Theory of Elastic Wave Propagation and its Application to Scattering Problems

Terumi Touhei

CRC Press
Taylor & Francis Group

Theory of Elastic Wave Propagation and its Application to Scattering Problems

Elastic wave propagation applies to a wide variety of fields, including seismology, non-destructive testing, energy resource exploration, and site characterization. New applications for elastic waves are still being discovered. *Theory of Elastic Wave Propagation and its Application to Scattering Problems* starts from the standpoint of continuum mechanics, explaining stress and strain tensors in terms of mathematics and physics, and showing the derivation of equations for elastic wave motions, to give readers a stronger foundation. It emphasizes the importance of Green's function for applications of the elastic wave equation to practical engineering problems and covers elastic wave propagation in a half-space, in addition to the spectral representation of Green's function. Finally, the MUSIC algorithm is used to address inverse scattering problems.

- Offers comprehensive coverage of fundamental concepts through to contemporary applications of elastic wave propagation
- Bridges the gap between theoretical principles and practical engineering solutions

The book's website provides the author's software for analyzing elastic wave propagations, along with detailed answers to the problems presented, to suit graduate students across engineering and applied mathematics.

Terumi Touhei is a Professor at the Tokyo University of Science, with extensive experience of teaching graduate students.

Taylor & Francis
Taylor & Francis Group
http://taylorandfrancis.com

Theory of Elastic Wave Propagation and its Application to Scattering Problems

Terumi Touhei

CRC Press
Taylor & Francis Group
Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group, an **informa** business

First edition published 2024
by CRC Press
2385 NW Executive Center Drive, Suite 320, Boca Raton FL 33431

and by CRC Press
4 Park Square, Milton Park, Abingdon, Oxon, OX14 4RN

CRC Press is an imprint of Taylor & Francis Group, LLC

ISBN: 978-1-032-17077-0 (hbk)
ISBN: 978-1-032-17078-7 (pbk)
ISBN: 978-1-003-25172-9 (ebk)

DOI: 10.1201/9781003251729

Typeset in Nimbus font
by KnowledgeWorks Global Ltd.

Publisher's note: This book has been prepared from camera-ready copy provided by the author.

Access the Support Material: www.routledge.com/9781032170770

To Kinuko

Contents

Preface

The theory of elastic wave propagation is open to a large variety of areas of applications such as seismology, site characterization, nondestructive testing, medical image processing, and among others. These vast areas of applications, which are sometimes connected with scattering problems, are the result of its long history and growing theory. The purpose of the book is to describe the theory and applications for graduate students, researchers and engineers interested in elastic wave propagation.

Chapter 1 serves as an introduction to the theory of elastic wave propagation. We first clarify the basic concept of the continuum mechanics together with the definition of the Euler and Lagrange approaches. In addition, we define the strain and stress tensors using the modern style found in textbooks (Schutz, 1990). We try to connect the physical understandings with mathematical definitions for these tensors. Even today, it seems that there are not so many textbooks for the continuum mechanics concerned with the modern mathematical viewpoint of tensors. That is the reason why we chose the above description of the strain and stress tensors. The tensor algebra needed to understand the above content is provided in Appendix A.

We also introduce the Reynolds transport theorem to derive the equation of motion for a continuum medium. We linearize the equation of motion assuming that the motions in continuum media exhibit small-amplitude vibrations. The elastic wave equation, the treatments and/or the investigations of which are the theme of the book, is derived from the linearized equation above under the assumption that the medium is isotropic.

At the end of this chapter, the reciprocity of the elastic wavefield is derived using the Gauss divergence theorem. It may be true that the contents of this chapter seem to be complex and difficult, readers are, however, just required to have the knowledge of the strength of material, general mechanics, fluid mechanics, elementary of calculus and linear algebra, which should be taught at the undergraduate level in the engineering course.

In Chapter 2, we discuss the solutions for the elastic wave equation for a 3-D full space. As preparations for the discussions in this chapter, we also derived solutions for the scalar wave and Helmholtz equations. Solutions of the elastic wave equation are discussed by decomposing the elastic wave equation into scalar wave equations for P and S waves. The presence of the P and S waves for the elastic wavefield is an important fact, which is verified by the properties of a vector field consisting of irrotational and divergence-free components.

