

MEASURE OF Noncompactness, fixed Point Theorems, and Applications

Edited by S. A. Mohiuddine, M. Mursaleen and Dragan S. Djordjević





Measure of Noncompactness, Fixed Point Theorems, and Applications

The theory of the measure of noncompactness has proved its significance in various contexts, particularly in the study of fixed point theory, differential equations, functional equations, integral and integrodifferential equations, optimization, and others. This edited volume presents the recent developments in the theory of the measure of noncompactness and its applications in pure and applied mathematics. It discusses important topics such as measures of noncompactness in the space of regulated functions, application in nonlinear infinite systems of fractional differential equations, and coupled fixed point theorem.

Key Highlights:

- Explains numerical solution of functional integral equation through coupled fixed point theorem, measure of noncompactness, and iterative algorithm
- Showcases applications of the measure of noncompactness and Petryshyn's fixed point theorem functional integral equations in Banach algebra
- Explores the existence of solutions of the implicit fractional integral equation via extension of the Darbo's fixed point theorem
- Discusses best proximity point results using measure of noncompactness and its applications
- Includes solvability of some fractional differential equations in the holder space and their numerical treatment via measures of noncompactness

This reference work is for scholars and academic researchers in pure and applied mathematics.



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Preface

This book features original chapters on the theory of measure of noncompactness, fixed point theorems and their involvement in finding the solution of differential and integral equations in the classical and fractional sense, as well as in obtaining the Darbo-type fixed point theorems. Each chapter describes the problem of current importance, summarizes ways of their solution and possible applications methods, and improves the current understanding pertaining to measure of noncompactness and fixed point theorems. The presentation of the chapters is clear and self-contained throughout the entire book. The list of chapters is arranged alphabetically by the last name of the first author of each chapter.

Chapter 1 is devoted to some generalized fixed theorems containing Darbo and other researchers via fixed point and coupled fixed point theorems, and measure of noncompactness. Application of results is given by an example in a functional integral equation. For validity of this work, an iterative algorithm to approximate the solution of the above problem with an acceptable accuracy is constructed.

Chapter 2 establishes the existence of solution of nonlinear functional integral equations in Banach algebra by using the concept of Petryshyn's fixed point theorem and measure of noncompactness. Some interesting examples to examine the validity of our results are provided.

Chapter 3 is devoted to prove some fixed theorems in Banach space via the measure of noncompactness and proves the existence of solutions to an implicit functional equation involving a fractional integral with respect to a few functions which generalizes the Riemann-Liouville fractional integral and the Hadamard fractional integral. In order to illustrate the result, an example is constructed with the help of an integral equation.

Chapter 4 presents a brief survey of cyclic (noncyclic) condensing operators and utilizes them to investigate the existence of the best proximity points (pair) by using the measure of noncompactness. Also, some applications in the existence of optimum solutions for differential equations are described.

Chapter 5 investigates a class of Volterra functional integral equations with fractional-order and Hadamard-type fractional integrals. The main objective is to establish the existence of solutions for these equations using Petryshyn's fixed point theorem in Banach algebra. The findings of this chapter provide important insights into the behavior of fractional-order Volterra functional integral equations and contribute to the ongoing research in this field.

Preface

Chapter 6 is devoted to generalized Darbo fixed point theorem via measure of noncompactness, α -admissible function, and coupled fixed point theorem. Applications of the proved theorems and results are given by some examples. Therefore, the numerical solution of quadratic integral equations system is given via an iterative convergent algorithm with an acceptable accuracy.

Chapter 7 proves best proximity point (pair) theorems for newly defined cyclic and noncyclic contractive operators. By using these results, the optimum solution of an integral equation is obtained including illustrative examples.

Chapter 8 is devoted to obtaining the general solution and investigating alternative H-U stability results for the finite dimensional additive functional equation in a modular space via direct method. Moreover, the stability results for the same additive functional equation in modular space by using fixed point method with the help of Fatou property are investigated.

Chapter 9 explores the Ulam stability results of the quadratic functional equation in Banach space and multi-normed space by means of two different approaches, namely, Hyers and fixed point techniques.

