Dirichlet Series and Holomorphic Functions in High Dimensions

Andreas Defant, Domingo García, Manuel Maestre and Pablo Sevilla-Peris
More than 100 years ago, Harald Bohr identified a deep problem about the convergence of Dirichlet series and introduced an ingenious idea relating Dirichlet series and holomorphic functions in high dimensions. Elaborating on this work almost 20 years later, Bohnenblust and Hille solved the problem posed by Bohr.

In recent years, there has been a substantial revival of interest in the research area opened up by these early contributions. This involves the intertwining of the classical work with modern functional analysis, harmonic analysis, infinite dimensional holomorphy and probability theory as well as analytic number theory. New challenging research problems have crystallized and been solved in recent decades.

The goal of this book is to describe in detail some of the key elements of this new research area to a wide audience. The approach is based on three pillars: Dirichlet series, infinite dimensional holomorphy and harmonic analysis.

Andreas Defant is Professor of Mathematics at Carl v. Ossietzky Universität Oldenburg, Germany.

Domingo García is Professor of Mathematics at Universitat de València, Spain.

Manuel Maestre is Professor of Mathematics at Universitat de València, Spain.

Pablo Sevilla-Peris is Associate Professor of Mathematics at Universitat Politècnica de València, Spain.
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ANDREAS DEFANT
Carl v. Ossietzky Universität Oldenburg, Germany

DOMINGO GARCÍA
Universitat de València, Spain

MANUEL MAESTRE
Universitat de València, Spain

PABLO SEVILLA-PERIS
Universitat Politècnica de València, Spain
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Introduction

This is not a text on Dirichlet series. Indeed, they form a key element of analytic number theory and its many different subfields, and over the past 100 years have been examined from almost every possible point of view. Many of the books dealing with Dirichlet series are simply excellent, and there is very little that we could add to them. We do not aim to develop and describe a comprehensive theory on Dirichlet series. This would be far too ambitious.

This text is about something else. We take as a starting point the work of Harald Bohr (1913a) more than a hundred years ago. There he posed a concrete problem about convergence of Dirichlet series (see below) and took the first steps towards finding a solution while introducing some new ideas. These ideas were taken over and elaborated on by Bohnenblust and Hille (1931), who finally solved the problem 18 years later. These ideas are the seeds from which all that we intend to present in this text grows. Many years later, this problem turned out to be closely related to some other problems in functional analysis (more precisely, holomorphic functions on infinite dimensional spaces) and harmonic analysis (mainly Hardy spaces on the infinite dimensional torus). Our goal in this text is to describe and understand this link in detail. We intend to show how the original problem motivated developments in analysis and how some of these ideas facilitated progress for Dirichlet series, while others lost their connection with the topic but were interesting in their own right from the point of view of analysis. This volume rests on three pillars: Dirichlet series, infinite dimensional holomorphy and harmonic analysis.

**Dirichlet series** These are series of the form $\sum a_n n^{-s}$, where the coefficients $a_n$ are complex numbers and $s$ is a complex variable. They are a fundamental tool in analytic number theory, the most famous one being the Riemann $\zeta$-function. At the beginning of the twentieth century (between the 1910s and the 1930s) this was a very fashionable topic that attracted the attention of mathematicians like Hardy, Landau, Littlewood and Riesz. The young Harald Bohr
was among them, and it was he who started a systematic study of the convergence of Dirichlet series. Unlike power series, these series converge and define holomorphic functions in half-planes. Bohr was looking at the largest half-plane on which a given Dirichlet series converges absolutely and wanted to describe it in terms of the properties of the holomorphic function that it defines. He realized around 1913, with a simple, groundbreaking idea, that Dirichlet series and formal power series were intimately related through prime numbers. Let us briefly describe this relationship. A formal power series is of the form \( \sum \frac{c_\alpha z^\alpha}{n!} \), where the \( \alpha \) are multi-indices \((\alpha_1, \ldots, \alpha_N)\) of arbitrary length and for a given sequence \( z = (z_n)_n \) and some such \( \alpha \), \( z^\alpha = z_1^{\alpha_1} \cdots z_N^{\alpha_N} \). Now, given some natural number \( n \), we take its decomposition into prime numbers (which we denote by \( p_k \)) \( n = p_1^{\alpha_1} \cdots p_k^{\alpha_k} \). Then, by the uniqueness of this decomposition, to each \( n \) corresponds a unique \( \alpha \), and vice versa. In other words, with the coefficients \((a_n)_n\), there corresponds (by defining \( c_\alpha := a_{p^\alpha} \)) a unique family of coefficients \((c_\alpha)\), and with each Dirichlet series \( \sum a_n n^{-s} \), there corresponds a unique power series \( \sum c_\alpha z^\alpha \). This correspondence, which we will call Bohr’s transform in this text, is one of the major themes for us. Bohr himself was very much aware of this connection and the importance of studying power series. Quoting Bohr (1914), ‘It becomes clear in the course of these investigations that the theory of the absolute convergence of Dirichlet’s series of the type \( \sum a_n n^{-s} \) is very closely connected with the theory of power series in an infinite number of variables’ and Bohr (1913a), ‘Um dies Problem zu erledigen, ist ein tieferes Eindringen in die Theorie der Potenzreihen unendlich vieler Variablen nötig, als es mir in §3 gelungen ist’.\(^1\) This leads us to the next topic.

**Infinite dimensional holomorphy** The problem of developing a theory of holomorphic functions in infinitely many variables was in its infancy when Bohr came up with his problem. Looking at the classical theory of one variable, there are two ways of approaching holomorphy: the Weierstraß approach through power series expansions or the Cauchy approach through differentiability. A basic result taught in any course on complex analysis in one variable is that in fact both approaches are equivalent and a function is differentiable (holomorphic) if and only if it has a power series expansion. In the 1890s von Koch started a theory of holomorphic functions based on the Weierstraß approach and defined holomorphic functions in infinitely many variables to be precisely those having a representation as a power series. Hilbert (1909) also took this path (see also Hibert 1935): ‘Es handelt sich weiterhin vor allem darum, die wichtigste Begriffe und Sätze der Theorie der analytischen Funktionen mit endlicher Variablenzahl auf die Theorie der Theorie der analytis-

\(^1\) To solve this problem, an understanding of the theory of power series in infinitely many variables is needed, deeper than what I managed to achieve in §3.
chen Funktionen mit unendlichevielen Variablen zu übertragen’. Later, in 1915, Fréchet, following Cauchy, gave his definition of a differentiable function (what is now called a Fréchet differentiable). The natural domain for holomorphic functions in $n$ variables is the polydisc $\mathbb{D}^n$. When we make the jump to infinitely many variables, the role of an infinite dimensional polydisc is played by the open unit ball of $c_0$ (the Banach space of all null sequences). In this context, a bounded function is holomorphic (in the sense of Fréchet) if and only if it is continuous and holomorphic on each finite dimensional $\mathbb{D}^n$. It soon became clear that these two approaches through analyticity and differentiability were not equivalent. Although remaining popular for some time, the approach to infinite dimensional holomorphy through analyticity was slowly forgotten, and the theory progressed in a totally different direction. Quoting Dineen, (1999, p. 231) ‘the paper of Bohr and the subsequent papers by O. Toeplitz and H. F. Bohnenblust–E. Hille on the same topic contain interesting results on infinite dimensional holomorphy which would have played, if they had not been overlooked, a role in the development of holomorphic functions on $c_0$’.

One of our aims in this text is to again bring this approach to the public, convinced as we are of its many valuable aspects.

**Harmonic analysis on the infinite dimensional torus** Hardy spaces $H_p(\mathbb{T})$ play a central role in classical Fourier analysis. These are defined as the spaces of those functions in $L_p(\mathbb{T})$ whose Fourier coefficients $\hat{f}(n)$ are 0 for $n < 0$. An important result in the classical theory of Hardy spaces is that these spaces can actually be realized as spaces of holomorphic functions on the disc. In particular, $H_\infty(\mathbb{T})$ is isometrically isomorphic to $H_\infty(\mathbb{D})$, the space of bounded, holomorphic functions on the unit disc. In this sense, Hardy spaces (and not only $H_\infty$ but also the other $H_p$’s) can be seen as a sort of bridge that connects complex and Fourier analysis. Later, with the modern development of harmonic analysis on compact abelian groups, we can take $\mathbb{T}^\infty$ (all complex sequences $(z_n)$ with $|z_n| = 1$) and define, for each multi-index $\alpha$ with arbitrary length (this time with entries in $\mathbb{Z}$), the corresponding Fourier coefficient as $\hat{f}(\alpha) = \int_{\mathbb{T}^\infty} f(w)w^{-\alpha}dw$. With this, Hardy spaces $H_p(\mathbb{T}^\infty)$ are defined as those spaces consisting of the functions $f \in L_p(\mathbb{T}^\infty)$ for which $\hat{f}(\alpha) \neq 0$ only if $\alpha_k \geq 0$ for every $k$. In this way, a function in such a Hardy space defines a power series (in the sense that we mentioned before) given by $\sum \hat{f}(\alpha)z^\alpha$, which, via the idea of Bohr, is linked to some Dirichlet series. This is the third pillar upon which our volume rests.

---

2 The aim is, from now on, to transfer the most important notions and theorems of the theory of analytic functions in finitely many variables to the theory of analytic functions in infinitely many variables.
After some years of splendour, around the late 1930s, interest in the analytic approach of Bohr to Dirichlet series waned, perhaps because complex analysis in one variable lost some of its influence, perhaps because (despite Bohr’s good intuition and brilliant ideas) it did not have the proper tools to handle the problem. The fact is that some years of relative silence ensued, until by the mid 1990s the subject again received the attention of researchers who, supplied with the modern tools of functional and harmonic analysis (but also probability and number theory), came back to the original problems and shed new light on them. One of the cornerstones, and the starting point of this new interest, was the fact that in this new infinite dimensional setting, there were bridges not only between complex and harmonic analysis but also between these two areas and Dirichlet series, and these three ‘worlds’ that we have just introduced turned out to be basically the same. To be more precise, if $H_\infty$ is the space of all Dirichlet series that define a bounded, holomorphic function on the half-plane $[\text{Re} > 0]$, $H_\infty(B_{c_0})$ is the space of bounded, holomorphic functions on $B_{c_0}$ (the open unit ball of $c_0$), and $H_\infty(\mathbb{T}^\infty)$ is the Hardy space that we have just described, then

$$H_\infty = H_\infty(B_{c_0}) = H_\infty(\mathbb{T}^\infty),$$

as Banach spaces (this was done by Cole and Gamelin (1986) and Hedenmalm et al. (1997)). This is a crucial equality that we try to explain carefully in this text. This equality brought new people to the subject and in some sense started a sort of renaissance of the old ideas of Bohr. It is this renaissance that we want to depict in these notes, showing how the bridges between these worlds are built and how this deep relationship has had consequences in all three fields. There are some items that will act as leitmotifs for our story. They will appear again and again, each time in a slightly more refined version. We will not always take the fastest route to solve the problems that arise, preferring a path leading us to different interesting aspects which will later motivate further developments. Πάντα στο νου σου να ‘χεις την Ιθάκη./ Το φθάσιμον εκεί είν’ ο προορισμός σου./ Αλλά μη βιάζεις το ταξείδι διόλου./ Καλύτερα χρόνια μεόσα κέρδισες στο δρόμο./ μη προσδοκώντας πλούτη να σε δώσει η Ιθάκη. This also means that we will take our time to explain certain details that nonexperts in the given area will, we hope, find helpful (although experts might find it too protracted; on the other hand, we hope that they will find them convenient when they arrive at areas with which they are not so familiar). We now briefly describe those items.

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3 ‘Keep Ithaka always in your mind/ Getting there is your destination/ But do not hurry the journey at all/ Better if it lasts for years/ and elderly you end up in the island/ wealthy with all you have gained on the way/ not expecting Ithaka to give you any richness’. Ithaka, C.P. Cavafy, kindly translated for us by Mariana Prieto and Maria Galati.
Bohr’s problem  As we explained earlier, the natural domains where Dirichlet series converge are half-planes, and for each series there is a largest half-plane in which it converges, converges uniformly or converges absolutely. These half-planes are determined by abscissas \( \sigma_c, \sigma_u \) and \( \sigma_a \) that (contrary to what happens with power series in one variable) may be different. One of Bohr’s main interests was to find out how far apart from each other these can be. He was particularly interested in the maximal distance between \( \sigma_a \) and \( \sigma_u \) for a given Dirichlet series. He considered the number

\[
S = \sup(\sigma_a - \sigma_u)
\]

where the supremum ranges over all Dirichlet series. This gives the maximal width of the vertical band on which a Dirichlet series can converge uniformly but not absolutely. It took several years (from Bohr (1913a) to Bohnenblust and Hille (1931)) to determine the precise value of this number, which happened to be \( S = 1/2 \). This is somehow the starting point of the whole theory. Later, \( S \) was reformulated in terms of the Banach space \( \mathcal{H}_\infty \). Following this, the Hardy spaces \( H_p(\mathbb{T}_\infty) \) defined new spaces \( \mathcal{H}_p \) of Dirichlet series that defined new strips \( S_p \), which were even generalized to Dirichlet series taking values in some Banach space \( X \).

Bohr’s vision  This refers to the idea of Bohr that relates (formal) power series in infinitely many variables to Dirichlet series (also considered only as formal series). This defines a bijection

\[
\mathbb{B} : \mathbb{P} \overset{a_{\alpha} \mapsto z_{\alpha}}{\longrightarrow} \mathbb{D} \quad \sum c_{\alpha} z_{\alpha}^s \overset{a_{\alpha} = c_{\alpha}}{\longrightarrow} \sum a_{n} n^{-s}
\]

between the space \( \mathbb{P} \) of all power series and the space \( \mathbb{D} \) of all Dirichlet series, which we call the Bohr transform. Bohnenblust and Hille (1931) claimed: ‘Bohr showed that, though actually functions of a single variable \( s \), the variables \( z_n = v_n^s \) behave in many ways as if they were independent of one another’. Making a precise statement out of this idea will be part of our work. This transform will become one of our main tools, one that transfers problems (and solutions) from one world to the other. Understanding it in depth and clarifying this currently vague statement will be foci of our attention.

Convergence of power series  As we have already pointed out (and will more carefully explain in the text), power series and differentiability give two essentially different approaches to holomorphy. Although every \( f \) in \( H_{\infty}(B_{c_0}) \) defines a unique formal series \( \sum c_{\alpha} f(z^\alpha) \) (we call it the monomial series expansion of \( f \)), for certain functions, this expansion does not converge at all points.
The natural question arises: ‘for which points does \( \sum c_\alpha (f)z^\alpha \) converge for every such function \( f \)?’ Describing this set (which we call the ‘set of monomial convergence’) for different classes of functions will also be one of our motivations. This goal is closely related to Bohr’s problem. If we denote by \( M \) the largest \( p \) for which the power series expansion of every holomorphic function converges at every \( z \in \ell_p \cap B_{c_0} \), then we have \( S = \frac{1}{M} \). ‘Philosophically’ speaking, this indicates that Dirichlet series and holomorphic functions are, in some vague sense, inverse to each other. We will try in this text to make this vague idea more precise but also to explain why characterizing this set of convergences is a far more delicate problem than computing \( S \).

**The Bohnenblust–Hille inequality**  This inequality was proved by Bohnenblust and Hille (1931) when trying finally to solve Bohr’s problem of determining the exact value of \( S \). It is an extension of an important inequality by Littlewood and states that for each \( m \geq 2 \), there is a constant \( C_m \) such that for every \( m \)-homogeneous polynomial \( \sum \alpha c_\alpha z^\alpha \) in \( n \) variables, we have

\[
\left( \sum_{\alpha \in \mathbb{N}_0^n, |\alpha| = m} |c_\alpha| \right)^{\frac{2m}{m+1}} \leq C_m \sup_{z \in \mathbb{D}^n} \left| \sum_{\alpha \in \mathbb{N}_0^n, |\alpha| = m} c_\alpha z^\alpha \right|.
\]

We are going to see how improvements in this inequality (for example in the control of the growth of the constant \( C_m \) with respect to the degree \( m \)) allow one to solve finer and finer problems. We will look at this inequality from many different points of view, which range from considering the supremum on different sets on the right-hand side to studying it for polynomials taking values in some Banach space.

**The Bohr radius**  It was clear to Bohr that, in order to approach his problem on Dirichlet series, he must better understand the absolute convergence of power series in infinitely many variables (what we would today call holomorphic functions in infinite dimensional spaces). Quoting Bohr (1914): ‘The solution of what is called the “absolute convergence problem” for Dirichlet’s series of the type \( \sum a_n u^n \) must be based upon a study of the relations between the absolute value of a power series in an infinite number of variables on the one hand, and the sum of the absolute values of the individual terms of this series on the other’. What he did then was to start looking at power series (that is, holomorphic functions) in one variable. Somehow he fought against the triangle inequality, and he tried to find out for which \( z \) in \( \mathbb{D} \) the following bound holds:

\[
\sum_{n=1}^{\infty} |c_n z^n| \leq \sup_{|u| < 1} \left| \sum_{n=1}^{\infty} c_n u^n \right|.
\]
Of course, the above inequality does not hold for every $z$, but one can ask how large the set on which it holds can be. Again Bohr (1914) writes: ‘It was in the course of this investigation that I was led to consider a problem concerning power series in one variable only, which we discussed last year, and which seems to be of some interest in itself’. The problem being referred to was showing

$$\sup_{|z|<1/3} \sum_{n=1}^{\infty} |c_n z^n| \leq \sup_{|z|<1} \sum_{n=1}^{\infty} c_n z^n$$

and that one cannot replace the disc of radius $1/3$ on the left-hand side with a larger disc. By the mid 1990s the question arose of how to apply this to functions of several complex variables. How big a polydisc can we take in the left-hand side of an analogous inequality? The number

$$K_n = \sup \{0 < r < 1 : \sup_{z \in D^n} \sum_{\alpha \in \mathbb{N}^n} |c_\alpha(f) z^\alpha| \leq \sup_{z \in D^n} \sum_{\alpha \in \mathbb{N}^n} c_\alpha(f) z^\alpha\}$$

was defined for all bounded holomorphic functions on $\mathbb{D}^n$ and was called the ‘$n$th Bohr radius’ (not to be confused with the Bohr radius of an atom, named after Harald’s elder brother Niels). Trying to correctly estimate the growth (actually, decay) for these numbers with the dimension $n$ became an interesting problem that took some time to solve. Determining this optimal behaviour and looking at further developments, such as taking the supremum on other subsets of $\mathbb{C}^n$, will also keep us busy for a while.

**Contents**

This text is divided into four parts but actually consists of two relatively independent units. The first one corresponds to Part I, while the second one covers Parts II, III and IV. The first unit is completely independent from the second one. There we introduce and develop the classical theory: scalar valued Dirichlet series and functions that are defined either on the unit ball of $c_0$ or on the infinite dimensional torus. The second unit is more specialized and covers recent developments in different directions. This unit begins with Part II, thought of as a sort of toolbox where we make brief introductions to the basic tools that will be needed later. Then we consider scalar valued functions that are defined on the unit ball of some sequence space different from $c_0$ (typically $\ell_p$). This is done in Part III. Finally, in Part IV we deal with Dirichlet series and functions taking values in some Banach space.

The text consists of 26 chapters (apart from this introduction). Most of them pivot around one or two results which we deem to be the highlights of the
chapter (they appear boxed) and which, in some sense, condense the most important information. The chapter then focuses on giving the proof of these results. This often requires some preparation and preliminary results. In some cases the highlight is stated right at the beginning of the chapter, whereas in others the statement needs some previous knowledge and/or notation.

As we have said, there are some concepts and ideas that form the skeleton of this manuscript. They appear over and over again, in different ways, interweaving with each other. We now describe where each one of these ideas appears, and in this way we get an idea of the structure and contents of the text.

