}, \frac{\gamma}{\sqrt{n}}, \frac{\delta}{n}\right].$$

Note In the following, we usually index the parameter space by η rather than $\eta^{(n)}$.

[3] The older name, "Koopman–Darmois–Pitman" families, came from the separate efforts of the three authors to show that, under mild conditions, *only* definition (1.2) allowed this kind of sufficiency property.

Notice that $\boldsymbol{y}$ is now a vector, and that the tilting factor $e^{\eta^{(n)}\bar{y}}$ is tilting the *multivariate* carrier density $\prod_1^n g_0(y_i)$. This is still a one-parameter exponential family because the tilting is in a single direction, along $\mathbf{1} = (1, \ldots, 1)$.

The sufficient statistic $\bar{y}$ also has a one-parameter exponential family of densities,

$$g_\eta^{(n)}(\bar{y}) = e^{n(\eta\bar{y}-\psi)} g_0^{(n)}(\bar{y}),$$

where $g_0^{(n)}(\bar{y})$ is the g_0 density of $\bar{y}$ with respect to $m^{(n)}(d\bar{y})$, the induced carrying measure.

The density (1.22) can also be written (ignoring the carrier) as

$$e^{\eta S - n\psi}, \qquad \text{where } S = \sum_{i=1}^{n} y_i.$$

This moves a factor of n from the definition of the natural parameter to the definition of the sufficient statistic. For any constant c we can re-express an exponential family $\{g_\eta(y) = \exp(\eta y - \psi) g_0(y)\}$ by mapping η to η/c and y to cy. This tactic will be useful when we consider multiparameter exponential families.

Homework 1.8 $y_1, \ldots, y_n \overset{\text{iid}}{\sim} \text{Poi}(\mu)$. Describe the distributions of $\bar{y}$ and S, and say what are the exponential family quantities $(\eta, y, \psi, g_0, m, \mu, V)$ in both cases.

1.4 Maximum Likelihood Estimation in Exponential Familes

This section briefly reviews some basic results on maximum likelihood estimation (also with a few words about testing). The methodology is particularly simple in exponential families, as we will see. A good reference is Lehmann and Casella (1998), *Theory of Point Estimation*.

Suppose we observe a random sample $\boldsymbol{y} = (y_1, \ldots, y_n)$ from a member $g_\eta(y)$ of an exponential family $\mathcal{G}$,

$$y_i \overset{\text{iid}}{\sim} g_\eta(y), \qquad i = 1, \ldots, n,$$

and wish to estimate η. According to (1.22) in Section 1.3, the density of $\boldsymbol{y}$ is

$$g_\eta^{(n)}(\boldsymbol{y}) = e^{n[\eta\bar{y}-\psi(\eta)]} \prod_{i=1}^{n} g_0(y_i), \tag{1.23}$$

where $\bar{y} = \sum_1^n y_i/n$. The log likelihood function $l_\eta(\boldsymbol{y}) = \log g_\eta^{(n)}(\boldsymbol{y})$, with $\boldsymbol{y}$ fixed and η varying, is

$$l_\eta(\boldsymbol{y}) = n\left[\eta\bar{y} - \psi(\eta)\right],$$

giving *score function* $\dot{l}_\eta(\boldsymbol{y}) = \partial/\partial\eta\, l_\eta(y)$ equaling

$$\dot{l}_\eta(\boldsymbol{y}) = n(\bar{y} - \mu) \tag{1.24}$$

(remembering that $\dot{\psi}(\eta) = \partial/\partial\eta\,\psi(\eta)$ equals μ, the expectation parameter).

The maximum likelihood estimate (MLE) of η is the value $\hat{\eta}$ satisfying

$$\dot{l}_{\hat{\eta}}(\boldsymbol{y}) = 0.$$

Looking at (1.24), $\hat{\eta}$ is that η such that $\mu = \dot{\psi}(\eta)$ equals $\bar{y}$, that is,

$$\hat{\eta} : E_{\eta=\hat{\eta}}\left\{\overline{Y}\right\} = \bar{y}.$$

In other words, the MLE matches the theoretical expectation of $\overline{Y}$ to the observed mean $\bar{y}$.

We can also take the score function with respect to μ,

$$\frac{\partial}{\partial\mu}\, l_\eta(\boldsymbol{y}) = \dot{l}_\eta(\boldsymbol{y})\frac{\partial\eta}{\partial\mu} = \frac{\dot{l}_\eta(\boldsymbol{y})}{V} = \frac{n(\bar{y}-\mu)}{V}. \tag{1.25}$$

This gives

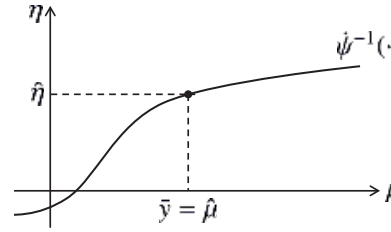
$$\frac{\partial}{\partial\mu}\, l_\eta(\boldsymbol{y})\Big|_{\mu=\bar{y}} = 0,$$

which shows that the MLE of μ is

$$\hat{\mu} = \bar{y}.$$

But $\mu = \dot{\psi}(\eta)$, a monotone one-to-one function; since MLEs map in the obvious way, we get

$$\hat{\eta} = \dot{\psi}^{-1}(\bar{y}).$$



For the Poisson $\hat{\eta} = \log\bar{y}$, and for the binomial, according to what we will see in Section 1.5,

$$\hat{\eta} = \log\left(\frac{\hat{\pi}}{1-\hat{\pi}}\right), \qquad \text{where } \hat{\pi} = \frac{y}{N}.$$

Fisher information is the expected square of the score function – which, since the expected score is always zero, is also its variance – denoted

$$i_\eta^{(n)} = nV$$

for the information for η. We write simply i_η for the case $n = 1$. The information for μ is

$$i_\eta^{(n)}(\mu) = n/V,$$

using (1.25), the notation being understood as the information for μ in a sample of size n, evaluated for $g_\eta(\boldsymbol{y})$. As always, V stands for V_η, the variance of a single observation y from $g_\eta(\cdot)$.