Green's functions for the scalar wave, as well as the elastic wave equations, have special standpoints for theoretical and numerical analyses since Green's function not only reveals the properties of wave propagation but also transforms the partial differential equation into an integral equation. We present the explicit forms as well as the derivation processes of Green's functions for a 3-D full space in detail in §2.4. The derivation of Green's function by the use of the Dirac delta function and Fourier transform can be thought of as a basic skill in mathematical physics for graduate engineering students. The basics for the Dirac delta function as well as the Fourier transform to understand the above procedures are explained in Appendix B.

The coupling of the reciprocity of the wavefield obtained in Chapter 1 and Green's function yields a representation theorem for the solution of the elastic wavefield. The representation theorem clarifies the relationship among the solution of the elastic wave equation, the boundary values and body forces. A practical application of the representation theorem for an engineering problem is presented in §4.1.

In Chapter 3, we discuss the solutions of the elastic wave equation for a 3D half-space. We show the presence of three types of waves: P, SV, and the SH waves. The interaction between P and SV waves is caused at the free boundary, whereas SH wave exists independently from the P and SV waves. The presence of the Rayleigh wave can be recognized based on the interaction of P and SV waves for a special case.

We also derived Green's function for an elastic half-space. We find that derivation of the closed form of Green's function for an elastic half-space is impossible. In stead of the closed form, we derive Green's function for an elastic half-space in the form of a Fourier-Hankel transform. We introduced two types of historical approximation methods for computing Green's function. The approximation methods we introduce are branch line integral and steepest descent path methods. Nowadays, it is not difficult to compute Green's function for an elastic half-space without using approximation methods. In this sense, the approximation methods for Green's function are sometimes thought to only have historical importance. We will see, however, that the historical approximation approach has the potential to improve modern computational methods. This is discussed in the next chapter.

At the end of this chapter in §3.5, we introduce the modern viewpoint of mathematics, which is the spectral theory of the operator to Green's function for an elastic half-space. According to §3.3, Green's function can be expressed by the residue term and branch-line integrals, respectively. These terms are unified in terms of the eigenfunctions of the point and continuous spectra using the spectral theory. We seek the representation of Green's function in terms of eigenfunctions. The discussions seem to be in a rather abstract manner. The representation of Green's function, however, enables us to formulate an efficient method for the scattering problem, which is also discussed in the next chapter.

The derivation of Green's function in the wavenumber domain, which is used in §3.2–§3.4, is very complicated. In addition, Green's function in the wavenumber domain yields the resolvent kernel used in §3.5, whose properties are very important for the spectral representation of Green's function. Therefore, the derivation of Green's function in the wavenumber domain as well as of the resolvent kernel together with its properties are separated from Chapter 3, and summarized in Appendix C. The comparisons of Green's functions obtained from various computational methods are summarized in Appendix D.

In Chapter 4, we develop numerical methods for the scattering of the elastic waves using Green's functions derived in Chapters 2 and 3. In §4.1, we apply the representation theorem to a solid-fluid interaction problem. Discretization of the representation theorem yields a boundary-element technique. We analyze the vibration of a virtual underground energy storage system and examine its properties.

In §4.2, we apply the spectral representation of Green's function and the generalized Fourier transform obtained in §3.5 to the equation of the type of the Lippmann-Schwinger equation. We compute the scattering wavefield caused by an underground fluctuation of the wavefield. It is true that the spectral representation of Green's function as well as the generalized Fourier transforms obtained in §3.5 are rather abstract. In spite of these mathematical forms, we show that the spectral representation of Green's function and the generalized Fourier transform can provide an efficient tool for analyzing engineering problems.

In §4.3, we compute the inversion of the point-like scatterers by means of the pseudo-projection and MUSIC algorithms. The MUSIC algorithm is presented in Appendix E. The pseudo-projection method is developed by the steepest descent path method, which is considered to be of historical importance. The discussions in §4.3 show a case in which the historical method can also improve the efficiency of modern computational methods.

In this book, we could not include the discussions for wave propagation in layered media, dispersion of guided waves, the time domain solutions via the Cagniard-de Hoop method and among others. The author, however, would like to disclose every computer program used for Chapter 4, so that the readers of this book can reproduce the same results. The author would appreciate it if many engineers were interested in the methods described in Chapter 4. The software is disclosed at `https://www.rs.tus.ac.jp/~ttouhei/main.pdf`.

