

# The History of Actuarial Science

Experience Studies and Estimation of Rates

Graduation of Decremental Rates

Edited by  
Steven Haberman and Trevor A. Sibbett



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# HISTORY OF ACTUARIAL SCIENCE



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*Edited by*  
STEVEN HABERMAN  
and  
TREVOR A. SIBBETT

VOLUME X

*Experience Studies and  
Estimation of Rates  
Graduation of  
Decremental Rates*

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T A B L E S  
O F  
*Interest, Discount, Annuities, &c.*

First published in the YEAR 1724,

B Y  
J O H N S M A R T,

A N D

NOW REVISED, ENLARGED, and IMPROVED,

B Y  
C H A R L E S B R A N D.

TO WHICH IS ADDED  
A N A P P E N D I X,  
CONTAINING  
Some OBSERVATIONS on the general PROBABILITY of LIFE.

---

Crescentem sequitur cura pecuniam  
Majorumque fames.

———— Bene est cui Deus obtulit  
Parcâ, quod fatis est, manu.

*Hor. Lib. 3, Ode 16.*

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L O N D O N :

Printed for T. LONGMAN, Pater-noster-Row; T. CADEL, in the Strand; and  
N. CONANT, Fleet-Street.

M DCC LXXX.

Charles Brand, 'Tables of Interest, Discount, Annuities &c. First published in the Year 1724, by John Smart, and now revised, enlarged, and improved by Charles Brand, 1780. To which is added an Appendix containing some observations on the general probability of life', London, T. Cadell and N. Conant, pp. 187-90.

Charles Brand published in the appendix to this work the first mortality data derived from the Amicable Society records. This company's mode of operation was not scientific at this time and had altered little from its start in 1706. Brand's calculations are the first attempt to calculate expectation of life from insurance company records and offer an interesting insight into the amount of data available and the actuarial approach of those days.

Compared with John Smart's original book, there is a new section of over fifty pages in Brand's revision giving examples of calculations for annuities upon lives and reversions. The interest tables are less accurate than the original tables of John Smart. Charles Brand also wrote *Treatise on Annuities and Assurances* (1775), but it contains no new matter.

Around 1811, the registrar of the Amicable Society Mr John Pensam prepared a life table from the data of the Society, which was not published, but which he said showed a significantly better mortality experience than that of the Northampton table. On 30 November 1841, Thomas Galloway, then Registrar of the Society published for members of the Society tables of the mortality experience from 5 April 1808 to 5 April 1841 on three bases, viz. that of the new entrants since 1808 when the Amicable Society adopted a sound scientific basis, the old members existing at 5 April 1808 and a combined experience.

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# A P P E N D I X.

*Containing some Observations, upon the general Probability of Life, according to Bills of Mortality for different Places, together with several Tables, for ascertained the Value of Annuities.*

THE probability of Life, to be deduced from Bills of Mortality, hath been considered by the curious Sir William Petty, in his Natural and; Political Observations on the Bills of Mortality of London, owned by Captain John Graunt: and since in a like Treatise on the Bills of Mortality of Dublin, as observed by Dr. Halley, in Philosophical Transactions, No. 196. " But the deduction from these Bills of Mortality," says he, " seemed even to their Authors, " to be defective: First, in that the Number, of the people were wanting; secondly, that the " Ages of the people dying was not to be had; and lastly, that both London and Dublin, by " reason of the great and casual accession of strangers who die therein, (as appeared in both, " by the great excess of the Funerals above the Births) rendered them incapable of being " standards for this purpose, which requires, if it were possible, that the people we treat of, " should not all be changed, but die where they were born, without any adventitious increase " from abroad, or decay by migration elsewhere." He farther adds, " This defect seems " in a great measure to be satisfied by the late curious Tables of the Bills of Mortality of the " city of Breslaw, lately communicated to the Royal Society by Mr. Jusstell, wherein both " the ages and sexes of all that die, are monthly delivered, and compared with the number " of the Births for five years last past, viz. 1687, 1688, 1689, 1690, and 1691, seeming to " be done with all the exactness and sincerity possible." Dr. Halley then proceeds to form his own Calculations upon those Observations, which he does with great accuracy and precision; but he afterwards expressed his doubts of what he had done, by saying, that " were this Cal- " culus founded upon the experience of a very great number of years, it would be very well " worth the while to think of methods for facilitating the computation of the value of two, " three, or more Lives:"—These remarks of Dr. Halley seem to have induced Mr. De Moivre to form an Hypothesis, by supposing the extremity of life to terminate with an age, which by taking the decrements of life in Arithmetic progression, might give answers to correspond nearly with the values in Dr. Halley's Tables.

We have likewise Tables of Observations published by the ingenious Mr. *Simpson*, upon 10 years Observations on the Bills of Mortality of *London*; — those Tables Mr. *Simpson* seemed very sensible were liable to objections, for in p. 7, of the Preface to his Book on the Doctrine of Annuities and Reversions, he supposes it will be observed in objection to his Tables, That *London* consists of too flux a body to admit of any certain measure for the probability of life, and that the accounts published by the parish Clerks are not to be depended upon; — the first objection he endeavours to remove, by saying, “ That tho’ the continual resort of people, from all parts, causes indeed a great increase, in the Bills of Mortality, it will no ways influence the values of the Annuities, thence deducted (if the numbers of persons coming up to town, at all ages be proportional, to the whole numbers of the living of the same ages) how far this reservation will assist, to remove this objection, must be left to those, who can form some idea of the accession and migression of the inhabitants; — to the second objection, he says, “ as to the difference that may arise, from any uncertainty or error, in the accounts of the Parish Clerks, it can be but very little, because if the age happens to be given in a little too high one time, there is the same chance of it being put down, much too low another;” — if this reasoning is to be adopted, there is an end to the necessity of having accurate returns, chance being to regulate their errors, by balancing them with those of equal variance.

