MEASURE OF NONCOMPACTNESS, FIXED POINT THEOREMS, AND APPLICATIONS

Edited by
S. A. Mohiuddine, M. Mursaleen
and Dragan S. Djordjević

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Measure of Noncompactness, Fixed Point Theorems, and Applications

The theory of the measure of noncompactness has proved its significance in various contexts, particularly in the study of fixed point theory, differential equations, functional equations, integral and integrodifferential equations, optimization, and others. This edited volume presents the recent developments in the theory of the measure of noncompactness and its applications in pure and applied mathematics. It discusses important topics such as measures of noncompactness in the space of regulated functions, application in nonlinear infinite systems of fractional differential equations, and coupled fixed point theorem.

Key Highlights:

- Explains numerical solution of functional integral equation through coupled fixed point theorem, measure of noncompactness, and iterative algorithm
- Showcases applications of the measure of noncompactness and Petryshyn's fixed point theorem functional integral equations in Banach algebra
- Explores the existence of solutions of the implicit fractional integral equation via extension of the Darbo's fixed point theorem
- Discusses best proximity point results using measure of noncompactness and its applications
- Includes solvability of some fractional differential equations in the holder space and their numerical treatment via measures of noncompactness

This reference work is for scholars and academic researchers in pure and applied mathematics.

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Contents

Preface

This book features original chapters on the theory of measure of noncompactness, fixed point theorems and their involvement in finding the solution of differential and integral equations in the classical and fractional sense, as well as in obtaining the Darbo-type fixed point theorems. Each chapter describes the problem of current importance, summarizes ways of their solution and possible applications methods, and improves the current understanding pertaining to measure of noncompactness and fixed point theorems. The presentation of the chapters is clear and self-contained throughout the entire book. The list of chapters is arranged alphabetically by the last name of the first author of each chapter.

Chapter 1 is devoted to some generalized fixed theorems containing Darbo and other researchers via fixed point and coupled fixed point theorems, and measure of noncompactness. Application of results is given by an example in a functional integral equation. For validity of this work, an iterative algorithm to approximate the solution of the above problem with an acceptable accuracy is constructed.

Chapter 2 establishes the existence of solution of nonlinear functional integral equations in Banach algebra by using the concept of Petryshyn's fixed point theorem and measure of noncompactness. Some interesting examples to examine the validity of our results are provided.

Chapter 3 is devoted to prove some fixed theorems in Banach space via the measure of noncompactness and proves the existence of solutions to an implicit functional equation involving a fractional integral with respect to a few functions which generalizes the Riemann-Liouville fractional integral and the Hadamard fractional integral. In order to illustrate the result, an example is constructed with the help of an integral equation.

Chapter 4 presents a brief survey of cyclic (noncyclic) condensing operators and utilizes them to investigate the existence of the best proximity points (pair) by using the measure of noncompactness. Also, some applications in the existence of optimum solutions for differential equations are described.

Chapter 5 investigates a class of Volterra functional integral equations with fractional-order and Hadamard-type fractional integrals. The main objective is to establish the existence of solutions for these equations using Petryshyn's fixed point theorem in Banach algebra. The findings of this chapter provide important insights into the behavior of fractional-order Volterra functional integral equations and contribute to the ongoing research in this field.

Chapter 6 is devoted to generalized Darbo fixed point theorem via measure of noncompactness, $\alpha-$admissible function, and coupled fixed point theorem. Applications of the proved theorems and results are given by some examples. Therefore, the numerical solution of quadratic integral equations system is given via an iterative convergent algorithm with an acceptable accuracy.

Chapter 7 proves best proximity point (pair) theorems for newly defined cyclic and noncyclic contractive operators. By using these results, the optimum solution of an integral equation is obtained including illustrative examples.

Chapter 8 is devoted to obtaining the general solution and investigating alternative H-U stability results for the finite dimensional additive functional equation in a modular space via direct method. Moreover, the stability results for the same additive functional equation in modular space by using fixed point method with the help of Fatou property are investigated.

Chapter 9 explores the Ulam stability results of the quadratic functional equation in Banach space and multi-normed space by means of two different approaches, namely, Hyers and fixed point techniques.