**Dirichlet series** This is one of the central topics of this text. We define them already in Chapter 1, where we show that half-planes are the natural domains on which these series converge. Then we define the abscissas $\sigma_c$, $\sigma_b$, $\sigma_u$ and $\sigma_a$ that define the maximal half-planes on which a given Dirichlet series (respectively) converges, defines a bounded holomorphic function, converges uniformly and converges absolutely. However, we do not study Dirichlet series as single objects; we take a more ‘functional analytic’ point of view and study classes (actually Banach spaces) of Dirichlet series. Already in Chapter 1 we define the space $H_\infty$ consisting of Dirichlet series that define a bounded, holomorphic function on the half-plane $[\text{Re} > 0]$. From this point on, Dirichlet series appear over and over again in the manuscript. In Chapter 11 we define and study Hardy spaces of Dirichlet series. These are defined as the image through the Bohr transform of the Hardy spaces $H_p(T\infty)$, but we immediately give an internal characterization, as the completion of the polynomials (finite sums) under a certain norm. For a given Dirichlet series $\sum a_n n^{-s}$, we define the translation by $\varepsilon > 0$ as the series $\sum a_n \varepsilon n^{-s}$. It is shown that a Dirichlet series is in $H_p$ if and only if every such translation is in $H_p$ and the norm of all these translations is uniformly bounded. As a tool, we briefly introduce Dirichlet series taking values in a Banach space $X$ (that is, with the coefficients $a_n$ belonging to $X$). We come back to this issue in more detail in Chapter 24, where we consider $D_\infty(X)$, the space of Dirichlet series that define a bounded, holomorphic function on $[\text{Re} > 0]$, and $H_p(X)$ (with $1 \leq p \leq \infty$), the space of Dirichlet series induced by the Hardy spaces of $X$-valued functions. Here we find some major differences from the scalar-valued case. The most important one is that it is not in general true that $D_\infty(X)$ and $H_\infty(X)$ are the same space, and this happens if and only if $X$ has the analytic Radon–Nikodym property (ARNP). This is very much related to the fact that the translation property that we have just stated (a Dirichlet series belongs to the Hardy space if and only if every translation does and the norms are uniformly bounded) is no longer true in general.
Introduction

**Infinite dimensional holomorphy**  In Chapter 2 we define holomorphic functions on $B_{c_0}$, the unit ball of $c_0$, and taking values on $\mathbb{C}$. We follow a strategy that becomes recurrent in the text. First we look at the problem for functions in finitely many variables (that is, defined on the $n$-dimensional polydisk $\mathbb{D}^n$ for arbitrary $n$) and then jump to the infinite dimensional setting, trying to isolate and overcome the difficulties. We take Fréchet differentiability as the definition of holomorphy. We then show that a Fréchet differentiable function defines a unique family of coefficients $(c_\alpha(f))_\alpha$. One of the highlights of this chapter is what we have called Hilbert’s criterion, which characterizes those families of coefficients that are associated to a bounded, holomorphic function. The power series defined by the coefficients of $f$ may not converge at some points, showing a major difference in the case of infinitely many variables: holomorphy (that is, differentiability) and analyticity (that is, having a convergent power series expansion) are not equivalent concepts. However, every bounded, analytic function is holomorphic. In Chapter 2 we also define one of our main tools; $m$-homogeneous polynomials, and we show how these are closely related to $m$-linear forms on $c_0$. We come back to the study of holomorphic functions on infinite dimensional spaces in Chapter 13, where we define Hardy spaces of holomorphic functions defined on $\ell_2 \cap B_{c_0}$. In Chapter 15 we give a complete introduction to holomorphic functions between Banach spaces. We start by transferring what was done before on $c_0$ to functions defined on an arbitrary Banach sequence space. We then consider arbitrary Banach spaces. We give different approaches to holomorphy, and we show that a holomorphic function always admits a representation as a series of homogeneous polynomials. We come back to the relationship between analyticity and holomorphy, showing that every analytic function defined on a Reinhardt domain in a Banach sequence space is holomorphic. As we said, an important tool for dealing with holomorphic functions are $m$-homogeneous polynomials. We look at them in detail in Chapter 21, where we make an important observation from the point of view of Banach space theory: the space of $m$-homogeneous polynomials on an infinite dimensional Banach sequence space never has an unconditional basis.

**Harmonic analysis on the infinite dimensional torus**  This is the third pillar upon which our building sits. It appears for the first time in Chapter 5, where we define in this area the main objects that we are going to use (Hardy spaces on $\mathbb{T}^\infty$) and our main tool (the Poisson kernel and transform). Our highlight here is that the spaces $H^\infty(\mathbb{T}^\infty)$ and $H^\infty(B_{c_0})$ are (as in the one-dimensional case) isometrically isomorphic. We follow again the strategy of looking first at the problem for finite dimensional functions before jumping to the infinite dimensional setting. This is again a central point in Chapter 13, where we extend this study to other Hardy spaces, showing that $H^p(\mathbb{T}^\infty)$ and $H^p(\ell_2 \cap B_{c_0})$ are
isometrically isomorphic. The Cole–Gamelin inequality and the brothers Riesz theorem \((H_1(T^\infty)\) is isomorphic to the space of analytic Borel measures of bounded variation on \(T^\infty)\) are the main tools. We prove these using our results on Dirichlet series from earlier in the text. In Chapter 24 we look at Hardy spaces of Banach-valued functions. We again have a Poisson transform, and we ask if, for \(X\)-valued functions, once again the Hardy space of measurable functions on \(T^\infty\) is equal (as a Banach space) to the Hardy space of holomorphic functions. We see that this is the case if and only if \(X\) has the ARNP. Let us finally point out that this harmonic analytic point of view appears also in Chapter 11, where the Hardy spaces of Dirichlet series are defined as the image under the Bohr transform of the spaces \(H_p(T^\infty)\).

**Bohr’s problem** The starting point of our approach is to determine \(S\), the maximal distance between \(\sigma_a\) and \(\sigma_u\). This is stated as a problem as soon as the abscissas are defined in Chapter 1. Our first move towards the solution is to define \(\mathcal{H}_\infty\) and reformulate the problem as that of determining \(S = \sup \mathcal{H}_\infty \sigma_a\). Then, in Chapters 2 and 3, we develop the theory that we need to apply in Chapter 4 to determine the solution of the problem, which is \(S = 1/2\). Later, in Chapter 9, we look at this in much more detail, in the following sense. What the solution of the problem tells us after the reformulation is that, for every Dirichlet series in \(\mathcal{H}_\infty\), we have \(\sum_{n=1}^\infty |a_n| \frac{1}{n^{1+\epsilon}} < \infty\) for every \(\epsilon > 0\). Then the question becomes: can we replace \(\epsilon\) with 0? The highlight in Chapter 9 is that not only can we let \(\epsilon = 0\) but, even more, we can put some term in the numerator that tends to \(\infty\). Bohr’s problem comes back briefly in Chapter 10, where it is linked to the set of monomial convergence of power series. In Chapter 12 the abscissas \(\sigma_{\mathcal{H}_p}\) defined by the Hardy spaces are introduced. This immediately motivates the definition of \(S_p\), the maximal difference between \(\sigma_u\) and \(\sigma_{\mathcal{H}_p}\). We show that in this case we also have \(S_p = 1/2\). We close this text with Chapter 26, where, after a long journey, we return to Bohr’s problem. Now we look at Bohr’s problem for Banach-valued Dirichlet series and its equivalent for Hardy spaces of vector-valued Dirichlet series, showing that this problem is closely related with the geometry of the Banach space, since in both cases the width of the maximal strip is \(1 - \frac{1}{\cot(X)}\) (where \(\cot(X)\) is the optimal cotype of the space).

**Bohr’s vision** This is probably our main tool and is certainly one of the engines that drives the text forward. Bohr’s transform is defined in Chapter 3, where the highlight is that this mapping defines an isometric isomorphism between \(\mathcal{H}_\infty\) and \(H_\infty(B_{c_0})\). Once again, we start by looking at functions in finitely many variables and Dirichlet series that depend on finitely many primes and later take the step to the general case. From this moment on, Bohr’s vision appears more or less implicitly all over the text. It is, for example, the idea behind the definition of the Hardy spaces of Dirichlet series in Chapter 11 and their
intimate connection with holomorphic functions on $\ell_2 \cap B_{c_0}$ in Chapter 13. In Chapter 24 we come back to the idea, this time for vector-valued Dirichlet series and functions.

**Convergence of power series**  This is in some sense a natural evolution of Bohr’s problem and vision and asks for which $z$ we have $\sum_{\alpha} |c_\alpha(f)z^\alpha| < \infty$ for all functions in a certain family (or, equivalently, every power series in a certain family). We call it the set of monomial convergence of the family of functions, and it appears for the first time in Chapter 10, where we consider $H_\infty(B_{c_0})$ and the space $P_m(c_0)$ of $m$-homogeneous polynomials. Bohr’s problem can be reformulated in terms of these sets: $S$ is the infimum of all $\sigma$ such that the sequence $(1/\nu^\sigma_n)$ is in the set of monomial convergence of $H_\infty(B_{c_0})$. It is then clear that trying to describe these sets is a far more delicate problem than computing $S$. The set of monomial convergence of $P_m(c_0)$ is completely characterized, whereas for $H_\infty(B_{c_0})$ we have only a partial description. In Chapter 12 we study and describe completely the set of monomial convergence of the Hardy spaces $H_p(T^\infty)$. In Chapter 20 we study the set of monomial convergence of functions that are defined, not on $B_{c_0}$, but on the open unit ball of an arbitrary $\ell_p$ or, even more generally, Reinhardt domains in some Banach sequence space.

**The Bohnenblust–Hille inequality**  This inequality, which relates the coefficients of an $m$-homogeneous polynomial with the supremum over the polydisc, is an important technical tool throughout this text. We are mostly interested in the inequality for polynomials. However, due to the close relationship between polynomials and multilinear mappings, we also have versions of the inequality for $m$-linear mappings. We begin by stating it in its original form (see above) in Chapter 6, where we also show that the exponent in the inequality is optimal. From that point on, we give several refinements and generalizations of the inequality. The first one appears in Chapter 8, where it is shown that the constant $C_m$ in the inequality can be taken as $C^m$. This is a crucial step towards finding the correct asymptotic decay of the Bohr radius. The Bohnenblust–Hille inequality is in some sense the final step in a series of inequalities involving mixed norms that are technically more involved to present but certainly no less interesting. A very refined version of these inequalities in Chapter 10 leads to the description of the set of monomial convergence. As a consequence of these inequalities we get that the constant $C_m$ can even be taken as $(1 + \varepsilon)^m$ for $\varepsilon > 0$ arbitrarily small. In Chapter 18 we replace the supremum over $D^n$ that appears on the right-hand side of the inequality with the supremum over the unit ball of $C^n$ with an $\|\|_p$-norm (that is, we replace polynomials on $c_0$ by polynomials on $\ell_p$). Finally, in Chapter 25, we give different versions of the inequality for vector-valued polynomials, relating it to the theory of summing operators.
The Bohr radius  This is defined in Chapter 8, where the highlight is to show that $K_n$ decays to 0 like $\sqrt{\log n/n}$. With the improvement of the Bohnenblust–Hille inequality in Chapter 10, we show that in fact the quotient of these two tends to 1. In Chapter 19 we again replace $D^n$ by the unit ball of $C^n$ with some $\|\cdot\|_p$-norm, showing that in this case the corresponding Bohr radius decays like $\left(\frac{\log n}{n}\right)^{\min\{p,2\}}$. With the techniques developed in Chapter 21 we come back to this problem in Chapter 22 with a more general point of view.

These are the main topics under consideration in this text. But there are others that, not being so central, still play an important role as the text progresses. One is probability in Banach spaces and random polynomials, which we tackle in Chapters 7 and 17. The second one is Banach space theory. We present no general theory in this regard, but it comes into play several times, and various concepts from the theory are used as important tools. We collect some of them (basis in a Banach space, unconditionality, cotype and operator ideals) in Chapter 14. We also make short introductions to two topics that are important techniques in the theory: tensor products (Chapter 16) and the analytic Radon–Nikodym property (Chapter 23). In all these cases we only introduce what we are going to use, trying to be as complete but also as concise as possible.

### Basic Definitions and Notation

We collect now the basic notation that we are going to use in this text. There are some concepts for which we use different notations, depending on the context. We try to be locally consistent with the notation, but overall, we prefer to keep the text readable over having a globally coherent notation. In many cases the notation will be introduced locally where it is used.

**Indices**  This is an important consideration throughout the whole text. We will give a more complete treatment in Section 2.7. Always, $\mathbb{N}_0$ denotes the set of nonnegative integers. We consider the index set

$$\mathbb{N}_0^{(\mathbb{N})} = \left( \bigcup_{n=1}^{\infty} \mathbb{N}_0^n \right) \times \{0\}$$

(here and later we mean $\{0\} = \{(0,0,0,\ldots)\}$). These are sequences of natural numbers that are 0 except for a finite number of entries. The elements of this set are going to be called multi-indices and denoted by $\alpha$. Such a multi-index is of the form $(\alpha_1, \alpha_2, \ldots, \alpha_n, 0, 0, \ldots)$. We write $|\alpha| = \sum_{k=1}^{\infty} \alpha_k$ (note that since there are only finitely many nonzero entries, this is actually a finite sum). For a multi-index $\alpha$ we define the support as $\text{supp } \alpha := \{k \in \mathbb{N} : \alpha_k \neq 0\}$. We will
always consider $\mathbb{N}^n_0$ as a subspace of $\mathbb{N}^n_0$ by identifying $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $(\alpha_1, \alpha_2, \ldots, \alpha_n, 0, 0, \ldots)$ without any further notice.

**Complex analysis** Arbitary complex numbers are going to be denoted either by $z$ or $s$. We let $\text{Re} z$ and $\text{Im} z$ denote the real and imaginary part, but we will also write $s = \sigma + it$. We write $\mathbb{D}$ and $\overline{\mathbb{D}}$ respectively for the open and closed discs in $\mathbb{C}$, and $\mathbb{T}$ for their boundary, which we call the torus. Thus
\[
\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \},
\overline{\mathbb{D}} = \{ z \in \mathbb{C} : |z| \leq 1 \},
\mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \}.
\]
For discs of arbitrary centre $a \in \mathbb{C}$ and radius $r > 0$, we write
\[
\mathbb{D}(a, r) = \{ z \in \mathbb{C} : |z - a| < r \}
\]
and
\[
\partial \mathbb{D}(a, r) = \{ z \in \mathbb{C} : |z - a| = r \}.
\]
In $\mathbb{C}^n$ we will consider the maximum norm
\[
\|(z_1, \ldots, z_n)\|_\infty = \max_{k=1,\ldots,n} |z_k|.
\]
This is the norm that we will use most often, and we will generally omit the subscript. However, we will also use the norms, defined for $1 \leq p < \infty$,
\[
\|(z_1, \ldots, z_n)\|_p = \left( \sum_{k=1}^{n} |z_k|^p \right)^{\frac{1}{p}}.
\]
Given $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$ and $r = (r_1, \ldots, r_n)$ with $r_j > 0$, for every $j$ we consider the open and closed $n$-dimensional polydiscs centred at $a$ and with polyradius $r$:
\[
\mathbb{D}^n(a, r) = \{ z = (z_j)_{j=1}^{n} : |z_j - a_j| < r_j \text{ for } j = 1, \ldots, n \},
\overline{\mathbb{D}}^n(a, r) = \{ z = (z_j)_{j=1}^{n} : |z_j - a_j| \leq r_j \text{ for } j = 1, \ldots, n \}.
\]
Note that the open polydisc $\mathbb{D}^n$ is $\mathbb{D}^n(a, r)$ for $a = 0$ and all $r_j = 1$ (and the same for the closed polydisc). In some cases, when we want to make a clear distinction, we will use the notation $r$ for polyradius and $r$ for a ‘usual’ radius (that is, a positive real number). We then write $r\mathbb{D}^n = \mathbb{D}^n(0, r)$ and, whenever $r = (r, \ldots, r)$, just $r\mathbb{D}^n$. We also consider the polytorus
\[
\mathbb{T}^n = \{ w \in \mathbb{C}^n : |w_j| = 1 \text{ for } j = 1, \ldots, n \}.
\]
Although contained in the topological boundary of $\mathbb{D}^n$, it is strictly smaller. It is often called the ‘distinguished boundary’ of $\mathbb{D}^n$. 

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*Section title* Introduction xxiii

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*Figure or table*
For a multi-index $\alpha \in \mathbb{N}_0^n$ and $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, we write

$$z^\alpha := z_1^{\alpha_1}z_2^{\alpha_2} \cdots z_n^{\alpha_n}.$$ 

In the same way, for a sequence $z = (z_n)_{n=1}^\infty$ and $\alpha \in \mathbb{N}_0^\infty$), we write

$$z^\alpha := z_1^{\alpha_1}z_2^{\alpha_2}z_3^{\alpha_3} \cdots .$$

Note that, since $\alpha$ has finite length, this product is actually finite.

For fixed $z_0 = |z_0|e^{i\theta} \in \mathbb{C} \setminus \{0\}$ we denote by $\log z_0$ the branch of the complex logarithm defined and holomorphic on $\mathbb{C} \setminus \{ rz_0 : r \leq 0 \}$ taking value in $\mathbb{R} + i[-\pi, \pi]$. The principal branch of the logarithm, defined and holomorphic on $\mathbb{C} \setminus (-\infty, 0]$ with value in $\mathbb{R} + i[-\pi, \pi]$, is denoted by $\log 1$.

Integration Each $w = (w_1, \ldots, w_n) \in \mathbb{T}^n$ can be uniquely written as $(e^{it_1}, \ldots, e^{it_n})$ with $t_j \in [0, 2\pi]$ (or in $[-\pi, \pi]$). Hence $\mathbb{T}^n$ can be identified with $[0, 2\pi]^n$ or $[-\pi, \pi]^n$, and each function $f : \mathbb{T}^n \to \mathbb{C}$ can be identified with a function on $[0, 2\pi]^n$:

$$f(w) = f(e^{it_1}, \ldots, e^{it_n}) \sim f(t_1, \ldots, t_n)$$

(or on $[-\pi, \pi]^n$ or $\mathbb{R}^n$ and $2\pi$–periodic in each variable). A function $f : \mathbb{T}^n \to \mathbb{C}$ is integrable on $\mathbb{T}^n$ if the mapping $(t_1, \ldots, t_n) \mapsto f(e^{it_1}, \ldots, e^{it_n})$ is Lebesgue integrable on $[0, 2\pi]^n$. For $f \in L_1(\mathbb{T}^n)$, we write

$$\int_{\mathbb{T}^n} f(w) d\sigma_n(w) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} f(e^{it_1}, \ldots, e^{it_n}) dt_1 \cdots dt_n .$$

That is, the notation $\sigma_n$ stands for the normalized Lebesgue measure on $\mathbb{T}^n$. We will only write $d\sigma_n(w)$ when there is some risk of confusion; otherwise, we will simply write $dw$.

Considering the infinite dimensional torus

$$\mathbb{T}^\infty = \{ w \in \mathbb{C}^\infty : |w_j| = 1 \text{ for } j \in \mathbb{N} \},$$

the countable product of copies of $\mathbb{T}$, we write $\sigma$ for the measure on $\mathbb{T}^\infty$ given by the countable product of copies of $\sigma_1$ on $\mathbb{T}$. That is, $\sigma$ is the Haar measure on the compact abelian group $\mathbb{T}^\infty$ (endowed with pointwise multiplication). As before, we shall write $dw$ instead of $d\sigma(w)$ when there is no possible confusion. Moreover, we consider $\mathbb{T}^n$ as a subset of $\mathbb{T}^\infty$ by means of identification with the first $n$ variables.

When we deal with functions taking values in some Banach space, we will always consider Bochner integration, as presented by Defant and Floret (1993 Appendix B), or Diestel and Uhl (1977).
On some occasions, we will also consider path integrals. If \( \gamma : [a, b] \to \mathbb{C} \) is a piecewise \( C^1 \) path, then, as usual, we write
\[
\int_{\gamma} f(\zeta) \, d\zeta = \int_{a}^{b} f(\gamma(t)) \gamma'(t) \, dt,
\]
where, if \( f \) takes values in an infinite dimensional Banach space, the latter integral is to be understood as a Bochner integral. If \( \zeta_0 \in \mathbb{C} \) and \( r > 0 \), we will write \( \int_{|\zeta - \zeta_0| = r} f(\zeta) \, d\zeta \) as a shorthand for \( \int_{\gamma} f(\zeta) \, d\zeta \), where \( \gamma : [0, 2\pi] \to \mathbb{C} \) is defined as \( \gamma(t) = \zeta_0 + re^{it} \).

