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Solutions Manual for Actuarial Mathematics for Life Contingent Risks

THIRD EDITION

David C. M. Dickson, Mary R. Hardy and Howard R. Waters

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Solutions Manual for Actuarial Mathematics for Life Contingent Risks

This must-have manual provides detailed solutions to all of the 300 exercises in Dickson, Hardy and Waters' *Actuarial Mathematics for Life Contingent Risks*, Third Edition. This groundbreaking text on the modern mathematics of life insurance is required reading for the Society of Actuaries' (SOA) LTAM Exam. The new edition treats a wide range of newer insurance contracts such as critical illness and long-term care insurance; pension valuation material has been expanded; and two new chapters have been added on developing models from mortality data and on changing mortality. Beyond professional examinations, the textbook and solutions manual offer readers the opportunity to develop insight and understanding through guided hands-on work, and also offer practical advice for solving problems using straightforward, intuitive numerical methods. Companion Excel spreadsheets illustrating these techniques are available for free download.

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SOLUTIONS MANUAL FOR ACTUARIAL MATHEMATICS FOR LIFE CONTINGENT RISKS

THIRD EDITION

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Contents

Preface

This manual presents solutions to all exercises from

> *Actuarial Mathematics for Life Contingent Risks*, third edition (AMLCR),
> by David C. M. Dickson, Mary R. Hardy, Howard R. Waters,
> Cambridge University Press, 2020
> ISBN 9781108478083

It should be read in conjunction with the spreadsheets posted at the website `www.cambridge.org\9781108747615` which contain details of the calculations required. However, readers are encouraged to construct their own spreadsheets before looking at the authors' approach. Spreadsheet solutions are provided for all the Excel-based exercises in AMLCR.

In some cases the answers in the manual will differ from answers calculated using tables such as those provided in Appendix D of AMLCR. The differences arise from rounding errors. The numbers given in this manual are calculated directly, without using rounded table values, unless otherwise indicated.

From time to time, updates to this manual may appear at `www.cambridge.org\9781108747615`.

Solutions for Chapter 1

1.1 The principle of charging in advance for life insurance is to eliminate the potential for policyholders to benefit from short-term life insurance cover without paying for it. Suppose premiums were payable at the end of the policy year. A life could sign up for the insurance, and lapse the contract at the end of the year. The life would have benefited from free insurance cover for that year.

In addition, life insurance involves significant acquisition expenses. The first premium is used to meet some or all of these expenses.

Background note: The fact that the insurance for a policyholder did not result in a claim does not make it free to the insurer. The insurer's view is of a portfolio of contracts. Suppose 100 people buy term life insurance for one year, with a sum insured of \$1000, at a premium of \$11 each. The insurer expects a mortality rate of 1%, which means that, on average, one life out of the 100 dies. If all the policyholders pay their first year's premiums in advance, and one life dies, then the insurer receives \$1100 (plus some interest) and pays out \$1000. On the other hand, if premiums were due at the year end, it is possible that many of the 99 expected to survive might decide not to pay. It would be difficult and expensive for the insurer to pursue payment. The policyholders have benefited collectively from the insurance and the insurer has not been appropriately compensated.

1.2 The purpose of whole life insurance is to provide tax efficient benefits on the death of the policyholder. However, the ability of the policyholder to maintain their policies, by paying the premiums on time, may decline with age, firstly because the policyholder's income may decline, so that finding the funds to pay premiums becomes harder, and secondly because at older ages individuals become less able to manage their financial affairs, due to age-related cognitive decline. So some insurers design products that obviate the need for policyholders to actively maintain their policies in old age by removing the need to pay premiums at older ages.

1.3 An insurance policy is lapse-supported if excess funds from lapsed contracts are used to subsidize the remaining policies. In term insurance, when a policyholder lapses there is no cash value payable, so there may be excess funds available from lapsing policies. However, for most common forms of term insurance, premiums are relatively low, and the funds released from lapsing policies are also small. Furthermore, because the premiums are low, and term insurance is typically bought to provide for the policyholder's family in the event of the policyholder's death, we do not expect a very significant number of these policies to lapse.

We also note that lapses in the early stages of the contract generate losses, not profits, as the insurer needs several years of premiums (typically three to seven) to recoup the initial expenses associated with writing the policy.

So pricing allowing for lapses may not significantly reduce the premium, and hence, although term insurance can be written and priced as a lapse-supported product, in practice the impact would be so small that it is typically not.

Note though that some niche term insurance policies sold for very long terms are lapse-supported. In Canada, some insurers sold term insurance policies terminating at age 100. These policies were essentially whole life policies without cash values, and were specifically designed to be lapse-supported.

1.4 Full life care provides for level fees (other than inflationary increases) throughout the resident's future lifetime in a CCRC, for residents who enter into Independent Living Units. The CCRC's costs associated with the care of each resident are expected to rise steeply as they progress through the different levels of care, so the full life care fee structure involves higher fees than costs in the early years, and lower fees than costs in later years.