Chapter 10 proves the existence of a solution of some nonlinear integral equations with the help of common fixed point theorems satisfying the generalized contractive condition in complete metric space for two pairs of weakly compatible mappings.

Chapter 11 introduces q-Pascal difference sequence spaces $c(P(q)\nabla) := c_{P(q)\nabla}$ and $c_0(P(q) := (c_0)_{P(q)\nabla}$. This chapter also obtains the Schauder bases and determines Alpha-(α-), Beta-(β-) and Gamma-(γ-) duals of $c(P(q)\nabla)$ and $c_0(P(q)\nabla)$. The final section has been devoted to compactness via Hausdorff measure of noncompactness on the space $c_0(P(q)\nabla)$.

Editors

S. A. Mohiuddine is a full professor of Mathematics at King Abdulaziz University, Jeddah, Saudi Arabia. An active researcher, he has coauthored five books: *Convergence Methods for Double Sequences and Applications* (Springer, 2014), *Advances in Summability and Approximation Theory* (Springer, 2018), *Soft Computing Techniques in Engineering, Health, Mathematical and Social Sciences* (CRC Press, Taylor & Francis Group, 2021), *Approximation Theory, Sequence Spaces and Applications* (Springer, 2022), *Sequence Space Theory with Applications* (CRC Press, Taylor & Francis Group, 2023), and a number of chapters in several edited books, and has contributed more than 170 research papers in various leading journals. He is the referee of many scientific journals and a member of the editorial board of various scientific journals, international scientific bodies and organizing committees. He has visited several international universities including Imperial College London, UK. He was a guest editor of a number of special issues in some journals. His research interests are in the fields of sequence spaces, statistical convergence, matrix transformation, measures of noncompactness and approximation theory. His name has appeared in the list of World's Top 2% Scientists (for the four consecutive years 2020–2023) prepared by Stanford University, California, through Scopus data provided by Elsevier.

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Dragan Djordjević was born on March 17, 1970. He received his Bachelor's (1992), Master's (1996) and Ph.D. (1998) degrees in mathematics at the University of Niš, Serbia. Dragan Djordjević has worked at the University of Niš continuously since 1992, and from 2006 as a full professor at the Faculty of Sciences and Mathematics. He taught several courses in various levels of education (Functional analysis, Operator theory, Complex analysis, Measure and integration, and Algebras of operators). He was Dean of the Faculty of Sciences and Mathematics, and a member of several scientific boards of the University of Niš (Senat, Scientific council for sciences and mathematics), Ministry of Education, Science and Technological Development of Serbia (Scientific council for mathematics, mechanics and computer science, Head of the council), as well as in the Republic of Serbia (National council for higher education). He was the head and a member of several scientific projects financed by the Ministry of Education, Science and Technological Development of Serbia. He is a member and vice president of the association Serbian Scientific Mathematical Society. Dragan Djordjević supervised 10 Ph.D. theses (eight in Niš, one in Belgrade and one in Skopje). He published 130 scientific papers, including 104 papers in JCR journals. He collaborated with more than 30 authors, most of whom were his students. His results are cited approximately 1400 times, and the Hirsch index is 22. Dragan Djordjević served as the invited speaker at several mathematical conferences (Korea, Japan, Turkey). He also organized conferences in mathematical or functional analysis in Niš and Novi Sad. He also reviewed papers for many mathematical journals, including *Mathematical Reviews*. His scientific work includes operator theory, with some aspects of functional analysis, linear algebra, general algebra and numerical analysis. His main focus is on operators on Banach and Hilbert spaces, their ordinary and generalized invertibility properties, Fredholm and spectral theory, as well as applications to general ring theory and operator equations in various structures. He also has some results related to majorization and doubly stochastic operators, theory of spaces with indefinite inner products and Hilbert C*-modules. Dragan Djordjević is a member of boards of some mathematical journals: *Filomat* (one of editors-in-chief), *Functional Analysis, Approximation and Computation* (founder and editor-in-chief), *Applied Mathematics and Computer Science* (founder and one of the editors-in-chief), *Advances in the Theory of Nonlinear Analysis and its Application* (member of the board), *Publications de l'Institute Mathematique* (member of the board) and *Advances in Operator Theory* (member of the board).