**Banach spaces** As a general rule and unless otherwise stated, capital letters such as \( X, Y \) or \( E \) will denote complex Banach spaces. Very occasionally (in Chapter 2, Section 15.1 and Chapter 16), we will consider normed or only linear spaces, or even real spaces instead of complex ones. When this is the case, it will be clearly stated. For a Banach space \( X \), we denote by \( B_X \) the open unit ball
\[
B_X = \{ x \in X : \|x\| < 1 \}.
\]
For \( U \subset X \) and \( r > 0 \), we write
\[
rU = \{ rx : x \in U \}.
\]
Then \( rB_X \) denotes the open ball which is centred at 0 and has radius \( r > 0 \), i.e. \( rB_X = \{ x \in X : \|x\| < r \} \). The ball with centre at some \( x_0 \) and radius \( r \) will be denoted
\[
B(x_0, r) = B_X(x_0, r) = x_0 + rB_X = \{ x \in X : \|x - x_0\| < r \}.
\]
The space of complex null sequences is denoted by \( c_0 \). With the norm \( \|z\| = \|z\|_{\infty} = \sup_{n \in \mathbb{N}} |z_n| \), it is a Banach space. Identifying \( \mathbb{C}^n = \mathbb{C}^n \times \{0\} \) and \( \mathbb{D}^n = \mathbb{D}^n \times \{0\} \), these two can be seen as subsets of \( c_0 \) and \( B_{c_0} \), respectively. We may make these identifications without saying so explicitly.

The segment joining two vectors \( x, y \in X \), is \( [x, y] = \{ (1 - \lambda)x + \lambda y : \lambda \in [0, 1] \} \).

As usual, \( \mathcal{L}(X, Y) \) stands for the linear space of all bounded and linear operators between the two Banach spaces \( X \) and \( Y \). Together with the operator norm \( \|T\| = \sup_{x \in B_X} \|Tx\| \), this forms a Banach space, and we write \( X^* = \mathcal{L}(X, \mathbb{C}) \) for the (topological) dual of \( X \). The canonical embedding of \( X \) into its bidual \( X^{**} \) is denoted by \( \kappa_X \).

**Number theory** The sequence of prime numbers \( p_1 = 2, p_2 = 3, p_3 = 5, \ldots \) is denoted by \( \mathbb{P} = (p_k)_{k=1}^\infty \). Sometimes \( p \) will also denote a single arbitrary prime number. For each \( x \geq 2 \) we denote the prime counting function by \( \pi(x) \), that is
\[
\pi(x) := \{|p : p \leq x|\}.
\]
Introduction

For a natural number \( n \in \mathbb{N} \), the function \( \Omega(n) \) counts the number of prime divisors of \( n \), counted with multiplicity. So if \( n = p_1^{\alpha_1} \cdots p_k^{\alpha_k} = p^\alpha \) is the unique prime number decomposition of \( n \), then \( \Omega(n) = \alpha_1 + \cdots + \alpha_k = |\alpha|. \) On the other hand, \( d(n) \) counts the number of divisors of \( n \), i.e. \( d(n) = |\{ k : k|n \}|. \) We will use the following version of the prime number theorem

\[ \pi(n) \sim \frac{n}{\log n}. \quad (0.1) \]

As a simple consequence, we have

\[ p_n \sim n \log n. \quad (0.2) \]

In particular, since \( \log n \ll n^\varepsilon \) (see below for the notation \( \ll \)), for every \( \varepsilon > 0 \), we have

\[ p_n \ll n^{1+\varepsilon} \text{ for all } \varepsilon > 0. \quad (0.3) \]

Using (0.2) and Cauchy’s condensation criterion, one easily has that

\[ \sum_{n=1}^{\infty} \frac{1}{p_n^r} \text{ converges if and only if } r > 1. \quad (0.4) \]

Miscellanea  We write \( \Sigma_n \) for the set of all permutations of \( \{1, 2, \ldots, n\} \) and \( \Sigma \) for the set of permutations of \( \mathbb{N} \). Unless otherwise stated, \( \log x \) will always denote the maximum of (the usual value of) \( \log x \) and 1.

Given two complex-valued functions \( f(x, y) \) and \( g(x, y) \), we write \( f(x, y) \ll g(x, y) \) if for each \( y \) there is a constant \( c(y) > 0 \) such that \( f(x, y) \leq c(y)g(x, y) \) for every \( x \). We write \( f(x, y) \sim g(x, y) \) if \( f(x, y) \ll g(x, y) \) and \( g(x, y) \ll f(x, y) \). The notation \( \ll \) and \( \sim \) is then self-explanatory. If there are universal constants (depending neither on \( x \) nor on \( y \)) satisfying the inequalities, we write \( f(x, y) \ll g(x, y) \) and \( f(x, y) \sim g(x, y) \).

For each \( x \geq 0 \) we define \( \lfloor x \rfloor = \max\{n \in \mathbb{N}_0 : n \leq x \} \) and \( \lceil x \rceil = \min\{n \in \mathbb{N}_0 : n \geq x \} \). The cardinality of a set \( A \) is denoted by \( |A| \). For \( 1 < p < \infty \), the conjugate number is denoted as \( p' \), that is \( \frac{1}{p} + \frac{1}{p'} = 1 \). We use the convention that 1 and \( \infty \) are conjugated to each other.

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PART ONE

BOHR’S PROBLEM AND COMPLEX ANALYSIS ON POLYDISCS
1
The Absolute Convergence Problem

As we have already pointed out, we have no aim of universality when it comes to Dirichlet series; we will present only those aspects that are crucial for stating our problem and for understanding the connection with analysis. And we will try to do this slowly. A Dirichlet series is a (in principle formal) series of the following type:

\[ D(s) = \sum a_n n^{-s}, \]  

(1.1)

where the \( a_n \) are complex coefficients and \( s = \sigma + it \) is the complex variable of the Dirichlet series. The Riemann \( \zeta \)-function is the most prominent example:

\[ \zeta(s) = \sum n^{-s}. \]

We denote the set of all such Dirichlet series by \( \mathcal{D} \). There is a natural linear structure on \( \mathcal{D} \), given by

\[ \sum_n a_n n^{-s} + \sum_n b_n n^{-s} = \sum_n (a_n + b_n)n^{-s}, \]

\[ \lambda \left( \sum_n a_n n^{-s} \right) = \sum_n (\lambda a_n)n^{-s}. \]

The so-called Dirichlet multiplication

\[ \left( \sum_n a_n n^{-s} \right) \left( \sum_n b_n n^{-s} \right) = \sum_n \left( \sum_{km=n} a_k b_m \right)n^{-s} \]

(1.2)

turns the linear space \( \mathcal{D} \) into an algebra. This algebra \( \mathcal{D} \) of Dirichlet series is commutative with unit \( \sum_n \delta_{n1} n^{-s} \). It can be checked easily that the invertible series \( \sum a_n n^{-s} \) in \( \mathcal{D} \) are those with \( a_1 \neq 0 \).

The theory of Dirichlet series constitutes one of the most useful tools in analytic number theory. Number theory is often concerned with arithmetical functions (complex-valued functions defined on \( \mathbb{N} \)) which are motivated by divisibility properties of integers, and their importance is usually due to their
contribution to a better understanding of the distribution of primes. Each arithmetical function $a: \mathbb{N} \to \mathbb{C}$ with $n \mapsto a_n$ induces a Dirichlet series $\sum a_n n^{-s}$, and the study of the analytic properties of this object in many cases leads to an increase in understanding of the arithmetic function itself. We give five classical examples, all of which define fundamental concepts in number theory.

(i) The counting functions $\Omega$ and $\omega$ of the total number of primes $p$ factoring $n$, taken with and without multiplicity, i.e. $\Omega(n) = |\alpha|$, where $n = p^\alpha$ is the prime number decomposition of $n$, and $\omega(n) = \sum_{p|n} 1$.

(ii) The divisor function

$$d(n) = \sum_{d|n} 1.$$ 

(iii) The Euler totient function

$$\varphi(n) = \sum_{1 \leq k \leq n \atop \gcd(k,n) = 1} 1.$$ 

(iv) The Möbius function

$$\mu(n) = \begin{cases} 1 & n = 1, \\ (-1)^k & n = p_1 p_2 \ldots p_k, \\ 0 & \text{otherwise}. \end{cases}$$ 

(v) The von Mangoldt function

$$\Lambda(n) = \begin{cases} \log p & n = p^m \text{ for some prime } p \text{ and some } m \in \mathbb{N}, \\ 0 & \text{otherwise}. \end{cases}$$

In number theory, there are numerous results which underline the importance of these arithmetic functions. For example, the following important formulas illustrate the deep connection of their associated Dirichlet series with the $\zeta$-function (so far, all equalities should be understood within the algebraic structure of $\mathbb{C}$):

$$\sum_n d(n) n^{-s} = \zeta^2(s), \quad \sum_n \varphi(n) n^{-s} = \frac{\zeta(s-1)}{\zeta(s)},$$

$$\sum_n \mu(n) n^{-s} = \frac{1}{\zeta(s)}, \quad \sum_n \Lambda(n) n^{-s} = -\frac{\zeta'(s)}{\zeta(s)}.$$

### 1.1 Convergence of Dirichlet Series

Given a Dirichlet series $\sum a_n n^{-s}$, the main purpose of this section is to establish three abscissas in the complex plane,
• the abscissa of convergence,
• the abscissa of uniform convergence, and
• the abscissa of absolute convergence,

which define the largest right half-planes on which the Dirichlet series converges, converges uniformly and converges absolutely. We also want to see that $\sum a_n n^{-s}$ defines a holomorphic function on its maximal right half-plane of convergence.

But let us first give a hint of why half-planes are the natural domains to deal with convergence of Dirichlet series. Ordinary Dirichlet series (those defined in (1.1)) can be seen as special examples of so-called general Dirichlet series. These are series of the following type:

$$\sum_{n} a_n e^{-\lambda_n s},$$

where the $\lambda_n$ (called ‘frequencies’) form a positive sequence of real numbers tending to infinity. Taking $\lambda_n = \log n$, we obtain the ordinary Dirichlet series as in (1.1). Taking $\lambda_n = n$, we obtain the series $\sum a_n e^{-ns}$. After the change of variable $z = e^{-s}$, these series turn out to be the usual power series $\sum a_n z^n$ in one complex variable.

In this sense the subject of power series (well known to every undergraduate student) can be interpreted as a sub-theory of general Dirichlet series. The latter series are a highly interesting objects of study in their one – but here we only recall this notion in order to point out the differences between the convergence of power series and the convergence of ordinary Dirichlet series.

As already mentioned, we aim at proving that ordinary Dirichlet series define maximal half-planes on which they converge and define holomorphic functions. Moreover, there are maximal half-planes on which these series converge uniformly or even absolutely. It may come as a surprise that in general all three of these domains differ. Why is this a surprise?

Let us compare this phenomenon with what we know from power series, that is, Dirichlet series of the type $\sum a_n e^{-ns}$. Let us have a look at how this change of variable $z = e^{-s}$ affects the regions of convergence. Power series in one variable of the form $\sum a_n z^n$ converge in discs, and the biggest (open) disc on which they converge is the same as the biggest (open) disc on which they converge absolutely. In other words, the radii that define the biggest disc where a power series converges or converges absolutely are the same. Even more, this radius gives the biggest $R$ for which the series converges uniformly on every disc of radius $R - \varepsilon$. To translate this in terms of generalized Dirichlet series,
we write \( s = \sigma + it \). Then we have
\[
e^{-\sigma}e^{-it} = e^{-\sigma-it} = e^{-s} = z = |z|e^{i\theta};
\]
that is, with the change of variable \( z = e^{-s} \), we transform \( |z| \) into \( e^{-\Re s} \). Therefore, if \( |z| < r \), then \( \Re s > \log 1/r \), or, in other words, discs in the variable \( z \) centred at 0 are transformed into right half-planes in the variable \( s \) and radii of convergence are substituted by abscissas defining a half-plane. Thus, the generalized Dirichlet with \( \lambda_n = n \) converge on right half-planes and the maximal half-planes for convergence, uniform convergence and absolute convergence are the same (or to put it in other words, the absicssas coincide). This is indeed the beginning of our story!

It is now not so hard to believe that the Dirichlet series \( \sum a_n n^{-s} \) also converge on half-planes. Our first duty is to prove that this is indeed the case, and that these half-planes can, in contrast with what happens with power series, be very different from each other. We start with the definition of the abscissa of convergence \( \sigma_c(D) \) given by
\[
\sigma_c(D) = \inf \{ \sigma \in \mathbb{R} : D \text{ converges in } \Re s > \sigma \} \in [-\infty, \infty]. \tag{1.3}
\]
The following result is going to be indispensable, and it is the starting point of all our story.

**Theorem 1.1** Let \( D = \sum a_n n^{-s} \) be a Dirichlet series (not everywhere divergent). Then it converges on the half-plane \( \Re s > \sigma_c(D) \) and diverges on \( \Re s < \sigma_c(D) \). Moreover, the following limit function \( f \) of \( D \) is holomorphic:
\[
f : [\Re > \sigma_c(D)] \to \mathbb{C} \text{ given by } f(s) = \sum_{n=1}^{\infty} a_n \frac{1}{n^s}.
\]

Note that no general statement on the convergence of the Dirichlet series on the abscissa of convergence itself is made. The preceding theorem is an immediate consequence of the following lemma.

**Lemma 1.2** If a Dirichlet series \( \sum a_n n^{-s} \) is convergent at \( s_0 \), and \( 0 \leq \alpha < \pi/2 \), then it converges uniformly in the angular set \( \{ s \in \mathbb{C} : \Re s > \Re s_0, |\arg(s - s_0)| < \alpha \} \) (here the argument \( \arg(u) \) of \( u = |u|e^{i\arg u} \) is taken in \([-\pi, \pi]\)).

Before we start with the proof of this lemma, let us briefly indicate why it in fact proves Theorem 1.1. Clearly, if \( \sum a_n n^{-s} \) converges at \( s_0 = \sigma_0 + it_0 \), then the lemma shows that it converges on every half-plane \( \Re s > \sigma \) with \( \sigma > \sigma_0 \). This leads to the ‘pointwise part’ of the theorem. But the ‘holomorphic part’ is immediate as well, once we make the simple observation that any compact set in \( [\Re > \sigma_c(D)] \) can be included in some angular set with vertex in any point of the abscissa \( [\Re = \sigma_c(D)] \) just taking a wide enough angle. Then the series
1.1 Convergence of Dirichlet Series

converges uniformly on every compact set in \([\text{Re} > \sigma_c(D)]\), and by Weierstraß convergence theorem, it defines a holomorphic function there.

For the proof of the lemma (as in several other moments later), Abel summation is going to be an important tool. If \(a_1, \ldots, a_N, b_1, \ldots, b_N \in \mathbb{C}\) and \(A_n = \sum_{k=1}^{n} a_k\) for \(1 \leq n \leq N\), then

\[
\sum_{n=1}^{N} a_n b_n = A_N b_N + \sum_{n=1}^{N-1} A_n (b_n - b_{n+1}). \tag{1.4}
\]

Indeed, since \(a_n = A_n - A_{n-1}\) for \(2 \leq n \leq N\), we have

\[
\sum_{n=1}^{N} a_n b_n = A_1 b_1 + \sum_{n=2}^{N} (A_n - A_{n-1}) b_n = A_N b_N + \sum_{n=1}^{N-1} A_n b_n - \sum_{n=2}^{N} A_{n-1} b_n
\]

\[
= A_N b_N + \sum_{n=1}^{N-1} A_n b_n - \sum_{n=1}^{N-1} A_n b_{n+1} = A_N b_N + \sum_{n=1}^{N-1} A_n (b_n - b_{n+1}).
\]

**Proof of Lemma 1.2** Let us assume first that \(s_0 = 0\). We fix \(0 \leq \alpha < \pi/2\) and then, as the series \(\sum_{n=1}^{\infty} a_n\) converges, given \(\varepsilon > 0\), there exists \(N_0\) such that for
all $N_0 \leq N < M$,

$$\left| \sum_{n=N}^{M} a_n \right| \leq \varepsilon \cos \alpha .$$

Consider $s = \sigma + it$ in the angular set $S = \{ s \in \mathbb{C} : \Re s > 0, |\arg(s)| < \alpha \}$.

Abel summation as in (1.4) implies

$$\sum_{n=N}^{M} a_n \frac{1}{n^s} = \left( \sum_{k=N}^{M} a_k \right) \frac{1}{M^\sigma} + \sum_{n=N}^{M-1} \left( \sum_{k=n}^{M-1} a_k \right) \left( \frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma} \right),$$

hence

$$\left| \sum_{n=N}^{M} a_n \frac{1}{n^s} \right| \leq \varepsilon \cos \alpha \frac{1}{M^\sigma} + \sum_{n=N}^{M-1} \varepsilon \cos \alpha \left| \frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma} \right| .$$

But

$$\left| \frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma} \right| = \left| \int_{n}^{n+1} \frac{s}{x^\sigma+1} \, dx \right| \leq \int_{n}^{n+1} \frac{|s|}{x^\sigma+1} \, dx = |s| \left( \frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma} \right),$$

and so we obtain

$$\left| \sum_{n=N}^{M} a_n \frac{1}{n^s} \right| \leq \varepsilon \cos \alpha \frac{|s|}{\sigma} \left( \frac{1}{M^\sigma} + \sum_{n=N}^{M-1} \left( \frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma} \right) \right) = \varepsilon \cos \alpha \frac{|s|}{\sigma} \frac{1}{N^\sigma} .$$

Since $\frac{\varepsilon}{|s|} = \cos(\arg s) \geq \cos \alpha$, we finally get that for all $s$ in the angular set $S$,

$$\left| \sum_{n=N}^{M} a_n \frac{1}{n^s} \right| < \varepsilon ,$$

and the conclusion follows if $s_0 = 0$. Now, if it is not the case, we may consider the Dirichlet series $\sum \frac{a_n}{n^u} n^{-s}$, which converges at $u = 0$ and, by what we have just shown, converges uniformly on $\{ u \in \mathbb{C} : \Re u > 0, |\arg(u)| < \alpha \}$. Making the change of variable $s_0 + u = s$ and observing that if $\Re u > 0$ and $|\arg(u)| < \alpha$, then $\Re s > \Re s_0$ and $|\arg(s-s_0)| < \alpha$, we finally have that the series $\sum a_n n^{-s}$ converges uniformly on $\{ s \in \mathbb{C} : \Re s > \Re s_0, |\arg(s-s_0)| < \alpha \}$.