We have also been favoured, in *Monsieur De Buffon’s* Natural History, with Tables, formed from the Observations of *Monsieur Du Pre de S. Mour*, of the French Academy, and there published by his authority; these Tables comprehended Observations on three Parishes in *Paris*, and twelve in the country; — from the nature of the Registers and Returns, upon which they have been formed, they seem less liable to those objections which have properly been attributed to the Bills of Mortality of *Breslaw* and *London*.

*Monsieur De Moivre*, in his Book on the Doctrine of Chances, has recommended a Register to be taken of Individuals, ascertaining their ages and occupations, in order to deduce from thence, some more certain grounds to obviate those difficulties which have justly been assigned to the imperfect and chimerical Returns, made as general Bills of Mortality; — this recommendation, as well as the defects, which appeared evident, from the manner in which the several Bills of Mortality have been kept, induced me to take up a Register of a number of Individuals, who were incorporated by her late Majesty QUEEN ANNE, Anno 1706, by the name of the “ *Amicable Society for a perpetual Assurance Office*.” This Society was then instituted in order to make provision for the Wives, Children, and other Relations of those who might become Members thereof; from the Register of the deaths of Members, I have taken the day on which each Member died, — and having the date when each of these became a Member, I have from thence computed the time each Member lived, and arranged them at the same time according to their respective ages, the Totals and Averages whereof have produced the Durations contained in the following Table.

T A B L E I.

Shewing the Durations of the Lives of 3826 Persons who have been Members of the AMICABLE SOCIETY, &c. and on whose Deaths Claims have been paid since the Institution in 1706 to the Year 1777, both inclusive, being a Period of 72 Years.

Ages.		Durations.		Ages.		Durations.		Ages.		Durations.	
		Years.	Decim. Parts.			Years.	Decim. Parts.			Years.	Decim. Parts.
12	}	15.7022		31	15.3369	42	14.8191	53	11.9013	}	
10				32	14.3315	43	13.5315	54	12.8410		
22	}	15.2301		33	13.4383	44	18.1835	55	10.8547	}	
23				31	15.1917	45	14.5613	56	8.2585		
24	}	14.5424		35	15.8246	46	15.1808	10		}	
25				36	16.5397	47	15.2821	60			
26	}	12.5121		37	14.8876	48	13.1452	62	}		
27				38	13.6493	49	15.0410	64			
28	}	12.1150		39	13.2849	50	15.6657	66	}	6.7671	
29				40	15.9506	51	15.0707	69			
30	}	15.9671		41	15.5561	52	16.4739	70	}		
											75

T A B L E II.

Shewing the Expectancy of Life, deduced from the foregoing Table of Durations.

Ages.		Expectancy.		Ages.		Expectancy.		Ages.		Expectancy.	
		Years.	Decim. Parts.			Years.	Decim. Parts.			Years.	Decim. Parts.
12	}	31.5941		31	27.1325	42	21.0991	53	12.1245	}	
10				32	26.5861	43	20.5178	54	11.1803		
22	}	31.0481		33	26.0763	44	20.0168	55	9.6267	}	
23				34	25.6029	45	19.1578	56	8.0128		
24	}	30.5131		35	25.0556	46	18.5406	10		}	
25				36	24.4732	47	17.8461	60			
26	}	29.5898		37	23.8530	48	17.1025	62	}		
27				38	23.3013	49	16.5422	64			
28	}	28.6325		39	22.8093	50	15.7208	66	}	6.7671	
29				40	22.3385	51	14.7390	69			
30	}	27.6980		41	21.7142	52	13.6827	70	}		
											75

\* There being but few Lives of those Ages, it became necessary to combine them, to form proper Averages.

† When the Society was first instituted, and until the year 1738, the Ages of those who became Members were not limited (as they now are) to between 12 and 43; but many instances happened afterwards, by exchanging older for younger Members, for those above the Age of 43.

This last Table of the Expectancy of Life, is formed by estimating the share of Life, which those of the succeeding ages would have enjoyed, had they been admitted into the Society, when of the given age;—it must be observed, that there were but few instances of Members admitted at the earlier and later periods of life, which has rendered it necessary to combine them, in order to form *one* average, which may more nearly correspond with the others in the true Expectancy; And as this Table gives an Expectancy of Life, agreeing so nearly with that contained in the Table taken from the Observations of *Monsieur Du Pre de S. Mour*, on three parishes in *Paris*, and twelve in the country, (and published by *Monsieur De Buffon* in his Natural History) I apprehend I shall be justified in forming a Table for the value of Annuities agreeable to those Observations, and considering the same as applicable to the value of *such* Lives as have been admitted into the Amicable Society, especially as those Observations comprehend an allowance for the risque which attends different periods of Life.

## T A B L E III.

*Shewing the Expectancy of Life, taken from Vol. I. of Monsieur de Buffon's Natural History.*

Age.	Expectancy.		Age.	Expectancy.		Age.	Expectancy.	
	Years. Decim. Parts.	$\frac{D}{100}$		Years. Decim. Parts.	$\frac{D}{100}$		Years. Decim. Parts.	$\frac{D}{100}$
1	33.0000	22	32.2333	44	19.7500	65	8.5000	
2	38.0000	23	31.8333	45	19.2500	66	8.0000	
3	40.0000	24	31.2500	46	18.7500	67	7.5000	
4	41.0000	25	30.7500	47	18.1667	68	7.0000	
5	41.5000	26	30.1667	48	17.6667	69	6.5833	
6	42.0000	27	29.5833	49	17.1667	70	6.1667	
7	42.2500	28	29.0000	50	16.5833	71	5.6667	
8	41.5000	29	28.5000	51	16.0000	72	5.3333	
9	40.8333	30	28.0000	52	15.5000	73	5.0000	
10	40.1667	31	27.5000	53	15.0000	74	4.7500	
11	39.5000	32	26.9167	54	14.5000	75	4.5000	
12	38.7500	33	26.2500	55	14.0000	76	4.2500	
13	38.0833	34	25.5833	56	13.4167	77	4.0833	
14	37.4167	35	25.0000	57	12.8333	78	3.9167	
15	36.7500	36	24.4167	58	12.2500	79	3.7500	
16	36.0000	37	23.8333	59	11.6667	80	3.5833	
17	35.3333	38	23.2500	60	11.0833	81	3.4167	
18	34.6667	39	22.6667	61	10.5000	82	3.2500	
19	34.0000	40	22.0833	62	10.0000	83	3.1667	
20	33.4167	41	21.5000	63	9.5000	84	3.0833	
21	32.9167	42	20.9167	64	9.0000	85	3.0000	
		43	20.3333					