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Chapter 1

The existence and numerical solution of functional integral equation via coupled fixed point theorem, measure of noncompactness and iterative algorithm

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1.1 Introduction and auxiliary facts

Darbo's fixed point theorem [11] is a very important generalization of Schauder's fixed point theorem and includes the existence part of Banach's fixed point theorem. To generalize some fixed theorems and coupled fixed point theorems with the help of measure of noncompactness, we introduce notations, definitions, and preliminary facts which are used throughout this chapter. Denote by $\mathbb{R}$ the set of real numbers and put $\mathbb{R}_+ = [0, +\infty)$. Let $(E, \|\cdot\|)$ be a real Banach space with zero element 0. Let $\overline{B}(x, r)$ denote the closed ball centered at x with radius r. The symbol $\overline{B}_r$ stands for the ball $\overline{B}(0, r)$. For X, a nonempty subset of E, we denote by $\overline{X}$ and $ConvX$ the closure and the closed convex hull of X, respectively. Moreover, let us denote by $\mathfrak{M}_E$ the family of nonempty bounded subsets of E and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact sets. We use the following definition of the measure of noncompactness given in [8].

Definition 1.1.1. *A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ is said to be a measure of noncompactness in E if it satisfies the following conditions:*

(1^0) The family $ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $ker\mu \subset \mathfrak{N}_E$,

(2^0) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$,

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(3^0*)* $\mu(\bar{X}) = \mu(X)$,

(4^0*)* $\mu(ConvX) = \mu(X)$,

(5^0*)* $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ *for* $\lambda \in [0,1]$,

(6^0*)* *If* (X_n) *is a sequence of closed sets from* m_E *such that* $X_{n+1} \subset X_n (n = 1, 2, ...)$ *and if* $\lim\limits_{n \longrightarrow \infty} \mu(X_n) = 0$, *then the set* $X_\infty = \bigcap_{n=1}^{\infty} X_n$ *is nonempty.*

The family $ker\mu$ defined in axiom (1^0) is called the kernel of the measure of noncompactness μ.

One of the properties of the measure of noncompactness is $X_\infty \in ker\mu$. Indeed, from the inequality $\mu(X_\infty) \leq \mu(X_n)$ for $n = 1, 2, 3, ...$, we infer that $\mu(X_\infty) = 0$. Further, facts concerning measures of noncompactness and their properties may be found in [3, 6, 7, 8, 9, 15].

Theorem 1.1.2. *(Schauder's theorem) [1] Let* C *be a closed and convex subset of a Banach space* E*. Then every compact and continuous map* $T : C \to C$ *has at least one fixed point.*

In the following, we state a fixed point theorem of Darbo type proved by Banaś and Goebel [8].

Theorem 1.1.3. *Let* C *be a nonempty, closed, bounded and convex subset of the Banach space* E *and* $T : C \to C$ *be a continuous mapping. Assume that there exists a constant* $k \in [0,1)$ *such that* $\mu(TX) \leq k\mu(X)$ *for any nonempty subset of* C*. Then* T *has a fixed point in the set* C*.*

1.2 New fixed point theorem for $\alpha-$admissible functions

Definition 1.2.1. *Let* Ω *be a nonempty subset of Banach space* E*. Let* $F : \Omega \longrightarrow \Omega$ *be a map and* $\alpha : \mathbb{R}_+ \to \mathbb{R}$ *be a function. Then we say that operator* F *is a* $\alpha-$*admissible if for any subset* X *of* Ω

$$\alpha(\mu(X)) \geq 1 \Longrightarrow \alpha(\mu(FX)) \geq 1. \tag{1.1}$$

where μ *is an arbitrary measure of noncompactness.*

We start this section with the first of our main theorems.