Acknowledgment

First of all, I would like to thank all of the faculty members of the Department of Civil Engineering, Tokyo University of Science for their support and time in completing the book. For the description of the concept of continuum mechanics, I would like to thank many discussions made by Professor Hiroshi Okada and Professor Takahiro Tsukahara, Department of Mechanical and Aerospace Engineering, Tokyo University of Science. I would like to thank Professor Keiichi Kato, Department of Mathematics, Tokyo University of Science. He invited me to participate in the Division of Research Alliance for Mathematical Analysis at the Research Institute for Science and Technology. The discussions in the Research Alliance were important for the spectral representation of Green's function for an elastic half-space and its application in scattering problems. I appreciate Professor Masayuki Nagano and Professor Yoshifumi Ohmiya, Department of Architecture, Tokyo University Science. We worked together for the Disaster Risk Management Course and conducted many discussions, which were also important for this study.

I would like to thank JOGMEC for the use of the analysis model shown in Fig. 4.1.3 in Chapter 4, as a commissioned project by the Agency for Natural Resources and Energy. I would appreciate Tokyo Electric Power Services Co., Ltd. and Hakusan Corporation in cooperation with the numerical results presented in §4.1.

I enjoyed the discussions with Professor Emeritus, Sohichi Hirose, Tokyo Institute of Technology, Professor Kazuhisa Abe, Professor Kazuhiro Koro, Niigata University, Dr. Takahiro Saitoh, Gunma University, Dr. Taizo Maruyama, Tokyo Institute of Technology, and Dr. Akira Furukawa, Hokkaido University in joint seminars on applied mechanics.

I am grateful to Mr. Tony Moore, editor, as well as Mr. Frazer Merritt and Ms. Aimee Wragg, editorial assistants at Taylor & Francis Group, for their invaluable support in initiating the book project and for their valuable advice regarding the manuscript.

Special thanks I would like to show to Professor Emeritus, Tatsuo Ohmachi, Tokyo Institute of Technology. I really appreciate his encouragement, careful reading of the draft of the manuscript, many suggestions and more.

I acknowledge that any remaining errors are solely my responsibility. I would greatly appreciate it if those are kindly pointed out.

Finally, I would like to thank my wife Kinuko. With her understanding and support, I could complete the book.

Author

Terumi Touhei received a B.S. in 1981 and an M.Eng. in 1983 from Waseda University, Japan. Following that, he served as an engineer at Sato Kogyo. Co., Ltd. until 1992. In 1991, he earned a Ph.D. from the Tokyo Institute of Technology in Japan. Since 1992, he has been a faculty member at Tokyo University of Science, currently holding the position of a professor. Professor Touhei's research interests primarily focus on solid mechanics and the representation of Green's function in an elastic wave field, particularly applied to scattering problems.

1 Introduction

1.1 BASIC CONCEPT OF CONTINUUM MECHANICS

Materials can take the form of solids, liquids, or gases. Liquids and gases, the shapes of which are easily deformed, are considered to be fluids. Gases, however, are different from liquids in that their volumes can easily change. We encounter many types of solids and fluids in our daily lives. However, the distinction is not always so obvious. For example, the Earth's mantle and glaciers are considered to be solids. However, both behave like fluids if they are observed over a long time period. Therefore, in order to investigate different phenomena, not only objects to be observed, but also the temporal and spatial scales need to be considered.

In general, the distance between molecules or atoms is about 1 nm. As a result, an enormous number of molecules or atoms are present, even in a cubic volume with side lengths of 1 μm. Therefore, a macroscopic approach to investigating deformation phenomena for solid or fluid flow is possible, in which the solid or fluid body can be assumed to be continuous. This is a starting point of continuum mechanics, and we refer to the investigated body as a *continuum body*. We also define a particle or spatial point in a continuum body as it still contains enormous number of molecules or atoms. As a consequence, the motion of a particle has to be understood as an averaged value for an enormous number of molecules or atoms in a particle under certain temporal and spatial scales.

In the context of continuum mechanics, we define physical quantities in a spatial and/or temporal coordinate system. For example, let ΔV and ΔM be the volume of a small element and the mass of the element, respectively. Then, the relationship between ΔV and ΔM can be expressed as

$$\Delta M = \rho(\boldsymbol{x})\Delta V \tag{1.1.1}$$

where ρ is the mass density at a spatial point $\boldsymbol{x}$.