A VIEW  
OF THE  
RISE AND PROGRESS  
OF THE  
**EQUITABLE SOCIETY,**

AND OF  
THE CAUSES WHICH HAVE CONTRIBUTED  
TO ITS SUCCESS.

WITH  
REMARKS ON SOME OF THE LATE MISREPRESENTATIONS  
RESPECTING  
THE RULES AND PRACTICE  
OF THE SOCIETY.

---

BY WILLIAM MORGAN, F. R. S.  
ACTUARY TO THE EQUITABLE SOCIETY.

---

**SECOND EDITION:**  
TO WHICH IS ADDED,  
**AN APPENDIX,**  
CONTAINING  
TABLES OF THE VALUES OF LIFE ANNUITIES AND ASSURANCES,  
COMPUTED FROM A CORRECTED TABLE OF THE PROBABILITIES  
OF LIFE IN THE SOCIETY.

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LONDON:  
PRINTED FOR  
LONGMAN, REES, ORME, BROWN, AND GREEN,  
PATERNOSTER-ROW.  
1829.

William Morgan, Appendix to the second edition of 'A View of the Rise and Progress of the Equitable Society and of the Causes which contributed to its success,' &c. London, Longman, Rees &c. (1828), pp. 67–75.

In the appendix here, William Morgan constructed a mortality table from the (sometimes sparse) data of the company and used it to calculate pure premiums. This is the first mortality table derived by a life insurance company transacting business on a scientific basis deduced from its own experience. The table was used in connection with the valuation of the Equitable Assurance as at 31 December 1829 and a comparison of the reserves with the usual Northampton table valuation revealed additional surplus of £700,000. Attempts at an experience table of mortality derived from very limited published information of Equitable Assurance had been made previously by Benjamin Gompertz 1825, Griffith Davies, 1825, and Charles Babbage, 1826. Davies's table gained considerable credence.

Joshua Milne successfully challenged inconsistencies in William Morgan's published data and reworked the conclusions derived from some of it – detail is given in the article 'Annuities', Vol. 3, *Encyclopaedia Britannica*, 7th edition, pp. 202–3 and 209–10 (1842). Morgan used the *number of deaths* with the *number of policies*, which resulted in much understated mortality since there were many lives with more than one policy. The treatment of lapses and surrenders was not explained.

Arthur Morgan, who succeeded William Morgan, published another, better table of Equitable mortality experience in 1834.

William Morgan, who in 1828 was getting old and had an inaccurate memory, retired in 1830.

## APPENDIX.

---

IN the account which was published in my Address to the General Court in the year 1800, of the probabilities of life in the Society, I had no other end in view, than to shew that they were so much higher than those in the table from which the premiums were computed, as to secure the Society against all danger from continuing those premiums in its future assurances. But I never considered those probabilities as affording sufficient *data* for the construction of a table of the values of Life Annuities *at all ages*, or thought that it would be prudent to found any important measures upon documents so scanty and imperfect. Having been regarded, however, as one of the many proofs of the increased longevity of mankind, which had of late become a very popular opinion, not only tables of Life Annuities, &c. have been computed from these defective materials, but the Society has been urged to adopt them as the basis of all its future operations. The following Tables, though computed from a corrected account of the probabilities of life in the Society, are still deduced from documents, particularly in the earlier parts of life, much too defective to be depended upon,

and have been constructed merely for the purpose of effecting some calculations directed to be made by the last General Court of the Society.

From the Fifth of these Tables it appears, if the computations are *wholly* made from the *Equitable Experience*, that is, if the single and annual premiums are both computed from it, that an assurance acquires a greater value after a given time, than if the computations were made from the Northampton Table, and, therefore, that if the state of the Society were to be determined by the former, the surplus would be rather *less* than if it were to be computed from the latter. The value of an assurance is generally increased, in proportion as the probabilities of life are higher from which it is computed; and, therefore, in determining the state of a Society, the apparent surplus will be less in this case, than if the computations were made from a table giving lower probabilities of life. But if the single premiums are computed from one set of observations, and the annual premiums from another, the results must necessarily be erroneous in proportion as these observations differ.\* Thus, in the case of the Equitable Society, if the single premiums are computed from the probabilities of life in the Society, and the annual premiums, as at present, from the Northampton Table, the

\* It can hardly be necessary to observe, that the value of an assurance is expressed by the difference between the single premium and the product of the annual premium multiplied into the value of the Life Annuity.

assurances, so far from acquiring any value during the earlier part of their existence, will be in *debt* to the Society \*, so that a person aged 35, after having paid his premiums for 10 years, for an assurance of 1000*l.* on his life, instead of *receiving* any thing for his interest on surrendering his policy, ought to *pay* the Society 13*l.* 12*s.* though, in reality, it is fairly worth 111*l.* 12*s.*† As most of the assurances are originally made between the ages of 25 and 50, and the probabilities of life in the Society at these ages are much higher than in the Northampton Table, it is evident, that, if this method of computation were adopted, the greater part of the assurances made since the year 1816 would appear to have acquired little or no value, and, therefore, that the apparent surplus would be greatly increased above its real amount, and afford a pretence for such a distribution of it as would be highly injurious to the new members, and ultimately to the security of the Society.

