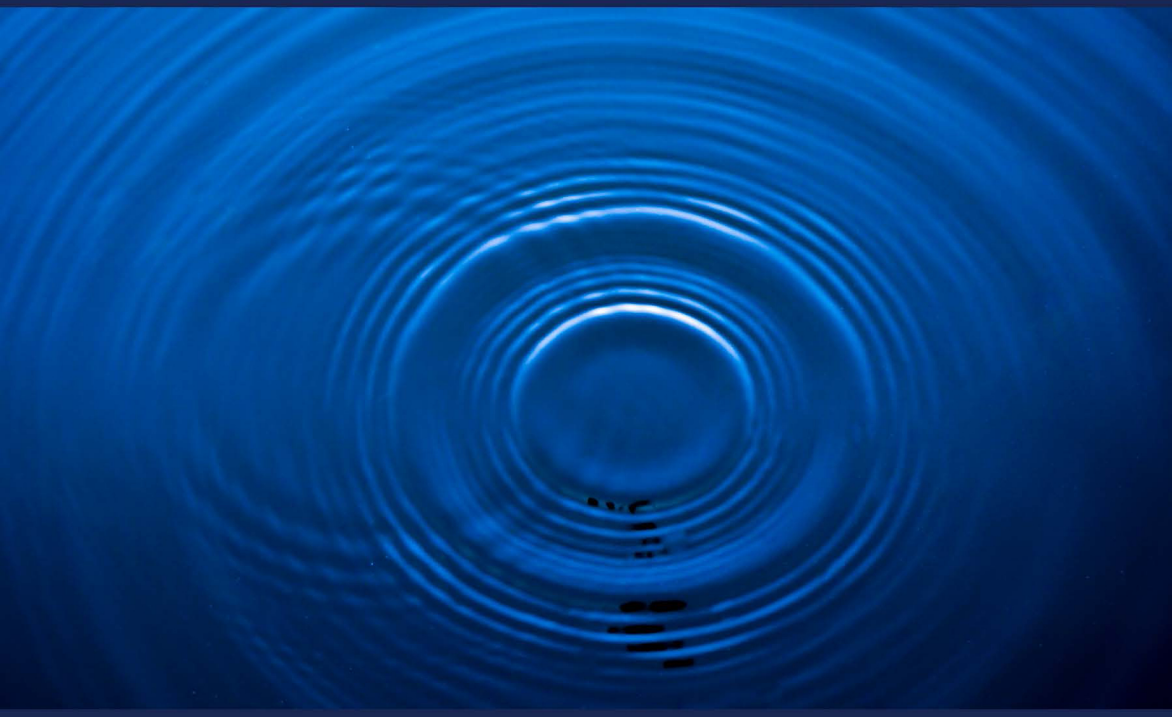

The Method of Moments in Electromagnetics

Third Edition

Walton C. Gibson

CRC Press
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Preface to the Third Edition

Shortly before the publication of the second edition of this book, I finished work on a new version of the *Serenity* RCS solver code. This new software added support for the Adaptive Cross Approximation (ACA), permitting a direct solution of the matrix system in compressed form. It also took advantage of the incredible computation power afforded by general purpose graphics processing units (GPUs), which can perform matrix operations much faster than traditional CPUs. Unfortunately, given the short timeframe, I was not able to add any material on the ACA to the second edition. Since that time, significant additional progress has been made on *Serenity*, as well as implementation of a Multi-Level Adaptive Cross Approximation (MLACA), which has recently begun to receive significant attention in the literature. Thus, the time now seems right to update the second edition with material on these very exciting, powerful algorithms.

This third edition adds a significant amount of new material. First, I have added a chapter that presents an in-depth treatment of the ACA as applied to the Method of Moments, as well as its software implementation on CPUs and GPUs. Next, I have added a chapter on the Multi-Level Adaptive Cross Approximation (MLACA), an advanced compression technique that uses a recursive, butterfly compression algorithm. This yields a far better level of compression than the regular ACA, at the expense of extra computation time. Finally, I have revisited and updated the existing material on the Fast Multipole Method (FMM), as well as the Multi-Level Fast Multipole Algorithm (MLFMA). This new treatment utilizes elements of the ACA approach to further reduce the the memory requirements of the FMM and MLFMA, and also presents compressed, block versions of the ILUT and SAI preconditioners. I have also done my best to correct known typographical errors from the second edition, and to incorporate feedback I've received from readers.