In this textbook, three-dimensional (3D) solid media are considered, and a spatial point is expressed in a Cartesian coordinate system, unless otherwise stated. We generally express the components of the position vector in the following form:

$$\boldsymbol{x} = (x_1, x_2, x_3) = (x_j), \ (j = 1, 2, 3) \tag{1.1.2}$$

where the subscripted index describes the component of the coordinate system. In this chapter, however, we express the components of a vector and clarify the chosen coordinate system. For example, we use the following expression for the components of a vector:

$$\boldsymbol{x} \underset{\mathscr{O}}{\rightarrow} (x_1, x_2, x_3) = (x_j), \ (j = 1, 2, 3) \tag{1.1.3}$$

DOI: 10.1201/9781003251729-1

where $\mathscr{O}$ denotes the coordinate system, which defines the location of the origin and the direction of the base vectors [1]. Note that a vector itself represents a physical quantity that is independent of the coordinate system. Equation (1.1.3) is convenient for determining the transformation rule for the components between two different coordinate systems, i.e., different locations of the origins as well as different directions of the base vectors. The tensor algebra necessary for continuum mechanics in this textbook is explained in Appendix A.

Here, we define the term "particle" in a continuum body. Based on the concept of a particle, we can formulate two different approaches in continuum mechanics, which are the Lagrange approach and the Euler approach. The Lagrange approach considers the trajectory of a particle in a continuum body, whereas the Euler approach considers the motion of a particle at its current coordinates. The Lagrange approach is used when the current state of a continuum body is strongly affected by its initial state, whereas, in the Euler approach, the effects of the initial state can be ignored. For wave problems in elastic solids, where the vibration amplitude is small compared to the wavelength, the difference between the Lagrange and Euler approaches is very small, which leads to a linear equation for elastic wave motion, which is the theme of this text book. This introduction describes the linear equation for elastic wave motion and its reciprocity, after the description of the strain and stress tensors. In addition, the Euler and Lagrange approaches, as well as the Reynolds transport theorem, are also discussed.

1.2 STRAIN TENSOR

1.2.1 DEFORMATION OF CONTINUUM BODY

Figure 1.2.1 shows a continuum body, which occupies a region V_0 at time $t = 0$. The body undergoes deformation and moves to a region V at time t. In this section, we deal with a finite region of the continuum body to investigate the deformation.

With respect to the Lagrange and Euler approaches, we use two coordinate systems to investigate the deformation, which are the original coordinate system denoted by $(X_i), (i = 1,2,3)$, and the current coordinate system denoted by $(x_i), (i = 1,2,3)$. We also use the following notations:

$$\boldsymbol{X} \underset{\mathscr{O}}{\rightarrow} (X_1, X_2, X_3) \tag{1.2.1}$$

$$\boldsymbol{x} \underset{\mathscr{O}}{\rightarrow} (x_1, x_2, x_3) \tag{1.2.2}$$

The original coordinate system is used to distinguish particles in the original state, whereas the current coordinate system is used to describe the location of particles at the current time. Note that the location of the origin as well as the direction of the

[1] We use the notation in Eq. (1.1.3) following Schutz, B.F. (1990). We discuss the concept of the tensor algebra by means of the above notation in Appendix A. We also try to provide the physical meaning of the strain and stress tensors in this chapter from the mathematical definition of tensors given in Appendix A.

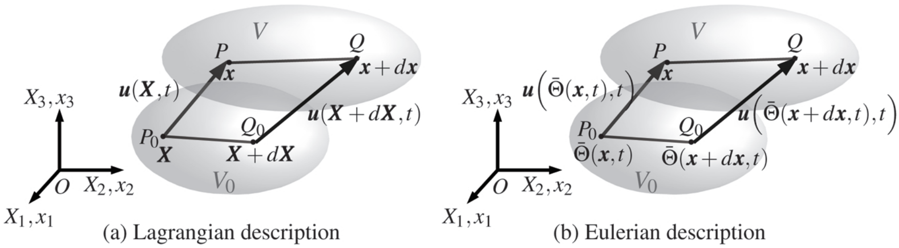

Figure 1.2.1 Deformation of a continuum body. Region V_0 undergoes deformation and moves to V. Consequently, the line element $\overline{P_0Q_0}$ in V_0 shifts to $\overline{PQ}$ in V. The original coordinate system X_i is used to distinguish particles in V_0, and the current coordinate system x_i is used to describe the location of the particle at the current time. We show the Lagrangian and Eulerian descriptions for the displacement field in (a) and (b), respectively.

base vectors for both the original and current coordinate systems can be identical, as we used the notations for Eqs. (1.2.1) and (1.2.2).