Modified life care allows fees to increase when the resident moves between stages, but fees are level within each stage. The increase in fees applied when the resident moves from a lower level of care to a higher level is less than the increase in costs, so that the increase is predictable and manageable for the resident.

The fee-for-service plan sets fees approximately equal to the CCRC's costs at each stage of care. In the early years, a life entering an Independent Living Unit would be expected to have the lowest level of fees, and as they move through to higher levels of care, the fees will rise substantially.

Hence, the initial fees for a life entering an Independent Living Unit will be highest under the full life care structure, and lowest under the fee-for-service structure, with modified life care falling in the middle.

1.5 (a) The insurer will calculate the premium for a term or whole life insurance policy assuming that the policyholder is in relatively good health; otherwise, if the insurer assumed that all purchasers were unhealthy, the cost of insurance would be prohibitive to those customers who are healthy. The assumption then is that claims will be relatively rare in the first few years of insurance, especially since most policies are sold to lives in their 30s and 40s.

This means that the price is too low for a life who is very unwell, for whom the risk of a claim shortly after purchase might be 10 or 100 times greater than for a healthy life. The insurer therefore needs evidence that the purchaser is in good health, to avoid the risk that insurance is bought too cheaply by lives who have a much higher probability of a claim.

The objective of underwriting is to produce a relatively homogeneous insured population when policies are issued. The risk that the policyholder purchases the insurance because they are aware that their individual risk is greater than that of the insured population used to calculate the premium is an example of adverse selection risk. Underwriting is a way of reducing the impact of adverse selection for life insurance.

Adverse selection for an annuity purchaser works in the other direction – a life might buy an annuity if they considered their mortality was lighter than the general population. But, since adverse selection is likely to affect all lives purchasing annuities, more or less, it does not generate heterogeneity, and the impact can be managed by assuming lower overall mortality rates for annuitants.

In addition, the difference in the net cost to the insurer arising from adverse selection will be smaller compared with the term insurance example.

(b) The insurer will be more rigorous with underwriting for term insurance than for whole life insurance because the potential financial consequence of adverse selection is greater. Note that the insurer expects few claims to arise from the term insurance portfolio. Premiums are small, relative to the death benefit, because the probability of payment of the death benefit is assumed to be small. For whole life insurance, premiums are substantially larger as payment of the death benefit is a certain event (ignoring surrenders). The only uncertainty is the timing of the benefit payment.

The main risk to the insurer is that a life with a very high mortality risk, much higher than the assumed insured population, purchases life insurance. It is likely in this case that the life will pay very few premiums, and the policy will involve a large death benefit payout with very little premium income. Since term insurance has much lower premiums for a given sum insured than whole life insurance, it is likely that such a policyholder would choose term insurance. Hence, the risk of adverse selection is greater for term insurance than for whole life insurance, and underwriting is used to reduce the adverse selection risk.

1.6 (a) Without term insurance, the homeowner's dependents may struggle to meet mortgage payments in the event of the homeowner's death. The lending company wishes to reduce as far as possible the risk of having to foreclose on the loan. Foreclosure is expensive for the lender and creates hardship for the homeowner's family at the worst possible time. Term insurance is used to pay off the mortgage balance in the event of the homeowner's death, thus avoiding the foreclosure risk for both the lender and the homeowner's family.

(b) If the homeowner is paying regular instalments of capital and interest to pay off the mortgage, then the term insurance sum insured will decrease as the loan outstanding decreases. The reduction in loan outstanding is slow in the early years of, say, a 25-year mortgage, but speeds up later. The reduction in the term insurance sum insured is therefore not linear. Different loan provisions, including interest-only loan periods, cliff-edge repayment schedules (where the interest is very low for some period and then increases substantially), fixed or variable interest rates, and fixed or variable repayment instalments will all affect the sum insured.

(c) In Section 1.5.2 it is noted that around 2%–3% of applicants for insurance are considered to be too high risk. If these lives are, in consequence, unable to purchase property, then that is a social cost for these lives that may not be acceptable.

1.7 In with-profit whole life insurance, the insurer invests the premiums, and excess investment returns over the minimum required to fund the original benefits are shared between the policyholders and the insurer.

With a cash bonus, the policyholder's share of profits can be paid out in cash, similar to a dividend on shares. In this case, the investments need to be realized (i.e. assets sold for cash). The payout is immediate.

With a reversionary bonus, the policyholder's share of profits is used to increase the sum insured. The assets can remain in the capital markets until the sum insured is due.

Cash Bonus System – Insurer Perspective
Advantages

- Bonuses are transparent and easy to explain to policyholders.
- It does not involve maintenance of records of payouts and does not impact schedules for surrender values.
- The prospect of cash bonuses may persuade policyholders to continue with their policies rather than surrender.

