The History of Actuarial Science

Life Tables and Survival Model: Part 1

Edited by
Steven Haberman and Trevor A. Sibbett

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HISTORY OF ACTUARIAL SCIENCE

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OF
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STEVEN HABERMAN
and
TREVOR A. SIBBETT

VOLUME I

Life Tables and Survival Model

Part 1

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PREFACE

As the introduction attempts to explain, this collection of material is intended to provide a guide to the history of actuarial science until the early part of the twentieth century. A difficulty immediately arises in defining the boundaries of a subject, especially when that subject is in its embryonic state. Given the constraints of time and space, we have decided to exclude items that specifically and exclusively deal with contiguous subject areas like demography and compound interest. However, the reader will find some comments on topics other than actuarial science. The range of material available, even in a specialist area like actuarial science, is very wide but this collection aims at being representative and capturing the major contributions to the subject, the major issues, developments and debates.

In deciding on which contributions to include in these volumes we have attempted to apply the following criteria consistently and have sought works that satisfy at least one of these criteria. Firstly, a work should be intellectually important. Secondly, it should have changed an aspect of theory or practice significantly. Thirdly, a work should be new and should have made a significant impact on theory or practice. We have also attempted to ensure that the items chosen are illustrative of the broad sweep of actuarial theory and practice.

In choosing a cut-off date for this project, we have been mindful of the need to stand back from the development of actuarial science. A date too close to the modern day would make the judgement of significance difficult not least because of the short span of time that would have elapsed. A date at the start of the century allows us to see the direct descendants of much of what today is accepted and commonplace.

Another problem has been the allocation of works to themes when many contributions clearly straddle a number of different aspects of actuarial science. To avoid reproducing a work more than once, we have allocated a work to the theme that seems to us to be most appropriate. In the introduction and textual notes, we have attempted to draw attention to the scope of each work.

In an ambitious project of this kind there are bound to be differences of view and judgement on what, and how much, should be included. No doubt each expert will find fault when it comes to his/her own area, but it is hoped that overall we have captured the major contributions and that this set of books will prove to be both a valuable primary source and a reference collection for workers in the future which will provide easy access to the most important contributions to this body of literature.

There is a preponderance of works that appeared originally in English or were translated into English: in making this selection we have striven against the possibility of bias that might arise from our familiarity with English ideas and methodologies. The reader will be able to judge how successful we have been in attaining this particular objective.

I have been fortunate in being able to work alongside Trevor Sibbett who is a respected "amateur" historian of matters actuarial and I should like to acknowledge his assistance, guidance and forbearance. However, any errors remaining in the introduction are entirely my responsibility.

Trevor and I should like also to acknowledge the sterling support and considerable assistance we received from the staff of the Institute of Actuaries' Library, in particular Sally Grover and Roy Park.

S.H.

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Thomas Drucker for permission to publish his translation of "De Usu Artis Conjectandi in Jure", ch. II, pp. 7-16.

THE HISTORY OF ACTUARIAL SCIENCE

The origins of actuarial science lie in the seventeenth century. During this period, commercial needs gave rise to transactions involving compound interest, marine insurance was commonplace and the algebra of life annuities came into existence. Indeed valuation techniques for financial transactions were known even in ancient Roman times and became of increasing importance as world trade developed after 1500.

During the sixteenth century, some of the European writers on arithmetic such as Simon Stevin and Jan Trenchant devoted some space to elementary problems in compound interest. There were no actuaries in the formal sense at that time. The nearest equivalent was the mathematical practitioner, a consultant who would tackle all kinds of problems on request from commercial matters to navigation. Richard Witt was one such; he practised in London in the early 1600s and wrote the first comprehensive book in English on compound interest. But Witt's book did not venture into life insurance mathematics and it was not until later in the seventeenth century that the necessary tools became available. One such was the developing science of probability; another was the concept of the life table based on mortality investigations, the first published example being that of John Graunt in 1662.

Thus, by 1670 two of the main foundations of actuarial science were firmly in place: compound interest, probability theory and the life table. These tools were employed almost immediately by the Dutch prime minister, Jan de Witt, to investigate the value of Government life annuities. But his treatise, although of considerable merit, remained unnoticed for many years and did not influence the development of actuarial science. Then in 1693 Edmund Halley, the British mathematician, published his famous Royal Society paper, describing the construction of the life table from observations. He also set down the method for valuing life annuities, which is, in essence, the same as that used today.

Actuarial science had thus arguably been born, and by 1700 the way was open for the techniques to be applied for commercial purposes. At this time, life annuities were bought and sold freely, although in the main their values were not properly calculated. Life assurance consisted mainly of short term risk-only contracts purchased by single premium from an individual underwriter, the premiums being assessed on the basis of the underwriter's experience and judgement, but with no formal scientific basis. Early in the eighteenth century the Amicable Society established life assurance on a basis which involved some build-up of funds or reserves.