The relationship between the original and current coordinate systems is described by a map defined as:

$$\boldsymbol{x} = \Theta(\boldsymbol{X}, t) \tag{1.2.3}$$

where t is the current time. We call Eq. (1.2.3) the *Lagrangian description*, since the map Θ enables us to trace the trajectory of a particle in a continuum body. Alternatively, we can also define a map from $(\boldsymbol{X}, t)$ to its original position of a particle as

$$\boldsymbol{X} = \bar{\Theta}(\boldsymbol{x}, t) \tag{1.2.4}$$

which is called the *Eulerian description*. Equations (1.2.3) and (1.2.4) have to be consistent each other in the sense that Eq. (1.2.4) can be derived by solving Eq. (1.2.3).

As shown in Fig. 1.2.1, a line element $\overline{P_0Q_0} \in V_0$ shifts to $\overline{PQ} \in V$ due to the deformation. The position vectors for $P_0, Q_0 \in V_0$ are denoted by $\boldsymbol{X}$ and $\boldsymbol{X} + d\boldsymbol{X}$, whereas $P, Q \in V$ are $\boldsymbol{x}$ and $\boldsymbol{x} + d\boldsymbol{x}$, respectively. Vectors $\overrightarrow{P_0P}$ and $\overrightarrow{Q_0Q}$ are the displacement vectors defined by the endpoints of the line elements. We introduce a displacement field to express these vectors such that

$$\begin{aligned}
\overrightarrow{P_0P} &= \boldsymbol{u}(\boldsymbol{X}, t) = \boldsymbol{x} - \boldsymbol{X} \\
&= \Theta(\boldsymbol{X}, t) - \boldsymbol{X} \\
\overrightarrow{Q_0Q} &= \boldsymbol{u}(\boldsymbol{X} + d\boldsymbol{X}, t) = \boldsymbol{x} + d\boldsymbol{x} - (\boldsymbol{X} + d\boldsymbol{X}) \\
&= \Theta(\boldsymbol{X} + d\boldsymbol{X}, t) - (\boldsymbol{X} + d\boldsymbol{X})
\end{aligned} \tag{1.2.5}$$

where $\boldsymbol{u}$ denotes the displacement field. We applied the Lagrangian description to the displacement field, where the independent variables $(\boldsymbol{X}, t)$ are used. We can also use the Eulerian description to the displacement field by

$$\begin{aligned}\overrightarrow{P_0P} &= \boldsymbol{u}\big(\bar{\Theta}(\boldsymbol{x},t),t\big) = \boldsymbol{x} - \bar{\Theta}(\boldsymbol{x},t)\\ \overrightarrow{Q_0Q} &= \boldsymbol{u}\big(\bar{\Theta}(\boldsymbol{x}+d\boldsymbol{x},t),t\big) = \boldsymbol{x}+d\boldsymbol{x} - \bar{\Theta}(\boldsymbol{x}+d\boldsymbol{x},t)\end{aligned} \tag{1.2.6}$$

where the independent variables $(\boldsymbol{x},t)$ are used. We also show the concept of the Lagrangian and Eulerian descriptions of the displacement field in Figs. 1.2.1 (a) and (b), respectively.

1.2.2 DEFINITION OF STRAIN TENSOR

It is necessary to have a quantitative measure of the degree of deformation that enables us to describe the internal states of the body. This is expressed in the following form:

$$\eta := \frac{|\overline{PQ}| - |\overline{P_0Q_0}|}{|\overline{P_0Q_0}|} = \frac{|d\boldsymbol{x}| - |d\boldsymbol{X}|}{|d\boldsymbol{X}|} \tag{1.2.7}$$

which is the ratio of the change in length to the original length of the line element (see Fig. 1.2.1). We will derive the expression for η in terms of the components of the coordinate system. By means of the properties of η, which is a physical quantity independent of the coordinate system, we will proceed to the concept of the strain tensor.