I began working on *Serenity* in 2000, and the first edition of this book in 2004. After 20 years of very hard work, I feel that it's finally time to take a rest. As a result, this will be the final edition of the book. Please send any errata to me via email at kalla@tripoint.org, and going forward I will maintain an up-to-date errata sheet for those interested.

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Preface to the Second Edition

The first edition of this book was well received, and since its publication I received a lot of feedback and questions about its content. When I began considering a second edition, I knew that a key element had to be the treatment of dielectric materials. After I started working on the new material, it became apparent that to accomplish this goal I would have to re-write a significant amount of the existing material. The result is a book that I believe significantly improves and expands upon the first edition in several key areas:

1. The first edition focused exclusively on integral equations for conducting problems. However, dielectric materials are an important aspect of practical electromagnetic devices, and the ability to include these in computational codes is of increasing importance. In this second edition, we will derive coupled surface integral equations that will treat conducting as well as composite conducting/dielectric objects. This treatment extends to objects having multiple dielectric regions with interfaces and junctions. Though many sections of this book have been changed or updated to reflect new material, we will continue to use the approach in the first edition, where we progress gradually from simple to more complex problems and topics.

2. As my background is in the design and programming of software codes for radar cross section (RCS) prediction, the first edition focused almost exclusively on far-zone radiated fields and RCS. In this second edition, I have added more material involving the calculation of near fields.

3. Technology continues to march ever onward, and I have attempted to keep the material in this edition as up to date as possible. As a result, existing material has been updated, and material deemed to be outdated or no longer relevant has been removed.

4. Known errors from the first edition have been corrected, and concepts that were unclear or poorly explained have been updated for clarity. Many equations have also been reformatted, and figures adjusted or regenerated to fix errors or improve their appearance.

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Preface

The Method of Moments (MoM) as applied to electromagnetic field problems was first described at length in Harrington's classic book[1], and was no doubt in use before that. Since then, computing technology has grown at a staggering pace, and the field of computational electromagnetics (CEM) has followed closely behind. Though many dissertations and journal papers have been dedicated to the MoM in electromagnetics, few textbooks have been written for those who are unfamiliar with it. For graduate students who are just beginning their study of CEM, or seasoned professionals solving real-world problems, this sets the barrier to entry fairly high.

Thus, this author felt that a new book describing the Moment Method as applied to electromagnetics was necessary. The first reason was because of the lack of a good introductory graduate-level text on the Moment Method. Though many universities offer courses in computational electromagnetics, the course material often comprises a disjoint collection of journal papers and copies of the instructor's personal notes. In the few books that do exist on the subject, the material is often presented at a high level with little to no attention paid to implementation details. Additionally, many of these often omit key information or refer to other papers (or worse, private communications), forcing the student to spend their time searching for missing information instead of focusing on course material. The second reason is the hope that a concise, up-to-date reference book will be of significant benefit to researchers and practicing professionals in the field of CEM. This book has several key features that set it apart from others of its kind:

1. *A straightforward, progressive introduction to field problems and the Moment Method*. This book begins by introducing the Method of Moments in the context of simple, electrostatic field problems. We then move to a review of frequency-domain electromagnetic theory, radiation equations, and the Green's functions for two- and three-dimensional objects. The surface equivalence theorem is then used to derive coupled surface integral equations of radiation and scattering for conducting and composite dielectric/conducting objects. Subsequent chapters are then dedicated to solving these integral equations for progressively more dif-

[1] R. F. Harrington *Field Computation by Moment Methods*, The Macmillan Company, 1968.

ficult problems involving thin wires, bodies of revolution, and two and three-dimensional bodies. With this material behind them, the student or researcher will be well-equipped for more advanced MoM topics encountered in the literature.

2. *A clear and concise summary of equations*. One of the fundamental problems encountered by this author is that clear expressions for the MoM matrix elements are almost never found in the literature. This book derives or summarizes the matrix elements used in every MoM problem, and all examples are computed using the expressions summarized in the text.

3. *A focus on radiation and scattering problems*. This book is primarily focused on scattering and radiation problems. Therefore, we will consider many practical examples such as antenna impedance calculation and radar cross section prediction, as well as calculation of near fields. Each example is presented in a straightforward manner with a careful explanation of the approach as well as explanation of the results.