Some of the pioneers in the 1700s and 1800s were eminent scientists and mathematicians who became interested in actuarial problems. Thus, we find Leonhard Euler, James and Daniel Bernoulli, Carl Friedrich Gauss, Abraham de Moivre, Benjamin Gompertz becoming involved in the science and making significant contributions. In the UK, many papers on actuarial matters were read before the Royal Society, there being no other formal theatre for discussion.

The 1800s saw the establishment of many life assurance companies in the UK and elsewhere, operating on a scientific basis following the pioneering work of James Dodson and Richard Price. A wide range of level annual premium contracts was available to the public. Annuities were calculated by proper methods and the dangers of inappropriate mortality tables became known. Life insurance mathematics, compound interest and probability theory were then at an advanced state of development, and mortality tables had developed from crude tabulations of deaths to properly calculated and considered works. The first table of life assurance mortality from pooled data was published in 1843, as also was English Life Table No. 1, for the national population. The collection and analysis of sickness statistics were under way, but the level of attainment was still fairly crude.

By the mid nineteenth century, a considerable amount of fundamental work had been completed. There then followed a rapid expansion in the refinement and practical application of actuarial theory. Life insurance business was expanding rapidly and many actuaries were employed in the industry on a full-time basis. The Institute of Actuaries was established in 1848 and the Faculty of Actuaries in 1856, providing the first dedicated actuarial publication media, respectively the Journal and the Transactions. The first text books began to appear for assisting students to pass professional examinations. A steady stream of significant papers on actuarial theory and practice was published in these journals, together with translations of significant contributions from continental Europe.

A statutory system of actuarial supervision of life insurance began with Massachusetts legislative session of 1858 and the Life Assurance Companies Act of 1870 in the UK. This latter piece of legislation was the first detailed statutory framework regulating the life assurance business, Parliament having chosen the "freedom with publicity" principle; this led to debates on the relative merits of the bonus reserve and the net premium valuation methods. The 1860s had earlier seen the invention of the contribution method of the distribution of surplus, almost universally used in life assurance practice in North America. In the US, the Actuarial Society of America was founded in 1880, the forerunner of the Society of Actuaries and the Casualty Actuarial Society in 1914.

Among the significant contributions of the period up to the early part of our current century is the appearance of Makeham's formula for the force of mortality which was found able to represent faithfully much published mortality data for the next 100 years. We also find the development of different approaches to graduation; the publication of premium conversion tables which eased the burden of calculating life insurance premiums in the days before calculating machines became generally available; the development of analysis of surplus, whereby the causes of the surplus or deficiency emerging in any type of valuation can be revealed and further analysed; the development of risk and credibility theory; the development of the mathematical theory of multiple decrement and multiple state models; the systematic presentation of pension fund valuation ideas and formulae; the formulation of the principles of institutional investment; the agreement on a standard actuarial notation which provided a language to facilitate international dialogue. We also note the important creation of the International Actuarial Association in 1895 which over the last 100 years has fostered and encouraged the development of actuarial ideas and the actuarial profession around the world, and has provided a forum for international meetings and the cross-fertilisation of ideas.

Life Tables and Survival Model

The first noteworthy writings on the survival model and what came to be called the life table is Ulpian's Table, dating from around 220. The calculations were intended for the valuation of annuities as relating to legacies. The interpretation of the numbers is unclear and it is possible that Ulpianus may have derived the figures from actual observations rather than guesswork alone, although Greenwood (1940) presents strong doubts with the conviction that Ulpianus merely interpolated between legal maximum and minimum estimates for the expectation of life. Still the Table was authorised for the valuation of life annuities by the Government of Tuscany in 1814. Mays (1971) provides some further analysis of this work.

The period before the nineteenth century was characterised by high levels of mortality in Europe. One of the main contributing factors was the incidence of epidemic diseases, among which the plague was the worst. After the Black Death in 1348-50, the plague recurred frequently for nearly four centuries. As Hald describes, an early "warning system" was set up in London in the 1530s by requesting the parish clerks to submit weekly reports on the number of plague deaths and all other deaths in case of an incipient outbreak. These weekly "bills" of mortality served to warn the authorities when measures should be taken against the epidemic and to warn the wealthy echelons of society when they should escape to the country. Starting in 1604, weekly bills of mortality for the parishes of London were published by the Company of Parish Clerks, and a bill for the whole year was published at the end of each year, in printed form from 1625. The ages of the deceased were not recorded before 1728. The weekly bills were published regularly until 1842 when they were superseded by the publication from the Registrar General.

As Hald (1990) notes, "this large amount of data had not been analysed statistically before John Graunt published his book in 1662", a remarkable work which "is widely regarded as a landmark in the descriptive and statistical analysis of demographic data". Graunt's critical appraisal of the rather unreliable data that had been collected, his development of concepts and techniques relevant for the analysis of that information, his consideration of errors and ambiguities in the data, his study of mortality by cause of death, his estimation of the same quantity by several different methods, his demonstration of the stability of statistical ratios and his creation of the life table all set new standards for reasoning. Graunt's work led to further investigations along three different avenues: "political arithmetic" or demography; testing the stability of ratios; the calculation of survivorship probabilities and expectations of life: Hald (1990). We are concerned with the third of these avenues. However, it is worth noting in passing that Graunt's work on political arithmetic was immediately taken up by William Petty who first coined this particular phrase (see Hull (1899)).