We start with an evaluation of the difference between the squared lengths of the line elements before and after deformation:

$$|d\boldsymbol{x}|^2 - |d\boldsymbol{X}|^2$$

because this is related to η by the following equation:

$$\begin{aligned}|d\boldsymbol{x}|^2 - |d\boldsymbol{X}|^2 &= \frac{|d\boldsymbol{x}| - |d\boldsymbol{X}|}{|d\boldsymbol{X}|}\,\frac{|d\boldsymbol{x}| - |d\boldsymbol{X}| + 2|d\boldsymbol{X}|}{|d\boldsymbol{X}|}\,|d\boldsymbol{X}|^2\\ &= \eta(\eta+2)\,|d\boldsymbol{X}|^2\end{aligned} \tag{1.2.8}$$

For $|\eta| \ll 1$, Eq. (1.2.8) can be simplified as

$$\eta = (1/2)\,\frac{|d\boldsymbol{x}|^2 - |d\boldsymbol{X}|^2}{|d\boldsymbol{X}|^2} \tag{1.2.9}$$

Now, let the components of $d\boldsymbol{x}$ and $d\boldsymbol{X}$ be expressed by

$$\begin{aligned}d\boldsymbol{x} &\underset{\mathscr{O}}{\rightarrow} (dx_1, dx_2, dx_3)\\ d\boldsymbol{X} &\underset{\mathscr{O}}{\rightarrow} (dX_1, dX_2, dX_3)\end{aligned} \tag{1.2.10}$$

Then, the difference in the squared lengths can be expressed as

$$\begin{aligned}|d\boldsymbol{x}|^2 - |d\boldsymbol{X}|^2 &= dx_i dx_i - dX_i dX_i\\ &= \left(\frac{\partial x_i}{\partial X_k}\frac{\partial x_i}{\partial X_l} - \delta_{kl}\right) dX_k dX_l\\ &= 2L_{kl}\, dX_k dX_l\end{aligned} \tag{1.2.11}$$

where

$$L_{kl} = (1/2)\left(\frac{\partial x_i}{\partial X_k}\frac{\partial x_i}{\partial X_l} - \delta_{kl}\right) \tag{1.2.12}$$

Note that the summation convention for the subscripted index (see Appendix A) is applied to Eq. (1.2.11). In the following, the summation convention is also applied to subscripted indices for all of the equations, unless otherwise stated. For the derivation of Eq. (1.2.11), we use the following equation:

$$dx_i = \frac{\partial x_i}{\partial X_k} dX_k \tag{1.2.13}$$

Equation (1.2.11) can be interpreted as a linear mapping of two vectors into a real number, if we distinguish dX_k and dX_l as different vectors. In this sense, L_{kl} is a component of a rank-2 tensor. We call L_{kl} the Lagrangian (Green) finite strain tensor. Alternatively, if we apply the following equation:

$$dX_i = \frac{\partial X_i}{\partial x_k} dx_k \tag{1.2.14}$$

to the difference between the squared lengths, then we have

$$\begin{aligned} |d\boldsymbol{x}|^2 - |d\boldsymbol{X}|^2 &= dx_i dx_i - dX_i dX_i \\ &= \left(\delta_{kl} - \frac{\partial X_i}{\partial x_k}\frac{\partial X_i}{\partial x_l}\right) dx_k dx_l \\ &= 2E_{kl} dx_k dx_l \end{aligned} \tag{1.2.15}$$

where E_{kl} is defined by

$$E_{kl} = (1/2)\left(\delta_{kl} - \frac{\partial X_i}{\partial x_k}\frac{\partial X_i}{\partial x_l}\right) \tag{1.2.16}$$

As in the case for L_{kl} in Eq. (1.2.11), E_{kl} can be regarded as a component of a rank-2 tensor. We refer to E_{kl} as the Eulerian (Almansi) finite strain tensor. The difference between Eqs. (1.2.12) and (1.2.16) is in the use of independent variables. Namely, the original coordinate system $(\boldsymbol{X}, t)$ is used for Eq. (1.2.12), whereas the current coordinate system $(\boldsymbol{x}, t)$ is used for Eq. (1.2.16), yielding the strain tensors based on the Lagrange and the Euler approaches, respectively.