4. *An up-to-date reference*. The material contained within is presented in the context of current-day computing technology, and includes up-to-date material on methods such as as the Fast Multipole Method and Adaptive Cross Approximation that are now commonly used in CEM.

5. *"Show your work."* In this work we attempt to avoid a so-called "it can be shown" approach, which can be detrimental to students, and instead focus on step-by-step derivations in key areas that are often glossed over or omitted in other texts.

This book is intended for a one- or two- semester course in computational electromagnetics and a reference for the practicing engineer. It is expected that the reader will be familiar with time-harmonic electromagnetic fields and vector calculus, as well as differential/integral equations and linear algebra. The reader should also have some basic experience with computer programming in a language such as C, C++ or FORTRAN, or a mathematical environment such as MATLAB®. Because some of the expressions in this book require the calculation of special functions, the reader should be aware of what they are and know how to calculate them.

This book comprises nine chapters:

Chapter 1 presents a very brief overview of computational electromagnetics and some of the commonly used numerical techniques in this field. This will show the reader how surface integral equations and the Method of Moments fits into the world of CEM algorithms.

Chapter 2 introduces the Method of Moments in the context of simple electrostatic field problems. We then formalize the MoM, discuss point matching

and Galerkin's Method, and present some commonly used two-dimensional basis functions.

Chapter 3 begins with Maxwell's Equations, and then discusses electromagnetic boundary conditions, formulations for radiation, vector potentials, Green's functions, and near and far fields. It then uses the equivalence principle to derive a set of coupled integral equations for scattering by multi-region, composite conducting/dielectric objects. These equations are then discretized in general form using the Method of Moments, and are refined and adapted to specific problems in later chapters.

Chapter 4 discusses the solution of matrix equations, including topics such as Gaussian Elimination, scalar and block-LU Decomposition, matrix condition numbers, and iterative solvers.

Chapter 5 considers radiation and scattering by thin wires. We derive the thin wire kernel and the Hallén and Pocklington thin wire integral equations, and show how these are solved. We then apply the Moment Method to thin wires of arbitrary shape, and then solve practical thin wire problems involving antennas and feedlines.

Chapter 6 applies the Moment Method to two-dimensional problems. The integral equation formulations in Chapter 3 are adapted to general, two-dimensional boundaries, and at TM and TE polarizations. The first half of the chapter considers conducting geometries, and the latter half then moves into dielectric and composite geometries and problems.

Chapter 7 considers three-dimensional objects which can be described as bodies of revolution (BoRs). Examples involving the radar cross section of conducting, dielectric, and composite objects are then presented. We will then discuss junctions between dielectric and conducting regions and their treatment, and consider some example problems having junctions.

Chapter 8 moves to three-dimensional (3D) surfaces of arbitrary shape. We discuss the modeling of surfaces using triangular facets, and devote significant effort to summarizing the expressions used to evaluate singular potential integrals over triangular elements. We will then present a method by which generalized junctions between regions can be treated in a systematic and straightforward manner. Next, we consider radar cross section problems involving conducting and composite objects, as well as some three-dimensional antenna problems.

Chapter 9 introduces the Adaptive Cross Approximation (ACA), which is a fast, kernel-agnostic algorithm that can be used to compress blocks of the MoM system matrix, as well as blocks in the corresponding LU-factored matrix. We will cover the ACA algorithm as well as software implementation details on regular CPUs as well as graphics processing units (GPUs). We will then consider problems of larger electrical size than were considered in Chapter 8.

Chapter 10 introduces the Multi-Level Adaptive Cross Approximation (MLACA), which comprises a hierarchical extension of the ACA presented

in Chapter 9, allowing for much greater levels of compression at the expense of more computation. The MLACA algorithm and its implementation details on regular CPUs as well as GPUs is covered in detail. We will then consider problems of much larger electrical size than were considered in Chapter 9.

Chapter 11 discusses the Fast Multipole Method (FMM) and how it is used to accelerate the matrix-vector product in the iterative solution of 3D problems. We will cover the addition theorem, wave translation, and single and multi-level fast multipole algorithms. The treatment is concise and contains all the information required to successfully implement the FMM for conducting or composite geometries in new or existing moment method codes. We will then consider consider problems of even larger electrical size than were considered in Chapter 10.