As some commentators have mentioned, there are some errors in Graunt's work on the life table (see Westergaard (1932) and Glass (1950, 1964)). For example, he did not fully appreciate the relationship between the life table and the age distribution and size of the corresponding stationary population and indeed confuses these concepts. We should however recognise the pioneering nature of this work, which had immense influence. Thus, bills of mortality were introduced in other cities (for example, Paris in 1667). His methods of statistical and demographic analysis were adopted in England, France, the Netherlands and Germany and led ultimately to the creation of government statistical offices. Some would argue that, although Graunt was the first to describe the dying out of a population cohort, he did not compile the first life table in the modern actuarial sense (Sutherland (1963)). This concept was nevertheless picked up by the Huygens brothers, improved upon by de Witt and Halley and subsequently became a key tool in medical statistics and demography as well as actuarial science.

There was speculation for some time that Graunt's book was in fact written by Petty but this argument has been effectively disposed of by Glass (1964) and Sutherland (1963), inter alia.

As is clear from the discussion, there exists a large literature about Graunt's work. Most recently, Kreager (1988) casts new light on Graunt's book and connects his techniques with common bookkeeping.

The first authors to have used Graunt's life table were Christiaan and Lodewijk Huygens who corresponded in 1669 on a probabilistic interpretation of the life table. They considered the calculation of the average age at death, the corresponding expectation of life and the median remaining lifetime, carefully noting the distinction between these concepts. They also considered joint-life expectations, including the expectation of the longest and shortest lifetime of a group. The details are discussed further by David (1962), Hald (1990), Schevichaven et al (1898) and Seal (1980).

Then followed applications of the life table to the problems of valuing annuities. In the sixteenth and seventeenth centuries, states and cities often raised money for public purposes by selling life annuities to their citizens. In 1671, Jan de Witt, the distinguished and long-serving Prime Minister, submitted a report to the States of Holland showing how to calculate the value of annuities by means of a piecewise linear life table combined with the age of the nominee and the rate of interest. De Witt's life table was hypothetical, although his report refers to some investigations of the mortality of annuitants. De Witt's approach to annuity calculations was through the distribution of the number of deaths in the underlying life table, i.e. via the formula in modern notation

$$a_x = \sum_{t=1}^{\infty} a_{\overline{t|}} \frac{d_{x+t}}{l_x}$$

although Hald (1990) believes that de Witt was familiar with the alternative approach of Halley. De Witt also demonstrated some appreciation of the effects of self selection and changes in mortality levels on the value of an annuity. De Witt's report was forgotten until Hendriks (1852) rediscovered it and provided an English translation and commentary.

Edmund Halley's paper of 1693 was seminal and of great importance to actuarial science. He constructed a life table from observations of the yearly number of deaths in Breslau (where the parish registers were among the first to contain age at death) and used this in seven different ways (Hald (1990)), including considering the "proportion of men able to bear arms" (as did Graunt) and the median remaining lifetime (as did Huygens). He used the odds ${}_tp_x/{}_tq_x$ as a measure of the "differing degrees of ... vitality in all ages" and he commented on the relationship between the price of a term assurance and the odds function. He then calculated the first table of values of annuities as a function of the nominee's age and developed formulae for calculating the value of joint life annuities (for two and three lives; with geometrical diagrams by way of explanation) and emphasised the benefit of using logarithms to reduce the volume of calculation. His approach to calculating the present value of annuities was through the distribution of the number of survivors, i.e.

via the formula in modern notation

$$a_x = \sum_{t=1}^{\infty} v^t \frac{l_{x+t}}{l_x}$$

- this is algebraically equivalent to de Witt's method although computationally more straightforward. However, we should note that, in a modern perspective, these calculations of present values actually relate to expected present values (of an annuity with a random term). De Witt's method can be easily adapted to considering higher moments, for example, the variance, whereas Halley's method cannot be progressed in this manner. Thus, de Witt's approach is also of lasting theoretical (and practical) importance.

Halley remarked that the government was selling annuities too cheaply and at a price independent of the age of the annuitant: his advice was ignored! As many commentators have noted, the life table function tabulated by Halley was what we would call L_{x-1} rather than l_x. It is noteworthy that the Breslau life table was reproduced in the updated version of 1737 of the abstract of the Amicable Society's charter and by-laws.

The first published work on life expectancy is due to Nicholas Bernoulli (1709), given that the correspondence between the Huygens brothers had not been published. In this work, Graunt's life table is used to provide illustrations of the calculation of the expectation of life and the median remaining lifetime. Bernoulli noted that expectations may be obtained by backward recursion and illustrates this process. There is no reference to the life tables of de Witt or of Halley, or to the methods described by Halley. An interesting problem solved by Bernoulli is the derivation of a formula for the expected lifetime of the last survivor of b lives dying within an interval of length a (under a uniform distribution of deaths assumption). In Chapter 6, Bernoulli considers and solves some simple problems from marine and life insurance by calculation of expectations.