Disadvantages

- It creates a liquidity risk – that assets need to be sold to meet bonus expectations, possibly at unfavourable times.
- Investment proceeds are volatile; volatility in cash bonuses may be difficult to explain to policyholders. There may be a temptation to over-distribute in an attempt to smooth, that could cause long-term losses.
- There may be problems determining equitable payouts, resulting in possible policyholder grievances.

Cash Bonus System – Policyholder Perspective
Advantages

- Cash is immediate and it is easy to understand the distribution.

Disadvantages

- May not be tax efficient.
- The risks to the insurer may lead to under-distribution to avoid risk.
- Possible volatility of bonuses.

Reversionary Bonus System – Insurer Perspective
Advantages

- Assets remain invested as long as a policy is in force, reducing liquidity risk.
- Bonuses appear larger as they are generally delayed many years.
- Bonuses may not be paid in full if a policy is surrendered subsequently, allowing higher rates of bonus to be declared for remaining policyholders.
- Over-distribution can be mitigated with lower bonuses between the declaration year and the claim event.

Disadvantages

- More complex to value, to keep records.
- Policyholders may not understand the approach, and there may be resentment (e.g. on surrender).
- Difficult to determine an equitable distribution.
- Easy to over-declare, as profits are based on asset values which may subsequently decrease.
- It is difficult to reduce bonus rates, even when justified. This may lead to loss of new and existing business.

Reversionary Bonus System – Policyholder Perspective
Advantages

- It may be tax efficient to receive profit share with sum insured.
- The system allows more investment freedom for the insurer, with higher upside potential for the policyholder.

Disadvantages

- Difficult to understand, especially ‘super-compound’ systems.

- Possible loss of profit share on surrender.
- Opaque system of distribution. It is difficult to compare how different companies perform.

1.8 Under a simple reversionary bonus of 5% per year, the sum insured in year t, for $t = 1, 2, \ldots, 5$, is

$$50\,000\,(1 + 0.05(t - 1)).$$

Under a compound reversionary bonus of 5% per year, the sum insured in year t, for $t = 1, 2, \ldots, 5$, is

$$50\,000\,(1.05^{t-1}).$$

Consider now the super-compound reversionary bonus and let S_t denote the total sum insured in the tth year for $t = 1, 2, \ldots, 5$, with $S_1 = 50\,000$. The sum insured in year $t + 1$ will consist of the sum insured in year t (S_t), the bonus on the original sum insured ($0.05S_1$) and the bonus on previously declared bonus ($0.1(S_t - S_1)$). Thus

$$S_{t+1} = S_t + 0.05S_1 + 0.1(S_t - S_1) = 1.1S_t - 0.05S_1\,,$$

and so we can calculate the total sum insured recursively in this case. The table below shows rounded values of the different sums insured.

Year	Simple	Compound	Super-compound
1	50 000	50 000	50 000
2	52 500	52 500	52 500
3	55 000	55 125	55 250
4	57 500	57 881	58 275
5	60 000	60 775	61 603

Note that in each case the sum insured is 52 500 in year 2 as the bonus at the end of year 1 is 5% of the sum insured in year 1 under each bonus structure, but after that the different bonus structures yield different results, with the simple bonus giving the lowest sums insured, and the super-compound giving the highest.

1.9 The first period of sickness is from time 1.00 to time 1.25, and the second is from time 2.00 to time 3.50. These are separate periods of sickness because the off period is six months, and the time between these periods of sickness is nine months.

Because of the one-year waiting period, no sickness benefit is payable for the period of sickness from time 1.00 to time 1.25.

The third period of sickness commences 0.25 years (three months) after the end of the second period of sickness, so the third period of sickness is treated as a continuation of the second. So, allowing for the one-year waiting period which applies from time 2.00 to time 3.00, sickness benefit is payable from time 3.00 to time 3.50, and this benefit payment continues throughout the entire third period of sickness (for a total of $0.50 + 4.25 = 4.75$ years, so the five-year term is not attained).

As the final period of sickness commences 0.75 years after the third period of sickness ended, this is treated as a new period of sickness (due to the six-month off period) to which the one-year waiting period applies, meaning that benefit is payable from time 9.75 to time 10.

The following table summarises the payments.

Time from inception	Benefit payments
3.00–3.50	6 months at $2000 per month
3.75–8.00	51 months at $2000 per month
9.75–10.00	3 months at $2000 per month

1.10 For a comprehensive answer, we need to understand Andrew's age, health and family responsibilities and support. The answers for an average 65-year-old retiree in good health would be different from those for a 50-year-old retiree in poor health. Also, we should consider the impact of governmental benefits (old age pension, social security, health costs), and any potential support from family in the event that he faces financial ruin.

In the absence of more detailed information, we assume that Andrew is a person in average health at an average retirement age of, say, 65. We also assume that the $500 000 represents the capital on which he wishes to live reasonably comfortably for the remainder of his life. We also ignore tax issues, though these are likely to be very significant in this kind of decision in practice.

Consider the risks Andrew faces at retirement.