We have from Theorem 1.1 that Dirichlet series converge on half-planes. Then we have considered the maximal half-plane and the abscissa that defines it (given by $\sigma_c(D)$). We go now one step further and wonder what happens when we think of absolute convergence of Dirichlet series. For any Dirichlet series $D = \sum_n a_n n^{-s}$, the abscissa of absolute convergence is given by

$$\sigma_a(D) = \inf \{ \sigma \in \mathbb{R} : D \text{ converges absolutely in } [\Re > \sigma] \} \in [-\infty, \infty] .$$

Observe that this is nothing else but the abscissa of convergence of the Dirichlet series $\sum |a_n| n^{-s}$. Then the half-plane $[\Re > \sigma_a(D)]$ of absolute convergence
1.1 Convergence of Dirichlet Series

defines the largest half-plane on which $D$ converges absolutely. Obviously, we have

$$-\infty \leq \sigma_c(D) \leq \sigma_a(D) \leq \infty.$$  

For certain Dirichlet series, these two abscissas may coincide (just take a Dirichlet series with positive coefficients), but in general this need not be the case. Take for example the series $\sum n(-1)^n n^{-s}$. This converges for every $\text{Re } s > 0$ but converges absolutely only on the plane $[\text{Re } > 1]$. In other terms,

$$\sigma_c\left(\sum n(-1)^n n^{-s}\right) = 0 \quad \text{and} \quad \sigma_a\left(\sum n(-1)^n n^{-s}\right) = 1.$$  

This shows that the situation for ordinary Dirichlet series is very much different from that of power series (i.e. generalized Dirichlet series with frequency $\lambda_n = n$). Here the abscissas $\sigma_c(D)$ and $\sigma_a(D)$ may not be equal, and one of the half-planes may be strictly bigger than the other. Now the question is, how big can it get? or how far apart can $\sigma_a(D)$ and $\sigma_c(D)$ be to each other? Actually not really too far away. Assume that $\sum a_n n^{-s}$ converges at some $s_0 = \sigma_0 + it$. Then the sequence $\left(\frac{|a_n|}{n^{\sigma_0+1+\epsilon}}\right)_n$ is bounded by, say, $K$, and we have, for every $\epsilon > 0$,

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma_0+1+\epsilon}} = \sum_{n=1}^{\infty} \frac{|d_n|}{n^{\sigma_0}} \leq K \sum_{n=1}^{\infty} \frac{1}{n^{1+\epsilon}} < \infty.$$  

Then $\sigma_a(D) \leq \sigma_0 + 1 + \epsilon$. This then shows that for every Dirichlet series, $\sigma_a(D) \leq \sigma_c(D) + 1$, and, together with the previous example, we obtain the following equality.

**Proposition 1.3**

$$\sup \{\sigma_a(D) - \sigma_c(D) : D \text{ Dirichlet series} \} = 1.$$  

As we pointed out, the first obvious example of Dirichlet series for which the abscissas of convergence and absolute convergence coincide are those with positive coefficients. But there are other examples. We will say that a Dirichlet series $\sum a_n n^{-s}$ ‘depends on finitely many primes’ if there is a finite set of primes appearing as factors of the $n$ for which $a_n \neq 0$. To be more precise, there are prime numbers $p_i_1, \ldots, p_i_N$ such that if $a_n \neq 0$, then $n = p_i_1^{\alpha_1} \cdots p_i_N^{\alpha_N}$ for some $\alpha$. In this case we will say that the Dirichlet series depends on $N$ primes. If the primes appearing are $p_1, \ldots, p_N$ (that is, the first ones), we will then say that the Dirichlet series ‘depends on the first $N$ primes’, and we will denote the class of such series by $\mathfrak{D}^{(N)}$. For these series, all these abscissas cannot be distinguished, as the next result shows.

**Proposition 1.4** Let $D = \sum a_n n^{-s}$ depend on finitely many primes; then

$$\sigma_c(D) = \sigma_a(D).$$
Proof. The argument is fairly simple; we give it just for series that depend on
two primes (the one for an arbitrary number of primes is then straightforward).
Assume that \( \sum a_n n^{-s} \) depends only on two primes; that is, there are primes \( p \) and \( q \) so that we can rewrite the series in the following way (keeping the natural
order)
\[
\sum_{k,\ell} a_{p^k q^\ell} (p^k q^\ell)^{-s}.
\]
Assume that the series converges at some \( s = \sigma + it \). Then there is some \( M > 0 \) such that for all \( k, \ell \),
\[
\frac{|a_{p^k q^\ell}|}{(p^k q^\ell)^{\sigma}} \leq M.
\]
Consequently, for every \( \varepsilon > 0 \) and \( N \),
\[
\sum_{k,\ell=1}^{N} |a_{p^k q^\ell}| \frac{1}{(p^k q^\ell)^{\sigma+\varepsilon}} \leq M \sum_{k=1}^{\infty} \frac{1}{(p^k)^{\varepsilon}} \sum_{\ell=1}^{\infty} \frac{1}{(q^\ell)^{\varepsilon}} < \infty,
\]
which in fact proves that \( \sigma_c(D) \geq \sigma_a(D) \). \( \square \)

We come to the third abscissa which will be of interest to us, the abscissa of
uniform convergence. Every Dirichlet series \( D = \sum a_n n^{-s} \) can be viewed as a
sequence of functions \( \left\{ \sum_{n=1}^{N} a_n n^{-s} \right\}_{N \in \mathbb{N}} \) on the complex plane. Then
\[
\sigma_u(D) = \inf \left\{ \sigma \in \mathbb{R} : \sum a_n n^{-s} \text{ converges uniformly in } \{ \text{Re} > \sigma \} \in [\infty, \infty] \right\}
\]
defines the largest half-plane on which \( \sum a_n n^{-s} \) converges uniformly on every
strictly smaller half-plane. The abscissa of uniform convergence of the series
is located in between the abscissa of convergence and the abscissa of absolute
convergence:
\[
-\infty \leq \sigma_v(D) \leq \sigma_u(D) \leq \sigma_a(D) \leq \infty.
\]
Obviously, \( \sigma_v(D) \leq \sigma_u(D) \) for every Dirichlet series. On the other hand, given
a Dirichlet series \( D = \sum a_n n^{-s} \) and \( \varepsilon > 0 \), for every \( M > N \) and \( t \in \mathbb{R} \) we have
\[
\left| \sum_{n=N}^{M} a_n \frac{1}{n^{\sigma_u(D)+\varepsilon+it}} \right| \leq \sum_{n=N}^{M} \frac{|a_n|}{n^{\sigma_u(D)+\varepsilon}}.
\]
Since the right-hand side converges, this shows that the series is uniformly
Cauchy on \( \{ \text{Re} > \sigma_u(D) + \varepsilon \} \) and, hence, \( \sigma_u(D) \leq \sigma_a(D) \). We wonder now
about the maximal possible distance between \( \sigma_u(D) \) and \( \sigma_c(D) \).

**Proposition 1.5**
\[
\sup \{ \sigma_u(D) - \sigma_c(D) : D \text{ Dirichlet series } \} = 1.
\]
1.1 Convergence of Dirichlet Series

Clearly, by Proposition 1.3, we have

\[
\sup_{D \in \mathcal{D}} \sigma_u(D) - \sigma_c(D) \leq \sup_{D \in \mathcal{D}} \sigma_a(D) - \sigma_c(D) = 1, \tag{1.8}
\]

and in order to get equality, we have to search for an appropriate example. Figure 1.2 illustrates the situation. Lemma 1.2 tells us that the Dirichlet series converges uniformly on any angular domain like the shaded one. We can make the angle as wide as we wish, making the line ‘almost vertical’. The intuition would then lead us to think that both abscissas are equal. We have to fight against this intuition, finding an example (which cannot be completely obvious) that these abscissas are not equal.

![Figure 1.2](image_url)

Figure 1.2 Nontrivial examples are needed.

To prove Proposition 1.5 (and at several other occasions), the following Bohr–Cahen formulas are going to be useful tools.
Proposition 1.6 Let \( D = \sum_n a_n n^{-s} \) be a Dirichlet series. Then
\[
\sigma_c(D) \leq \limsup_{N \to \infty} \frac{\log \left( \left| \sum_{n=1}^{N} a_n \right| \right)}{\log N},
\]
\[
\sigma_u(D) \leq \limsup_{N \to \infty} \frac{\log \left( \sup_{t \in \mathbb{R}} \left| \sum_{n=1}^{N} a_n n^{-it} \right| \right)}{\log N},
\]
\[
\sigma_a(D) \leq \limsup_{N \to \infty} \frac{\log \left( \sum_{n=1}^{N} |a_n| \right)}{\log N}.
\]
In each case, if the abscissa is not negative, then we have equality.

Note that the condition for equality is not superfluous. Take \( D = \sum_n n n^{-2} n^{-s} \), which converges at \( s = 0 \). For this example, \( \sigma_c(D) = \sigma_u(D) = \sigma_a(D) = -1 \), but the limits are all 0.

Proof We prove first the estimate/formula for \( \sigma_c(D) \). Let us write \( L \) for the lim sup in the right-hand side. If \( L = \infty \), then the inequality is anyway clear; let us assume \( L \in \mathbb{R} \) (we will consider later the case \( L = -\infty \)). We want to show that, given \( D = \sum_n a_n n^{-s} \), we have \( \sigma_c(D) \leq L \). Fix some \( \varepsilon > 0 \) and set \( \sigma_0 = L + \varepsilon \). Our aim is to prove that the series \( \sum_{n=1}^{\infty} a_n n^{-\sigma_0} \) converges. For \( N \geq 2 \) we write \( A_N = \sum_{n=1}^{N} a_n \). By Abel summation (1.4) with \( a_n \) and \( 1/n^{\sigma_0} \), we have
\[
\sum_{n=1}^{N} a_n \frac{1}{n^{\sigma_0}} = \sum_{n=1}^{N-1} A_n \left( \frac{1}{n^{\sigma_0}} - \frac{1}{(n+1)^{\sigma_0}} \right) + A_N \frac{1}{N^{\sigma_0}},
\]
and hence for each \( M > N + 1 \),
\[
\sum_{n=N+1}^{M} a_n \frac{1}{n^{\sigma_0}} = \sum_{n=N}^{M-1} A_n \left( \frac{1}{n^{\sigma_0}} - \frac{1}{(n+1)^{\sigma_0}} \right) + A_M \frac{1}{M^{\sigma_0}} - A_N \frac{1}{N^{\sigma_0}}, \quad (1.9)
\]
which gives
\[
\left| \sum_{n=N+1}^{M} a_n \frac{1}{n^{\sigma_0}} \right| \leq \sum_{n=N}^{M-1} |A_n| \left| \frac{1}{n^{\sigma_0}} - \frac{1}{(n+1)^{\sigma_0}} \right| + |A_M| \frac{1}{M^{\sigma_0}} + |A_N| \frac{1}{N^{\sigma_0}}. \quad (1.10)
\]
Remember that \( \varepsilon > 0 \) is fixed; then we can find \( n_0 \) such that for every \( n \geq n_0 \),
\[
\frac{\log |A_n|}{\log n} < L + \frac{\varepsilon}{2},
\]
which clearly implies
\[
|A_n| < n^{L+\frac{\varepsilon}{2}}. \quad (1.11)
\]
Since \( \sigma_0 = L + \varepsilon \), we have, for every \( M > N + 1 \geq n_0 \),
\[
\frac{|A_M|}{M^{\sigma_0}} + \frac{|A_N|}{N^{\sigma_0}} \leq M^{L+\frac{\varepsilon}{2}-\sigma_0} + N^{L+\frac{\varepsilon}{2}-\sigma_0} = \frac{1}{M^{\frac{\varepsilon}{2}}} + \frac{1}{N^{\frac{\varepsilon}{2}}}. \quad (1.12)
\]
On the other hand, we have (recall (1.5))

\[
\left| \frac{1}{n^{\sigma_0}} - \frac{1}{(n+1)^{\sigma_0}} \right| = \left| \int_n^{n+1} \frac{|\sigma_0|}{x^{\sigma_0+1}} \, dx \right| \leq \int_n^{n+1} \frac{|\sigma_0|}{x^{\sigma_0+1}} \, dx \leq \frac{|\sigma_0|}{\min\{n^{\sigma_0+1}, (n+1)^{\sigma_0+1}\}} \leq 2|\sigma_0| \frac{1}{n^{\sigma_0+1}}
\]

(note that for \( \sigma_0 = 0 \) the left inequality is anyway trivial). This, together with (1.11), gives

\[
|A_n| \left| \frac{1}{n^{\sigma_0}} - \frac{1}{(n+1)^{\sigma_0}} \right| \leq 2|\sigma_0| |n^{L+\varepsilon}| \frac{1}{n^{L+\varepsilon+1}} = 2|\sigma_0| \frac{1}{n^{1+\varepsilon}}.
\]

Then (1.10) shows that for every \( M > N + 1 \geq n_0 \),

\[
\left| \sum_{n=N+1}^{M} a_n \frac{1}{n^{\sigma_0}} \right| \leq 2|\sigma_0| \sum_{n=N}^{M-1} \frac{1}{n^{1+\varepsilon}} + \frac{1}{M^{\varepsilon}} + \frac{1}{N^{\varepsilon}}.
\]

Obviously, \( n_0 \) can be chosen so that the last term is as small as we want.

This shows that the sequence \( \left( \sum_{n=1}^{N} a_n \frac{1}{n^{\sigma_0}} \right) \) is Cauchy, and then the series \( \sum_{n=1}^{\infty} a_n \frac{1}{n^{\sigma_0}} \) converges, which implies that \( \sigma_c(D) \leq \sigma_0 = L + \varepsilon \). Since \( \varepsilon > 0 \) was arbitrary, this yields \( \sigma_c(D) \leq L \).

If \( L = -\infty \), then we fix \( R \in \mathbb{R} \) and set \( r_0 = R + \varepsilon \). Proceeding exactly as we did in the previous case, we show that \( \sigma_c(D) \leq R \), and since this holds for every \( R \), we conclude that \( \sigma_c(D) = -\infty \).

Let us assume additionally that \( \sigma_c(D) \geq 0 \) and see that in this case we even have the equality. Given \( \varepsilon > 0 \), we have that \( \sigma_0 = \sigma_c(D) + \varepsilon > 0 \). Since now the series \( \sum_{n=1}^{\infty} a_n \frac{1}{n^{\sigma_0}} \) converges, there is some \( B > 0 \) such that for every \( N \),

\[
\left| \sum_{n=1}^{N} a_n \frac{1}{n^{\sigma_0}} \right| \leq B.
\]

Writing \( B_N = \sum_{n=1}^{N} a_n \frac{1}{n^{\sigma_0}} \) for \( N \geq 2 \), we have (again by (1.4))

\[
\sum_{n=1}^{N} a_n \frac{1}{n^{\sigma_0}} n^{\sigma_0} = \sum_{n=1}^{N-1} B_n (n^{\sigma_0} - (n+1)^{\sigma_0}) + B_N n^{\sigma_0}.
\]

Then

\[
\left| \sum_{n=1}^{N} a_n \right| \leq B \sum_{n=1}^{N-1} ((n+1)^{\sigma_0} - n^{\sigma_0}) + BN^{\sigma_0} \leq 2BN^{\sigma_0},
\]

and as a consequence,

\[
\log \left| \sum_{n=1}^{N} a_n \right| \leq \log(2B) + \sigma_0 \log N
\]
The Absolute Convergence Problem

for every \( N \geq 2 \). This gives \( L \leq \sigma_0 = \sigma_c(D) + \epsilon \), and hence

\[
\limsup_{N \to \infty} \frac{\log |\sum_{n=1}^N a_n|}{\log N} \leq \sigma_c(D),
\]

which completes the proof of the formula for \( \sigma_c(D) \).

The formula for \( \sigma_a(D) \) follows simply by applying the one for \( \sigma_c(D) \) to the series \( \sum_n |a_n| n^{-s} \), and the proof for \( \sigma_u(D) \) will be given after the following lemma.

The basic structure of the proof of the formula for \( \sigma_u(D) \) is the same. We just have to observe that this abscissa has to do with uniform convergence on half-planes, so it deals with the supremum on those half-planes, but what we see in the formula for \( \sigma_a(D) \) is a supremum only on a vertical abscissa. This suggests that a sort of maximum modulus principle that reduces the supremum on a half-plane \([\text{Re} > \sigma]\) to the supremum on the abscissa \([\text{Re} = \sigma]\) should hold for Dirichlet polynomials. This is indeed the case and follows easily from the case \( \sigma = 0 \).

Lemma 1.7  For \( a_1, \ldots, a_N \in \mathbb{C} \), we have

\[
\sup_{\text{Re} s > 0} \left| \sum_{n=1}^N a_n n^{-s} \right| = \sup_{t \in \mathbb{R}} \left| \sum_{n=1}^N a_n n^{-it} \right|.
\]

Before we give the proof, note that for arbitrary \( \sigma_0 \in \mathbb{R} \) we can apply this lemma to \( \frac{a_n}{n^{\sigma_0}} \) (as in the proof of Lemma 1.2) for \( n = 1, \ldots, N \) to have this sort of maximum modulus principle for arbitrary half-planes:

\[
\sup_{\text{Re} s > \sigma_0} \left| \sum_{n=1}^N a_n n^{-s} \right| = \sup_{\text{Re} z > 0} \left| \sum_{n=1}^N a_n \frac{1}{n^{\sigma_0 + z}} \right| = \sup_{\text{Re} z > 0} \left| \sum_{n=1}^N a_n \frac{1}{n^{\sigma_0 + z}} \right| = \sup_{t \in \mathbb{R}} \left| \sum_{n=1}^N a_n \frac{1}{n^{\sigma_0 + it}} \right| = \sup_{t \in \mathbb{R}} \left| \sum_{n=1}^N a_n \frac{1}{n^{\sigma_0 + it}} \right|.
\]  (1.15)

Proof  Let us write

\[
A = \sup_{t \in \mathbb{R}} \left| \sum_{n=1}^N a_n \frac{1}{n^{\sigma_0 + it}} \right| \quad \text{and} \quad B = \sup_{\text{Re} s > 0} \left| \sum_{n=1}^N a_n \frac{1}{n^{s}} \right|.
\]

By continuity, \( A \leq B \). For the converse inequality, let us fix \( \epsilon > 0 \) and consider the function \( g_\epsilon : \mathbb{C} \setminus [\text{Re} < 0] \to \mathbb{C} \) defined as

\[
g_\epsilon(s) := e^{-\epsilon \sqrt{s}} \sum_{n=1}^N a_n \frac{1}{n^{\sigma_0 + \epsilon}}.
\]

where \( \sqrt{s} = e^{\frac{1}{2} \log s} \) denotes the principal square root of \( s \in \mathbb{C} \setminus [\text{Re} \leq 0] \) and
\( \sqrt{0} = 0 \). Then \( g_\varepsilon \) is a holomorphic function on \( \text{Re} > 0 \) and continuous up to the boundary. Taking now \( s = re^{it} \in \mathbb{C} \) with \( \text{Re} s \geq 0 \), we have

\[
|g_\varepsilon(s)| = e^{-\varepsilon \text{Re} \sqrt{r}} \left| \sum_{n=1}^{N} a_n \frac{1}{n^s} \right| = e^{-\varepsilon \sqrt{r} \cos \frac{t}{2}} \left| \sum_{n=1}^{N} a_n \frac{1}{n^s} \right| \leq Be^{-\varepsilon \sqrt{r} \cos \frac{t}{2}},
\]

and this tends to 0 as \( r \to \infty \). Then there exists \( R > 0 \) such that \( |g_\varepsilon(re^{it})| \leq A \) for \( r \geq R \). Taking now \( \Delta = \{s \in \mathbb{C} : \text{Re} s \geq 0, |s| < R\} \), we have \( |g_\varepsilon(s)| \leq A \) for every \( s \) in the boundary of \( \Delta \), since again, for each \( t \in \mathbb{R} \),

\[
|g_\varepsilon(it)| = e^{-\varepsilon \text{Re} \sqrt{r}} \left| \sum_{n=1}^{N} a_n \frac{1}{n^{it}} \right| \leq Ae^{-\varepsilon \sqrt{r} \cos \frac{t}{2}} \leq A.
\]

By the maximum modulus principle, \( |g_\varepsilon(s)| \leq A \) for all \( s \in \Delta \). This altogether gives \( |g_\varepsilon(s)| \leq A \) for every \( \text{Re} s \geq 0 \). Letting \( \varepsilon \to 0 \) gives the conclusion. \( \square \)

We can now prove the formula for the abscissa of uniform convergence.

**Proof of the formula for \( \sigma_u(D) \) in Proposition 1.6** We follow the proof for the abscissa \( \sigma_e(D) \). Again we write \( L \) for the limit superior, and again we want to show first that \( \sigma_u(D) \leq L \). Assume that \( L \neq \pm \infty \) and, as above, choose \( \varepsilon > 0 \) and set \( \sigma_0 = L + \varepsilon \). The aim is to show that \( \left( \sum_{n=1}^{N} a_n n^{-s} \right) \) converges uniformly on \( \text{Re} > \sigma_0 \). For \( N \geq 2 \) and \( s \in \mathbb{C} \) we write \( A_N(s) = \sum_{n=1}^{N} a_n \frac{1}{n^s} \).