Chapter 12 discusses some methods of numerical integration including the trapezoidal and Simpson's rule, and then area coordinates and Gaussian quadrature over planar triangular elements.

Throughout this text, an e^{jwt} time convention is assumed and suppressed throughout. We use SI units except in some examples where the test articles had dimensions in inches or feet. Numbers in parentheses ( ) refer to equations, and numbers in brackets [ ] are citations. Scalar quantities are written in italic font (a), vectors in bold lowercase font (**a**) with unit vectors denoted using the caret ($\hat{\mathbf{a}}$), and matrices in bold uppercase font (**A**).

Acknowledgments

This book came about because of my experience as a student in electromagnetics and as a developer of computational electromagnetics codes. My walk down this path undoubtedly began very early with my interest in the power of the computer, as well as the mysteries behind radio waves and antennas. I got my first taste as an undergraduate student at Auburn University, where my professor of electromagnetics, Dr. Thomas Shumpert, encouraged me to pursue the subject at the graduate level. It progressed further during my study at the University of Illinois, where I learned many great things about electromagnetics from Dr. Weng Chew and my graduate advisor Dr. Jianming Jin. During my subsequent work on the code suite known as *lucernhammer*, I encountered many hard problems in mathematics and software algorithms, programming, and optimization. Without the help of friends and colleagues, it is likely I would not have completed this path at all. Therefore, I would like to thank the following people who provided me with useful assistance, advice, and information when it was needed:

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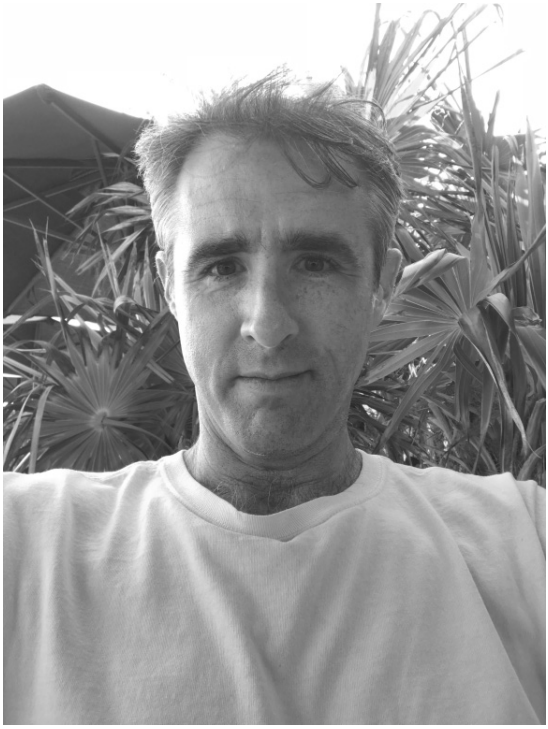
About the Author



Walton C. Gibson was born in Birmingham, Alabama in 1975. He received the Bachelor of Science degree in electrical engineering from Auburn University in 1996, and the Master of Science degree from the University of Illinois Urbana-Champaign in 1998. His professional interests include electromagnetic theory, computational electromagnetics, radar cross section prediction, and computer graphics. He is the author of the industry-standard *lucernhammmer* electromagnetic signature prediction software. He is a licensed amateur radio operator (callsign K4LLA), with interests in antennas and low-frequency HF propagation.

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Chapter 1

Computational Electromagnetics

[1] In the beginning, the design and analysis of electromagnetic devices and structures was largely experimental. However, once the computer and numerical programming languages were developed, people immediately began using them to solve electromagnetic field problems of ever-increasing complexity. This pursuit of *computational electromagnetics* (CEM) has yielded many innovative, powerful analysis algorithms, and it now drives the development of electromagnetic devices people use every day. As the power of the computer continues to grow, so do the number of available algorithms as well as the size and complexity of the problems that can be solved. While the data gleaned from experimental measurements is invaluable, the entire process can be costly in terms of money and the manpower required to do the machine work and assembly, and to collect data at the measurement range. One of the fundamental drives behind reliable computational electromagnetics algorithms is the ability to simulate the behavior of devices and systems before they are actually built. This allows engineers to engage in levels of optimization that would be painstaking or even impossible if done experimentally. CEM also helps to provide fundamental insights into electromagnetic problems through the power of computation and computer visualization, making it one of the most important areas of engineering today.