Suppose 3000 members, each assured in 1000*l.* at the common age of 30, to consist of three classes of 1000 each. Those of the first class to have been assured 15 years ago; those of the second, 10 years ago; and those of the third class, 5 years ago. Computing the values of

\* See Table V.

† Two eminent counsellors have given it as their opinion, that computations of this kind may be legally *made*; but we have not been favoured with their opinion whether any person is legally obliged to abide by the result of them.

these assurances, by combining the Equitable Experience with the Northampton Table, in the manner now proposed, the amount of these values will not exceed 70,000*l.*; but if they are computed by the Northampton Table, it will exceed 890,000*l.*, and if *wholly* by the Equitable Experience, 418,000*l.* As the single premiums, or, which is the same thing, the values of the reversions of the sums assured, are diminished, by computing from the Equitable Experience, the persons selling their additions, like those surrendering their policies, will receive less than the real value for them. But to prevent this loss, it has been suggested that it might be proper, on these occasions, to use the Northampton Table, by which means, a member, whose additions are stated in the decennial investigation to be worth only 43*l.*, may receive 50*l.* for the present value of them *after that account has been settled*, and the member who ought to *pay* the Society on discontinuing his assurance 87*l.* may receive 53*l.* for the value of his interest in it, so that after giving a fallacious statement of the surplus in the decennial investigation, the evil is to be aggravated by a measure still more ruinous and unjust. But I cannot believe that any person, upon due consideration, would advise the Society to deviate into such a course of error and delusion, and, regardless of all knowledge on the subject, to forsake those sound principles of honour and justice, which can alone secure to it that high degree of credit and prosperity by which it is at present distinguished.

TABLE I.

Showing the Probabilities of the Duration of Human Life,  
according to the Experience of the Equitable Society.

Age.	Living.	Decr.	Age.	Living.	Decr.	Age.	Living.	Decr.
20	3000	21	46	2354	32	72	966	72
21	2979	22	47	2322	32	73	894	72
22	2957	23	48	2290	33	74	822	72
23	2934	23	49	2257	33	75	750	72
24	2911	23	50	2224	36	76	678	69
25	2888	23	51	2188	39	77	609	66
26	2865	23	52	2149	40	78	543	61
27	2842	23	53	2109	42	79	482	59
28	2819	23	54	2067	44	80	423	57
29	2796	24	55	2023	47	81	366	54
30	2772	24	56	1976	49	82	312	51
31	2748	24	57	1927	52	83	261	50
32	2724	24	58	1875	54	84	211	43
33	2700	25	59	1821	57	85	168	37
34	2675	25	60	1764	59	86	131	31
35	2650	25	61	1705	60	87	100	25
36	2625	25	62	1645	61	88	75	19
37	2600	26	63	1584	63	89	56	14
38	2574	26	64	1521	65	90	42	11
39	2548	26	65	1456	66	91	31	9
40	2522	26	66	1390	68	92	22	7
41	2496	27	67	1322	69	93	15	6
42	2469	27	68	1253	71	94	9	5
43	2442	28	69	1182	72	95	4	3
44	2414	29	70	1110	72	96	1	1
45	2385	31	71	1038	72			

TABLE II.

Showing the Expectations of Human Life, from 20 to 96, deduced from the Mortality among the Members of the Equitable Society.

Ages.	Expectations.	Ages.	Expectations.
20	41.11	59	14.19
21	40.40	60	13.63
22	39.70	61	13.09
23	39.00	62	12.55
24	38.31	63	12.01
25	37.61	64	11.49
26	36.82	65	10.99
27	36.20	66	10.48
28	35.49	67	9.99
29	34.78	68	9.69
30	34.08	69	9.06
31	33.37	70	8.61
32	32.66	71	8.17
33	31.95	72	7.75
34	31.24	73	7.33
35	30.53	74	6.93
36	29.82	75	6.55
37	29.10	76	6.20
38	28.39	77	5.83
39	27.67	78	5.49
40	26.95	79	5.12
41	26.23	80	4.76
42	25.51	81	4.42
43	24.79	82	4.10
44	24.07	83	3.81
45	23.36	84	3.60
46	22.66	85	3.39
47	21.96	86	3.21
48	21.26	87	3.05
49	20.57	88	2.90
50	19.80	89	2.71
51	19.18	90	2.45
52	18.52	91	2.14
53	17.86	92	1.81
54	17.22	93	1.43
55	16.58	94	1.05
56	15.96	95	0.750
57	15.36	96	0.500
58	14.77		

TABLE III.

Showing the Value of an Annuity on a Single Life, according to the Probabilities of Life in the Equitable Society, computed at Three per Cent.

Age.	Annuity.	Age.	Annuity.	Age.	Annuity.
20	21.572	46	14.895	72	6.1037
21	21.376	47	14.565	73	5.7938
22	21.181	48	14.212	74	5.4911
23	20.980	49	13.853	75	5.1997
24	20.772	50	13.481	76	4.9254
25	20.566	51	13.114	77	4.6519
26	20.352	52	12.778	78	4.3725
27	20.132	53	12.412	79	4.0772
28	19.953	54	12.045	80	3.7815
29	19.725	55	11.652	81	3.4903
30	19.493	56	11.287	82	3.2293
31	19.253	57	10.921	83	2.9823
32	19.055	58	10.561	84	2.7937
33	18.810	59	10.200	85	2.6201
34	18.555	60	9.846	86	2.4618
35	18.292	61	9.553	87	2.3124
36	18.020	62	9.264	88	2.1852
37	17.741	63	8.968	89	2.0131
38	17.459	64	8.646	90	1.7948
39	17.166	65	8.337	91	1.5010
40	16.854	66	7.994	92	1.1903
41	16.540	67	7.682	93	.8390
42	16.222	68	7.367	94	.5363
43	15.893	69	7.051	95	.2427
44	15.581	70	6.7342	96	
45	15.244	71	6.4179		

TABLE IV.