Chapter 10 proves the existence of a solution of some nonlinear integral equations with the help of common fixed point theorems satisfying the generalized contractive condition in complete metric space for two pairs of weakly compatible mappings.

Chapter 11 introduces q-Pascal difference sequence spaces $c(P(q)\nabla) := c_{P(q)\nabla}$ and $c_0(P(q) := (c_0)_{P(q)\nabla}$. This chapter also obtains the Schauder bases and determines Alpha- $(\alpha$ -), Beta- $(\beta$ -) and Gamma- $(\gamma$ -) duals of $c(P(q)\nabla)$ and $c_0(P(q)\nabla)$. The final section has been devoted to compactness via Hausdorff measure of noncompactness on the space $c_0(P(q)\nabla)$.

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Chapter 1

The existence and numerical solution of functional integral equation via coupled fixed point theorem, measure of noncompactness and iterative algorithm

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1.1 Introduction and auxiliary facts

Darbo's fixed point theorem [11] is a very important generalization of Schauder's fixed point theorem and includes the existence part of Banach's fixed point theorem. To generalize some fixed theorems and coupled fixed point theorems with the help of measure of noncompactness, we introduce notations, definitions, and preliminary facts which are used throughout this chapter. Denote by \mathbb{R} the set of real numbers and put $\mathbb{R}_+ = [0, +\infty)$. Let $(E, \|\cdot\|)$ be a real Banach space with zero element 0. Let $\overline{B}(x, r)$ denote the closed ball centered at x with radius r. The symbol \overline{B}_r stands for the ball $\overline{B}(0, r)$. For X, a nonempty subset of E, we denote by \overline{X} and ConvX the closure and the closed convex hull of X, respectively. Moreover, let us denote by \mathfrak{M}_E the family of nonempty bounded subsets of E and by \mathfrak{N}_E its subfamily consisting of all relatively compact sets. We use the following definition of the measure of noncompactness given in [8].

Definition 1.1.1. A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

(1⁰) The family $ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $ker\mu \subset \mathfrak{N}_E$,

 $(2^0) \ X \subset Y \Rightarrow \mu(X) \le \mu(Y),$

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(3⁰)
$$\mu(X) = \mu(X),$$

$$(4^0) \ \mu(ConvX) = \mu(X),$$

(5⁰) $\mu(\lambda X + (1-\lambda)Y) \le \lambda \mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$,

(6⁰) If (X_n) is a sequence of closed sets from m_E such that $X_{n+1} \subset X_n$ (n = 1, 2, ...) and if $\lim_{n \to \infty} \mu(X_n) = 0$, then the set $X_{\infty} = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

The family $ker\mu$ defined in axiom (1⁰) is called the kernel of the measure of noncompactness μ .

One of the properties of the measure of noncompactness is $X_{\infty} \in ker\mu$. Indeed, from the inequality $\mu(X_{\infty}) \leq \mu(X_n)$ for n = 1, 2, 3, ..., we infer that $\mu(X_{\infty}) = 0$. Further, facts concerning measures of noncompactness and their properties may be found in [3, 6, 7, 8, 9, 15].

Theorem 1.1.2. (Schauder's theorem) [1] Let C be a closed and convex subset of a Banach space E. Then every compact and continuous map $T : C \to C$ has at least one fixed point.

In the following, we state a fixed point theorem of Darbo type proved by Banaś and Goebel [8].

Theorem 1.1.3. Let C be a nonempty, closed, bounded and convex subset of the Banach space E and $T: C \to C$ be a continuous mapping. Assume that there exists a constant $k \in [0, 1)$ such that $\mu(TX) \leq k\mu(X)$ for any nonempty subset of C. Then T has a fixed point in the set C.

1.2 New fixed point theorem for α -admissible functions

Definition 1.2.1. Let Ω be a nonempty subset of Banach space E. Let $F : \Omega \longrightarrow \Omega$ be a map and $\alpha : \mathbb{R}_+ \to \mathbb{R}$ be a function. Then we say that operator F is a α -admissible if for any subset X of Ω

$$\alpha(\mu(X)) \ge 1 \Longrightarrow \alpha(\mu(FX)) \ge 1. \tag{1.1}$$

where μ is an arbitrary measure of noncompactness.