The first graphical representation of a life table function is attributed to Isaac de Graaf (1729) in the extract shown: the graph seems to represent the number alive in a life table, l_x, calculated on some hypothetical basis.

John Smart's life table of 1738 was the first specific to London and was based on the numbers of deaths occurring in London in the period 1728-37. At his request in 1726, the bills of mortality had included the numbers dying at each age from 1728 onwards on a weekly basis.

The earliest life tables developed for males and females are due to Nicholas Struyck (1740), based on registers of annuitants (as were for example those of de Witt and Deparcieux). He also noted, in passing, the effects of self selection and calculated annuity values for the two sexes and compared his results with those of Halley. Like Halley, he pointed out that the government was selling annuities too cheaply. He also gave examples of increasing, decreasing and deferred annuities. Struyck was also a writer on probability and demography.

Among the other great investigations of annuitant mortality are those of William Kersseboom (1742: not reproduced here) and Antoine Deparcieux who produced the first French life table in 1746, based on the data from the operation of two tontines. He constructed the life table directly from observed deaths - methodological errors were reduced because the tontine populations were not generally subject to loss of observations due to withdrawal or migration. This life table was used extensively by French life insurance companies for many years well into the next century. Thus, at about the same time, Struyck, Kersseboom and Deparcieux appreciated the significant of constructing a life table from the registers of annuitants.

The concept of the annual rate of mortality was introduced by Thomas Watkins in 1761 (writing as TW). Watkins commented critically on Halley's work, noting problems with extrapolation of the life table to the oldest ages and with the use of a small radix. He pointed out that Halley's and Brackenridge's life tables appeared smooth from the progression of d_x values but not when the values of the annual rate of mortality were examined. Watkins drew attention to the need for smoothing (i.e. graduating) crude data and advocated considering the annual rate of mortality and the probable expectation of life and their first differences in the context of the life table. The annual rate of mortality was not widely adopted until the next century, with the work of Emmanuel Duvillard, John Finlaison inter alia. It is noteworthy that Johann Lambert (in volume III of his 1772 book) followed Halley's use of the odds function and used the reciprocal of q_x as a measure of *"vitality"* (Daw (1980)).

In 1749, the Swedish General Register Office was established and the first national set of population statistics started to be collected from that date. Per Wargentin in his paper of 1766 presented an analysis of mortality and population data for Sweden using these statistics. Wargentin combined the death registration data for 1755-63 with the triennial *"censuses"* of 1757, 1760 and 1763 to calculate the inverse death rate for the two sexes for quinquennial age groups (except at the youngest ages). He noted the lower levels of mortality for females and commented on the likely errors in the data (in terms of individuals not counted and ages misstated). He criticised Halley's approach on the grounds that the underlying population would need to be stationary, noting that the population of Sweden was not stationary.

Wargentin's groundwork was taken up by Richard Price in his construction of a life table for Sweden using an arbitrary radix and a smoothed set of inverse mortality rates. Price followed Wargentin's methodology and used an extended version of his data i.e., death registration data for 1755-76 with the seven triennial *"censuses"* from 1757 to 1775. The construction of this life table represents an important breakthrough. It appears in the fourth edition (1783) of Price's masterpiece "Observations on Reversionary Payments". Price prepared life tables for Sweden, for Stockholm on its own, for males and females separately. He also constructed a persons life table using the incorrect approach of taking a simple average of the consituent male and female life tables. This book also contains the second Northampton life table based on the experience from 1735-80 (see later).

An interesting, although virtually unknown, article of 1767 from the great mathematician Leonhard Euler adapted the life table to solve a number of problems requiring inference from incomplete data. The work anticipated "important parts of modern stable population theory for a one-sex population closed to migration. Its ideas have been published many times during the subsequent two centuries by writers who independently rediscovered them" (Nathan Keyfitz writing in Smith and Keyfitz (1977)). Euler's aim was to use the life table model to study real populations through the progression of cohorts, allowing for population increase or decrease. He thereby considered the age structure of a stationary population and a stable population (in the demographic sense - see Keyfitz (1985)), thus generalising the usual life table formulae.

The first life table for the United States was due to Edward Wigglesworth (1789), based on deaths in the states of Massachusetts and New Hampshire. His methodology used observed deaths only, ignoring the developments that had been made by Wargentin and Price.

Joshua Milne's textbook of 1815 dealt with a number of aspects of life insurance mathematics. It also considered the construction of life tables. In particular, Milne described the construction of the Carlisle life table based on the 1779-87 experience in two parishes of that town, following the methods that Price used for his Swedish life table. The data were sparse (e.g. only 406 deaths at ages 20-59) and the attempts at smoothing the grouped numbers of deaths and population counts were unsatisfactory. The life table was little heeded for some years after its publication but was later adopted enthusiastically by actuaries and became a standard table. Further discussion on the Carlisle table can be found in King (1884).