Showing the Single and Annual Premiums for assuring 100*l.* on a Single Life, and computed from the Probabilities of Life in the Equitable Society.

Age.	Single Premium.	Annual Premium.	Age.	Single Premium.	Annual Premium.
20	34.246	1.517	46	53.688	3.378
21	34.816	1.556	47	54.648	3.511
22	35.368	1.595	48	55.677	3.600
23	35.970	1.636	49	56.513	3.805
24	36.575	1.680	50	57.799	3.992
25	37.175	1.724	51	58.874	4.171
26	37.798	1.770	52	59.852	4.344
27	38.438	1.819	53	60.918	4.542
28	38.960	1.859	54	61.986	4.752
29	39.624	1.912	55	63.131	4.990
30	40.299	1.967	56	64.194	5.225
31	40.998	2.025	57	65.259	5.474
32	41.575	2.078	58	66.308	5.735
33	42.288	2.135	59	67.359	6.014
34	43.130	2.201	60	68.389	6.306
35	43.796	2.270	61	69.244	6.561
36	44.590	2.344	62	70.084	6.828
37	45.401	2.423	63	70.955	7.120
38	46.230	2.504	64	71.884	7.452
39	47.075	2.591	65	72.783	7.795
40	47.984	2.687	66	73.782	8.204
41	48.909	2.788	67	74.691	8.603
42	49.824	2.893	68	75.630	9.039
43	50.782	3.006	69	76.550	9.508
44	51.690	3.117	70	77.474	10.017
45	52.671	3.243			





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**INVESTIGATION**

**MORTALITY**

**IN THE**

**I N D I A N A R M Y .**

**BY**

**W. S. B. WOOLHOUSE, F.R.A.S., &c.,**

**ACTUARY TO THE NATIONAL LOAN FUND LIFE ASSURANCE SOCIETY.**



**LONDON:**  
**A. H. BAILY, 83, CORNHILL.**

**1839.**

Wesley Stoker Barker Woolhouse, 'Investigation of Mortality in the Indian Army,' London, A. H. Baily (1839), pp. 1–9 and graphs.

Wesley Stoker Barker Woolhouse, in this work, introduced the concepts of exposure time and of conventional exposed to risk formulae. Although exposure had been discussed previously, see e.g. Thomas Watkin's 'Letter to Dr Brackenridge' (1760), Woolhouse put the idea into proper mathematical form and considered fractional exposures in respect of entrants and exits. This is the start of a more formal detailed scientific approach to evaluation of data before the start of analysis and graduation.

Woolhouse was well known for his high standard of mathematics and his considerable capacity for attention to detail.

Mortality rates of members of armed forces were very high in the nineteenth century – often 5% p.a. and in some stations up to 10% p.a. in time of war and wars were frequent. Much information is available, e.g. article 'Army' in Vol. 1 of C. Walford's *The Insurance Cyclopaedia* (1871) (also includes information on crude army sickness rates. Between 4 and 5% of an army could be constantly incapacitated by sickness and smallpox was rife.) A. H. Smee and Thomas G. Ackland's 'On the Assurance Risks incident to Professional Military and Naval Lives; &c.', *Journal of the Institute of Actuaries*, Vol. 35 (1899), gives data drawn from military campaigns over a period of 30 years mainly in respect of European wars, but also includes the US Army experience in the American Civil War (1861–6). The book form of this *Journal of the Institute of Actuaries* article, *On the Assurance of Lives liable to service in Military and Naval Operations at Home and Abroad*, London, C. & E. Layton (1890), gives extensive references.

# LAW OF MORTALITY

IN THE

## INDIAN ARMY.

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THE efficiency, stability, and general advantages of a Life Assurance business, depend very materially on the extent and value of the experience from which the table of mortality is calculated, since this table is to be abided by on all occasions, as furnishing the original data for the calculation of premiums, and the valuation of every species of life contingencies. Every collection of facts which may essentially contribute to our knowledge on the subject of human mortality, is, therefore, of the utmost importance to Life Assurance Societies. I am not aware of the publication of any Table, to be relied upon, showing the law of mortality observed amongst the army in India, although the benefits of life assurance are peculiarly applicable to that class of persons; but a work has been recently published, involving abundant and valuable materials for the calculation of such a table, and entitled "Alphabetical List of the Officers of the India Army; with the dates of their respective promotion, retirement, resignation, or death, whether in India or in Europe, from the year 1760 to the year 1834 inclusive, corrected to September 30, 1837; compiled and edited by Messrs. Dodwell and Miles, East-India Army Agents, 69, Cornhill." This volume, which gives a precise account of more than twelve thousand lives, over a period of nearly 80 years, bears evidence of great care and accuracy; and, at the desire of the Directors of the National Loan Fund Society, who are ever anxious to secure in all cases the best information that can be derived, on the mortality of human

life, I have, with the inviting and excellent means so afforded, gone into a calculation of the law of mortality experienced in the Indian Army.

I shall first give an account of the process followed in the calculation, and take the opportunity of considering the question in its general form, as the method may be useful on other occasions:—

In any proposed current year of age, let us denote,  $N$ , the number in existence at the commencement of the period;

$n$ ,  $r$ ,  $D$ , } the number of { new members } in the course of the  
retirements } period;  
deaths }

$R$ , the number remaining in existence on the list, at the termination of the period;

$\epsilon$ , the total number in the course of the period that pass the point of commencement of the proposed year of age, or the total number that enter at the commencement; and,  $N'$ ,  $n'$ ,  $r'$ ,  $D'$ ,  $R'$ ,  $\epsilon'$ , the same numbers for the next higher year of age, &c.