1.1 CEM Algorithms

The range of electromagnetic problems is extensive, and this has led to the development of different classes of CEM algorithms, each with its own benefits and limitations. In the "early days" of CEM, many problems of practical size could not be solved unless some assumptions were made about the underlying physics and approximations made, usually under the asymptotic or high-frequency limit. These approximate algorithms are now commonly known as "high-frequency" methods. Algorithms that do not make these sorts of approximations are more demanding in terms of CPU and system memory, and historically have been limited to problems of small electrical size. These are usually

referred to as "exact" or "low-frequency" algorithms. Both classes of algorithms can be further subdivided into time or frequency-domain algorithms. We will now summarize some of the most commonly used methods to provide some context in how the Moment Method fits in the CEM environment.

1.1.1 Low-Frequency Methods

Low-frequency (LF) methods are so-named because they solve Maxwell's Equations with no implicit approximations, and are typically limited to problems of small electrical size due to limitations of computation time and system memory. Though computers continue to grow more powerful and can solve problems of ever-increasing size, this nomenclature will likely remain common in the literature for the foreseeable future.

1.1.1.1 Finite Difference Time Domain Method

The Finite Difference Time Domain (FDTD) method [2, 3] uses finite differences to solve Maxwell's Equations in the time domain. The application of FDTD is usually very straightforward: the solution domain is discretized into small rectangular or curvilinear elements, and a "leap frog" in time is used to compute electric and magnetic fields from one another at discrete time steps. FDTD excels at analysis of inhomogeneous and nonlinear media, though its demands for system memory are high due to discretization of the entire solution domain, and it suffers from dispersion issues and the need to artificially truncate of the solution boundary. FDTD is typically applied in EM packaging and waveguide problems, as well as the study of wave propagation in complex (often composite) materials.

1.1.1.2 Finite Element Method

The Finite Element Method (FEM) [4, 5] is a method used to solve frequency-domain boundary-valued electromagnetic problems via variational techniques. It can be used with two- and three-dimensional canonical elements of differing shape, allowing for a highly accurate discretization of the solution domain. FEM is often used in the frequency domain for computing the field distribution in complex, closed regions such as cavities and waveguides. As in the FDTD, the solution domain must be discretized and truncated, making the FEM approach often unsuitable for radiation or scattering problems unless combined with a boundary integral equation approach such as the Method of Moments [4].

1.1.1.3 Method of Moments

The Method of Moments (MOM) is a technique used to solve electromagnetic surface[1] or volume integral equations in the frequency domain. MOM differs from FDTD and FEM as the electromagnetic sources (surface or volume currents) are the quantities of interest, and so only the surface or volume of the antenna or scatterer must be discretized. As a result, the MOM is widely used in solving radiation and scattering problems. In this book, we focus on the practical solution of surface integral equations of radiation and scattering using MOM.

1.1.2 High-Frequency Methods

Electromagnetic problems of large size have existed long before the computers that could solve them. Common examples of larger problems are those involving radar cross section prediction, or the calculation of an antenna's radiation pattern when in close proximity to a large structure. Many approximations have been made to the equations of radiation and scattering to make these problems tractable. Most of these treat the fields in their asymptotic limit, and employ ray-optics and edge diffraction. When the problem is electrically large, many asymptotic methods produce results that are accurate enough on their own, or can be used as a "first pass" before a more accurate though computationally demanding method is applied.

1.1.2.1 Geometrical Theory of Diffraction

The Geometrical Theory of Diffraction (GTD) [6, 7] uses ray-optics to determine electromagnetic wave propagation. The spreading and amplitude intensity and decay in a ray bundle are computed using Fermat's principle and the radius of curvature at the bounce points. GTD attempts to account for the fields diffracted by edges, allowing for a calculation of the fields in shadow regions. GTD is fast but the results are often fair to poor for complex geometries.

1.1.2.2 Physical Optics

Physical Optics (PO) [8] is a method for approximating the surface currents, allowing a boundary integration to be performed to obtain the fields. As we will see, PO and the MOM are closely related as they use the same equations to integrate the surface currents, however the MOM calculates the surface currents directly instead approximating them. While robust, the PO does not account for the fields diffracted by edges or those from multiple reflections, so supplemental corrections are typically added to it. The PO method is used

[1] In this context, it is also referred to as the Boundary Element Method (BEM).

extensively in high-frequency reflector antenna analysis, as well as many radar cross section prediction codes such as *lucernhammer MT*.