John Finlaison was the first actuary to be described as Government Actuary although the government had received advice prior to this, for example from William Morgan. (Finlaison was also the first President of the Institute of Actuaries in 1848). He was Actuary of the National Debt Office from 1822-51 and carried out a number of important duties. His most important contribution to actuarial science was his work on the life tables for government annuities. William Morgan had been consulted when the National Debt office began the sale of annuities in 1800. Morgan had made the error of adopting a life table with suitably prudent margins for life insurance premiums (Richard Price's Northampton Table): this meant that the government was significantly undercharging for its annuities for about 20 years. (We recall that earlier de Witt and Halley had faced similar problems). In 1819, Finlaison pointed out the error. He was then commissioned by the Chancellor of the Exchequer to carry out an investigation into annuitants' mortality and to produce a new set of annuity tables. He carried out a major mortality study, which included the records of various tontines from 1695 to 1789 and which was published in 1829. In this work, Finlaison criticised earlier writers who had been dependent on others for the data used, for example Deparcieux and Kersseboom, and he recognised the importance of treating the sexes separately. He took painstaking care in the collection and preliminary tabulation of the data to eliminate various types of errors. In particular, with tontines and government annuities, it was common for a subscriber to possess many shares via nominees. Finlaison was careful to avoid the duplication caused by counting the same person more than once. He thus produced a complete set of tables for single lives (by 1823) and for joint lives, which formed the basis for the pricing of government life annuities until 1884.

Benjamin Gompertz's paper of 1825 marked the "beginning of a new era" for actuarial science (Hooker (1965)). The well-known "law" of mortality that he proposed was an enormous improvement on previous attempts to represent life table functions by a mathematical formula and it thereby opened up a new approach to the life table. Hitherto, the table had been regarded as a record of observed numbers surviving from an initial cohort - Gompertz now introduced the idea that l_x was a continuous, mathematical function, connected by (what we would call) the underlying force of mortality. At this stage, the force of mortality had not been identified: Gompertz worked in terms of l_x and its fluxion (out of respect for Newton, he persisted in using the language of fluxions rather than differentials throughout his life). His objective in doing the background research was to find a general form to facilitate interpolation. He analysed actual experience before proposing his hypothesis and he demonstrated how his formula could be applied with a good deal of accuracy to the Carlisle and Northampton life tables and to Deparcieux's observations. He explained that his "law" could be interpreted in terms of the average exhaustion's of an individual's resistance to death. As Hooker (1965) points out, Gompertz does mention "two generally coexisting causes" of death; "the one, chance, without previous deposition to death or deterioration; the other, a deterioration, or an increased inability to withstand destruction". It now seems strange that his notion of two causes of death did not lead Gompertz to Makeham's later modification: $\mu_x = A+Bc^x$. However, Gompertz's presentation of ideas and train of thought were not completely clear (as noted by Makeham (1890)) and indeed his paper did not receive the wide recognition it merited, although it was subsequently championed by many eminent thinkers of the time, including De Morgan, Herschel, Sprague and Woolhouse. (The paper's reception may also by explained by Gompertz's use of the obsolete ideas of fluxions and the number of errata).

The curve of deaths, or a graphical presentation of the d_x function was introduced in 1826 by Thomas Young, the eminent physicist. He considered a number of well-known life tables and averaged the numbers of deaths and then introduced an obscure combination of polynomials of higher order (involving for example x^{40}) for different parts of the age range to represent this "average" curve of deaths.

Johann Lambert (volume III of his 1772 work) discussed the force of vitality, recognisable as the reciprocal of the force of mortality (Daw (1980)). It was T R Edmonds who introduced in 1832 the term "force of mortality" and showed its algebraic form. The regular use of the force of mortality (or hazard rate) in actuarial mathematics and statistics (for example, survival analysis) dates from this book. Edmonds used three Gompertz type curves to represent the force of mortality over different age ranges: up to age 9, from 9 to 55, and 55 and over.

Francis Corbaux was the first to argue that population life tables are constructed from an aggregate of different life tables for lives of separate subgroups (1833). He identified a number of factors (or covariables) including sex and occupation that should be allowed for to avoid class selection. This work covered a number of topics. Thus, Corbaux recommended graduation of third or fourth differences of $\log q_x$ in order "that a rectification of any irregularities, incident to the data supplied by experience may thus be arrived at". He also discussed increasing, stationary and decreasing populations, expectation of life, initial and class selection of assured lives, fertility rates specific for age, and other demographic issues.

The idea of variability of life table calculations was first mentioned by Augustus de Morgan in his 1838 essay which, inter alia, surveyed life insurance mathematics. He derived an expression for the "probable error" of the expectation of life at high ages.