Then the estimated number that have existed in the proposed year of age, exclusive of the survivors  $\epsilon$  of the preceding year, is evidently,

$$\omega = (N+n) - (R+r) \quad \text{---} \quad (1),$$

and the total number that reach the end of the year of age is  $\epsilon + \omega - D$ . Hence, as this must be the number that enter on the next succeeding year, we have

$$\epsilon' = \epsilon + \omega - D$$

similarly,  $\epsilon'' = \epsilon' + \omega' - D'$

$$\epsilon''' = \epsilon'' + \omega'' - D''$$

&c.      &c.

These relations being carried on from the first year, we obviously have,

$$\epsilon = \Sigma (\omega - D) \quad \text{---} \quad (2),$$

in which  $\Sigma$  includes the values for all the preceding years of age.

Now the terms composing  $\omega$  are distributed through the year at indiscriminate periods in the year of age, and may, with large numbers, be practically considered as having reference to the middle of the year, or as having undergone only half the chance of death in one year. The *risk* may therefore be regarded as that of  $\epsilon + \frac{\omega}{2}$  persons entering upon the year and continuing exposed to its termination, and as  $D$  is the number of deaths that occur in the same year, the number of survivors may be considered to be  $\epsilon + \frac{\omega}{2} - D$ .

Thus the proportionate mortality in the year is  $\frac{D}{\epsilon + \frac{\omega}{2}}$ , and the portion that survive or the probability of surviving the year, is

$$a = \frac{\epsilon + \frac{\omega}{2} - D}{\epsilon + \frac{\omega}{2}} \quad \dots \quad (3).$$

By substituting the values of  $\epsilon, \omega$ , we have,

$$\begin{aligned} \text{Risk} &= \Sigma \left\{ (N+n) - (R+r) - D \right\} + \frac{(N+n) - (R+r)}{2} \\ \text{Survivors} &= \Sigma \left\{ (N+n) - (R+r) - D \right\} + \frac{(N+n) - (R+r)}{2} - D \\ a &= \frac{\Sigma \left\{ (N+n) - (R+r) - D \right\} + \frac{(N+n) - (R+r)}{2} - D}{\Sigma \left\{ (N+n) - (R+r) - D \right\} + \frac{(N+n) - (R+r)}{2}} \end{aligned}$$

which express the values in terms of the original data ; but the expressions (1), (2), (3), point out the mode of calculation in a more simple form, and admit of a very convenient practical arrangement, as we shall afterwards find.

Let the successive probabilities throughout the several years of age, calculated in this manner, be denoted by  $a, a_1, a_2, a_3$ , &c. ; and suppose a table of mortality of the usual form to commence with *unity* as the number living. Then the proportion surviving the first year =  $a$  ; the proportion surviving the 2d year =  $a a_1$  ; the proportion surviving the 3d year =  $a a_1 a_2$  ; and similarly the proportion which survive  $m$  years =  $a a_1 a_2 \dots a_{m-1}$ .

So far it will be found that the results, as they immedi-

ately come out from the calculation, however great the number of lives may be, will generally, as a series, involve many small and irregular fluctuations, which it will be absolutely necessary to rectify before the table can be fit for final use. The usual practice in the formation of tables of mortality, is to distribute the values of  $a$ ,  $a_1$ ,  $a_2$ , &c. into as regular a progression as possible, by small corrections ascertained by means of averages of certain groups or sets, as, for instance, in quinquennial periods. There does not appear to be any very serious objection to this method of adjustment, except in the care and patience required in its *proper* employment. I would, however, recommend another expedient for effecting the required object, which possesses the advantage of being a direct process, and of doing the least possible violence to the results actually arising out of the experience. In the first place, calculate the numbers living at periods of four years, *viz.*

$$\begin{aligned} a_0 &= 1.00000 \\ a_1 &= a_0(a_1, a_2, a_3) \\ a_2 &= a_1(a_1, a_2, a_3) \\ a_{11} &= a_1(a_1, a_2, a_{11}) \\ &\text{\&c.} \quad \text{\&c.} \end{aligned}$$

Then difference these values out algebraically to the fourth order of differences, and apply such corrections as may be found necessary to compensate the principal irregularities of the fourth differences, observing that the addition of each *unit* to any quantity, imparts the variations  $+1, -4, +6, -4, +1$  to the five differences directly opposite to it, and that the subtraction of each *unit* produces the same variations only with contrary signs. In this process, it is advisable to apply the necessary variations to the differences immediately after each correction is assigned to a quantity, then to proceed to another adjustment, and so to continue, until the differences are rendered either sufficiently uniform in their progression, or sufficiently small in their magni-

tudes. The corrected quantities being then differenced, will check with the corrected fourth differences, and thus secure against the occurrence of an error in this process of preparatory adjustment. This being done, the values for the intermediate years of age are to be deduced by interpolation, using the differences as far as the fourth order. Perhaps the simplest way of effecting this is to determine, first the middle values  $a_{10}, a_{20}, a_{30}$ , &c., to re-difference the quantities  $a_{10}, a_{20}, a_{30}, a_{40}$ , &c., and again to determine, in the same manner as before, the middle values  $a_1, a_2, a_3$ , &c. which will complete the series, and exhibit the required law of mortality.

The practical method of executing this purpose, which may be done in red ink on the difference paper, is to form values of  $\Delta_1 - \frac{1}{2} \left( \frac{\text{sum of } \Delta_2}{8} - \frac{\text{sum of } \Delta_4}{16} \right)$ ; in which  $\Delta_1$  denotes the first difference standing opposite the place of a quantity sought; also sum of  $\Delta_2$  and sum of  $\Delta_4$  denote the sums of the pairs of second and fourth differences which cross the same horizontal line: then half of the values so found will be the corrections to apply successively to the immediately preceding values of  $a$  to obtain the required series of intermediate values.