(1) Outliving his assets – this is the risk that at some point the funds are all spent and Andrew must live on whatever government benefit or family support that might be available.

(2) Inflation risk – that is, that his standard of living is gradually eroded by increases in the cost of living that are not matched by increases in his income.

(3) Catastrophe costs – this is the risk that a large liability arises and Andrew does not have the assets (or cannot access the assets) to meet the costs. Examples might include the cost of health care for Andrew or a dependent (where health care is not freely available); catastrophic uninsured liability; cost of long-term care in older age.

Andrew may also have some ‘wants’ – for example

(1) Bequest – Andrew may want to leave some assets to dependents if possible.

(2) Flexible spending – Andrew may want the freedom of full access to all his capital at all times.

We now consider the options listed in the question in light of the risks and potential ‘wants’ listed.

(a) With a level life annuity, Andrew is assured of income for his whole life, and eliminates the risk of outliving his assets. However, he retains the inflation risk, and he may not have sufficient assets to meet catastrophe costs. If he uses all his capital for an annuity, there will be no bequest funds available on his death, and no flexibility in spending during his lifetime.

(b) As in (a), Andrew will not outlive his assets, and this option also covers inflation risk to some extent. There may be some residual inflation risk, as the cost of living increases that Andrew is exposed to may differ from the inflation adjustments applied to his annuity. In order to purchase the cost of living cover, Andrew will receive a significantly lower starting annual payment than under option (a). All other issues are similar to those under option (a).

(c) A 20-year annuity-certain will offer a similar or slightly higher benefit to a life annuity for a 65-year-old man in average health. Andrew’s life expectancy might be around 18 years, so on average the annuity will be sufficient to give Andrew a life income and allow a small bequest. An annuity-certain can be reasonably easily converted to cash in the event of a catastrophe or a change in circumstances. However, there is a significant risk that Andrew will live more than 20 years, and it will be difficult to manage the dramatic change in income at such an advanced age.

(d) Investing the capital and living off the interest would involve much risk. The interest income will be highly variable, and will be insufficient to live on in some years. If Andrew invests the capital in safe, stable long-term bonds, he might make only 2%–3% after expenses (or less, this figure has been highly variable over the last 20 years) which would be insufficient if it is his only income. There is also reinvestment risk, as he could live longer than the longest income he could lock-in in the market, and there will be counterparty risk (that is, the risk that the borrower will default on the interest and capital) if his investment is not in solid risk-free assets.

If Andrew needs a higher income, he will have to take more risk. For example, he might invest in corporate bonds with counterparty risk, or he might put some of his capital in stocks, which have upside potential but downside risk. Using riskier investments would increase the volatility of his income and threaten his capital. If he invests heavily in shares, he may see negative returns in some years. This strategy just might not be sustainable.

Income would also not be inflation hedged, in general.

On the other hand, the capital would be accessible in the event of a catastrophe or for flexible spending (although that would raise the risk of outliving assets). This system would allow for a significant bequest, assuming that Andrew managed to live on the investment, but at the expense of income level and stability for Andrew. Also, Andrew would have the added complication of managing a portfolio of assets, or paying someone to manage them for him. On the other hand, purchasing an annuity involves substantial hidden expenses that would not be incurred under this option.

(e) \$40 000 is 8% of the capital. If this rate is higher than the interest rate achievable on capital, then Andrew will be drawing down the capital and risks outliving his assets. The income is not inflation hedged, but the system does allow spending flexibility. Other issues are as for option (d).

Solutions for Chapter 2

2.1 (a) $p_{x+3} = 1 - q_{x+3} = 0.98$.

(b) ${}_2p_x = p_x\ p_{x+1} = 0.99 \times 0.985 = 0.97515$.

(c) As ${}_3p_{x+1} = {}_2p_{x+1}\ p_{x+3}$, we have ${}_2p_{x+1} = 0.95/0.98 = 0.96939$.

(d) ${}_3p_x = p_x\ {}_3p_{x+1}\ /p_{x+3} = 0.95969$.

(e) ${}_{1|2}q_x = p_x\ (1 - {}_2p_{x+1}) = 0.03031$.

2.2 We have

$$\begin{aligned} e_x &= \sum_{t=1}^{\infty} {}_tp_x = p_x + \sum_{t=2}^{\infty} {}_tp_x \\ &= p_x + p_x \sum_{t=2}^{\infty} {}_{t-1}p_{x+1} \\ &= p_x \left(1 + \sum_{t=1}^{\infty} {}_tp_{x+1} \right) \\ &= p_x\,(1 + e_{x+1}). \end{aligned}$$

So $p_x = e_x/(1 + e_{x+1})$ and hence

$${}_3p_{60} = p_{60}\ p_{61}\ p_{62} = \frac{e_{60}}{1+e_{61}}\ \frac{e_{61}}{1+e_{62}}\ \frac{e_{62}}{1+e_{63}} = 0.93834.$$

2.3 (a) From Example 2.3,

$${}_{10}p_{40} = \exp\left\{ \frac{-B}{\log c}\, c^{40} \left(c^{10} - 1 \right) \right\} = 0.995078.$$

(b) Noting that ${}_tp_{40} = 1 - F_{40}(t)$, we have

$$\frac{d}{dt}\,{}_tp_{40} = -f_{40}(t) = -{}_tp_{40}\,\mu_{40+t}.$$

Evaluating this at $t = 10$ gives -0.000567.