Theorem 1.2.2. *Let* Ω *be a nonempty, bounded, closed and convex subset of a Banach space* E*,* μ *is an arbitrary measure of noncompactness and* $F : \Omega \longrightarrow \Omega$ *be a map. Suppose that the following conditions are satisfied:*

(1) F is $\alpha-$admissible;

(2) F is continuous;

(3) there exists $\Omega_0 \subseteq \Omega$ such that $\alpha(\mu(\Omega_0)) \geq 1$;

(4) for any nonempty subset X of Ω

$$\alpha(\mu(X))\psi(\mu(FX)) \leq \varphi(\mu(X)), \tag{1.2}$$

where $\psi, \varphi : [0, \infty) \to [0, \infty)$ are continuous, $\varphi(t) < \psi(t)$ for each $t > 0$ and $\varphi(0) = \psi(0) = 0$. Then F has at least one fixed point, and the set of all fixed points of F in Ω is compact.

Proof. Let $\Omega_0 \subseteq \Omega$ such that $\alpha(\mu(\Omega_0)) \geq 1$. Define a sequence $\{\Omega_n\}$ by $\Omega_n = Conv(F\Omega_{n-1})$, $n \geq 1$. $F\Omega_0 = F\Omega \subseteq \Omega = \Omega_0, \Omega_1 = Conv(F\Omega_0) \subseteq \Omega = \Omega_0$, therefore by continuing this process, we have

$$\Omega_0 \supseteq \Omega_1 \supseteq \Omega_2 \supseteq \cdots . \tag{1.3}$$

If there exists an integer $N \geq 0$ such that $\mu(\Omega_N) = 0$, then Ω_N is relatively compact and since $F\Omega_N \subseteq Conv(F\Omega_N) = \Omega_{N+1} \subseteq \Omega_N$. Thus theorem 1.1.2 implies that F has a fixed point. So we assume that $\mu(\Omega_n) \neq 0$ for $n \geq 0$. So, from (6.13) and (2^0), we deduce that $\{\mu(C_n)\}$ is a non-negative non increasing sequence and consequently there exists $\delta \geq 0$ such that

$$\lim_{n\to\infty} \mu(C_n) = \delta.$$

We claim that $\delta = 0$. On the contrary, assume that

$$\lim_{n\to\infty} \mu(C_n) = \delta > 0. \tag{1.4}$$

By condition (3), we have $\alpha(\mu(\Omega_0)) \geq 1$. Since, by hypothesis, F is $\alpha-$admissible, we obtain

$$\alpha(\mu(\Omega_1)) = \alpha(\mu(conv(F(\Omega_0)))) = \alpha(\mu(F(\Omega_0))) \geq 1 \Longrightarrow \alpha(\mu(F(\Omega_1))) \geq 1,$$
$$\alpha(\mu(\Omega_2)) = \alpha(\mu(conv(F(\Omega_1)))) = \alpha(\mu(F(\Omega_1))) \geq 1 \Longrightarrow \alpha(\mu(F(\Omega_2))) \geq 1.$$

By induction, we get

$$\alpha(\mu(\Omega_n)) \geq 1 \ for \ all \ n \in \mathbb{N} \cup \{0\}.$$

Now, by (1.2), we have

$$\begin{aligned} \psi(\mu(\Omega_{n+1})) &= \psi(\mu(Conv(F(\Omega_n)))) = \psi(\mu(F(\Omega_n))) \qquad (1.5) \\ &\leq \alpha(\Omega_n)\psi(\mu(F(\Omega_n))) \\ &\leq \varphi(\mu(\Omega_n)). \qquad (1.6) \end{aligned}$$

Since

$$\varphi(\mu(\Omega_n)) \leq \psi(\mu(\Omega_n)), \tag{1.7}$$

then from (1.5) and (1.7), we get

$$\psi(\mu(\Omega_{n+1})) \leq \varphi(\mu(\Omega_n)) \leq \psi(\mu(\Omega_n)). \tag{1.8}$$

Since φ and ψ are continuous, then from (1.4) and (1.8), we have

$$\psi(\delta) = \lim_{n\to\infty} \psi(\mu(\Omega_n)) = \lim_{n\to\infty} \varphi(\mu(\Omega_n)) = \varphi(\delta),$$

and so $\delta = 0$, a contradiction. Thus

$$\lim_{n\to\infty} \mu(\Omega_n) = 0.$$

Since $\Omega_{n+1} \subseteq \Omega_n$ and $F(\Omega_n) \subseteq \Omega_n$ for all $n \geq 1$, then from (A_6), $\Omega_\infty = \bigcap_{n=1}^{\infty} \Omega_n$ is a nonempty convex closed set, invariant under T, and belongs to $Ker\mu$. Therefore Theorem 1.2.2 completes the proof. □

From Theorem 1.2.2 if the function $\alpha : \mathbb{R}_+ \to \mathbb{R}$ is such that $\alpha(t) = 1$ for all $t \in \mathbb{R}$, we deduce the following theorem.