Gompertz's Law was given a more general structure by William Makeham in 1860, in what became known as Makeham's Law. However, Makeham regarded his own contribution only as a modification to Gompertz's Law. (Indeed, his final published paper of 1890 was entitled "Further improvements of Gompertz's Law"). Though, as we have seen, Gompertz had considered causes of death as being of two kinds, chance and deterioration, he did not finally link them to mathematical expressions in his work. Makeham proposed that causes could be approximately divided between what would be roughly independent of age and what would be increasing with age. We can write his modification in the well-known form: $\mu_x = A + Bc^x$. He then demonstrated how convenient this assumption was in the calculation of joint life annuity values, where a version of de Morgan's Law of Uniform Seniority applies. At the time it was proposed, Makeham's Law appeared to fit existing data well (e.g. the Seventeen Offices' Experience quoted in his paper) and numerous life tables have been graduated on the basis of Makeham's Law: the most recent major table was probably the CSO 1941 (US) table, which follows this "law" from ages 15 to 95. It is also noteworthy that Makeham presented this paper before he had passed any of the examinations of the Institute of Actuaries. In later work he proposed further modifications to Gompertz's Law involving polynomial terms in attempts to represent assured lives' experience: see Makeham (1890).

The fact that the Gompertz and Makeham laws could not be expected to represent the mortality experience throughout life led to the investigation of formulae which might be expected to do so. So, Gompertz (1860 paper reprinted in 1872) himself suggested a formula based on an amalgamation of several of his curves with different constants. The Danish mathematician Thorvald Thiele proposed a combination of three terms for interpolation purposes: a decreasing Gompertz curve to represent the mortality of infancy, a normal curve of error to represent mortality at young adult ages and an increasing Gompertz curve to represent old-age mortality viz

$$\mu_x = a_1 \exp(-b_1 x) + a_2 \exp\left(-\tfrac{1}{2}\left(\frac{x-c}{b_2}\right)^2 \right) + a_3 \exp(b_3 x).$$

A translated version of this work appears in 1872 in the Journal of the Institute of Actuaries. Thiele's senior colleague, Ludvig Oppermann had proposed a law in 1870 to represent the force of mortality up to its "first point of inflection (or to the age of about 20)":

$\mu_x = ax^{-1/2} + b + cx^{1/3}$. Oppermann was the first to consider that the rate of decline of the force of mortality in infancy may not be exponential with respect to age, and that a transformation of age (say the square root) might be helpful. Thiele commented that "Oppermann has gained for himself lasting credit by this formula".

A different perspective on the life table and survival model was offered by the eminent statistician Wilhelm Lexis. His first contribution (1875) was to devise the well-known demographic diagram that displays the population by age and time: each individual at any moment is represented by a point; the collection of points for any single individual represents his life-line through time; the end of the line represents the moment and age of his/her death. His second contribution (1877) was to represent the empirical distribution of deaths by age by a normal curve, noting an observed surplus of early deaths, which, after excluding deaths at childhood ages, he classed as "premature deaths". Those deaths that are represented by the normal curve, he described as "normal deaths". This approach was rediscovered by Clarke (1950) in his separation of deaths into "anticipated" and "senescent", the ages at death for the latter being measures of the natural lifespan.

These particular contributions from Makeham, Thiele, Oppermann and Lexis were not progressed further within the time frame of this study. It was not until the suggestions by Perks (1932) that the logistic family of curves be used and then by Heligman and Pollard (1980) that a combination of double-exponential and lognormal curves be used to represent the odds q_x/p_x that progress towards a parametric mortality curve for the full age range was achieved.

Life Insurance Mathematics

The first textbook on life insurance mathematics was Abraham de Moivre's "Annuities on Lives" published in 1725. As Hald (1990) explains, at this time, "economic contracts that depended on the lifetimes of the parties involved were important parts of everyday life" in Europe, particularly in the UK. "Besides life annuities sold by the government, there were pensions granted by the government, the Church, municipalities, parishes and so on; life interests and reversions specified by wills and marriage settlements"; and complex contracts involving property. Such contracts were difficult to evaluate and the need for a more thorough mathematical analysis of these problems than that provided by Halley was clear. This is the "challenge taken up by de Moivre" (Hald (1990)).

As Hald notes, de Moivre had a "genius for developing mathematical approximations". He suggested approximating Halley's life table for Breslau by a piecewise linear function and proved that the value of an annuity under this hypothesis would be a linear function of an annuity-certain. So it was not necessary to tabulate the value of single life annuities since these could be derived directly from existing tables of annuities-certain. This avoided the difficult computation of the many products and sums that had bothered Halley. Although this hypothesis is sometimes quoted as de Moivre's Law, he realised that it was defective as a representation of human mortality over all ages. But the point was that de Moivre considered the assumption to be adequate for the purpose intended i.e. the evaluation of annuities within the range of ages then commonly required in practice.

De Moivre also gave a simple method for tabulating annuity values, a_x, by means of the backward recursion formula

$$a_x = vp_x(1+a_{x+1})$$

which is a generalization of Nicholas Bernoulli's formula for the calculation of the expectation of life (Hald (1990)). It seems clear that de Moivre appreciated that this formula had general application in the calculation of complete annuity tables; however, Young (1908) puts forward the claims of Euler as the first writer to appreciate the significance of this formula. Similarly, de Moivre considered the value of a temporary annuity using both his approximate method and a backward recursion method.