Let $\zeta = h(\eta)$ be any smooth function of η, also expressed as, say,

$$\zeta = H(\mu) = h\left(\dot{\psi}^{-1}(\mu)\right).$$

Then ζ has MLE $\hat{\zeta} = h(\hat{\eta}) = H(\hat{\mu})$ and score

$$\frac{\partial}{\partial \zeta} l_\eta(\boldsymbol{y}) = \frac{\dot{l}_\eta(\boldsymbol{y})}{\dot{h}(\eta)}.$$

Figure 1.2 and Table 1.1 show the MLE and information relationships.

In general, the Fisher information i_θ for a one-parameter family $f_\theta(x)$ has two expressions in terms of the first and second derivatives of the log likelihood,

$$i_\theta = E\left\{\left(\frac{\partial l_\theta}{\partial \theta}\right)^2\right\} = -E\left\{\frac{\partial^2 l_\theta}{\partial \theta^2}\right\}. \tag{1.26}$$

For $i_\eta^{(n)}$, the Fisher information for η in $\boldsymbol{y} = (y_1, \ldots, y_n)$, we have

$$-\ddot{l}_\eta(\boldsymbol{y}) = -\frac{\partial^2}{\partial \eta^2}\, n(\eta \bar{y} - \psi) = -\frac{\partial}{\partial \eta}\, n(\bar{y} - \mu) = nV_\eta = i_\eta^{(n)}, \tag{1.27}$$

so in this case $-\ddot{l}_\eta(\boldsymbol{y})$ gives $i_\eta^{(n)}$ without requiring an expectation over $\boldsymbol{y}$.

Homework 1.9 (a) Does

$$i_\eta^{(n)}(\mu) = -\frac{\partial^2}{\partial \mu^2}\, l_\eta(\boldsymbol{y})\ ?$$

(b) Does

$$i_{\eta=\hat{\eta}}^{(n)}(\mu) = -\frac{\partial}{\partial \mu^2}\, l_\eta(\boldsymbol{y})\Big|_{\eta=\hat{\eta}} \qquad (\hat{\eta} \text{ the MLE})\ ?$$

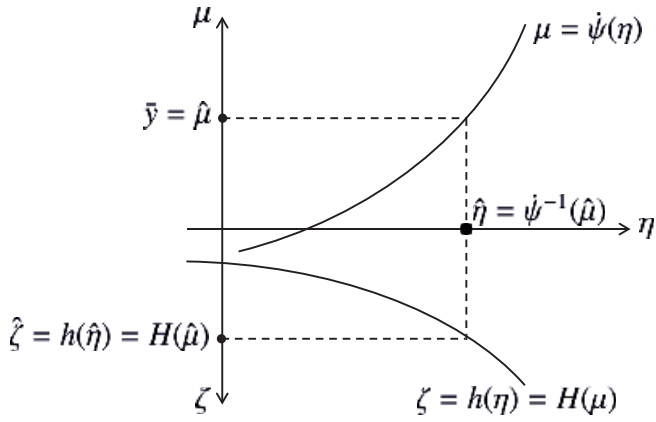


Figure 1.2 Maximum likelihood estimates.

Table 1.1 *Score functions and Fisher information.*

Score functions		Fisher information
$\eta:$	$\dot{l}_\eta(\mathbf{y}) = n(\bar{y} - \mu)$	$i_\eta^{(n)} = \mathrm{Var}_\eta\left[\dot{l}_\eta(\mathbf{y})\right] = nV = ni_\eta$
$\mu:$	$\dfrac{\partial l_\eta(\mathbf{y})}{\partial \mu} = \dfrac{n(\bar{y} - \mu)}{V}$	$i_\eta^{(n)}(\mu) = \dfrac{n}{V} = ni_\eta(\mu)$
$\zeta:$	$\dfrac{\partial l_\eta(\mathbf{y})}{\partial \zeta} = \dfrac{n(\bar{y} - \mu)}{\dot{h}(\eta)}$	$i_\eta^{(n)}(\zeta) = \dfrac{nV}{\dot{h}(\eta)^2} = ni_\eta(\zeta)$

Cramér–Rao Lower Bound

The Cramér–Rao lower bound (CRLB) for an unbiased estimator $\bar{\zeta}$ of a general parameter $\zeta = h(\eta)$ is

$$\mathrm{Var}_\eta(\bar{\zeta}) \geq \frac{1}{i_\eta^{(n)}(\zeta)} = \frac{\dot{h}(\eta)^2}{nV_\eta}. \tag{1.28}$$

For $\zeta \equiv$ the expectation parameter μ we get

$$\mathrm{Var}(\bar{\mu}) \geq \frac{V_\eta^2}{nV_\eta} = \frac{V_\eta}{n}. \tag{1.29}$$

In this case the MLE $\hat{\mu} = \bar{y}$ is unbiased and achieves the CRLB. This happens only for μ or linear functions of μ, and not for η, for instance. The regularity conditions necessary for the CRLB are almost always satisfied