1.1.2.3 Physical Theory of Diffraction

The Physical Theory of Diffraction (PTD) [9, 10] is a means for supplementing the PO solution by adding to it the fields radiated by nonuniform currents along diffracting edges of an object. PTD is commonly used in high-frequency radar cross section and scattering analysis.

1.1.2.4 Shooting and Bouncing Rays

The Shooting and Bouncing Ray (SBR) method [11, 12] was developed to predict the multiple-bounce scattered fields from complex objects. It uses the ray-optics model to determine the path and amplitude of a ray bundle, but uses a physical optics-based scheme that integrates the surface currents induced by the ray at each bounce point. SBR is often used in scattering codes to account for multiple reflections on a surface or those inside a cavity, and as such it supplements the fields computed by PO and the PTD. SBR is also used to predict wave propagation and scattering in complex urban environments to determine the coverage for cellular telephone service.

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Chapter 2

The Method of Moments

In this book, we are concerned with solving surface integral equations. Because no analytic solutions exist for most of these problems, in practice we must apply numerical methods. In this chapter we will introduce the Method of Moments (MoM), a technique that is used to convert integral equations into a linear system that can then be solved numerically via computer. To begin, we will analyze a few simple electrostatic problems to provide a bit of context, and then formally define the MoM. We will next discuss the expansion of an unknown function by a sum of weighted basis functions, and compare and contrast point matching and the Method of Galerkin. In the chapters that follow, we will then discuss electrodynamic problems and numerical methods for solving a general system of linear equations. With this material behind us, we will then be ready to apply the MoM to the more challenging integral equation problems in later chapters.

2.1 Electrostatic Problems

Because electrostatic problems are relatively simple compared to the electrodynamic case, they provide a good context for introducing algorithms used to solve integral equations. Recall that the electric potential ϕ_e at a point $\mathbf{r}$ due to an electric charge density q_e is given by the integral

$$\phi_e(\mathbf{r}) = \int_V \frac{q_e(\mathbf{r}')}{4\pi\epsilon|\mathbf{r}-\mathbf{r}'|}\, d\mathbf{r}'. \tag{2.1}$$

If we know $q_e(\mathbf{r}')$, we can obtain the electric potential everywhere. If we instead know the electric potential $\phi_e(\mathbf{r})$ but not the charge density, (2.1) becomes an integral equation for $q_e(\mathbf{r}')$. We will now solve this problem numerically for a pair of practical examples, a charged wire and plate.

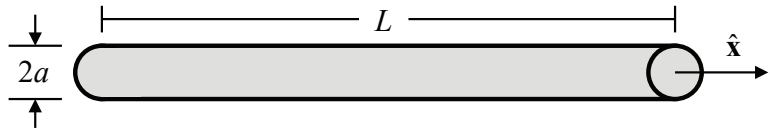


FIGURE 2.1: Thin Wire Dimensions

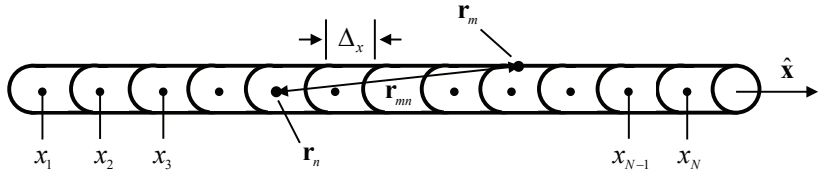


FIGURE 2.2: Thin Wire Segmentation

2.1.1 Charged Wire

Consider a thin, conducting wire of length L and radius a oriented along the x axis, as shown in Figure 2.1. If the radius of the wire is very small compared to the length ($a \ll L$), the electric potential on the wire can be expressed via the integral

$$\phi_e(\mathbf{r}) = \int_0^L \frac{q_e(x')}{4\pi\epsilon|\mathbf{r}-\mathbf{r}'|}\,dx' \tag{2.2}$$

where

$$|\mathbf{r}-\mathbf{r}'| = \sqrt{(x-x')^2+(y-y')^2}\,. \tag{2.3}$$

We intend to convert (2.2) into a linear system of equations, so we first subdivide the wire into N subsegments, each having length Δ_x, as shown in Figure 2.2. Within each subsegment, we assume that the charge density has a constant value such that $q_e(x')$ is piecewise constant over the length of the wire. Mathematically, this can be written as

$$q_e(x') = \sum_{n=1}^{N} a_n f_n(x') \tag{2.4}$$

where a_n are unknown coefficients, and $f_n(x')$ is a pulse function having a constant value on segment n but zero on all others, i.e.