We start this section with the first of our main theorems.

Theorem 1.2.2. Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E, μ is an arbitrary measure of noncompactness and F: $\Omega \longrightarrow \Omega$ be a map. Suppose that the following conditions are satisfied:

- (1) F is α -admissible;
- (2) F is continuous;
- (3) there exists $\Omega_0 \subseteq \Omega$ such that $\alpha(\mu(\Omega_0)) \ge 1$;
- (4) for any nonempty subset X of Ω

$$\alpha(\mu(X))\psi(\mu(FX)) \le \varphi(\mu(X)), \tag{1.2}$$

where $\psi, \varphi : [0, \infty) \to [0, \infty)$ are continuous, $\varphi(t) < \psi(t)$ for each t > 0 and $\varphi(0) = \psi(0) = 0$. Then F has at least one fixed point, and the set of all fixed points of F in Ω is compact.

Proof. Let $\Omega_0 \subseteq \Omega$ such that $\alpha(\mu(\Omega_0)) \geq 1$. Define a sequence $\{\Omega_n\}$ by $\Omega_n = Conv(F\Omega_{n-1}), n \geq 1$. $F\Omega_0 = F\Omega \subseteq \Omega = \Omega_0, \Omega_1 = Conv(F\Omega_0) \subseteq \Omega = \Omega_0$, therefore by continuing this process, we have

$$\Omega_0 \supseteq \Omega_1 \supseteq \Omega_2 \supseteq \cdots . \tag{1.3}$$

If there exists an integer $N \geq 0$ such that $\mu(\Omega_N) = 0$, then Ω_N is relatively compact and since $F\Omega_N \subseteq Conv(F\Omega_N) = \Omega_{N+1} \subseteq \Omega_N$. Thus theorem 1.1.2 implies that F has a fixed point. So we assume that $\mu(\Omega_n) \neq 0$ for $n \geq 0$. So, from (6.13) and (2⁰), we deduce that { $\mu(C_n)$ } is a non-negative non increasing sequence and consequently there exists $\delta \geq 0$ such that

$$\lim_{n \to \infty} \mu(C_n) = \delta.$$

We claim that $\delta = 0$. On the contrary, assume that

$$\lim_{n \to \infty} \mu(C_n) = \delta > 0. \tag{1.4}$$

By condition (3), we have $\alpha(\mu(\Omega_0)) \geq 1$. Since, by hypothesis, F is α -admissible, we obtain

$$\begin{aligned} \alpha(\mu(\Omega_1)) &= \alpha(\mu(conv(F(\Omega_0)))) = \alpha(\mu(F(\Omega_0))) \ge 1 \Longrightarrow \alpha(\mu(F(\Omega_1))) \ge 1, \\ \alpha(\mu(\Omega_2)) &= \alpha(\mu(conv(F(\Omega_1)))) = \alpha(\mu(F(\Omega_1))) \ge 1 \Longrightarrow \alpha(\mu(F(\Omega_2))) \ge 1. \end{aligned}$$

By induction, we get

$$\alpha(\mu(\Omega_n)) \ge 1 \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

Now, by (1.2), we have

$$\psi(\mu(\Omega_{n+1})) = \psi(\mu(Conv(F(\Omega_n)))) = \psi(\mu(F(\Omega_n)))$$
(1.5)

$$\leq \alpha(\Omega_n)\psi(\mu(F(\Omega_n))) \leq \varphi(\mu(\Omega_n)).$$
(1.6)

4 Measure of Noncompactness, Fixed Point Theorems, and Applications

Since

$$\varphi(\mu(\Omega_n)) \le \psi(\mu(\Omega_n)), \tag{1.7}$$

then from (1.5) and (1.7), we get

$$\psi(\mu(\Omega_{n+1})) \le \varphi(\mu(\Omega_n)) \le \psi(\mu(\Omega_n)).$$
(1.8)

Since φ and ψ are continuous, then from (1.4) and (1.8), we have

$$\psi(\delta) = \lim_{n \to \infty} \psi(\mu(\Omega_n)) = \lim_{n \to \infty} \varphi(\mu(\Omega_n)) = \varphi(\delta),$$

and so $\delta = 0$, a contradiction. Thus

$$\lim_{n \to \infty} \mu(\Omega_n) = 0.$$

Since $\Omega_{n+1} \subseteq \Omega_n$ and $F(\Omega_n) \subseteq \Omega_n$ for all $n \ge 1$, then from (A_6) , $\Omega_{\infty} = \bigcap_{n=1}^{\infty} \Omega_n$ is a nonempty convex closed set, invariant under T, and belongs to $Ker\mu$. Therefore Theorem 1.2.2 completes the proof.