Further, de Moivre showed how the value of a joint-life annuity (for a group of independent lives) could be expressed approximately by means of the values of the corresponding single-life annuities so that joint-life annuities could be easily evaluated and manipulated. He began this investigation with his linear hypothesis but when the results for three lives became unwieldy he switched to a different hypothesis, viz. assuming that the lives have geometrically decreasing probabilities of survival i.e. ${}_tp_x = p^t$ which corresponds to assuming a constant force of mortality. He applied the same approach to joint life assurances (on three lives). However, he did not present any systematic investigation of the error involved in using this approximation.

Importantly, he also gave a systematic exposition of formulae for the value of last survivor annuities for any number of lives, reversionary annuities and annuities on successive lives which were used in leasehold property contracts (where, for example, on the death of the annuitant a successor enjoys the annuity for the duration of his/her subsequent lifetime). He also considered joint life survival and contingent probabilities under his linear mortality hypothesis. He discussed single life and joint life expectations of life under his linear mortality hypothesis, using the trapezoidal rule to approximate the integrals. He also considered the general formula for the expectation of life for the longest of two lives. Here, he was obviously unaware of the earlier work of the Huygens brothers. De Moivre also considered the present value of contingent (or survivorship) assurances but his approach was in error; this was identified and corrected subsequently by Simpson.

Overall, the 1725 book laid the foundations of modern life insurance mathematics and can be regarded as one of the major landmarks in the development of the subject.

The book appeared in four separate editions. De Moivre also wrote the important text on probability "The Doctrine of Chances", which appeared in three editions: 1718, 1738 and a posthumous one in 1756. There is some overlap with material from "Annuities on Lives" appearing in the 1738 edition of the Doctrine with the 1756 edition effectively containing much of his work on life insurance mathematics.

The 1738 edition of the Doctrine contained de Moivre's first full full table of a_x values and some new material on successive lives (correcting an earlier error - we note that de Moivre's derivation of the formulae is rather artificial and is improved upon by the more direct proof given later by Simpson in his 1742 book) and on the value of annuities for children. Importantly, de Moivre included formulae for reversionary annuities involving up to four lives with complex last survivor statuses. He derived the formula for the present value of (or single premium for) a temporary life insurance benefit and calculated the value for a particular age and term using Halley's life table. This important result was ignored by the market which persisted with single premiums for one year policies that were not age dependent until 1762.

In a subsequent letter to William Jones, de Moivre gave an approximation to the value of a complete life annuity and provided an improved explanation (compared to the 1725 version) of the use of his piecewise linear life table hypothesis for calculating the value of annuities.

Following de Moivre, several books and tables on annuities were published in the decades after 1725, the most significant being Richard Hayes' book and the major contributions of Thomas Simpson.

Richard Hayes published the first book (1727) devoted solely to tables of annuities on lives, although he did not give any details of his methods of computation. The book offered no original ideas but one of the examples referred to what we would recognise as a whole life insurance policy paid for by level annual premiums. This idea was perfected later by James Dodson (1755) and it is not clear whether Dodson was aware of Hayes' earlier contribution.

Thomas Simpson's book of 1742 began with a favourable comment on de Moivre's "Annuities on Lives" and was largely built on de Moivre's. The structure and the expository problems were similar. But his presentation and explanations were clearer and more concise, he corrected the errors he had discovered in de Moivre's work, many of his proofs were more general and he made some important new contributions. Firstly, he constructed a life table based on the London bills of mortality in which he modified Smart's life table at ages under 25 because of Smart's failure to take migration into account: but the description of his methodology is too vague and unclear to follow (there are some helpful suggestions in Westergaard (1932)). Secondly, he used his life table for calculating tables of values of single life and joint life annuities for lives of the same age, adapting de Moivre's backward recursion formula to the case of joint life annuities. Thirdly, he devised computational rules (by trial and error) for calculating joint life annuities for different ages from the tabulated joint life annuity values. Fourthly, he demonstrated that de Moivre went too far in some of his simplifications and his formula for valuing joint life annuities was not sufficiently accurate. Thus, he provided the first satisfactory solution to the problem of calculating values of annuities for two and three lives. He also proved four general theorems on reversionary annuities involving two groups of three lives (involving the statuses $\overline{xyz}|\overline{abc}$, $xyz|abc$, $\overline{xyz}|abc$, $xyz|\overline{abc}$): from these formulae, all of de Moivre's results are easily found as well as some new results. He indicated how these results might be extended to apply to any number of lives and gives some examples of applications. Lastly, he considered the adjustments to be made to life annuity values if payments are to be made every half year or quarterly and derived the first order approximations in common use.

Simpson followed de Moivre closely in his discussion of contingent probabilities; but for the evaluation of the integrals involved he used the Newton-Cotes formulae for numerical integration rather than the trapezoidal rule. Instead of discussing expectations of life, he solved the related problem of defining the stationary population corresponding to a given life table and determining its size as the annual number of births multiplied by the expectation of life at age zero: Hald (1990).