Abel summation (as in (1.4) with \( \frac{a_n}{n^\sigma} \) and \( \frac{1}{n^\sigma} \)) gives, for each \( t \in \mathbb{R} \) and \( N \),

\[
\sum_{n=1}^{N} a_n \frac{1}{n^{\sigma_{0}+it}} = \sum_{n=1}^{N} \frac{a_n}{n^t} \frac{1}{n^{\sigma_0}} = \sum_{n=1}^{N-1} A_n(it) \left( \frac{1}{n^{\sigma_0}} - \frac{1}{(n+1)^{\sigma_0}} \right) + A_N(it) \frac{1}{N^{\sigma_0}}.
\]

Proceeding as in (1.9), we have that for every \( t \in \mathbb{R} \) and every \( M > N + 1 \),

\[
\left| \sum_{n=N+1}^{M} \frac{a_n}{n^{\sigma_{0}+it}} \right| \leq \sum_{n=N}^{M-1} |A_n(it)| \left| \frac{1}{n^{\sigma_0}} - \frac{1}{(n+1)^{\sigma_0}} \right| + |A_M(it)| \frac{1}{M^{\sigma_0}} + |A_N(it)| \frac{1}{N^{\sigma_0}}.
\]

As in (1.11), by the very definition of \( L \), there is \( n_0 \) such that, for every \( N \geq n_0 \),

\[
\sup_{t \in \mathbb{R}} |A_N(it)| \leq N^{L + \frac{\varepsilon}{2}}.
\]

Using this fact instead of (1.11) and (1.13), proceeding as in (1.12) and (1.14), we end up with

\[
\sup_{t \in \mathbb{R}} \left| \sum_{n=N}^{M} \frac{a_n}{n^{\sigma_{0}+it}} \right| \leq 2 |\sigma_0| \sum_{n=N}^{M} \frac{1}{n^{1+\frac{\varepsilon}{2}}} + \frac{1}{M^{\frac{\varepsilon}{2}}} + \frac{1}{N^{\frac{\varepsilon}{2}}}
\]

for every \( M > N + 1 \geq n_0 \). Then (1.15) shows that the sequence \( \left( \sum_{n=1}^{N} a_n \frac{1}{n^s} \right) \) is uniformly Cauchy on \( \text{Re} > \sigma_0 \), which implies that \( \sigma_u(D) \leq L \). If \( L = -\infty \), the inequality follows with a modification of the previous argument, just as we did for \( \sigma_e(D) \).
Assume conversely that $\sigma_u(D) \geq 0$; then for $\sigma_0 = \sigma_u(D) + \varepsilon > 0$ the series $(\sum_{n=1}^{N} \frac{a_n}{n^s})_N$ converges uniformly on $[\text{Re} > \sigma_0]$, and there exists $B > 0$ such that for all $N$ by (1.15),

$$\sup_{\text{Re } s = \sigma_0} \left| \sum_{n=1}^{N} \frac{a_n}{n^s} \right| = \sup_{\text{Re } s \gg \sigma_0} \left| \sum_{n=1}^{N} \frac{a_n}{n^s} \right| \leq B.$$ 

We write $B_N(it) = \sum_{n=1}^{N} \frac{a_n}{n^{\sigma_0+it}}$ for $t \in \mathbb{R}$ and $N \geq 2$. For every such $t$ and $N$ we have, using Abel summation (1.4),

$$\sum_{n=1}^{N} \frac{1}{n^it} = \sum_{n=1}^{N} \frac{1}{n^{\sigma_0+it}} n^{\sigma_0} = \sum_{n=1}^{N-1} B_n(it)(n^{\sigma_0} - (n + 1)^{\sigma_0}) + B_N(it)n^{\sigma_0},$$

and consequently, the proof finishes exactly in the same way as for $\sigma_c(D)$. □

We can now determine the maximal distance between the abscissas of convergence and uniform convergence of a Dirichlet series.

**Proof of Proposition 1.5** For the lower estimate, take $(p_n)_n$ the sequence of prime numbers, and consider the Dirichlet series $D = \sum (-1)^n p_n^{-s}$; note first that $D = \sum a_n n^{-s}$, where $a_n = (-1)^k$ if $n = p_k$, and 0 otherwise. It is easy to check that $\sigma_c(D) = 0$ (either by Proposition 1.6 or by Leibniz’s criterion for alternate series). We are going to estimate $\sigma_u(D)$ by using the formula in Proposition 1.6.

The key point is to choose for every $N$ a sequence $(t_k)_k$ in $\mathbb{R}$ such that $\lim_k t_k^{-it} = (-1)^n$ for every $n = 1, \ldots, N$; Proposition 3.4 (Kronecker’s theorem) shows that this simultaneous approximation is possible, a fact that, although it comes later in this text, does not require in the proof anything we do not know at this moment. Then for all $N$,

$$N = \lim_{k \to \infty} \left| \sum_{n=1}^{N} (-1)^n \frac{1}{p_n^{it}} \right| \leq \sup_{t \in \mathbb{R}} \left| \sum_{n=1}^{N} (-1)^n \frac{1}{p_n^t} \right| = \sup_{t \in \mathbb{R}} \left| \sum_{n=1}^{p_N} \frac{a_n}{n^s} \right|,$$

and hence

$$\sigma_u(D) = \limsup_{N \to \infty} \frac{\log \left( \sup_{t \in \mathbb{R}} \left| \sum_{n=1}^{N} a_n n^{it} \right| \right)}{\log N} \geq \limsup_{N \to \infty} \frac{\log \left( \sup_{t \in \mathbb{R}} \left| \sum_{n=1}^{p_N} a_n n^{it} \right| \right)}{\log p_N} \geq \limsup_{N \to \infty} \frac{\log N}{\log p_N}.$$

But by the prime number theorem $\lim_{N \to \infty} \frac{\log N}{\log p_N} = 1$, hence $\sigma_u(D) \geq 1$. On the other hand, $\sigma_c(D) = 0$ so that $\sigma_u(D)$ by (1.8) cannot exceed 1. Hence $\sigma_u(D) = 1$, and the proof is complete. □

**Remark 1.8** A trick that we have already used, for example in (1.15) (and that we will use several times from now on), is to consider the Dirichlet series

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1.1 Convergence of Dirichlet Series

defined by dividing the coefficients by a factor $n^{z_0}$ with $z_0 = \sigma_0 + i\theta_0 \in \mathbb{C}$, that is, given a Dirichlet series $D = \sum a_n n^{-s}$, to consider

$$D_{z_0} = \sum \frac{a_n}{n^{z_0}} n^{-s}. \quad (1.16)$$

What we do with this is to move the abscissas of convergence by $\sigma_0$. Indeed, the series $\sum_{n=1}^{\infty} a_n \frac{1}{n^s}$ converges if and only if $\sum_{n=1}^{\infty} \frac{a_n}{n^{z_0}} \frac{1}{n^{s-z_0}}$ converges. This gives

$$\sigma_c\left(\sum \frac{a_n}{n^{z_0}} n^{-s}\right) = \sigma_c\left(\sum a_n n^{-s}\right) - \sigma_0. \quad (1.17)$$

With exactly the same argument, we have

$$\sigma_u\left(\sum \frac{a_n}{n^{z_0}} n^{-s}\right) = \sigma_u\left(\sum a_n n^{-s}\right) - \sigma_0 \quad \text{and} \quad \sigma_a\left(\sum \frac{a_n}{n^{z_0}} n^{-s}\right) = \sigma_a\left(\sum a_n n^{-s}\right) - \sigma_0.$$

In this way we shift the convergence abscissas of a Dirichlet series. We will often call this a ‘(simple) translation argument’. Since our main interest is going to be the distance between abscissas (and more precisely between the abscissas of absolute and uniform convergence), when we shift a Dirichlet series in this way, we do not modify this distance, or to put it more clearly, for every Dirichlet series $D$ and every $z_0 \in \mathbb{C}$ we have

$$\sigma_a(D) - \sigma_u(D) = \sigma_a(D_{z_0}) - \sigma_u(D_{z_0}).$$

In Theorem 1.1 we are dealing with holomorphic functions on half-planes $f : [\text{Re} > \sigma_c(D)] \to \mathbb{C}$ that can be represented by Dirichlet series $\sum a_n n^{-s}$.

We wonder now to what extent the function is uniquely determined by the coefficients, and vice versa (as it happens for holomorphic functions on discs and power series). Cauchy’s integral formula shows that the Taylor coefficients of a holomorphic function $f$, on the open unit disc $\mathbb{D}$, say, determine the Taylor coefficients uniquely. Something similar happens here. The coefficients of a Dirichlet series are uniquely determined by the limit function if we integrate along vertical lines (which, as we noted before, are the reflex of the circles).

**Proposition 1.9** Let $\sum a_n n^{-s}$ be a Dirichlet series (not everywhere divergent) and $f$ its limit function. Then for every $\kappa > \sigma_a(D)$ and $N \in \mathbb{N}$,

$$a_N = \lim_{R \to \infty} \frac{1}{2Ri} \int_{\kappa-iR}^{\kappa+iR} f(s) N^s ds.$$

The integral in the statement has to be understood as a path integral along the segment from $\kappa - iR$ to $\kappa + iR$.

**Proof** Let us fix $N \in \mathbb{N}$. Note first that an easy computation shows that for each $R > 0$,

$$\frac{1}{2R} \int_{-R}^{R} \left(\frac{N}{n}\right)^{it} dt = \begin{cases} 1 & \text{if } n = N, \\ \frac{\sin(R \log(N/n))}{R \log(N/n)} & \text{if } n \neq N. \end{cases} \quad (1.17)$$
Now we fix $R > 0$, and since the limit function $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ converges uniformly on $[\kappa + it : -R \leq t \leq R]$, we have

$$\frac{1}{2Ri} \int_{\kappa+iR}^{\kappa-iR} f(s)Nsds = \frac{1}{2R} \int_{-R}^{R} f(\kappa + it)N^{\kappa + it}dt = \frac{1}{2R} \int_{-R}^{R} \sum_{n=1}^{\infty} a_n N^{\kappa + it}dt$$

$$= \frac{N^\kappa}{2R} \sum_{n=1}^{\infty} \frac{a_n}{n^\kappa} \int_{-R}^{R} \left( \frac{N}{n} \right)^{it} dt = a_N + N^\kappa \sum_{n=1}^{\infty} \frac{a_n}{n^\kappa} \frac{\sin(R \log(N/n))}{R \log(N/n)} .$$

But

$$\left| \sum_{n=1}^{\infty} \frac{a_n}{n^\kappa} \frac{\sin(R \log(N/n))}{R \log(N/n)} \right| \leq \sum_{n=1}^{\infty} \frac{|a_n|}{n^\kappa R |\log(N/n)|} \leq \frac{C}{R} \sum_{n=1}^{\infty} |a_n|$$

for some constant $C > 0$. Since $\sum_{n=1}^{\infty} \frac{|a_n|}{n^\kappa} < \infty$ (because we are choosing $\kappa > \sigma_a(D)$), this latter term tends to 0 whenever $R \to \infty$. As a consequence, we obtain

$$\lim_{R \to \infty} \frac{1}{2R} \int_{-R}^{R} f(\kappa + it)N^{\kappa + it}dt = a_N .$$

### 1.2 Statement of the Problem

We know now that the width of the largest possible strip on which a Dirichlet series converges but does not converge absolutely or uniformly, respectively, equals 1 (Proposition 1.3 and 1.5). The question is then, what happens with the other abscissas, $\sigma_a(D)$ and $\sigma_u(D)$? Are they always the same? If not, how far apart can they be from each other? This was a problem Harald Bohr was interested in and that took years to solve. It will keep us busy for a large part of this text.

We now pose the first big question we have to face. Try to determine the exact value of the following number that describes the maximal width of the strips in the plane where a Dirichlet series may converge uniformly but not absolutely.

$$S = \sup \{ \sigma_a(D) - \sigma_u(D) : D \text{ Dirichlet series} \} = ? \quad (1.18)$$

Obviously, from Proposition 1.3, we have $S \leq 1$, but we can immediately reduce this upper bound.

**Proposition 1.10** Given a Dirichlet series $\sum a_n n^{-s}$, the width of the strip on which it converges uniformly but not absolutely does not exceed $1/2$, i.e.

$$S \leq \frac{1}{2} .$$
We are going to prove this in two different ways. We need the following Parseval-type formula due to Carlson for finite Dirichlet polynomials $\sum_{n=1}^{N} a_n n^{-s}$.

**Proposition 1.11** For every finite family of complex numbers $a_1, \ldots, a_N$ the following holds:

$$\sum_{n=1}^{N} |a_n|^2 = \lim_{R \to \infty} \frac{1}{2R} \int_{-R}^{R} \left| \sum_{n=1}^{N} a_n n^it \right|^2 dt.$$

**Proof** From (1.17) we have that for every $n$ and $m$,

$$\lim_{R \to \infty} \frac{1}{2R} \int_{-R}^{R} n^it m^{-it} dt = \delta_{n,m}.$$

Then

$$\lim_{R \to \infty} \frac{1}{2R} \int_{-R}^{R} \left| \sum_{n=1}^{N} a_n n^it \right|^2 dt = \lim_{R \to \infty} \frac{1}{2R} \int_{-R}^{R} \left( \sum_{n=1}^{N} a_n n^it \right) \left( \sum_{m=1}^{N} \overline{a}_m m^{-it} \right) dt$$

$$= \sum_{n,m=1}^{N} a_n \overline{a}_m \left( \lim_{R \to \infty} \frac{1}{2R} \int_{-R}^{R} n^it m^{-it} dt \right) = \sum_{n,m=1}^{N} a_n \overline{a}_m \delta_{n,m} = \sum_{n=1}^{N} |a_n|^2. \ \Box$$

**First proof of Proposition 1.10** Let $\sum a_n n^{-s}$ be a Dirichlet series, and let us
show that
\[ \sigma_u(D) \leq \sigma_u(D) + 1/2. \]

We take \( \sigma_0 > \sigma_u(D) \), and we have to see that \( \sum_{n=1}^{\infty} \left| a_n \right| n^{-\sigma_0 + 1/2} < \infty \). We choose \( \sigma_u(D) < \sigma < \sigma_0 \) and set \( \varepsilon = \sigma_0 - \sigma \). By the Cauchy–Schwarz inequality, we have
\[
\sum_{n=1}^{\infty} \left| a_n \right| n^{\frac{-\sigma + \varepsilon}{2}} \leq \left( \sum_{n=1}^{\infty} \left| a_n \right|^2 n^{2(\sigma + \varepsilon/2)} \right)^{1/2} \left( \sum_{n=1}^{\infty} \left( n^{1/2 + \varepsilon/2} \right)^2 \right)^{1/2}.
\]

Since the Dirichlet series converges uniformly on \( \text{Re} s > \sigma \) (because \( \sigma > \sigma_u(D) \)), there is some \( K > 0 \) such that
\[
\sup_{\text{Re} s=\sigma + \varepsilon/2} \left| \sum_{n=1}^{N} a_n \frac{1}{n^s} \right| < K
\]
for every \( N \). Then, by Proposition 1.11, we have, for each \( N \),
\[
\sum_{n=1}^{N} \left| a_n \right| n^{\sigma + \varepsilon/2} = \lim_{R \to \infty} \frac{1}{2R} \int_{-R}^{R} \left| \sum_{n=1}^{N} a_n \frac{1}{n^{\sigma + \varepsilon/2}} \right|^2 dt \leq \lim_{R \to \infty} \frac{1}{2R} \int_{-R}^{R} \left| \sum_{n=1}^{N} a_n \frac{1}{n^{\sigma + \varepsilon/2 - it}} \right|^2 dt \leq K^2.
\]

Since \( N \) is arbitrary, the series \( \sum_{n=1}^{\infty} \left| a_n \right| n^{\sigma + \varepsilon} \) converges, and this completes the proof.

Second proof of Proposition 1.10  We take a Dirichlet series \( D = \sum a_n n^{-s} \) with \( \sigma_u(D) \geq 0 \) and use the Cauchy–Schwarz inequality and Proposition 1.11 to get
\[
\sum_{n=1}^{N} \left| a_n \right| \leq N^{1/2} \left( \sum_{n=1}^{N} \left| a_n \right|^2 \right)^{1/2} \leq N^{1/2} \left( \lim_{R \to \infty} \frac{1}{2R} \int_{-R}^{R} \left| \sum_{n=1}^{N} a_n n^{s-it} \right|^2 dt \right)^{1/2} \leq N^{1/2} \sup_{t \in \mathbb{R}} \left| \sum_{n=1}^{N} a_n n^{it} \right|. \quad (1.19)
\]

Now we deduce with the Bohr–Cahen formula (Proposition 1.6) and
\[
\frac{\log \sum_{n=1}^{N} \left| a_n \right|}{\log N} \leq \frac{\log \left( N^{1/2} \sup_{t \in \mathbb{R}} \left| \sum_{n=1}^{N} a_n n^{it} \right| \right)}{\log N} = \frac{\log N^{1/2}}{\log N} + \frac{\log \left( \sup_{t \in \mathbb{R}} \left| \sum_{n=1}^{N} a_n n^{it} \right| \right)}{\log N} = \frac{1}{2} + \frac{\log \left( \sup_{t \in \mathbb{R}} \left| \sum_{n=1}^{N} a_n n^{it} \right| \right)}{\log N}.
\]
that $\sigma_a(D) \leq 1/2 + \sigma_u(D)$. Finally, if $\sigma_u(D) < 0$, then the simple translation argument from Remark 1.8 gives the result. □

**Remark 1.12** By Proposition 1.4, if a Dirichlet series $D$ depends only on finitely many primes, then $\sigma_c(D) = \sigma_u(D) = \sigma_a(D)$. In particular, we have

$$\sup \{\sigma_a(D) - \sigma_u(D) : \text{$D$ depends on finitely many primes}\} = 0.$$ 

So, these Dirichlet series are useless to try to solve Bohr’s problem – if we want to find a Dirichlet series that separates the abscissas $\sigma_a(D)$ and $\sigma_u(D)$, then we will have to look for one in which all primes come into play. □

### 1.3 Bohr’s Theorem

One of our main goals from now on is to complete the study of (1.18) and to show that indeed $S = 1/2$. But this is going to require a lot of work. Given a Dirichlet series $D = \sum a_n n^{-s}$, we so far have seen three abscissas of convergence determined by $\sigma_c(D)$, $\sigma_u(D)$ and $\sigma_a(D)$. Our first aim is to try to understand $\sigma_u(D)$ by means of the analytic properties of the limit function of the Dirichlet series. In fact, we are going to see that boundedness of the function and uniform convergence of the series are closely related. It helps to invent another abscissa, the abscissa of boundedness, given by

$$\sigma_b(D) = \inf \{\sigma \in \mathbb{R} : \text{the limit function of $D$ is bounded on $\text{Re } s > \sigma$}\}.$$ 

(1.20)

Clearly, every $\sigma$ for which $D$ on $[\text{Re } s > \sigma]$ converges uniformly leads to a $\sigma$ for which $f$ on $[\text{Re } s > \sigma]$ is bounded. This shows that $\sigma_b(D) \leq \sigma_u(D)$. In view of Proposition 1.4 and Remark 1.12 we clearly have that if a Dirichlet series depends on finitely many primes, then $\sigma_u(D) = \sigma_b(D)$. This is in fact true in general. It is a deep result (Theorem 1.13), which we call ‘Bohr’s theorem’, and it is the highlight of this chapter.

**Theorem 1.13** Let $D = \sum a_n n^{-s}$ be a Dirichlet series (not everywhere divergent). Assume that its limit function extends to a bounded, holomorphic function $f$ on $[\text{Re } s > 0]$. Then $\sum a_n n^{-s}$ converges uniformly on $[\text{Re } s > \varepsilon]$ for every $\varepsilon > 0$, that is, $\sigma_u(D) \leq 0$.