In order to analyze wave propagation in a continuum body, the governing equation uses the displacement field as an unknown function to be solved. Therefore, it is necessary to determine the relationship between the strain tensor and the displacement field. According to Eq. (1.2.5), we have

$$x_i = u_i + X_i \tag{1.2.17}$$

and as a result,

$$\frac{\partial x_i}{\partial X_k} = \frac{\partial u_i}{\partial X_k} + \delta_{ik} \tag{1.2.18}$$

Therefore, the Green finite strain tensor can be expressed as

$$L_{kl} = (1/2)\left(\frac{\partial u_k}{\partial X_l} + \frac{\partial u_l}{\partial X_k} + \frac{\partial u_i}{\partial X_l}\frac{\partial u_i}{\partial X_k}\right) \tag{1.2.19}$$

Likewise, from

$$X_i = x_i - u_i \tag{1.2.20}$$

we obtain the expression for the Eulerian strain tensor in the following form:

$$E_{kl} = (1/2)\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} - \frac{\partial u_i}{\partial x_l}\frac{\partial u_i}{\partial x_k}\right) \tag{1.2.21}$$

The relationship of the gradient of the displacement field between the Lagrange and Euler approaches is expressed as

$$\begin{aligned}\frac{\partial u_k}{\partial x_l} &= \frac{\partial u_k}{\partial X_j}\frac{\partial X_j}{\partial x_l} \\ &= \frac{\partial u_k}{\partial X_j}\left(\delta_{jl} - \frac{\partial u_j}{\partial x_l}\right) \\ &= \frac{\partial u_k}{\partial X_l} - \frac{\partial u_k}{\partial X_j}\frac{\partial u_j}{\partial x_l}\end{aligned} \tag{1.2.22}$$

Therefore, for a case that the gradient of the displacement field is very small, namely

$$\left|\frac{\partial u_k}{\partial X_j}\right| \ll 1, \quad \left|\frac{\partial u_k}{\partial x_l}\right| \ll 1 \tag{1.2.23}$$

we can proceed our discussions by assuming that

$$\frac{\partial u_k}{\partial x_l} = \frac{\partial u_k}{\partial X_l} \tag{1.2.24}$$

Equation (1.2.24) shows that the differences between the Lagrange and Euler approaches are very small if Eq. (1.2.23) holds. For this case, Eqs. (1.2.19) and (1.2.21) yield:

$$\varepsilon_{ij} = (1/2)\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right) \tag{1.2.25}$$

which is referred to as the infinitesimal strain tensor. As for the Green and Almansi strain tensors, ε_{ij} can be also regarded as a rank-2 tensor. The difference between the squared lengths of the line elements in terms of the infinitesimal strain tensor is expressed as

$$|d\boldsymbol{x}|^2 - |d\boldsymbol{X}|^2 = 2\varepsilon_{ij}dx_i dx_j \tag{1.2.26}$$

1.2.3 CHARACTERIZATION OF INFINITESIMAL STRAIN TENSOR

Now, let us discuss the properties of the infinitesimal strain tensor. For the discussion, we do not distinguish between the original and current coordinate systems, because we consider a situation in which the difference between the results of the Euler and Lagrange approaches is very small. Under these circumstances, we simply use the spatial derivative $\partial/\partial x_j$. According to Eqs. (1.2.9) and (1.2.26), the relationship between the parameter η and the infinitesimal strain tensor is

$$\eta = \frac{\varepsilon_{ij} dx_i dx_j}{|d\boldsymbol{x}|^2} \tag{1.2.27}$$

At this point, we have two tasks related to Eq. (1.2.27). One is to connect the tensor algebra discussed in Appendix A with Eq. (1.2.27). We have to determine the transformation rule for the components of the infinitesimal strain tensor for the different coordinate systems. Note that we do not distinguish between the original and current coordinate systems for the discussion of the infinitesimal strain tensor. At this point, the transformation rule is considered for different base vectors that span the continuum body. The second task is to connect the infinitesimal strain tensor with the concept of strain learned in undergraduate engineering mechanics.

The starting point of the discussion is Eq. (1.2.27). We consider two coordinate systems $O\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$, $O'\{\boldsymbol{e}'_1, \boldsymbol{e}'_2, \boldsymbol{e}'_3\}$ and we express the components of the direction vector in the following form:

$$\begin{aligned} \frac{d\boldsymbol{x}}{|d\boldsymbol{x}|} &\underset{\mathscr{O}}{\rightarrow} (d_1, d_2, d_3) \\ \frac{d\boldsymbol{x}}{|d\boldsymbol{x}|} &\underset{\mathscr{O}'}{\rightarrow} (d'_1, d'_2, d'_3) \end{aligned} \tag{1.2.28}$$

Note that the origin of the two coordinate systems $\mathscr{O}$ and $\mathscr{O}'$ is identical for the discussion.