2.4 The one-year survival probability is

$$\begin{aligned} p_x &= \exp\left\{-\int_0^1 (0.002 + 0.001t)\,dt\right\} \\ &= \exp\{-0.002 - 0.0005\} \\ &= 0.99750, \end{aligned}$$

giving $q_x = 0.00250$.

2.5 We have

$$\begin{aligned} e_{x:\overline{n}|} &= \mathrm{E}[\min(K_x, n)] \\ &= \sum_{k=0}^{n-1} k\ \Pr[K_x = k] + \sum_{k=n}^{\infty} n\ \Pr[K_x = k] \\ &= \sum_{k=0}^{n-1} k\,({}_{k|}q_x) + n\ \Pr[K_x \ge n] \\ &= \sum_{k=0}^{n-1} k\,({}_kp_x - {}_{k+1}p_x) + n\,{}_np_x \\ &= ({}_1p_x - {}_2p_x) + 2({}_2p_x - {}_3p_x) + \cdots + (n-1)({}_{n-1}p_x - {}_np_x) + n\,{}_np_x \\ &= {}_1p_x + {}_2p_x + \cdots + {}_np_x \\ &= \sum_{k=1}^{n} {}_kp_x\,. \end{aligned}$$

2.6 (a) $G(x)$ can be written as

$$G(x) = \frac{(90 - x)(x + 200)}{18\,000}\,.$$

At the limiting age, ω, we have $G(\omega) = 0$, and since $x \ge 0$, the limiting age is $\omega = 90$.

A more complete statement of the resulting survival function $S_0(x)$ then is

$$S_0(x) = \begin{cases} \dfrac{18\,000 - 110x - x^2}{18\,000} & \text{for } 0 \leq x < 90, \\ 0 & \text{for } x \geq 90. \end{cases}$$

(b) First, we have $S_0(0) = 1$. Next, we see that $S_0(x) = 0$ for all $x \geq 90$. Third, the derivative of $S_0(x)$ is

$$\begin{cases} \dfrac{-110 - 2x}{18\,000} & \text{for } 0 < x < 90, \\ 0 & \text{for } x > 90, \end{cases}$$

which is negative for $0 < x < \omega$, indicating that the survival function decreases from age 0 to the limiting age of 90.

Hence all three conditions for a survival function are satisfied.

(c) $S_0(20)/S_0(0) = 0.8556$.

(d) The survival function for $t < 70$ is

$$\begin{aligned} S_{20}(t) &= \frac{S_0(20+t)}{S_0(20)} \\ &= \frac{18\,000 - 110\,(20+t) - (20+t)^2}{18\,000 - 110\,(20) - 20^2} \\ &= \frac{15\,400 - 150\,t - t^2}{15\,400} \\ &= 1 - \frac{3t}{308} - \frac{t^2}{15\,400}. \end{aligned}$$

(e) $(S_0(30) - S_0(40))/S_0(20) = 0.1169$.

(f) $\mu_x = -S_0'(x)/S_0(x)$. Using part (b), for $x < 90$ we obtain

$$\begin{aligned} \mu_x &= \left(\frac{110 + 2x}{18\,000}\right)\left(\frac{18\,000}{18\,000 - 110x - x^2}\right) \\ &= \frac{110 + 2x}{18\,000 - 110x - x^2} \end{aligned}$$

so that $\mu_{50} = 0.021$.

2.7 (a) The survival function S_x is given by

$$\begin{aligned} S_x(t) &= \frac{S_0(x+t)}{S_0(x)} \\ &= \frac{e^{-\lambda(x+t)}}{e^{-\lambda x}} \\ &= e^{-\lambda t}. \end{aligned}$$

(b) $\mu_x = -\frac{d}{dx}\log S_0(x) = -\frac{d}{dx}\log e^{-\lambda x} = \frac{d}{dx}\lambda x = \lambda.$

(c) As ${}_tp_x = e^{-\lambda t}$, which is independent of x, we have

$$\begin{aligned} e_x &= \sum_{t=1}^{\infty} {}_tp_x = \sum_{t=1}^{\infty} e^{-\lambda t} \\ &= e^{-\lambda} + e^{-2\lambda} + e^{-3\lambda} + \cdots \\ &= \frac{e^{-\lambda}}{1-e^{\lambda}} \\ &= \frac{1}{e^{\lambda}-1}. \end{aligned}$$

(d) This lifetime distribution is unsuitable for human mortality as survival probabilities, and therefore expected future lifetimes, are independent of attained age. The force of mortality for this lifetime distribution is constant. The force of mortality for humans increases significantly with age.