Moreover, there is a universal constant $C > 0$ such that for every such Dirichlet series and every $x \geq 2$ we have

$$\sup_{\text{Re } s > 0} \left| \sum_{n \leq x} a_n \frac{1}{n^s} \right| \leq C \log x \sup_{\text{Re } s > 0} |f(s)|. $$

(1.21)

Note that the inequality (1.21) can be seen as a quantified variant of the
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first statement. Before we start the proof we, as announced, prove that for each Dirichlet series the abscissa of uniform convergence and the abscissa of boundedness coincide.

**Corollary 1.14** For every Dirichlet series \( D \),

\[ \sigma_b(D) = \sigma_u(D). \]

**Proof** Let us note first that if the Dirichlet series is everywhere divergent, then \( \infty = \sigma_c(D) = \sigma_b(D) = \sigma_u(D) \). We may then assume that the series converges at some point. Above we have already argued that \( \sigma_b(D) \leq \sigma_u(D) \), and hence it only remains to show that \( \sigma_u(D) \leq \sigma_b(D) \). Assume that \( D = \sum a_n n^{-s} \) converges and defines a bounded function on \( \text{Re} > \sigma \). Then the translated series \( \sum a_n n^{-s} \) converges and defines a bounded function on \( \text{Re} > 0 \), and therefore this series, by Bohr’s theorem 1.13, converges uniformly on \( \text{Re} > \varepsilon \) for every \( \varepsilon > 0 \). This gives that \( \sum a_n n^{-s} \) converges uniformly on \( \text{Re} > \sigma + \varepsilon \) for every \( \varepsilon > 0 \), and we obtain as desired \( \sigma_u(D) \leq \sigma_b(D) \).

Alternatively, we can prove this result as a consequence of (1.21). For \( \varepsilon > 0 \), define

\[ D_{\sigma_b(D)+\varepsilon} = \sum_n a_n n^{\sigma_b(D)+\varepsilon} n^{-s}. \]

Then \( \sigma_b(D_{\sigma_b(D)+\varepsilon}) = \sigma_b(D) - (\sigma_b(D) + \varepsilon) = -\varepsilon \) and \( \sigma_u(D_{\sigma_b(D)+\varepsilon}) = \sigma_u(D) - (\sigma_b(D) + \varepsilon) \) (see also Remark 1.8). In particular, \( D_{\sigma_b(D)+\varepsilon} \) has a bounded limit function \( f_\varepsilon \) on \( \text{Re} > 0 \). Hence Lemma 1.7 and (1.21) imply

\[ \sigma_u(D_{\sigma_b(D)+\varepsilon}) \leq \limsup_N \log \left( \sup_{s \in \mathbb{R}} \left| \sum_{n=1}^N \frac{a_n}{n^{\sigma_b(D)+\varepsilon}} n^{s} \right| \right) \leq \limsup_N \log \left( C \log N \sup_{\text{Re},s>0} |f_\varepsilon(s)| \right) = 0, \]

and this gives the conclusion. \( \square \)

Let us start preparing the proof of Theorem 1.13. One of its main ingredients is going to be the following classical formula for Dirichlet series \( \sum a_n n^{-s} \) and their limit functions \( f \) due to Perron (it will be proved in Proposition 1.16; see there the details of the statement):

\[ \sum_{n \leq x} a_n = \frac{1}{2\pi i} \int_{\kappa-iT}^{\kappa+iT} f(s) \frac{x^s}{s} \, ds + O\left( \frac{x^\kappa}{T} \sum_{n=1}^\infty \frac{|a_n|}{n^\kappa \log(x/n)} \right). \quad (1.22) \]

Let us give first a naive approach to this formula. We begin by integrating the
limit function

\[ \frac{1}{2\pi i} \int_{\kappa - iT}^{\kappa + iT} f(s) \frac{x^s}{s} ds = \frac{1}{2\pi i} \int_{\kappa - iT}^{\kappa + iT} \sum_{n=1}^{\infty} a_n \frac{1}{n^s} \frac{x^s}{s} ds = \sum_{n=1}^{\infty} a_n \frac{1}{2\pi i} \int_{\kappa - iT}^{\kappa + iT} \left( \frac{x^s}{n} \right) \frac{1}{s} ds. \]

What we need now is to show that the integrals in the last term give 1 plus some error term whenever \( n \leq x \), and 0 plus some error term otherwise. We do this in the following lemma.

**Lemma 1.15** Let us consider the function \( h \) defined by \( h(y) = 0 \) for \( 0 < y < 1 \) and 1 for \( y > 1 \). Then, for every \( \kappa > 0 \), every \( y > 0 \) with \( y \neq 1 \) and every \( T > 0 \), we have

\[ \left| \frac{1}{2\pi i} \int_{\kappa - iT}^{\kappa + iT} \frac{y^s}{s} ds - h(y) \right| \leq \frac{1}{\pi T |\log y|}. \]

**Proof** We take first \( 0 < y < 1 \) and choose some \( c > \kappa \). We integrate along the rectangle defined by \( \kappa - iT \), \( \kappa + iT \), \( c + iT \), \( c - iT \):

By Cauchy’s theorem we have

\[ \int_{\kappa - iT}^{\kappa + iT} \frac{y^s}{s} ds + \int_{c - iT}^{c + iT} \frac{y^s}{s} ds + \int_{c - iT}^{c + iT} \frac{y^s}{s} ds + \int_{\kappa - iT}^{\kappa + iT} \frac{y^s}{s} ds = 0, \]

hence

\[ \left| \int_{\kappa - iT}^{\kappa + iT} \frac{y^s}{s} ds \right| \leq \left| \int_{c - iT}^{c + iT} \frac{y^s}{s} ds \right| + \left| \int_{\kappa - iT}^{\kappa + iT} \frac{y^s}{s} ds \right| + \left| \int_{c - iT}^{c + iT} \frac{y^s}{s} ds \right| . \]

We have to bound each one of these three terms. For the first one, we have

\[ \left| \int_{c - iT}^{c + iT} \frac{y^s}{s} ds \right| \leq \int_{c - iT}^{c + iT} |\frac{y^{u+iT}}{u+iT}| du = \int_{c - iT}^{c + iT} \frac{y^u}{\sqrt{u^2 + T^2}} du \leq \frac{1}{T} \int_{c - iT}^{c + iT} y^u du = \frac{1}{T} \left( \frac{y^c}{\log y} - \frac{y^c}{\log y} \right). \] (1.23)
For the last term, we have the same bound. We bound the second term in the following way:

$$\left| \int_{c+iT}^{\infty} \frac{y^s}{s} ds \right| \leq \int_{-T}^{T} \left| y^{c+it} \frac{1}{c+it} \right| dt \leq \int_{-T}^{T} y^c \frac{1}{\sqrt{c^2 + t^2}} dt = y^c \log \left( \frac{\sqrt{c^2 + T^2} + T}{c^2} \right)^2 \tag{1.24}$$

(recall that \( \int \frac{1}{\sqrt{x^2 + y^2}} dx = \log(x + \sqrt{x^2 + y^2}) \)). Altogether this gives

$$\left| \int_{k-iT}^{k+iT} \frac{y^s}{s} ds \right| \leq y^c \log \left( \frac{\sqrt{c^2 + T^2} + T}{c^2} \right)^2 + \frac{2}{T} \left( \frac{y^c}{\log y} - \frac{y^t}{\log y} \right).$$

This tends to \(-2 \frac{y^t}{T \log y} = 2 \frac{y^t}{T \log y} \) as \( c \to \infty \); therefore, for \( 0 < y < 1 \), we have

$$\left| \frac{1}{2\pi i} \int_{k-iT}^{k+iT} \frac{y^s}{s} ds \right| \leq \frac{1}{\pi} \frac{y^t}{T \log y}.$$

Let us take now \( y > 1 \) and choose again \( c > k \) to integrate along the rectangle given by \( k - iT, k + iT, -c + iT, -c - iT \):

Now the function \( \frac{y^s}{s} \) has a pole at \( s = 0 \) with residue 1; then

$$\int_{k-iT}^{k+iT} \frac{y^s}{s} ds + \int_{c+iT}^{\infty} \frac{y^s}{s} ds + \int_{-c+iT}^{k+iT} \frac{y^s}{s} ds + \int_{-c-iT}^{c-iT} \frac{y^s}{s} ds = 2\pi i.$$

Hence

$$\left| \int_{k-iT}^{k+iT} \frac{y^s}{s} ds - 2\pi i \right| \leq \left| \int_{c+iT}^{\infty} \frac{y^s}{s} ds \right| + \left| \int_{-c+iT}^{k+iT} \frac{y^s}{s} ds \right| + \left| \int_{-c-iT}^{c-iT} \frac{y^s}{s} ds \right| + \left| \int_{c-iT}^{k-iT} \frac{y^s}{s} ds \right|.$$

We bound the integrals using again (1.23) and (1.24):

$$\left| \int_{k+iT}^{c+iT} \frac{y^s}{s} ds \right| \leq \frac{1}{T} \left( \frac{y^k}{\log y} - \frac{y^{-c}}{\log y} \right),$$

$$\left| \int_{c+iT}^{\infty} \frac{y^s}{s} ds \right| \leq y^{-c} \log \left( \frac{\sqrt{c^2 + T^2} + T}{c^2} \right)^2$$
1.3 Bohr’s Theorem

(1.25)

\[ \left| \sum_{n \leq x} a_n - \frac{1}{2\pi i} \int_{\kappa-iT}^{\kappa+iT} f(s) \frac{x^s}{s} ds \right| \leq \frac{1}{\pi} \frac{\chi^x}{T} \sum_{n=1}^{\infty} \frac{|a_n|}{n^x \log(x/n)}. \]

\[ |\int_{\kappa-iT}^{\kappa+iT} \frac{y^s}{s} ds - 2\pi i| \leq \frac{2y^\kappa}{T|\log y|}, \]

which gives

\[ \left| \frac{1}{2\pi i} \int_{\kappa-iT}^{\kappa+iT} \frac{y^s}{s} ds - 1 \right| \leq \frac{1}{\pi} \frac{y^\kappa}{T|\log y|}, \]

and completes the proof.

The following estimate is the precise formulation of Perron’s formula (1.22).

**Proposition 1.16**  Let \( D = \sum a_n n^{-s} \) be a Dirichlet series with limit function \( f \). Then for every \( \kappa > \max\{0, \sigma_a(D)\} \), every \( T > 0 \) and every \( x \geq 1 \) with \( x \notin \mathbb{N} \), we have

\[ \left| \sum_{n \leq x} a_n - \frac{1}{2\pi i} \int_{\kappa-iT}^{\kappa+iT} f(s) \frac{x^s}{s} ds \right| \leq \frac{1}{\pi} \frac{\chi^x}{T} \sum_{n=1}^{\infty} \frac{|a_n|}{n^x \log(x/n)}. \]

**Proof**  Given \( \kappa, x \) and \( T \), we use Lemma 1.15 (with the function \( h \) defined there) to have, for every \( n \neq x \),

\[ \left| a_n h\left( \frac{x}{n} \right) - \frac{1}{2\pi i} \int_{\kappa-iT}^{\kappa+iT} a_n \left( \frac{x}{n} \right)^s \frac{1}{s} ds \right| \leq \frac{1}{\pi} \frac{|a_n|}{T|\log(x/n)|}. \]

Since \( \kappa > \sigma_a(D) \), the series \( \sum_{n=1}^{\infty} a_n \frac{1}{n^x} \) converges uniformly on \( [\kappa - iT, \kappa + iT] \) to \( f \), and then

\[ \left| \sum_{n \leq x} a_n - \frac{1}{2\pi i} \int_{\kappa-iT}^{\kappa+iT} f(s) \frac{x^s}{s} ds \right| = \left| \sum_{n=1}^{\infty} \left( a_n h\left( \frac{x}{n} \right) - a_n \frac{1}{2\pi i} \int_{\kappa-iT}^{\kappa+iT} \left( \frac{x}{n} \right)^s \frac{1}{s} ds \right) \right| \leq \frac{1}{\pi} \frac{\chi^x}{T} \sum_{n=1}^{\infty} \frac{|a_n|}{n^x \log(x/n)}. \]

We are now ready to give the proof of our highlight. In order to keep the notation as simple as possible, we write \( ||f||_\infty = \sup_{\text{Re} s > 0} |f(s)| \).

**Proof of Theorem 1.13**  We divide the proof into three steps. In the first step we show that for a given Dirichlet series \( D \) satisfying the hypothesis, the equation (1.21) holds up to a constant \( C = C(D) \). Then in the second step, we will establish the first assertion of the theorem. Finally, in the third step, we will show how this implies that the constant from step one can be chosen independently from \( D \).

Step 1. Let \( D = \sum a_n n^{-s} \) be a Dirichlet series with a limit function that extends to a bounded and holomorphic function \( f \) on \( [\text{Re} > 0] \). We show that there is a constant \( C(D) \geq 1 \) for which

\[ \sup_{\text{Re} s > 0} \left| \sum_{n \leq x} a_n \frac{1}{n^s} \right| \leq C(D) \log x ||f||_\infty \]

(1.26)
for all $x \geq 2$. First we only handle those $x \geq 3/2$ with $x = N + 1/2$ for some $N \in \mathbb{N}$. Fix such $x$, some $\delta > 0$ and $\kappa > \max \{1/\log(3/2), \sigma_a(D) + 1 + 2\delta\}$. Then by Perron’s formula (Proposition 1.16), for each $T \geq 1$,

$$
\left| \sum_{n \leq x} a_n \right| \leq \left| \sum_{n \leq x} a_n - \frac{1}{2\pi i} \int_{k-iT}^{k+iT} f(s) \frac{x^s}{s} ds \right| + \left| \frac{1}{2\pi i} \int_{k-iT}^{k+iT} f(s) \frac{x^s}{s} ds \right| \\
\leq \frac{x^{\kappa}}{\pi T} \sum_{n=1}^\infty \frac{|a_n|}{n^{\kappa} \log(x/n)} + \frac{1}{2\pi} \left| \int_{k-iT}^{k+iT} f(s) \frac{x^s}{s} ds \right|.
$$

(1.27)

Later we will see that $T = x^{\kappa + 1}$ is the choice that serves to our purposes. We begin by bounding the first term, which controls the error. Let us note that if $n > x$ (that is, $n \geq N + 1$), then we have

$$
\left| \log \frac{x}{n} \right| = \log \frac{x}{n} \geq \log \frac{N + 1/2}{N + 1/2} \geq \frac{1}{4(N + 1/2)} = \frac{1}{4x};
$$

(1.28)

the last inequality holds because $\log \frac{y + 1/2}{y} \geq \frac{1}{4y}$ for $y \geq 1$, which is equivalent to $(1 + \frac{1}{2y})^{2y} \geq e^{\frac{1}{2}}$ (true for $y = 1$ and since $(1 + \frac{1}{2y})^{2y}$ increases to $e$). On the other hand, if $n \leq x$ (that is, $n \leq N$), then

$$
\left| \log \frac{x}{n} \right| = \log \frac{x}{n} \geq \log \frac{N + 1/2}{N} \geq \frac{1}{4N} \geq \frac{1}{4(N + 1/2)} = \frac{1}{4x}.
$$

(1.29)

Integrating in Proposition 1.9 along the line $\text{Re} = \sigma_a(D) + \delta$ and bounding, we have that $|a_n| \leq n^{\sigma_a(D) + \delta} ||f||_\infty$ for every $n$. Combining this with (1.28) and (1.29), we are able to control the error term in (1.27). Indeed, for each $T \geq 1$, we have

$$
\frac{x^\kappa}{\pi T} \sum_{n=1}^\infty \frac{|a_n|}{n^{\kappa} \log(x/n)} \leq \frac{x^\kappa}{\pi T} \sum_{n=1}^\infty \frac{||f||_\infty n^{\sigma_a(D) + \delta}}{n^{\sigma_a(D) + 1 + 2\delta} \log(x/n)}
$$

$$
\leq \frac{x^\kappa}{\pi T} ||f||_\infty \sum_{n=1}^\infty \frac{1}{n^{{1+\delta}} \log(x/n)} \leq \frac{4x^{\kappa + 1}}{\pi T} \sum_{n=1}^\infty \frac{1}{n^{1+\delta} ||f||_\infty}.
$$

We choose $T = x^{\kappa + 1}$ and obtain

$$
\frac{x^\kappa}{\pi T} \sum_{n=1}^\infty \frac{|a_n|}{n^{\kappa} \log(x/n)} \leq \frac{4}{\pi} \sum_{n=1}^\infty \frac{1}{n^{1+\delta} ||f||_\infty}.
$$

(1.30)

In order to bound the integral in (1.27), we choose $0 < \varepsilon < \kappa$ and integrate on the following rectangle:
Then, by Cauchy’s theorem, we have

\[
\int_{\kappa-iT}^{\kappa+iT} f(s) \frac{x^s}{s} ds = \int_{\varepsilon-iT}^{\varepsilon+iT} f(s) \frac{x^s}{s} ds + \int_{\varepsilon+iT}^{\kappa+iT} f(s) \frac{x^s}{s} ds + \int_{\kappa-iT}^{\varepsilon-iT} f(s) \frac{x^s}{s} ds.
\]

We bound each one of these integrals separately. We bound first the integral \(A\) using (1.24):

\[
|A| = \left| \int_{-T}^{T} f(\varepsilon + it) \frac{x^{\varepsilon + it}}{\varepsilon + it} dt \right| \leq \|f\|_{\infty} \int_{-T}^{T} \left| \frac{x^{\varepsilon + it}}{\varepsilon + it} \right| dt = x^{\varepsilon} \log \left( \frac{\sqrt{1 + \left( \frac{T}{\varepsilon} \right)^2 + \frac{T}{\varepsilon}}} {\varepsilon} \right) \|f\|_{\infty} = 2x^{\varepsilon} \log \left( \sqrt{1 + \left( \frac{T}{\varepsilon} \right)^2 + \frac{T}{\varepsilon}} \right) \|f\|_{\infty}.
\]

The integral \(B\) is bounded using (1.23)

\[
|B| = \left| \int_{\varepsilon}^{\kappa} f(u + iT) \frac{x^{u+iT}}{u + iT} du \right| \leq \|f\|_{\infty} \int_{\varepsilon}^{\kappa} \left| \frac{x^{u+iT}}{u + iT} \right| du \leq \|f\|_{\infty} \frac{1}{T} \frac{x^{\varepsilon} - x^{\varepsilon}} {\log x},
\]

and the same bound holds for \(C\). Then

\[
\left| \int_{\kappa-iT}^{\kappa+iT} f(s) \frac{x^s}{s} ds \right| \leq \left( 2x^{\varepsilon} \log \left( \sqrt{1 + \left( \frac{T}{\varepsilon} \right)^2 + \frac{T}{\varepsilon}} \right) + \frac{2}{T} x^{\varepsilon} - x^{\varepsilon} \right) \|f\|_{\infty}.
\]

With this, together with (1.27) and (1.30), we have for every \(0 < \varepsilon < \kappa\)

\[
\left| \sum_{n \leq x} a_n \right| \leq 2 \left( \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}} + x^{\varepsilon} \log \left( \sqrt{1 + \left( \frac{T}{\varepsilon} \right)^2 + \frac{T}{\varepsilon}} \right) + \frac{1}{T} x^{\varepsilon} - x^{\varepsilon} \right) \|f\|_{\infty}.
\]

Recall that we already fixed \(T = x^{\varepsilon+1}\), and note that since \(x \geq 3/2\) and \(1/ \log(3/2) < \kappa\), we have that \(\varepsilon := 1/ \log x < \kappa\). Then \(x^{\varepsilon} = e\) and \(T/\varepsilon = \)
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\[ x^{\kappa+1} \log x, \text{ and hence} \]

\[ \left| \sum_{n \leq x} a_n \right| \leq 2\left( \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}} + e \log \left( \sqrt{1 + (x^{\kappa+1} \log x)^2 + x^{\kappa+1} \log x} + \frac{1}{x \log x} \left(1 - \frac{e}{x^\kappa}\right) \right) \right) \|f\|_\infty. \]  

(1.31)