As shown in Appendix A, the transformation rule for the components of the direction vector is

$$d_j = a_{ij} d'_i \tag{1.2.29}$$

where a_{ij} is defined by

$$a_{ij} = \boldsymbol{e}'_i \cdot \boldsymbol{e}_j \tag{1.2.30}$$

In addition, since the left-hand side of Eq. (1.2.27) is independent of the coordinate system, the parameter η can also be expressed as

$$\eta = \varepsilon'_{ij} d'_i d'_j \tag{1.2.31}$$

As a result, we have

$$\eta = \varepsilon'_{ij} d'_i d'_j = \varepsilon_{kl} a_{ik} a_{lj} d'_i d'_j \tag{1.2.32}$$

which yields

$$d'_i d'_j (\varepsilon'_{ij} - \varepsilon_{kl} a_{ik} a_{jl}) = 0 \Longleftrightarrow \varepsilon'_{ij} = a_{ik} a_{jl} \varepsilon_{kl} \tag{1.2.33}$$

from which we find that ε_{ij} follows the transformation for a rank-2 tensor. Therefore, we can define a linear mapping of two vectors $\boldsymbol{\varepsilon}(\cdot,\cdot)$, the components of which are defined by

$$\boldsymbol{\varepsilon}(\boldsymbol{e}_i,\boldsymbol{e}_j) = \varepsilon_{ij} \tag{1.2.34}$$

which is called the strain tensor. For the case in which $\boldsymbol{\varepsilon}$ takes the arguments of direction vectors, $\boldsymbol{\varepsilon}$ provides a clear physical meaning of

$$\boldsymbol{\varepsilon}\left(\frac{d\boldsymbol{x}}{|d\boldsymbol{x}|},\frac{d\boldsymbol{x}}{|d\boldsymbol{x}|}\right) = \eta \tag{1.2.35}$$

Here, we consider the components of the strain tensor in more detail. At an undergraduate level, normal strain is defined as the ratio of the change in length to the original length, whereas shear strain is defined as the decrease in angle from $\pi/2$. Figure 1.2.2 shows the concept of normal and shear strains in the $x_1 - x_2$ plane. We now connect these concepts with the components of the strain tensor using ε_{11} and ε_{12}. According to Eq. (1.2.25), the strain tensor component ε_{11} is

$$\varepsilon_{11} = \lim_{\Delta x_1 \to 0} \frac{u_1(\boldsymbol{x}+\Delta x_1\boldsymbol{e}_1) - u_1(\boldsymbol{x})}{\Delta x_1} \tag{1.2.36}$$

We see that $u_1(\boldsymbol{x}+\Delta x_1\boldsymbol{e}_1) - u_1(\boldsymbol{x})$ in Eq. (1.2.36) is the change in length, and Δx_1 is the original length of the line element, which are δ and L, respectively, in Fig. 1.2.2. Therefore, ε_{11} corresponds to the normal strain for the x_1 direction.

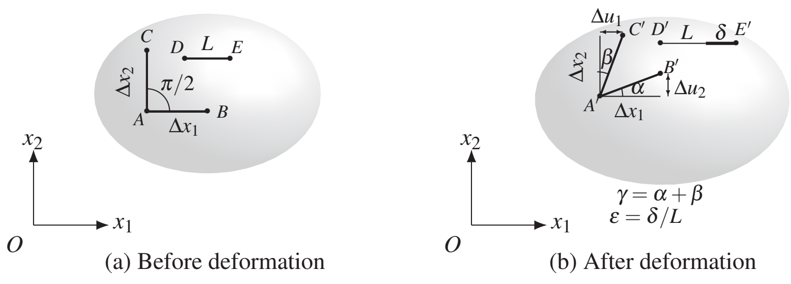

Figure 1.2.2 Normal and shear strains commonly taught at the undergraduate level. Normal strain ε is defined as the ratio of the change in length δ to the original length L. Shear strain γ is defined as the decrease in angle from $\pi/2$. Note that points from A to E shift to A' to E', respectively, due to deformation.