Simpson's 1752 work dealt with a number of subjects in an accessible style. Part 6 considered annuities, providing some notable additions to his 1742 book and many new examples. He considered the single premium for a deferred annuity and the value of a continuous single life annuity (under de Moivre's linear hypothesis); he also provided a table of the value of annuities on two lives of different ages, a computational rule connecting the value of an annuity on three lives to the values of annuities on two lives and rules for calculating contingent assurances (under de Moivre's linear hypothesis) in which he corrected one of de Moivre's mistakes.

Simpson's contributions as a whole represent an important step forward. In particular, his clearly presented method of mathematical argument and proof became of great importance for the following generation of actuaries (Hald

(1990)).

De Moivre accused Simpson of plagiarism. There is a close similarity between the texts and further, according to Hald (1990), Simpson had previously plagiarised de Moivre's Doctrine of Chances in his 1740 book on the Law of Chance. On both occasions, Simpson referred to de Moivre in the preface but not in the text even though he used all of de Moivre's results. In his later editions, de Moivre gave Simpson the same treatment of omission. Simpson reacted to the accusation with a vigorous defence. It is worth noting Hald's comments that Simpson's various writings were controversial and "aroused many accusations of plagiarism". However, he was a successful author of elementary textbooks and it is clear that he did make important contributions to actuarial science in his own right. Simpson also made some important contributions to statistical theory, recognising the importance of the idea of observational error. To quote Stigler (1986), he took the "crucial step towards error".

Hald (1990) notes that de Moivre advocated using an approximation to the life table to facilitate calculations whereas Simpson based his calculations on an observed life table. The next generation of actuaries used Simpson's approach although they abandoned his life table (which referred to the general population of London rather than to annuitants or to insured lives) and based their calculations on better life tables derived from the observed experience of life insurance companies. At the same time, many attempts were made to follow up de Moivre's approach and, as we have seen, it was not until 1825 that Benjamin Gompertz succeeded in formulating a mortality hypothesis that would be widely accepted.

In the strong competition between de Moivre and Simpson, a comprehensive theory of insurance mathematics for valuing annuities was forged, and the necessary tools for practical application were thereby provided (Hald (1990)). It is noteworthy that much of the theory developed by de Moivre and Simpson referred to annuities. It is strange that neither took the step of adapting these ideas to set up a theory of life assurances. Perhaps, as Hald comments, "they did not feel the need for such a theory because they usually expressed the benefits in terms of annuities".

James Dodson's work of 1755 demonstrated how permanent life insurance could be operated with level annual premiums calculated on the basis of age at entry and how this level charge for an increasing risk would lead to the build up of a fund. This work is a major landmark and marks the birth of the ideas of scientific life insurance which underpin the subsequent development of this industry to the present day. Indeed, Dodson is often described as the "father of modern life insurance".

The translation of these principles into practice introduced new problems for which solutions had to be found if the life office was to remain financially sound. Formerly, it was necessary only to consider the office's finances one year at a time with short term contracts being issued. With premiums being fixed at the outset, and hence not adjustable during the term of the policy in the light of subsequent experience, it was essential to take a long view in estimating experience. Thus, these principles profoundly affected not only the development of life insurance contracts available to the public but also the need to ensure that the life office fully understood the long term financial consequences of issuing these new contracts. This led to the development of the actuarial theory and practice of life insurance, discussed in the next section.

Richard Price's book, *"Observations of Reversionary Payments"* first published in 1771, and eventually running to seven editions, was a further major work in this area. Price dealt with a number of important subjects. He constructed the Northampton life table based on deaths in 1735-70 and then in 1735-80 and used this as the basis of premium calculations (Dodson's scale of premiums was based on a life table constructed from the deaths in London in 1728-50. This period included two years with mortality rates higher than the average by about 25% leading to premium rates that were too high in commercial terms). In his methodology, Price followed Halley and assumed that the underlying population was stationary, making adjustments for migration and the deaths of immigrants. The Northampton life table was in practical use for many years, having been adopted by the Equitable for its premium basis. Price's book also discussed problems in life insurance mathematics involving life insurance policies paid for by single and annual premiums dependent on age at entry, contingent probabilities and reversionary annuities. He considered insurance and annuity portfolios and then analysed the financially unsound position of some recently established annuity societies selling reversionary annuities: as a result some of these societies were closed. He provided a commentary on the status of the Equitable Life Assurance Society and the Amicable Society. He also considered a scheme providing for old age pensions and set down proposals for a sickness insurance scheme (see later) and commented on the National Debt, as well as on many other matters.

The fourth edition (1783) of Richard Price's masterpiece is important because it contains his second Northampton life table based on the experience from 1735-80. With this life table, a set of tables of life contingency functions including assurance, annuity and annual premiums and expectations of life for the single life and joint life cases was also included. This life table and the derived financial and functions were to form the basis of much life insurance practice for the next century. Importantly, Price also constructed a life table for Sweden using a radix and a graduated set of inverse mortality rates following Wargentin's methodology: this represented an important breakthrough.