All in all we have proved that for every \( x = N + \frac{1}{2}, \)

\[ \left| \sum_{n \leq x} a_n \right| \leq C(\kappa, \delta) \log x \|f\|_\infty, \]

where \( \delta > 0, \kappa > \max \left\{ 1/ \log(3/2), \sigma_a(D) + 1 + 2\delta \right\} \) and 

\[ C(\kappa, \delta) = 2 \sup_{u \geq 3/2} \frac{1}{\log u} \left[ \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}} \right. \]

\[ + e \log \left( \sqrt{1 + (u^{\kappa+1} \log u)^2 + u^{\kappa+1} \log u} + \frac{1}{u \log u} \left(1 - \frac{e}{u^\kappa}\right) \right) < \infty \]

(1.32)

(from the three summands into which the fraction is decomposed, the first and last one tend to 0 as \( u \to \infty, \) and the middle one tends to \( \kappa + 1). \)

Now the proof of (1.26) is easily finished. Let \( D = \sum a_n n^{-s} \) be a Dirichlet series with a limit function that extends to a bounded and holomorphic function \( f \) on [Re > 0]. Take again \( \delta > 0 \) and \( \kappa > \max \left\{ 1/ \log(3/2), \sigma_a(D) + 1 + 2\delta \right\}. \)

Now fix some \( s \in [Re > 0], \) and define the Dirichlet series \( D_s = \sum n^{-s} n^{-z}. \)

Then the limit function of \( D_s \) is given by \( f_s(z) = \sum n^{-s} \frac{1}{n^{-z}} = f(s + z) \) for Re \( z > \sigma_a(D) - \text{Re } s = \sigma_a(D_s) \) (see Remark 1.8). That is, \( D_s \) has a limit function that extends to a bounded and holomorphic function on [Re > 0]. Then, using what we just have shown, for each \( x = N + \frac{1}{2}, \) we have

\[ \left| \sum_{n \leq x} \frac{a_n}{n^{s}} \right| \leq C(\kappa, \delta) \log x \|f_s\|_\infty = C(\kappa, \delta) \log x \sup_{\text{Re } z > \text{Re } s} |f(z)| \leq C(\kappa, \delta) \log x \|f\|_\infty; \]

here the constant \( C(\kappa, \delta) \) is that of (1.32) since with \( \sigma_a(D_s) = \sigma_a(D) - \text{Re } s \)

\( \kappa > \max \left\{ 1/ \log(3/2), \sigma_a(D) + 1 + 2\delta \right\} > \max \left\{ 1/ \log(3/2), \sigma_a(D_s) + 1 + 2\delta \right\}. \)

Finally, for arbitrary \( x \geq 2, \) we have

\[ \left| \sum_{n \leq x} \frac{a_n}{n^s} \right| = \left| \sum_{n \leq |x|} \frac{a_n}{n^s} \right| + \left| \sum_{n < |x| + \frac{1}{2}} \frac{a_n}{n^s} \right| \]

\[ \leq C(\kappa, \delta) \log \left( |x| + \frac{1}{2} \right) \|f\|_\infty \leq 2C(\kappa, \delta) \log x \|f\|_\infty, \]

which completes the proof of (1.26).
1.4 The Banach Space $H_\infty$

Step 2. Let $\sum a_n n^{-s}$ be a Dirichlet series with a limit function which extends to a bounded and holomorphic function $f$ on $\{\text{Re} > 0\}$. Then using the Bohr–Cahen formulas (Proposition 1.6), Lemma 1.7 and (1.26), we conclude that

$$\sigma_u(D) \leq \limsup_{N \to \infty} \frac{\log \left( \sup_{Re s > 0} \left| \sum_{n=1}^{N} a_n n^{is} \right| \right)}{\log N} \leq \limsup_{N \to \infty} \frac{\log (C(D) \log N ||f||_{\infty})}{\log N} = 0,$$

and the first statement of the theorem is proved.

Step 3. It remains to show (1.21), that is, the constant $C(\kappa, \delta)$ from (1.32) can in fact be chosen to be independent from $D$. Note that in the formula for $C(\kappa, \delta)$ the only parameter that depends on $D$ is $\kappa$ (which is taken to be $\sigma_a(D) + 1 + 2\delta$). But $\delta > 0$ was anyway arbitrary, and we could fix it at e.g. $\delta = 1/2$. Then we need $\kappa > \sigma_a(D) + 2$. But by Step 2, we know that $\sigma_u(D) \leq 0$, which by Proposition 1.10 implies $\sigma_a(D) \leq 1/2$. We can then choose $\kappa = 3 > \max\{\sigma_a(D) + 2, 1/\log(3/2)\}$ for every Dirichlet series $D$, and the conclusion follows. \hfill \square

1.4 The Banach Space $H_\infty$

We are now ready to define what is going to be one of the most important objects in this text. It is a Banach space (even a Banach algebra) of Dirichlet series, which we are going to call $H_\infty$. It will help us to study the abscissa of uniform convergence of a Dirichlet series using functional analytical (Banach space) methods. We consider $H_\infty = \{ \sum a_n n^{-s} : \text{converges on } \{\text{Re} > 0\}, \text{where its limit function is bounded} \}$, and there we define

$$\| \sum a_n n^{-s} \|_{\infty} = \sup_{\text{Re } s > 0} \left| \sum_{n=1}^{\infty} a_n \frac{1}{n^s} \right| .$$

Note first that Proposition 1.9 in fact shows that this defines a norm on $H_\infty$. We denote by $H_\infty([\text{Re} > 0])$ the linear space of all bounded and holomorphic functions $f$ on the positive half-plane $[\text{Re} > 0]$, which together with

$$\|f\|_{\infty} = \sup_{\text{Re } s > 0} |f(s)|$$

forms a Banach algebra (the completeness of $H_\infty([\text{Re} > 0])$ is a standard consequence of the Weierstraß’s convergence theorem). From Theorem 1.1 we
know that limit functions of Dirichlet series are holomorphic, hence the identification

\[ \mathcal{H}_\infty \hookrightarrow H_\infty([\text{Re}>0]) \] given by

\[ f(s) = \sum_{n=1}^{\infty} a_n \frac{1}{n^s} \] (1.33)

is an isometric embedding. Consequently, \((\mathcal{H}_\infty, \|\cdot\|_\infty)\) can be viewed as an isometric subspace of \(H_\infty([\text{Re}>0])\) (later, in Remark 1.20, we will see that the embedding in (1.33) is even strict). But we can say much more: \(\mathcal{H}_\infty\) is a Banach space, and a Banach algebra when we endow it with the Dirichlet multiplication defined in (1.2). This is the content of our next result. Summarizing, the embedding in (1.33) is multiplicative, and \(\mathcal{H}_\infty\) is a closed proper subalgebra of \(H_\infty([\text{Re}>0])\).

**Theorem 1.17** \((\mathcal{H}_\infty, \|\cdot\|_\infty)\) is a Banach algebra.

The argument for the proof is based on Bohr’s theorem 1.13 and therefore not trivial. We first need to collect two results of independent interest.

**Remark 1.18** If the limit function of \(D = \sum a_n n^{-s}\) has a bounded and holomorphic extension \(f\) to \([\text{Re}>0]\), then \(\sigma_\mu(D) \leq 0\) by Theorem 1.13, and the limit function of \(\sum a_n n^{-s}\) and \(f\) are identical on \(\text{Re}>0\) since they coincide on the smaller half-plane. This proves that \(\sum a_n n^{-s} \in \mathcal{H}_\infty\). Since the reverse implication is obvious, we have that a Dirichlet series belongs to \(\mathcal{H}_\infty\) if and only if its limit function extends (from some a priori smaller half-space) to a bounded holomorphic function on \([\text{Re}>0]\). \(\blacksquare\)

The second important tool for the proof of Theorem 1.17, again based on Bohr’s theorem 1.13, will also be of great use later on.

**Proposition 1.19** For every \(\sum a_n n^{-s} \in \mathcal{H}_\infty\), we have

\[ \left( \sum_{n=1}^{\infty} |a_n|^2 \right)^{1/2} \leq \| \sum a_n n^{-s} \|_\infty. \]

In particular, for all \(N\),

\[ |a_N| \leq \| \sum a_n n^{-s} \|_\infty. \] (1.34)

**Proof** We fix \(\sigma > 0\) and \(\varepsilon > 0\). By Bohr’s theorem 1.13, the sequence \(D_N = \sum_{n=1}^{N} a_n n^{-s}\) converges uniformly on \([\text{Re} = \sigma]\) to the limit function \(f\) of \(\sum a_n n^{-s}\); hence there is \(N_0\) such that for every \(N \geq N_0\),

\[ \sup_{t \in \mathbb{R}} \left| \sum_{n=1}^{N} a_n \frac{1}{n^{\sigma+it}} - f(\sigma + it) \right| < \varepsilon, \]
1.4 The Banach Space $\mathcal{H}_\infty$

and as a consequence,

$$\sup_{t \in \mathbb{R}} \left| \sum_{n=1}^{N} a_n \frac{1}{n^{\sigma + it}} \right| \leq \left\| \sum a_n n^{-s} \right\|_\infty + \varepsilon.$$ 

But then Proposition 1.11 implies that for every $N \geq N_0$,

$$\left( \sum_{n=1}^{N} \left| a_n \frac{1}{n^\sigma} \right|^2 \right)^{1/2} = \left( \lim_{R \to \infty} \frac{1}{2R} \int_{-R}^{R} \left| \sum_{n=1}^{N} a_n \frac{1}{n^{\sigma + it}} \right|^2 dt \right)^{1/2} \leq \left\| \sum a_n n^{-s} \right\|_\infty + \varepsilon.$$ 

Hence, if $\sigma$ and $\varepsilon$ both tend to 0, the claimed inequality is proved. $\square$

Finally, we are ready for the proof of Theorem 1.17.

**Proof of Theorem 1.17** We show first that $\mathcal{H}_\infty$ is closed in the Banach space $H_\infty([\text{Re} > 0])$. Take a sequence $(D_k)_k = \left( \sum a_n n^{-s} \right)_k$ in $\mathcal{H}_\infty$, and assume that in $H_\infty([\text{Re} > 0])$ it converges to $f \in H_\infty([\text{Re} > 0])$. It has to be shown that $f \in \mathcal{H}_\infty$. According to Remark 1.18, we prove that $f$ is the limit function of a Dirichlet series $D$ on some half-plane. By Proposition 1.19, for every $k$ and $l$ and for all $n$,

$$|a_k^l - a_l^k| \leq \|D_k - D_l\|_\infty.$$  \hfill (1.35)

Hence, for each $n$, the sequence $(a_n^k)_k$ is Cauchy and therefore converges to some $a_n$. Define the Dirichlet series $\sum a_n n^{-s}$. As announced, we prove that $f(s) = \sum a_n \frac{1}{n^s}$ on some appropriate half-plane. To do that, we again apply Proposition 1.19, which assures that for every $k, n$ we have $|a_k^l| \leq \|D^k\|_\infty$. Taking limits in $n$, we see that the boundedness of $(D^k)$ in $\mathcal{H}_\infty$ implies the boundedness of $(a_n)$ in $\mathbb{C}$, hence $\sigma_{an}(D) \leq 1$. We finally check that $f(s) = \sum a_n \frac{1}{n^s}$ on $[\text{Re} > 1]$. To do this, take some $s \in [\text{Re} > 1]$ and $\varepsilon > 0$. By assumption and (1.35), there is $k$ such that

$$|f(s) - D^k(s)| < \varepsilon \quad \text{and} \quad |a_k^l - a_l^k| < \epsilon \quad \text{for all } n.$$ 

Moreover, there is $N$ such that

$$D^k(s) - \sum_{n=1}^{N} a_n \frac{1}{n^s} < \varepsilon \quad \text{and} \quad D(s) - \sum_{n=1}^{N} a_n \frac{1}{n^s} < \varepsilon.$$ 

But then

$$|f(s) - D(s)| \leq |f(s) - D^k(s)| + \left| D^k(s) - \sum_{n=1}^{N} a_n \frac{1}{n^s} \right|$$

$$+ \left| \sum_{n=1}^{N} a_n \frac{1}{n^s} - \sum_{n=1}^{N} a_n \frac{1}{n^s} \right| + \left| \sum_{n=1}^{N} a_n \frac{1}{n^s} - D(s) \right| \leq 2\varepsilon + \varepsilon \sum_{n=1}^{\infty} \frac{1}{n^{\text{Re}s}} + \varepsilon.$$ 

This finishes the argument for the completeness of $\mathcal{H}_\infty$, and it remains to check
that $\mathcal{H}_\infty$ is a Banach algebra. Given $D_1 = \sum a_n n^{-s}$ and $D_2 = \sum b_n n^{-s}$ in $\mathcal{H}_\infty$, with associated limit functions $f_1, f_2 \in H_\infty[\text{Re } 0]$, we need to show that

$$D = D_1 D_2 = \sum_{n=1}^\infty \left( \sum_{km=n} a_k b_m \right) n^{-s} \in \mathcal{H}_\infty$$

(see (1.2)) and $\|D\|_\infty \leq \|D_1\|_\infty \|D_2\|_\infty$. From Corollary 1.14 we know that $\sigma_\alpha(D_j) = \sigma_\beta(D_j) \leq 0$. Then $\sigma_\alpha(D_j) \leq 1$ for $j = 1, 2$ (use for example Proposition 1.3), and consequently, for $\sigma > 1$,

$$\sum_{n=1}^\infty \left( \sum_{km=n} |a_k b_m| \right) \frac{1}{n^{\sigma}} = \left( \sum_{k=1}^\infty |a_k| \frac{1}{k^{\sigma}} \right) \left( \sum_{m=1}^\infty |b_m| \frac{1}{m^{\sigma}} \right) < \infty.$$

If $g$ now denotes the limit function of $D$, then this proves that $g = f_1 f_2$ on $[\text{Re } 1]$. But $f_1 f_2$ is a bounded and holomorphic function on $[\text{Re } 0]$ which extends $g$, and hence another application of Remark 1.18 implies that $D = D_1 D_2 \in \mathcal{H}_\infty$, as well as $\|D\|_\infty = \|g\|_\infty \leq \|f_1\|_\infty \|g\|_\infty = \|D_1\|_\infty \|D_2\|_\infty$. \qed

**Remark 1.20** We now give a simple argument which shows that $\mathcal{H}_\infty$ in fact is a strict subalgebra of $H_\infty([\text{Re } 0])$. This will rely on the following fact. For every $N \geq 1$ and every Dirichlet series $\sum a_n n^{-s}$ (not everywhere divergent) with limit function $f$, we have

$$a_N = \lim_{\text{Re } s \to \infty} \frac{N^s \left( f(s) - \sum_{n=1}^{N-1} a_n \frac{1}{n^s} \right)}{N^{s+1}}$$

(here the sum is defined to be zero whenever $N = 1$). Observe that this again shows that the limit function of a Dirichlet series determines its coefficients uniquely (a fact which already followed from Proposition 1.9). To check (1.36), we may assume $D = \sum a_n n^{-s} \in \mathcal{H}_\infty$ (if this is not the case, just take some $\sigma_0 > \sigma_\alpha(D)$ and consider the translation $\sum \frac{a_n}{n^{\sigma_0}} n^{-s} \in \mathcal{H}_\infty$). Take $s \in \mathbb{C}$ with $\sigma = \text{Re } s > 2$. Using (1.34), we have

$$\left| N^s \left( f(s) - \sum_{n=1}^{N-1} a_n \frac{1}{n^s} \right) - a_N \right| = \left| \sum_{n=N+1}^\infty a_n \frac{N^n}{n^n} \right|$$

$$\leq \left\| \sum a_n n^{-s} \right\|_\infty \sum_{n=N+1}^\infty \left( \frac{N}{n} \right)^{\sigma-2} \frac{N^2}{n^2} \leq \frac{\pi^2}{6} N^2 \left( \frac{N}{N+1} \right)^{\sigma-2} \left\| \sum a_n n^{-s} \right\|_\infty,$$

and then (1.36) holds because the last term tends to 0 as $\sigma \to \infty$. Let us now use this information to show that (1.33) is a strict inclusion. We have to find some function in $H_\infty([\text{Re } 0])$ that cannot be represented as a Dirichlet series. As a matter of fact, we give not only one but two such functions. First we take the function $f(s) = \sum_{n=1}^{s-1} \frac{1}{n^s}$, which is holomorphic and bounded on $[\text{Re } 0]$. Suppose that it could be represented by a Dirichlet series, that is, $f(s) = \sum_{n=1}^\infty a_n \frac{1}{n^s}$ on
1.4 The Banach Space $H_\infty$

[Re > 0]. Then (1.36) gives $a_1 = \lim_{\text{Re } s \to \infty} \frac{s-1}{s+1} = 1$, but

$$|a_2| = \lim_{\text{Re } s \to \infty} \left| 2^s \left( \frac{s-1}{s+1} - 1 \right) \right| = \lim_{\text{Re } s \to \infty} \frac{2^{\text{Re } s+1}}{|s+1|} = \infty,$$

and this is a contradiction. Our second example is given by $f(s) = e^{-s}$. If $f(s) = \sum_{n=1}^{\infty} a_n \frac{1}{n}$ on [Re > 0], then $a_1 = 0$ and also $a_2 = \lim_{\text{Re } s \to \infty} 2^s/e^s = 0$, but $|a_3| = \lim_{\text{Re } s \to \infty} |3^s/e^s| = \infty$. \(\Box\)

We add a couple of remarks that reformulate some facts which we already proved earlier in terms of this new Banach space.

**Remark 1.21** First of all, Lemma 1.7 tells us that for Dirichlet polynomials $\sum_{n=1}^{N} a_n n^{-s}$, we have

$$\left\| \sum_{n=1}^{N} a_n n^{-s} \right\|_\infty = \sup_{t \in \mathbb{R}} \left| \sum_{n=1}^{N} a_n n^{-it} \right|.$$

In other words, for Dirichlet polynomials, the $H_\infty$-norm can be computed just as a supremum on the imaginary axis. \(\Box\)

**Remark 1.22** An obvious reformulation of the inequality given in Bohr’s theorem (see (1.21)) shows that there is a constant $C > 0$ such that for every Dirichlet series $\sum a_n n^{-s}$ in $H_\infty$ and all $x \geq 2$, we have

$$\left\| \sum_{n \leq x} a_n n^{-s} \right\|_\infty \leq C \log x \left\| \sum_{n} a_n n^{-s} \right\|_\infty.$$

In passing we note that in terms of Banach space theory, this result means that, given $N \in \mathbb{N}$, the basis constant (see (14.1)) of the span of the first $N$ many functions $(n^{-s})_n$ in $H_\infty$ is bounded by $\log N$. \(\Box\)

Let us finally come back to the guiding spirit of these notes. One of the major goals is to determine the precise value of the number $S$ defined in (1.18), the largest possible width of the strip on which a Dirichlet series converges uniformly but not absolutely.

**Remark 1.23** From Corollary 1.14 we know that for each Dirichlet series $D = \sum a_n n^{-s}$ the abscissas of uniform and bounded convergence coincide, i.e. $\sigma_u(D) = \sigma_b(D)$. Just using the definitions, both abscissas can be reformulated in terms of the Banach space $H_\infty$:

$$\sigma_u(D) = \inf \{ \sigma \in \mathbb{R}: \left( \sum_{n=1}^{N} a_n \frac{n}{n^\sigma} n^{-s} \right)_N \text{ converges in } H_\infty \}$$

and

$$\sigma_b(D) = \inf \{ \sigma \in \mathbb{R}: \sum a_n \frac{n}{n^\sigma} n^{-s} \in H_\infty \}.$$

\(\Box\)
We finish this chapter by noting that $S$ is nothing else than the supremum of all $\sigma_a(D)$ taken over all possible Dirichlet series in $H_\infty$, and then we can reformulate it in the following terms.