Theorem 1.2.3. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and $F : \Omega \longrightarrow \Omega$ be a continuous operator satisfying*

$$\psi(\mu(FX)) \leq \varphi(\mu(X))$$

for any subset X of Ω where μ is an arbitrary measure of noncompactness on E and $\psi, \varphi : [0,\infty) \to [0,\infty)$ are continuous, $\varphi(t) < \psi(t)$ for each $t > 0$ and $\psi(0) = \varphi(0) = 0$. Then F has at least one fixed point.

From Theorem 1.2.2 if the function $\alpha : \mathbb{R}_+ \to \mathbb{R}$ is such that $\alpha(t) = 1$ for all $t \in \mathbb{R}$, $\psi(t) = \psi_1(t)$ and $\varphi(t) = \psi_1(t) - \varphi_1(t)$ for each $t \in \mathbb{R}_+$ where $\psi_1, \varphi_1 : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are continuous and increasing function such that $\psi_1(t) = \varphi_1(t) = 0$ if and only if $t = 0$, we deduce the following theorem.

Theorem 1.2.4. *[4] Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let $F : \Omega \to \Omega$ be continuous operator satisfying*

$$\psi_1(\mu(FX)) \leq \psi_1(\mu(X)) - \varphi_1(\mu(X)),$$

for any nonempty $X \subseteq \Omega$, where μ is an arbitrary measure of noncompactness and $\psi_1, \varphi_1 : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are continuous and increasing function such that $\psi_1(t) = \varphi_1(t) = 0$ if and only if $t = 0$. Then F has at least one fixed point in Ω.

From Theorem 1.2.2 if the function $\alpha : \mathbb{R}_+ \to \mathbb{R}$ is such that $\alpha(t) = 1$ for all $t \in \mathbb{R}$, $\psi(t) = t$ and $\varphi(t) = \beta(t)t$ which $\beta \in S$, we deduce the following theorem.

Theorem 1.2.5. *[2] Let C be a nonempty, bounded, convex and closed subset of a Banach space E and let $T : C \longrightarrow C$ be a continuous operator such that*

$$\mu(TX) \leq \beta(\mu(X))\mu(X)$$

for any subset X of C where μ is an arbitrary measure of noncompactness on E and $\beta \in S$. Then T has at least one fixed point.

From Theorem 1.2.2 if the function $\alpha : \mathbb{R}_+ \to \mathbb{R}$ is such that $\alpha(t) = 1$ for all $t \in \mathbb{R}$, $\psi(t) = t$, we deduce the following theorem.

Theorem 1.2.6. *[4] Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let $T : \Omega \longrightarrow \Omega$ be a continuous mapping such that*

$$\mu(TX) \leq \varphi(\mu(X)) \tag{1.9}$$

for any nonempty subset X of Ω where μ is an arbitrary measure of noncompactness and $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is non decreasing and continuous from the right. Then T has at least one fixed point in the set Ω.

The following corollary gives us a fixed point theorem with a contractive condition of integral type.

Corollary 1.2.7. *Let C be a nonempty, bounded, closed and convex subset of a Banach space E, $k \in (0, 1)$ and $T : C \to C$ be a continuous operator such that for any $X \subseteq C$ one has*

$$\int_0^{\mu(T(X))} f(s)\ ds \leq k \int_0^{\mu(X)} f(s)\ ds,$$

where μ is an arbitrary measure of noncompactness and $f : [0, \infty) \to [0, \infty)$ is a Lebesgue-integrable mapping which is summable (i.e., with finite integral) on each compact subset of $[0, \infty)$, non-negative and such that for each $\epsilon > 0$, $\int_0^\epsilon f(s)\ ds > 0$. Then T has at least one fixed point in C.