$$f_n(x') = \begin{cases} 0 & x' < (n-1)\Delta_x \\ 1 & (n-1)\Delta_x \le x' \le n\Delta_x \\ 0 & x' > n\Delta_x \end{cases}. \tag{2.5}$$

We now set the potential of the wire to $\phi_e = 1$V. Substituting (2.4) into (2.2)

then yields

$$1 = \int_0^L \sum_{n=1}^{N} a_n f_n(x') \frac{1}{4\pi\epsilon|\mathbf{r}-\mathbf{r}'|}\, dx'. \tag{2.6}$$

Using the previous definition of the pulse function, we can rewrite this as

$$1 = \frac{1}{4\pi\epsilon} \sum_{n=1}^{N} a_n \int_{(n-1)\Delta_x}^{n\Delta_x} \frac{1}{|\mathbf{r}-\mathbf{r}'|}\, dx' \tag{2.7}$$

where we now have a sum of integrals, each over the domain of a single pulse function. Let us now fix the source points so that they are on the wire axis, and the field point to be on the wire's surface. This ensures that we do not encounter a singularity in the integrand. The denominator of the integrand now becomes

$$|\mathbf{r}-\mathbf{r}'| = \sqrt{(x-x')+a^2} \tag{2.8}$$

and (2.7) can be written as

$$\begin{aligned} 4\pi\epsilon = a_1 \int_0^{\Delta_x} \frac{1}{\sqrt{(x-x')+a^2}}\, dx' + a_2 \int_{\Delta_x}^{2\Delta_x} \frac{1}{\sqrt{(x-x')+a^2}}\, dx' + \cdots \\ + a_{N-1} \int_{(N-2)\Delta_x}^{(N-1)\Delta_x} \frac{1}{\sqrt{(x-x')+a^2}}\, dx' + a_N \int_{(N-1)\Delta_x}^{N\Delta_x} \frac{1}{\sqrt{(x-x')+a^2}}\, dx' \end{aligned} \tag{2.9}$$

which comprises one equation in N unknowns. If we can now somehow convert this equation to N equations in N unknowns, we can solve it by common matrix algebra routines. To do so, let us choose N independent field points x_m on the surface of the wire, each at the center of a wire segment. Doing so yields

$$\begin{aligned} 4\pi\epsilon &= a_1 \int_0^{\Delta_x} \frac{1}{\sqrt{(x_1-x')+a^2}}\, dx' + \cdots + a_N \int_{(N-1)\Delta_x}^{N\Delta_x} \frac{1}{\sqrt{(x_1-x')+a^2}}\, dx' \\ &\vdots \\ 4\pi\epsilon &= a_1 \int_0^{\Delta_x} \frac{1}{\sqrt{(x_N-x')+a^2}}\, dx' + \cdots + a_N \int_{(N-1)\Delta_x}^{N\Delta_x} \frac{1}{\sqrt{(x_N-x')+a^2}}\, dx' \end{aligned} \tag{2.10}$$

which comprises a matrix system of the form

$$\begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & \dots & Z_{1N} \\ Z_{21} & Z_{22} & Z_{23} & \dots & Z_{2N} \\ Z_{31} & Z_{32} & Z_{33} & \dots & Z_{3N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Z_{N1} & Z_{N2} & Z_{N3} & \dots & Z_{NN} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_N \end{bmatrix} \tag{2.11}$$

where the matrix elements Z_{mn} are

$$Z_{mn} = \int_{(n-1)\Delta_x}^{n\Delta_x} \frac{1}{\sqrt{(x_m - x') + a^2}}\, dx' \tag{2.12}$$

and the right-hand side (RHS) vector elements b_m are

$$b_m = 4\pi\epsilon\,. \tag{2.13}$$

2.1.1.1 Matrix Element Evaluation

The integral in (2.12) can be evaluated in closed form. Performing this integration yields [1] (200.01)

$$Z_{mn} = \log\left[\frac{(x_b - x_m) + \sqrt{(x_b - x_m)^2 - a^2}}{(x_a - x_m) + \sqrt{(x_a - x_m)^2 - a^2}}\right] \tag{2.14}$$