From Theorem 1.2.2 if the function $\alpha : \mathbb{R}_+ \to \mathbb{R}$ is such that $\alpha(t) = 1$ for all $t \in \mathbb{R}$, we deduce the following theorem.

Theorem 1.2.3. Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and $F : \Omega \longrightarrow \Omega$ be a continuous operator satisfying

$$\psi(\mu(FX)) \le \varphi(\mu(X))$$

for any subset X of Ω where μ is an arbitrary measure of noncompactness on E and $\psi, \varphi : [0, \infty) \to [0, \infty)$ are continuous, $\varphi(t) < \psi(t)$ for each t > 0 and $\psi(0) = \varphi(0) = 0$. Then F has at least one fixed point.

From Theorem 1.2.2 if the function $\alpha : \mathbb{R}_+ \to \mathbb{R}$ is such that $\alpha(t) = 1$ for all $t \in \mathbb{R}$, $\psi(t) = \psi_1(t)$ and $\varphi(t) = \psi_1(t) - \varphi_1(t)$ for each $t \in \mathbb{R}_+$ where $\psi_1, \varphi_1 : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are continuous and increasing function such that $\psi_1(t) = \varphi_1(t) = 0$ if and only if t = 0, we deduce the following theorem.

Theorem 1.2.4. [4] Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let $F : \Omega \to \Omega$ be continuous operator satisfying

$$\psi_1(\mu(FX)) \le \psi_1(\mu(X)) - \varphi_1(\mu(X)),$$

for any nonempty $X \subseteq \Omega$, where μ is an arbitrary measure of noncompactness and $\psi_1, \varphi_1 : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are continuous and increasing function such that $\psi_1(t) = \varphi_1(t) = 0$ if and only if t = 0. Then F has at least one fixed point in Ω .

From Theorem 1.2.2 if the function $\alpha : \mathbb{R}_+ \to \mathbb{R}$ is such that $\alpha(t) = 1$ for all $t \in \mathbb{R}$, $\psi(t) = t$ and $\varphi(t) = \beta(t)t$ which $\beta \in S$, we deduce the following theorem.

Theorem 1.2.5. [2] Let C be a nonempty, bounded, convex and closed subset of a Banach space E and let $T : C \longrightarrow C$ be a continuous operator such that

$$\mu(TX) \le \beta(\mu(X))\mu(X)$$

for any subset X of C where μ is an arbitrary measure of noncompactness on E and $\beta \in S$. Then T has at least one fixed point.

From Theorem 1.2.2 if the function $\alpha : \mathbb{R}_+ \to \mathbb{R}$ is such that $\alpha(t) = 1$ for all $t \in \mathbb{R}, \psi(t) = t$, we deduce the following theorem.

Theorem 1.2.6. [4] Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let $T : \Omega \longrightarrow \Omega$ be a continuous mapping such that

$$\mu(TX) \le \varphi(\mu(X)) \tag{1.9}$$

for any nonempty subset X of Ω where μ is an arbitrary measure of noncompactness and $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is non decreasing and continuous from the right. Then T has at least one fixed point in the set Ω .

The following corollary gives us a fixed point theorem with a contractive condition of integral type.

Corollary 1.2.7. Let C be a nonempty, bounded, closed and convex subset of a Banach space $E, k \in (0,1)$ and $T: C \to C$ be a continuous operator such that for any $X \subseteq C$ one has

$$\int_0^{\mu(T(X))} f(s) \ ds \le k \ \int_0^{\mu(X)} f(s) \ ds,$$

where μ is an arbitrary measure of noncompactness and $f: [0, \infty) \to [0, \infty)$ is a Lebesgue-integrable mapping which is summable (i.e., with finite integral) on each compact subset of $[0, \infty)$, non-negative and such that for each $\epsilon > 0$, $\int_0^{\epsilon} f(s) \, ds > 0$. Then T has at least one fixed point in C.