**Proposition 1.24**

$$ S = \sup_{D \in H_\infty} \sigma_a(D). $$

**Proof** From Bohr’s theorem 1.13 we know that $\sigma_a(D) \leq 0$ for each $D \in H_\infty$; hence, for each such $D$, we have that $\sigma_a(D) \leq \sigma_a(D) - \sigma_u(D) \leq S$. Conversely, given $\varepsilon > 0$, there is some Dirichlet series $D = \sum a_n n^{-s}$ such that

$$ S - \frac{\varepsilon}{2} \leq \sigma_a(D) - \sigma_u(D), $$

and then

$$ S - \varepsilon \leq \sigma_a(D) - \sigma_u(D) - \frac{\varepsilon}{2}. $$

Now we consider the translation $D_{\sigma_u(D)+\frac{\varepsilon}{2}} = \sum n^{\sigma_u(D) + \frac{\varepsilon}{2}}$, which by Remark 1.8, satisfies

$$ \sigma_a(D_{\sigma_u(D)+\frac{\varepsilon}{2}}) = \sigma_a(D) - \left( \sigma_u(D) + \frac{\varepsilon}{2} \right), $$

and

$$ \sigma_u(D_{\sigma_u(D)+\frac{\varepsilon}{2}}) = \sigma_u(D) - \left( \sigma_u(D) + \frac{\varepsilon}{2} \right) = -\frac{\varepsilon}{2}. $$

Hence $D_{\sigma_u(D)+\frac{\varepsilon}{2}} \in H_\infty$ and

$$ S - \varepsilon \leq (\sigma_a(D) - \sigma_u(D)) - \frac{\varepsilon}{2} = \sigma_a(D_{\sigma_u(D)+\frac{\varepsilon}{2}}), $$

which completes the argument. \hfill \Box

So far we know from Proposition 1.10 that $\sup_{D \in H_\infty} \sigma_a(D) \leq \frac{1}{2}$. Bohr’s absolute convergence problem from (1.18) hence asks whether or not we even have $\sup_{D \in H_\infty} \sigma_a(D) = \frac{1}{2}$.

### 1.5 Notes/Remarks

We cite from the book of Hardy and Riesz (1915): ‘Dirichlet’s series were, as their name implies, first introduced into analysis by Dirichlet, primarily with a view to applications in the theory of numbers. A number of important theorems concerning them were proved by Dedekind, and incorporated by him in his later editions of Dirichlet’s Vorlesungen über Zahlentheorie. Dirichlet and Dedekind, however, considered only real values of the variable $s$. The first theorems involving complex values of $s$ are due to Jensen (1884, 1888), who determined the nature of the region of convergence of the general series; and
the first attempt to construct a systematic theory of the function \( f(s) \) was made by Cahen (1894) in a memoir which, although much of the analysis which it contains is open to serious criticism, has served and possibly just for that reason as the starting point of most of the later researches in the subject’.

All results in this chapter are classical and can be found in many books. Quoting Helson (2005, p. vi), ‘The classical literature is enormous, accessible and marvelous. Any analyst will find pleasure in browsing the older titles’. Some of them are Apostol (1976, 1990); Hardy and Riesz (1915); Landau (1909a,b); Mandelbrojt (1944); Queffélec and Queffélec (2013); Tenenbaum (1995); Titchmarsh (1932).

The content of Theorem 1.1 and Lemma 1.2 was proved by Jensen (1884) and Cahen (1894).

The proof of Proposition 1.9 we present here follows Titchmarsh (1932 Section 9.7, p. 313).

The formulas for \( \sigma_c(D) \) and \( \sigma_d(D) \) in Proposition 1.6 are due to Jensen (1888) and Cahen (1894), whereas the formula for \( \sigma_u(D) \) was proved by Bohr (1913, Satz III).

The example of an alternate series in Proposition 1.5 was considered for the first time by Bohr (1913a, p. 486).

Proposition 1.10 goes back to Bohr (1913a, Satz III, Satz IX). The proof we give here that follows by now up to some point standard arguments is not the one Bohr gave. It can be found e.g. in Boas (1997). We will go back later to the original arguments of Bohr in Proposition 10.8 and Theorem 10.13.

The proof of Theorem 1.13 follows Maurizi and Queffélec (2010); (1.21) is due to Balasubramanian et al. (2006, Lemma 1.1); and Corollary 1.14 to Bohr (1913b, Satz 1). See also Hardy and Riesz (1915, Theorem 52).

Bohr’s absolute convergence problem (1.18) appeared for the first time in the introduction of Bohr (1913a): ‘Ob hier die Zahl \( \frac{1}{2} \) durch eine kleinere Zahl (vielleicht sogar durch die Zahl 0, was \( T = 0, S = \infty \) bedeuten würde) ersetzt werden kann, weiß ich nicht’.

Proposition 1.16 is taken from Tenenbaum (1995, Section II.2, p. 130ff.), who calls it ‘Perron’s first effective formula’. According to Landau (1909b, §86) and Landau (1909a, §231–233), this formula (also (1.22) and (1.25)) goes back to Riemann, Hadamard and Phragmen. The result is generally attributed to Perron (1908), who showed it for general Dirichlet series.

Proposition 1.11 is a particular case of a more general theorem due to Carlson (1922). We will give the general result in Theorem 11.2.

To the best of our knowledge, the first time that the Banach space \( \mathcal{H}_\infty \) was considered (although not with this notation) was by Hedenmalm et al. (1997)

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1 ‘Whether or not the number \( \frac{1}{2} \) can be replaced by a smaller number (maybe even by 0, which would mean \( T = 0, S = \infty \)) I do not know’
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A Dirichlet series in $\mathcal{H}_\infty$ converges uniformly on $[\text{Re} > \sigma]$ for every $\sigma > 0$, but it may not converge uniformly on $[\text{Re} > 0]$. Bonet (2018) has defined the space $\mathcal{H}_+^\infty$ of all Dirichlet series that converge uniformly on $[\text{Re} > 0]$, showing that it is a Fréchet–Schwartz nonnuclear space.
2

Holomorphic Functions on Polydiscs

In Chapter 1 we studied some basic facts about the convergence theory of Dirichlet series, and we defined the Banach space $\mathcal{H}_\infty$ of all those Dirichlet series which generate bounded holomorphic functions on the right half-plane. Our next step is to see, following an ingenious idea of H. Bohr, how Dirichlet series turn out to be intimately related with holomorphic functions on finite and infinite dimensional polydiscs.

Again we look at the simplest case first. We begin by linking bounded, holomorphic functions on the one-dimensional unit disc with Dirichlet series which only depend on the first prime number (that is, of the form $\sum a_{2^k} 2^{-ks}$). We denote by $\mathcal{H}_\infty^{(1)}$ the class of all these Dirichlet series that, as a consequence of Proposition 1.19, are a closed subspace of $\mathcal{H}_\infty$, and by $H_\infty(D)$, the Banach space of all bounded and holomorphic functions on $D$. The following elementary result will serve as a model for several similar but more involved results in higher dimensions that we will address later.

**Proposition 2.1** The mapping

$$H_\infty(D) \rightarrow \mathcal{H}_\infty^{(1)}$$

given by $f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k \leadsto \sum_{n=1}^{\infty} a_n n^{-s}$,

where $a_n = \frac{f^{(k)}(0)}{k!}$ for $n = 2^k$ and $a_n = 0$ else, defines an isometric isomorphism. In short, $H_\infty(D) = \mathcal{H}_\infty^{(1)}$ as Banach spaces, and the identification $f \leadsto \sum a_n n^{-s}$ identifies Taylor with Dirichlet coefficients. Moreover, for each $s \in \mathbb{R} > 0$,

$$f\left(\frac{1}{2^s}\right) = \sum_{n=1}^{\infty} a_n \frac{1}{n^s}.$$

Before we come to the proof, let us discuss a couple of simple facts about series. Let $\sum_k c_k z^k$ be a power series and $\sum_n a_n n^{-s}$ its corresponding Dirichlet series, i.e. $a_n = c_k$ for $n = 2^k$ and $a_n = 0$ else. Given $s \in \mathbb{C}$, the series $\sum_n a_n n^{-s}$
converges at $s$ if and only if $\sum_k c_kz^k$ converges at $z = 1/2^s$, and in this case
\[
\sum_{n=1}^{\infty} a_n n^{-s} = \sum_{k=0}^{\infty} a_{2^k} 2^{-ks} = \sum_{k=0}^{\infty} c_k z^k.
\] (2.1)

But more can be said, since, for each $\sigma \in \mathbb{R}$, the transformation
\[
[\text{Re} > \sigma] \longrightarrow \frac{1}{2}\sigma \mathbb{D} \setminus \{0\} \text{ defined by } s \rightsquigarrow z = \frac{1}{2^s}
\] (2.2)
is surjective, we see that in this simple case, convergence of $\sum_n a_n n^{-s}$ on half-planes $[\text{Re} > \sigma]$ transfers to convergence of $\sum_k c_k z^k$ on discs $\frac{1}{2^s} \mathbb{D}$, and vice versa. Let us go one step further and ask what happens if we also have boundedness. Again by (2.2) (and the maximum modulus theorem),
\[
\sup_{\text{Re} s > \sigma} \left| \sum_{n=1}^{\infty} \frac{a_n}{n^s} \right| = \sup_{z \in \frac{1}{2^s} \mathbb{D}} \left| \sum_{k=0}^{\infty} c_k z^k \right| = \sup_{|z| = \frac{1}{2^s}} \left| \sum_{k=0}^{\infty} c_k z^k \right| = \sup_{\text{Re} s = \sigma} \left| \sum_{k=0}^{\infty} a_{2^k} 2^{-ks} \right|.
\] (2.3)

**Proof of Proposition 2.1**  By (2.1) and (2.3), we see that the mapping given is well defined, and it is obviously linear and injective. A second look at (2.3) shows that it is even isometric, hence it only remains to prove that it is surjective. Take $\sum a_n n^{-s} \in H^{(1)}_\infty$ and consider the power series $\sum_k c_k z^k$ where $c_k = a_n$ for $n = 2^k$ and $c_k = 0$ else. Since $\sum a_n n^{-s}$ converges on $[\text{Re} > 0]$, we conclude from (2.2) that this series converges (absolutely) on $\mathbb{D}$, and therefore the function $f : \mathbb{D} \to \mathbb{C}$ defined as $f(z) = \sum_{k=0}^{\infty} c_k z^k$ is holomorphic. Again by (2.3), this function is bounded, and hence it is the object we were looking for. Additionally, we see that
\[
f\left(\frac{1}{2^s}\right) = \sum_{k=0}^{\infty} c_k \left(\frac{1}{2^s}\right)^k = \sum_{n=1}^{\infty} a_n \frac{1}{n^s},
\]
for every $\text{Re} s > 0$. \hfill \qed

So our aim now is to explore a similar connection between Dirichlet series and holomorphic functions with more than just one variable. We will do so in detail in Chapter 3. Before we get there we need some basics about complex analysis both for functions in finitely and infinitely many variables. We devote this chapter to introducing these.
2.1 Finitely Many Variables

If $U$ is an open subset of $\mathbb{C}^N$, then a function $f: U \to \mathbb{C}$ is holomorphic whenever for each $z \in U$ there is a (unique) vector $\nabla f(z) \in \mathbb{C}^N$ such that

$$\lim_{h \to 0} \frac{f(z + h) - f(z) - \langle \nabla f(z), h \rangle}{\|h\|} = 0,$$  \hspace{1cm} (2.4)

where $\langle x, y \rangle = \sum_{j=1}^{N} x_j y_j$ for $x, y \in \mathbb{C}^N$.

Take some holomorphic $f: U \to \mathbb{C}$; for every $a = (a_1, \ldots, a_N) \in U$ and each $j = 1, \ldots, N$, the function defined by

$$f_{ja}(z) = f(a_1, \ldots, a_{j-1}, z, a_{j+1}, \ldots, a_N)$$  \hspace{1cm} (2.5)

is clearly holomorphic as a function of $z$ in its domain of definition, and $f_{ja}'(z) = \nabla f(a_1, \ldots, a_{j-1}, z, a_{j+1}, \ldots, a_N)$. In other words, every holomorphic function is separately holomorphic. This means that for a holomorphic function, all partial derivatives exist; then for each multi-index $\alpha \in \mathbb{N}_0^N$, we write

$$(\partial^\alpha f)(a) = (\partial^{\alpha_N}_N \cdots \partial^{\alpha_1}_1 f)(a).$$

In fact, the converse of this statement is true (i.e. every separately holomorphic function is holomorphic). This is a result due to Hartogs that we do not need for the moment and will be proved in Theorem 15.7.

**Remark 2.2** The first thing that one expects from a holomorphic function $f: U \to \mathbb{C}$ is that it is continuous. This is indeed the case here. Given $x \in U$, choose $r > 0$ such that $D^N(x, r) \subset U$, and define $\eta: D^N(0, r) \to \mathbb{C}$ by

$$\eta(h) = \begin{cases} \frac{f(x+h) - f(x) - \langle \nabla f(x), h \rangle}{\|h\|} & \text{if } 0 < \|h\| < r, \\ 0 & \text{if } h = 0. \end{cases}$$

Clearly, by (2.4), $\eta$ is continuous at 0 and

$$f(x + h) - f(x) = \eta(h)\|h\| + \langle \nabla f(x), h \rangle$$

for every $h \in D^N(0, r)$. This implies that $f$ is continuous at $x$. ☑

**Remark 2.3** Suppose $f: U \to \mathbb{C}$ is holomorphic and $g: \Omega \to U$, where $\Omega \subseteq \mathbb{C}$ is open and such that for every $\lambda \in \Omega$, there is $g'(\lambda) \in \mathbb{C}^N$ with

$$\lim_{h \to 0} \frac{g(\lambda + h) - g(\lambda)}{h} = g'(\lambda)$$

(we then also say that $g$ is holomorphic). Then the composition $l = (f \circ g): \Omega \to \mathbb{C}$ is holomorphic and $l'(\lambda) = \langle \nabla f(g(\lambda)), g'(\lambda) \rangle$ for every $\lambda$. This is a particular case of the chain rule that will be proved in full generality in Proposition 15.1. ☑
Although our main interest is going to be the $N$th-dimensional polydisc $D^N$ (that is, in terms of the modulus, a cube with centre at 0 where all sides have length 1), at some moments we will consider also prisms (centred at the origin) with sides of different length. To be more precise, given a multiradius $r = (r_1, \ldots, r_N)$ with $0 < r_j < \infty$, we establish the following notation:

$$rD^N = D^N(0, r) := r_1D \times \cdots \times r_ND = \{(z_1, \ldots, z_N) : |z_j| < r_j, j = 1, \ldots, N\}.$$ 

An important fact in order to build the theory is that the linear space $H_\infty(rD^N)$ of all bounded, holomorphic functions on $rD^N$ together with the sup norm

$$\|f\|_\infty = \|f\|_{rD^N} := \sup_{z \in rD^N} |f(z)|$$

forms a Banach space. This is a consequence of the following Weierstraß theorem for functions in $N$ variables, which easily follows from the classical Weierstraß theorem for functions in one variable.

**Theorem 2.4** Let $(f_n)_n$ be a sequence of holomorphic functions on $rD^N$ that converges uniformly on all compact subsets of $rD^N$ to some $f : rD^N \to \mathbb{C}$. Then $f$ is holomorphic.

**Proof** Let us fix some $a \in rD^N$. For each $k = 1, \ldots, N$ and $n \in \mathbb{N}$, we define $f_{n,k} : r_kD \to \mathbb{C}$ by $f_{n,k}(\lambda) = f_n(a_1, \ldots, \lambda, \ldots, a_N)$, where $\lambda$ is in the $k$th position. Then $\nabla f_n(a) = (f'_{n,1}(a_1), \ldots, f'_{n,N}(a_N))$. It is an easy exercise to check that each sequence $(f_{n,k})_n$ converges uniformly on all compact subsets of $r_kD$ to $f_k$ (defined in the obvious way), and then, by the classical Weierstraß theorem, $f_k$ is holomorphic and $f''_{n,k} \to f''_k$ uniformly on all compact sets of $r_kD$. We define $\nabla f(a) = (f'_{1}(a_1), \ldots, f'_{N}(a_N))$ and we want to check that

$$\lim_{h \to 0} \frac{f(a + h) - f(a) - \langle \nabla f(a), h \rangle}{\|h\|} = 0. \quad (2.6)$$

We choose $s > 0$ such that $\overline{D}^N(a, s) \subseteq rD^N$ and $h \in \mathbb{C}^N$ with $\|h\| < s$. Then for each $n$ we define $g_n : sD \to \mathbb{C}$ by

$$g_n(\lambda) = f_n(a + \lambda \frac{h}{\|h\|}).$$

These $g_n$ are holomorphic functions on $sD$ and satisfy $g_n(0) = f_n(a)$, $g_n(\|h\|) = f_n(a + h)$, and $g'_{n}(\lambda) = \langle \nabla f_n(a + \lambda \frac{h}{\|h\|}), \frac{h}{\|h\|} \rangle$ for every $\lambda \in sD$ (recall Remark 2.3). In particular, for each $n$, we have

$$g'_{n}(0) = \langle \nabla f_n(a), \frac{h}{\|h\|} \rangle.$$
Now, for each $|\lambda| < \rho < s$, we apply the maximum modulus theorem to obtain

$$|g_n(\lambda) - g_n(0) - g'_n(0)\lambda| \leq |\lambda|^2 \sup_{|\eta| = \rho} \left| \sum_{m=2}^{\infty} \frac{g_n^{(m)}(0)}{m!} \eta^{m-2} \right|$$

$$= \frac{|\lambda|^2}{\rho^2} \sup_{|\eta| = \rho} \left| \sum_{m=2}^{\infty} \frac{g_n^{(m)}(0)}{m!} \eta^{m-2} \right| = \frac{|\lambda|^2}{\rho^2} \sup_{|\eta| = \rho} \left| g_n(\eta) - g_n(0) - g'_n(0)\eta \right|$$

$$\leq \frac{3}{\rho^2} |\lambda|^2 \sup_{|\eta| \leq \rho} |g_n(\eta)| \leq \frac{3}{\rho^2} |\lambda|^2 \sup_{z \in \mathbb{D}^N(a,s)} |f_n(z)|.$$

But the sequence $(f_n)_n$ converges uniformly on $\overline{\mathbb{D}}^N(a,s)$, thus

$$\sup_n \sup_{z \in \mathbb{D}^N(a,s)} |f_n(z)| = M < \infty.$$

Taking this into account, choosing $\lambda = ||h||$ (recall that the previous inequality holds for every $||h|| < \rho < s$ and make $\rho \to s$), we get

$$\left| f_n(a + h) - f_n(a) - \langle \nabla f_n(a), \frac{h}{||h||} \rangle ||h|| \right| \leq \frac{3M}{s^2} ||h||^2,$$

and hence

$$\left| \frac{f_n(a + h) - f_n(a) - \langle \nabla f_n(a), h \rangle}{||h||} \right| \leq \frac{3M}{s^2} ||h||.$$

Now, if we here let $n$ tend to infinity, then for every $0 < ||h|| < s$, we have (note that, by construction, $\nabla f_n(a) \to \nabla f(a)$)

$$\left| \frac{f(a + h) - f(a) - \langle \nabla f(a), h \rangle}{||h||} \right| \leq \frac{3M}{s^2} ||h||.$$

This gives (2.6) and proves that $f$ is holomorphic. □

**Theorem 2.5** $(H_\infty(\mathbb{D}^N), ||\cdot||_\infty)$ is a Banach space.

**Proof** Consider a Cauchy sequence $(f_n)_n$ in $H_\infty(\mathbb{D}^N)$, and denote its pointwise limit by $f: \mathbb{D}^N \to \mathbb{C}$. Then $(f_n)$ converges to $f$ uniformly on $\mathbb{D}^N$, and it remains to show that $f \in H_\infty(\mathbb{D}^N)$. But by Theorem 2.4, $f$ is holomorphic, and it is of course bounded since all $f_n$ are. □

Holomorphic functions in one variable can be defined in two ways. One can follow the Cauchy approach and define holomorphy through differentiability (as in (2.4)), or the Weierstraß approach through analyticity, saying that a function is holomorphic at a point if it can be developed as a power series on some disc centred at the point. One of the cornerstones of complex analysis for functions of one variable is that these two approaches lead to the same result: a function is differentiable if and only if it is analytic (and then we call it holomorphic). We have chosen the Cauchy approach to holomorphy in several