Proof. Let $\alpha(t) = 1$, $\psi(t) = \int_0^t f(s)\ ds$ and $\varphi(t) = k\ G(t)$ and apply Theorem 1.2.2. □

1.3 New fixed point theorems

In this section, we introduce a new notion of a contraction and establish new results for such mappings.

Recently Onsod et al. [19] introduced the notion $\widetilde{\Theta}$, the set of all the functions $\widetilde{\theta} : (0, +\infty) \to (1, +\infty)$ satisfying the following conditions:

$(\widetilde{\theta_1})$ $\widetilde{\theta}$ is nondecreasing and continuous;

$(\widetilde{\theta_2})$ $\inf_{t\in(0,\infty)} \widetilde{\theta}(t) = 1$.

Also, let S denote the class of real functions $\beta : [0, +\infty) \to [0, 1)$ satisfying the condition

$$\beta(t_n) \longrightarrow 1 \, implies \, t_n \longrightarrow 0.$$

Now, we are ready to state and prove our main result.

Definition 1.3.1. *Let $\mathcal{G}$ be a nonempty subset of Banach space $\mathcal{B}$. Let $\mathcal{H} : \mathcal{G} \longrightarrow \mathcal{G}$ be a function. Then we say that operator $\mathcal{H}$ is a $\theta-$contraction, if there exist $\theta \in \widetilde{\Theta}$ and $k \in (0, 1)$ and $\beta \in S$ such that for any subset G of $\mathcal{G}$ with $\mu(G) > 0$,*

$$\frac{1}{2}\mu(\mathcal{H}(G)) < \mu(G)) \Longrightarrow \theta(\mu(\mathcal{H}(G))) \le [\theta(\beta(\mu(G))\mu(G))]^k \tag{1.10}$$

where μ is an arbitrary measure of noncompactness.

Theorem 1.3.2. *Let $\mathcal{G}$ be a nonempty, bounded, closed and convex subset of a Banach space $\mathcal{B}$ and $\mathcal{H} : \mathcal{G} \to \mathcal{G}$ be a continuous operator and $\theta-$contraction. Then $\mathcal{H}$ has at least one fixed point in $\mathcal{G}$.*

Proof. Let $G_0 \subseteq \mathcal{G}$. Define a sequence $\{G_n\}$ by $G_n = Conv(\mathcal{H}G_{n-1})$, $n \ge 1$. $\mathcal{H}G_0 = \mathcal{H}G \subseteq G = G_0, G_1 = Conv(\mathcal{H}G_0) \subseteq G = G_0$, therefore by continuing this process, we have

$$G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n \supseteq G_{n+1} \supseteq \cdots . \tag{1.11}$$

If there exists a natural number N such that $\mu(G_N) = 0$, then G_N is compact. In this case, Theorem 1.2.2 implies that $\mathcal{H}$ has a fixed point. So we assume that $\mu(G_n) > 0$ for $n = 0, 1, 2, ...\cdot$ Therefore ,

$$\frac{1}{2}\mu(\mathcal{H}(G_n)) < \mu(\mathcal{H}(G_n)) = \mu(G_{n+1}) \le \mu(G_n)), \forall n \ge 1. \tag{1.12}$$

Hence from (1.12), for all $n \ge 1$, we have

$$\begin{aligned} \theta(\mu(G_{n+1})) &= \theta(\mu(Conv(\mathcal{H}(G_n)))) = \theta(\mu(\mathcal{H}(G_n))) && (1.13)\\ &\le [\theta(\beta(\mu(G_n))\mu(G_n)]^k \\ &< [\theta(\mu(G_n))]^k \\ &< [\theta(\mu(G_{n-1}))]^{k^2} \\ &\vdots \\ &< [\theta(\mu(G_0))]^{k^{n+1}} && (1.14) \end{aligned}$$

Taking $n \longrightarrow \infty$, we obtain

$$\theta(\mu(G_{n+1})) \longrightarrow 1.$$