The volume of Messrs. Dodwell and Miles would have been perfect for the present investigation had the respective ages on going out as Cadets been specified; but as the average age at that time has been found to be very nearly eighteen years, and as the number of lives is so great, I adopted eighteen as the age common to all at the middle of the year in which they received their appointment. The ages at death and retirement or discontinuance, as well as the ages of those on service at the end of the period, were thence readily deduced by means of the dates, and inserted with the pen, in the printed volume, at the same time distinguishing the deaths by a conspicuous mark.

The numbers for the respective ages were derived by going through the list first with the discontinuances, and then with the deaths, and successively drawing a vertica

stroke with the pen opposite the corresponding age in the Table, observing to bind each four strokes with the next one drawn horizontally, in order to afford greater facility afterwards in the counting. By dividing the work into portions, each occupying twenty pages of the printed volume, and in each instance checking the resulting totals with a general counting from the list without reference to the ages, the accuracy of the calculation becomes more satisfactorily secured; and any portion of it can, if necessary, be easily subjected to an after examination, without occasioning any inconvenience. Also, if in the investigation of any large experience, it should be desirable to classify the lives in any proposed manner, it will only be necessary in each case to prepare a sufficient number of vertical columns, opposite the ages, for the insertion of the respective classes.

In general, it will be requisite to give separate Tables of  $N+n$ ,  $D$ ,  $R$  and  $r$ ; but in the case before us,  $N=0$  for every age;  $n=$ the total number for the age of 18, and  $0$  for all other ages; also in collecting together the numbers of the several ages, I have thought it unnecessary to distinguish the officers on service at the end of the period from those who, by previous retirement, resignation, or removal, had been discontinued from the list; that is, instead of giving  $R$  and  $r$  separately, I have at once formed  $R+r$ , since by dividing the lives into only two classes, those who have died and those who have discontinued from the experience without death, the calculation is much simplified, less subject to error, and quite adequate to the purposes of this enquiry. But in discussing the experience of a progressing society, it will be requisite to distinguish the numbers composing  $R$  as essential data to be preserved for the calculation of a future experience, since it is evident that the  $R$  at the end of one experience will in reality be the  $N$  of that which consecutively follows it. The following table exhibits the several numbers obtained for each Presidency:

## No. I.

ENUMERATION of DEATHS, and DISCONTINUANCES, at  
each Age, and in each Presidency.

Year of Age.	BENGAL.		MADRAS.		BOMBAY.	
	Deaths. <i>D.</i>	Discon- tinuances. <i>R + r.</i>	Deaths. <i>D.</i>	Discon- tinuances. <i>R + r.</i>	Deaths. <i>D.</i>	Discon- tinuances. <i>R + r.</i>
18	95	70	16	13	6	1
19	142	102	109	44	45	25
20	142	90	114	44	78	20
21	132	107	139	63	57	36
22	141	84	118	70	59	28
23	132	105	132	116	62	36
24	144	83	105	55	42	32
25	93	76	108	61	52	27
26	117	89	108	52	36	29
27	106	147	90	71	42	48
28	104	183	91	83	48	77
29	109	123	93	115	32	67
30	102	217	90	135	36	51
31	83	157	77	84	26	50
32	80	159	65	80	30	44
33	70	85	63	53	23	27
34	66	113	50	78	20	42
35	69	106	62	116	25	66
36	56	110	47	106	22	47
37	65	99	42	55	14	52
38	58	74	49	44	23	25
39	52	27	40	27	14	13
40	47	26	34	16	8	3
41	44	19	57	20	13	3
42	47	33	38	19	16	12
43	52	46	26	44	16	14
44	37	57	31	33	14	22
45	38	65	30	46	6	24
46	42	66	21	38	16	19
47	32	80	16	38	5	20
48	26	79	17	41	9	9
49	22	44	16	33	9	7
50	22	62	10	20	7	7
51	21	60	16	27	3	13
52	12	42	11	10	2	12
	2600	3085	2131	1950	916	1008

No. I.—*continued.*

ENUMERATION of DEATHS, and DISCONTINUANCES, at each Age, and in each Presidency.

Year of Age.	BENGAL.		MADRAS.		BOMBAY.	
	Deaths. D.	Discon- tinuances. R + r.	Deaths. D.	Discon- tinuances. R + r.	Deaths. D.	Discon- tinuances. R + r.
	2600	3085	2131	1950	916	1008
53	15	11	5	8	1	2
54	17	10	9	1	4	4
55	12	14	3	21	2	5
56	14	20	3	10	3	14
57	10	23	4	7	3	—
58	15	6	4	8	—	10
59	12	8	3	9	1	7
60	7	14	4	4	—	4
61	4	13	1	4	—	2
62	5	4	2	4	—	—
63	7	—	2	5	2	—
64	11	—	3	2	1	—
65	2	1	3	7	—	—
66	7	—	3	2	—	—
67	4	—	3	2	—	—
68	5	—	3	1	2	3
69	7	—	3	—	1	—
70	5	—	1	—	—	—
71	4	—	1	—	—	—
72	4	3	1	—	1	—
73	4	4	3	2	2	1
74	3	2	1	2	—	—
75	3	4	2	—	—	2
76	3	2	3	2	—	1
77	1	3	—	2	—	—
78	2	—	2	—	—	—
79	—	—	—	—	—	—
80	—	—	1	2	—	—
81	—	—	—	—	1	—
82	3	—	—	—	—	—
83	1	—	1	—	—	—
84	—	3	—	—	—	—
85	—	—	—	—	—	—
	2787	3230	2205	2055	940	1063
		2787		2205		940
Totals	—	6017	—	4260	—	2003

Having prepared these data, the most natural course to be pursued, was, either to calculate separate tables for each of the three Presidencies, or, which would be more convenient, to unite all together in one general experience for the whole of India ; but to ascertain whether the latter arrangement could be justified by the circumstance of a like mortality pervading the three Presidencies, it was necessary first to calculate the rough table for each separately.