Proof. Let $\alpha(t) = 1$, $\psi(t) = \int_0^t f(s) \, ds$ and $\varphi(t) = k \ G(t)$ and apply Theorem 1.2.2.

1.3 New fixed point theorems

In this section, we introduce a new notion of a contraction and establish new results for such mappings.

Recently Onsod et al. [19] introduced the notion Θ , the set of all the functions $\tilde{\theta}: (0, +\infty) \to (1, +\infty)$ satisfying the following conditions:

 $(\widetilde{\theta_1})$ $\widetilde{\theta}$ is nondecreasing and continuous;

 $(\widetilde{\theta_2}) \inf_{t \in (0,\infty)} \widetilde{\theta}(t) = 1.$

Also, let S denote the class of real functions $\beta : [0, +\infty) \to [0, 1)$ satisfying the condition

$$\beta(t_n) \longrightarrow 1 \ implies \ t_n \longrightarrow 0.$$

Now, we are ready to state and prove our main result.

Definition 1.3.1. Let \mathcal{G} be a nonempty subset of Banach space \mathcal{B} . Let \mathcal{H} : $\mathcal{G} \longrightarrow \mathcal{G}$ be a function. Then we say that operator \mathcal{H} is a θ -contraction, if there exist $\theta \in \widetilde{\Theta}$ and $k \in (0,1)$ and $\beta \in S$ such that for any subset G of \mathcal{G} with $\mu(G) > 0$,

$$\frac{1}{2}\mu(\mathcal{H}(G)) < \mu(G)) \Longrightarrow \theta(\mu(\mathcal{H}(G))) \le [\theta(\beta(\mu(G))\mu(G))]^k$$
(1.10)

where μ is an arbitrary measure of noncompactness.

Theorem 1.3.2. Let \mathcal{G} be a nonempty, bounded, closed and convex subset of a Banach space \mathcal{B} and $\mathcal{H} : \mathcal{G} \to \mathcal{G}$ be a continuous operator and θ -contraction. Then \mathcal{H} has at least one fixed point in \mathcal{G} .

Proof. Let $G_0 \subseteq \mathcal{G}$. Define a sequence $\{G_n\}$ by $G_n = Conv(\mathcal{H}G_{n-1}), n \geq 1$. $\mathcal{H}G_0 = \mathcal{H}G \subseteq G = G_0, G_1 = Conv(\mathcal{H}G_0) \subseteq G = G_0$, therefore by continuing this process, we have

$$G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n \supseteq G_{n+1} \supseteq \cdots . \tag{1.11}$$

If there exists a natural number N such that $\mu(G_N) = 0$, then G_N is compact. In this case, Theorem 1.2.2 implies that \mathcal{H} has a fixed point. So we assume that $\mu(G_n) > 0$ for $n = 0, 1, 2, \dots$ Therefore,

$$\frac{1}{2}\mu(\mathcal{H}(G_n)) < \mu(\mathcal{H}(G_n)) = \mu(G_{n+1}) \le \mu(G_n)), \, \forall n \ge 1.$$

$$(1.12)$$

Hence from (1.12), for all $n \ge 1$, we have

$$\begin{aligned}
\theta(\mu(G_{n+1})) &= \theta(\mu(Conv(\mathcal{H}(G_n)))) = \theta(\mu(\mathcal{H}(G_n))) & (1.13) \\
&\leq [\theta(\beta(\mu(G_n))\mu(G_n)]^k \\
&< [\theta(\mu(G_n))]^k \\
&< [\theta(\mu(G_{n-1}))]^{k^2} \\
&\vdots \\
&< [\theta(\mu(G_0))]^{k^{n+1}} & (1.14)
\end{aligned}$$

Taking $n \longrightarrow \infty$, we obtain

$$\theta(\mu(G_{n+1})) \longrightarrow 1.$$