2.8 (a) $f_0(x) = -S_0'(x) = 0.002\,x\,e^{-0.001x^2}$.

(b) $\mu_x = f_0(x)/S_0(x) = 0.002\,x$.

(c) ${}_{5|15}q_{65} = \frac{S_0(70) - S_0(85)}{S_0(65)} = 0.45937.$

2.9 Write

$${}_tp_x = \exp\left\{-\int_0^t \mu_{x+s}\,ds\right\} = \exp\left\{-\int_x^{x+t} \mu_s\,ds\right\}.$$

In this case, we are treating t as fixed and x as variable. Let

$$h(x) = -\int_x^{x+t} \mu_s\,ds\,.$$

Then

$$ {}_tp_x = e^{-h(x)} \Rightarrow \frac{d}{dx}\,{}_tp_x = -h'(x)\,e^{-h(x)} = -h'(x)\,{}_tp_x\,. $$

Now

$$ h'(x) = \frac{d}{dx}\int_x^{x+t} \mu_s\,ds = \frac{d}{dx}\left(\int_0^{x+t} \mu_s\,ds - \int_0^{x} \mu_s\,ds\right) = \mu_{x+t} - \mu_x\,, $$

giving the required result that

$$ \frac{d}{dx}\,{}_tp_x = {}_tp_x\,(\mu_x - \mu_{x+t})\,. $$

2.10 Writing ${}_tp_x = s^t\,g^{c^x(c^t-1)}$, we have

$$ \begin{aligned} \log {}_{10}p_{50} &= \log 0.974054 = 10\log s + c^{50}\,(c^{10}-1)\,\log g\,, \\ \log {}_{10}p_{60} &= \log 0.935938 = 10\log s + c^{60}\,(c^{10}-1)\,\log g\,, \\ \log {}_{10}p_{70} &= \log 0.839838 = 10\log s + c^{70}\,(c^{10}-1)\,\log g\,. \end{aligned} $$

Then

$$ \begin{aligned} \frac{\log {}_{10}p_{70} - \log {}_{10}p_{60}}{\log {}_{10}p_{60} - \log {}_{10}p_{50}} &= \frac{(c^{70}-c^{60})(c^{10}-1)\,\log g}{(c^{60}-c^{50})(c^{10}-1)\,\log g} \\ &= c^{10} \\ &= 2.7141. \end{aligned} $$

Hence $c = 1.105$.

2.11 (a) We have

$$ \mathring{e}_x = \int_0^{\infty} {}_tp_x\,dt = \int_0^{1} {}_tp_x\,dt + \int_1^{\infty} {}_tp_x\,dt\,. $$

For the first term, we use the fact that ${}_tp_x \le 1$ for all $t \ge 0$, so that

$$ \int_0^{1} {}_tp_x\,dt \le 1. $$

For the second term, by changing the limits of integration we have

$$\int_1^\infty {}_tp_x\,dt = \int_0^\infty {}_{r+1}p_x\,dr = \int_0^\infty p_x\,{}_rp_{x+1}\,dr$$
$$= p_x\,\mathring{e}_{x+1} \le \mathring{e}_{x+1}\,.$$

So $\mathring{e}_x \le 1 + \mathring{e}_{x+1}$ as required.

(b) We have

$$\mathring{e}_x = \int_0^\infty {}_tp_x\,dt = \int_0^1 {}_tp_x\,dt + \int_1^2 {}_tp_x\,dt + \int_2^3 {}_tp_x\,dt + \cdots\,.$$

Since ${}_tp_x$ is a decreasing function of t,

$$\int_{s-1}^{s} {}_tp_x\,dt \ge {}_sp_x\,, \text{for } s = 1, 2, \ldots,$$

so

$$\mathring{e}_x \ge p_x + {}_2p_x + {}_3p_x + \cdots = e_x.$$

Alternatively, let $\lfloor\ \rfloor$ denote the floor function. Then

$$\begin{aligned} K_x = \lfloor T_x \rfloor \Rightarrow K_x &\le T_x \\ \Rightarrow\ \mathrm{E}[K_x] &\le \mathrm{E}[T_x] \\ \Rightarrow\ e_x &\le \mathring{e}_x\,. \end{aligned}$$

(c) We know that $\mathring{e}_x$ is the expected value of the future lifetime of (x), i.e. $\mathrm{E}[T_x]$, and e_x is the expected number of complete years of future life of (x), i.e. $\mathrm{E}[K_x]$. If we assume that the difference between T_x and K_x is 0.5 years, on average (on the principle that it is roughly equally likely to be any number of years between 0 and 1), then we have

$$\mathring{e}_x \approx \frac{1}{2} + e_x\,.$$

For a more mathematical explanation, we can use the repeated trapezium rule for numerical integration as follows:

$$\begin{aligned}\mathring{e}_x &= \int_0^\infty {}_t p_x \, dt \\ &\approx \frac{1}{2}(1 + {}_1 p_x + {}_1 p_x + {}_2 p_x + {}_2 p_x + \cdots) \\ &\approx \frac{1}{2} + e_x \,.\end{aligned}$$