By an examination of these rough preliminary tables, the particulars of which would unnecessarily extend the limits of this paper, I was induced to treat Bengal by itself, and to unite Madras and Bombay, for the following reasons :—

1. Because the numbers for Bengal differed very materially from those of Madras and Bombay.

2. Because the results for Madras and Bombay exhibited a marked similarity and a near approximation to each other.

3. Because the numbers for Bombay alone were too circumscribed to claim a sufficient degree of confidence in their value as a separate experience.

4. Because the total number of lives for Madras and Bombay being so nearly the same as that for Bengal, every element of calculation for the three Presidencies conjoined, or for the whole of India, may be inferred with tolerable accuracy, by taking the mean or half sum of those in the two departments thus assigned.

In the following specimen for Bengal, which will serve to show how the numbers are next combined, I have indicated the arrangement, which it would be convenient to adopt with the experience of a Society.

A Committee of Actuaries, 'Tables exhibiting the Law of Mortality deduced from the combined experience of seventeen Life Assurance Offices, embracing 83,905 policies; of which 40,616 are distinguished by denoting the sex of the lives assured and classing them into Town, Country, and Irish Assurances,' London, J. King (1843).

A committee of experienced and eminent actuaries held a meeting on 19 March 1838 and resolved to contribute data to afford means of determining the law of mortality which prevailed among assured lives. Ansell, Davies, Downes, Gompertz, Kirkpatrick, Milne, Rainbow and Woolhouse supervised the returns with Christie and Ingall as secretaries. 13 proprietary and 4 mutual offices pooled their data. Among these, the Equitable Society contributed 21,398 lives and the Amicable 4,618 lives. These tables are the result. They are the first mortality tables from pooled life office data. The 58 offices contributing to the finance each received a copy. Jenkin Jones seized the initiative and republished the main results with comment and monetary tables in *A Series of Tables of Annuities and Assurances calculated from a New Rate of Mortality amongst Assured Lives* (1843).

The tables were not greatly used in Great Britain, but were widely used elsewhere, particularly on the continent of Europe. Elizur Wright used the table in 1859 in Massachusetts, USA. They are known variously as the '17 Offices' Tables', the 'Experience Tables' and the 'Actuaries' Tables'. These tables were still being used by insurance companies in some countries early in the twentieth century.

E. J. Farren, 'Life Contingency Tables, Part I. – The Chances of Premature Death, and the value of Selection among Assured Lives' (1850), used the data to examine selection in the calendar year after entry and concluded that initial selection was merely of a temporary character.

T A B L E S

EXHIBITING THE

LAW OF MORTALITY,

DEDUCED FROM THE

COMBINED EXPERIENCE OF SEVENTEEN

LIFE ASSURANCE OFFICES,

EMBRACING 83,905 POLICIES;

OF WHICH

40,616 ARE DISTINGUISHED BY DENOTING THE SEX OF THE LIVES ASSURED,

AND BY CLASSING THEM INTO

TOWN, COUNTRY, AND IRISH

*Assurances.*

LONDON:

PRINTED BY J. KING, COLLEGE HILL, CITY.

1843.

MALE LIVES,

TABLE A (1).—Showing, out of the number of Assurances effected in each Current Year of Age, the respective numbers,

Current Age at Entry	YEARS OF DURATION																																		
	0			1			2			3			4			5			6			7			8			9			10				
	Dis.	Died	Ex.	Dis.	Died	Ex.	Dis.	Died	Ex.	Dis.	Died	Ex.	Dis.	Died	Ex.	Dis.	Died	Ex.	Dis.	Died	Ex.	Dis.	Died	Ex.	Dis.	Died	Ex.	Dis.	Died	Ex.	Dis.	Died	Ex.		
1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..		
2	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..		
3	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..		
4	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..		
5	..	..	..	..	..	..	..	..	..	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..		
6	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..		
7	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..		
8	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..		
9	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..		
10	..	..	5	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	2	..	..	..	..	..	..	..	..	..	..	..	..	..	..		
11	..	..	1	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..		
12	..	..	1	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..		
13	..	..	..	2	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..		
14	..	..	1	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	
15	..	..	2	3	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	
18	..	..	2	4	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	
17	..	..	2	3	..	10	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	
18	..	..	4	4	..	6	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	
19	..	1	6	5	..	3	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	
20	1	..	3	18	2	10	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	
21	4	..	8	57	1	4	..	10	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	
22	3	..	21	30	2	17	..	18	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	
23	5	..	25	26	5	18	..	20	4	16	..	16	1	12	..	9	..	14	4	2	9	..	2	..	7	..	..	7	..	..	..	..	..		
24	3	..	19	45	1	17	..	22	1	14	..	12	..	8	12	1	7	..	9	1	12	4	1	11	..	2	1	11	..	3	1	..	..		
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26	3	2	26	53	4	31	..	26	5	23	..	23	1	22	..	5	1	15	..	10	..	20	6	..	8	5	3	15	..	3	..	22	2	2	
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33	4	..	40	57	4	44	..	22	7	35	..	22	4	35	..	9	..	40	..	13	4	32	8	4	23	12	2	14	..	7	3	23	1	3	
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39	5	1	40	41	5	39	..	31	5	43	..	13	..	30	..	13	3	26	..	3	3	26	7	6	11	9	3	25	..	3	1	10	1	1	
40	3	2	46	64	3	34	..	20	7	40	..	15	3	29	..	16	3	11	..	4	1	15	4	4	26	13	2	24	..	2	3	12	5	2	
Carry over	70	27	805	977	68	632	..	581	73	644	..	394	46	524	..	203	39	441	..	191	45	420	141	46	379	206	47	353	..	75	27	347	44	31	
	0	1	2	3	4	5	6	7	8	9	10																								