(d) It is almost always the case in practice that $\mathring{e}_x$ is a decreasing function of x, but, in principle, it need not be. Consider a hypothetical population where people die only at ages 1 or 50. Of all those born, precisely one half die at age 1 and the remainder all die at age 50. Then

$$\mathring{e}_0 = \frac{1}{2}(1 + 50) = 25.5 \qquad \text{and} \qquad \mathring{e}_2 = 48.$$

2.12 (a) We have

$$\begin{aligned}\mathring{e}_x &= \int_0^\infty {}_r p_x \, dr \\ &= \int_0^\infty \frac{S_0(x+r)}{S_0(x)} \, dr \\ &= \frac{1}{S_0(x)} \int_x^\infty S_0(t) \, dt.\end{aligned}$$

(b) Use the product rule to differentiate the answer to part (a):

$$\begin{aligned}\frac{d}{dx} \mathring{e}_x &= \frac{-S_0'(x)}{S_0(x)^2} \int_x^\infty S_0(t) \, dt + \frac{1}{S_0(x)} (-S_0(x)) \\ &= \left(\frac{-S_0'(x)}{S_0(x)} \right) \left(\frac{1}{S_0(x)} \int_x^\infty S_0(t) \, dt \right) - 1 \\ &= \mu_x \, \mathring{e}_x - 1\end{aligned}$$

since $\mu_x = -S_0'(x)/S_0(x)$.

(c) We have

$$\frac{d}{dx}\left(x + \overset{\circ}{e}_x\right) = 1 + \frac{d}{dx}\,\overset{\circ}{e}_x = \mu_x\,\overset{\circ}{e}_x > 0,$$

so that $x + \overset{\circ}{e}_x$ increases with x.

Note that $x + \overset{\circ}{e}_x$ is the expected age at death of a life, given that the life has survived to age x. This is clearly an increasing function of x.

2.13 (a) For fixed $n > 0$, note that $T \le t$ if $T_x \le n + t$ for $t \ge 0$. Hence for $t \ge 0$,

$$\Pr[T \le t] = \Pr[T_x \le n + t] = F_x(n + t).$$

For $t < 0$, $\Pr[T \le t] = 0$. Note that T has a mixed distribution with mass of probability $F_x(n)$ at $t = 0$; if (x) does not survive to age $x + n$ then (x) does not live for any period after age $x + n$.

(b) The probability density function of T at $t > 0$ is $f_x(n + t)$. Thus,

$$\begin{aligned}
\mathrm{E}[T] &= \int_0^\infty t\, f_x(n+t)\, dt \\
&= \int_n^\infty (r-n)\, f_x(r)\, dr \\
&= \int_n^\infty r\, {}_r p_x\, \mu_{x+r}\, dr - n\, S_x(n) \\
&= -\int_n^\infty r\left(\frac{d}{dr}\, {}_r p_x\right) dr - n\, S_x(n) \\
&= -\left(r\, {}_r p_x \Big|_n^\infty - \int_n^\infty {}_r p_x\, dr \right) - n\, S_x(n) \\
&= \int_n^\infty {}_r p_x\, dr \\
&= \overset{\circ}{e}_x - \overset{\circ}{e}_{x:\overline{n}|}
\end{aligned}$$

where we have used Assumption 2.2 from Chapter 2 of AMLCR and the fact that $S_x(n) = {}_n p_x$.

2.14 (a) We start by finding $S_x(t)$ for $0 < t \le \omega - x$ as

$$S_x(t) = \exp\left\{-\int_x^{x+t} \mu_r\,dr\right\} = \exp\left\{-\int_x^{x+t} \frac{dr}{\omega - r}\right\}$$

$$= \exp\left\{\int_x^{x+t} \frac{d}{dr}\log(\omega - r)\,dr\right\}$$

$$= \exp\{\log(\omega - x - t) - \log(\omega - x)\}$$

$$= \exp\left\{\log\frac{\omega - x - t}{\omega - x}\right\}$$

$$= \frac{\omega - x - t}{\omega - x}\,.$$

So

$$F_x(t) = 1 - S_x(t) = \frac{t}{\omega - x}\,,$$

which is the distribution function for the $U(0, \omega - x)$ distribution.

(b) As $T_x \sim U(0, \omega - x)$, we have $\mathring{e}_x = \mathrm{E}[T_x] = (\omega - x)/2$. We find e_x as

$$e_x = \sum_{t=1}^{\omega-x} {}_tp_x = \sum_{t=1}^{\omega-x}\left(1 - \frac{t}{\omega - x}\right)$$

$$= \omega - x - \frac{1}{\omega - x}\sum_{t=1}^{\omega-x} t\,.$$

As the sum of the first n integers is $n\,(n+1)/2$,

$$e_x = \omega - x - \frac{(\omega - x)(\omega - x + 1)}{2\,(\omega - x)} = \frac{1}{2}(\omega - x) - \frac{1}{2}.$$

Hence $\mathring{e}_x - e_x = \frac{1}{2}$. (Remark: we could have obtained the same result by setting the upper limit of summation in our expression for e_x as $\omega - x - 1$ since ${}_{\omega-x}p_x = 0$.)

Alternatively, note that

$$\Pr[K_x = k] = \frac{1}{\omega - x} \quad \text{for } k = 0, 1, 2, \ldots, \omega - x - 1$$

and so

$$e_x = \sum_{k=0}^{\omega-x-1} k\,\Pr[K_x = k] = \frac{1}{\omega - x}\sum_{k=0}^{\omega-x-1} k.$$

2.15 (a) For $0 < t \le 10$, the survival function is

$$S(t) = \exp\left\{-\int_0^t \mu_r \, dr\right\} = \exp\left\{-\int_0^t \frac{dr}{4\,(10-r)}\right\}$$

$$= \exp\left\{\frac{1}{4}\int_0^t \frac{d}{dr}\log(10-r)\,dr\right\}$$

$$= \exp\left\{\frac{1}{4}\left(\log(10-t) - \log 10\right)\right\}$$

$$= \exp\left\{\frac{1}{4}\log\frac{10-t}{10}\right\}$$

$$= \left(1 - \frac{t}{10}\right)^{1/4}.$$

(b) First,

$$\mathrm{E}[T] = \int_0^{10} S(t)\,dt = \int_0^{10}\left(1 - \frac{t}{10}\right)^{1/4} dt.$$

Making the substitution $y = 1 - t/10$ gives

$$\mathrm{E}[T] = 10\int_0^1 y^{1/4}\,dy = 10 \times \frac{4}{5}\,y^{5/4}\Big|_0^1 = 8.$$

Next, following Example 2.6,

$$\mathrm{E}[T^2] = 2\int_0^{10} t\,S(t)\,dt = 2\int_0^{10} t\left(1 - \frac{t}{10}\right)^{1/4} dt,$$

and the same substitution gives

$$\mathrm{E}[T^2] = 20\int_0^1 10\,(1-y)\,y^{1/4}\,dy$$

$$= 200\int_0^1 (y^{1/4} - y^{5/4})\,dy$$

$$= 200\left(\frac{4}{5}\,y^{5/4} - \frac{4}{9}\,y^{9/4}\right)\Big|_0^1$$

$$= 640/9.$$

Hence

$$\mathrm{V}[T] = \frac{640}{9} - 8^2 = \frac{64}{9},$$

giving $\mathrm{SD}[T] = 2\frac{2}{3}$.

2.16 (a) We can check that S_0 is a survival function as follows:

$$S_0(0) = \exp\{0\} = 1,$$
$$\lim_{x\to\infty} S_0(x) = \exp\{-\infty\} = 0,$$

and the derivative is

$$\frac{d}{dx} S_0(x) = -(A + Bx + CD^x)\exp\left\{-\left(Ax + \frac{1}{2}Bx^2 + \frac{C}{\log D}D^x - \frac{C}{\log D}\right)\right\}$$
$$< 0 \quad \text{for } x > 0, \text{ as } A, B, C, D > 0\,.$$

Here we have used the result that

$$\frac{d}{dx}D^x = \frac{d}{dx}\exp\{x \log D\} = (\log D)\exp\{x \log D\} = (\log D)\, D^x.$$

(b) The survival function S_x is given by

$$\begin{aligned} S_x(t) &= \frac{S_0(x+t)}{S_0(x)} \\ &= \exp\left\{-A(x+t) - \frac{1}{2}B(x+t)^2 - \frac{C}{\log D}D^{x+t} + \frac{C}{\log D}D \right. \\ &\qquad \left. + Ax + \frac{1}{2}Bx^2 + \frac{C}{\log D}D^x - \frac{C}{\log D}\right\} \\ &= \exp\left\{-At - \frac{1}{2}B(2xt + t^2) - \frac{C}{\log D}D^x(D^t - 1)\right\}. \end{aligned}$$

(c) The force of mortality at age x is

$$\begin{aligned} \mu_x &= -\frac{1}{S_0(x)}\frac{d}{dx}S_0(x) = -\frac{d}{dx}\log S_0(x) \\ &= \frac{d}{dx}\left(Ax + \frac{1}{2}Bx^2 + \frac{C}{\log D}D^x - \frac{C}{\log D}\right) \\ &= A + Bx + CD^x. \end{aligned}$$

(d) The results below are obtained by numerical integration:

(i) ${}_{20|10}q_{30} = 0.1082$.

(ii) $e_{70} = 13.046$.