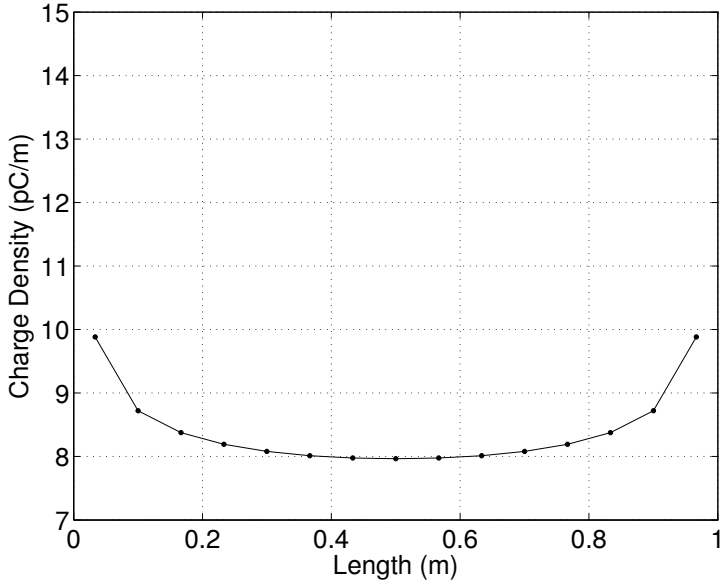
where $x_b = n\Delta_x$ and $x_a = (n-1)\Delta_x$. Note that the linear geometry of this problem yields a matrix that has a symmetric Toeplitz structure of the form

$$\mathbf{Z} = \begin{bmatrix} Z_1 & Z_2 & Z_3 & \dots & Z_N \\ Z_2 & Z_1 & Z_2 & \dots & Z_{N-1} \\ Z_3 & Z_2 & Z_1 & \dots & Z_{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Z_N & Z_{N-1} & Z_{N-2} & \dots & Z_1 \end{bmatrix} \tag{2.15}$$

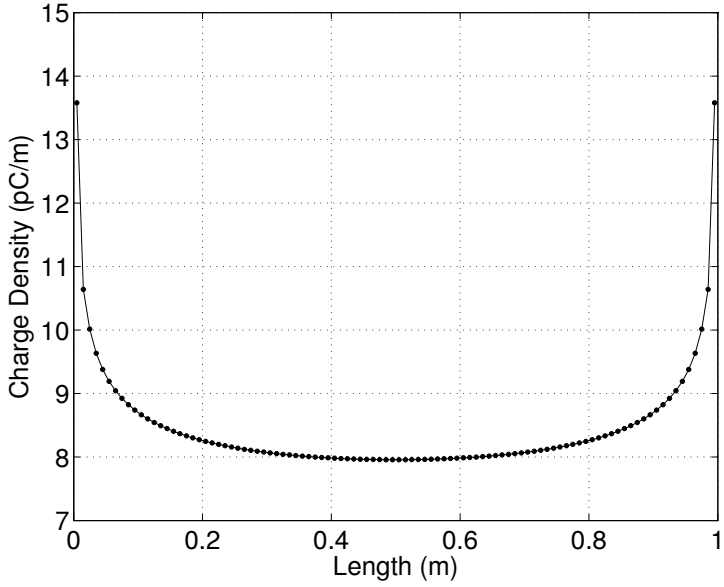
where the elements from just the first row can be used to populate the entire matrix.

2.1.1.2 Solution

In Figures 2.3a and 2.3b is plotted the computed charge density on the wire using 15 and 100 segments, respectively. The density at the lower discretization level is somewhat crude, as expected. The increase to 100 unknowns improves greatly the fidelity of the result. Using this charge density, we then compute the potential at 100 points along the wire using (2.2). The potential using 15 charge segments is shown in Figure 2.4a. While the voltage is near the expected value of 1V, it is not of constant value, particularly near the ends of the wire. Figure 2.4b shows the potential obtained using 100 segments. The voltage is now nearly constant across the entire wire, except at the endpoints. As a uniform segment size was used, the charge density tends to be oversampled in the middle of the wire and undersampled near the ends. As a result, the variation of the charge near the ends of the wire is not represented as accurately as in

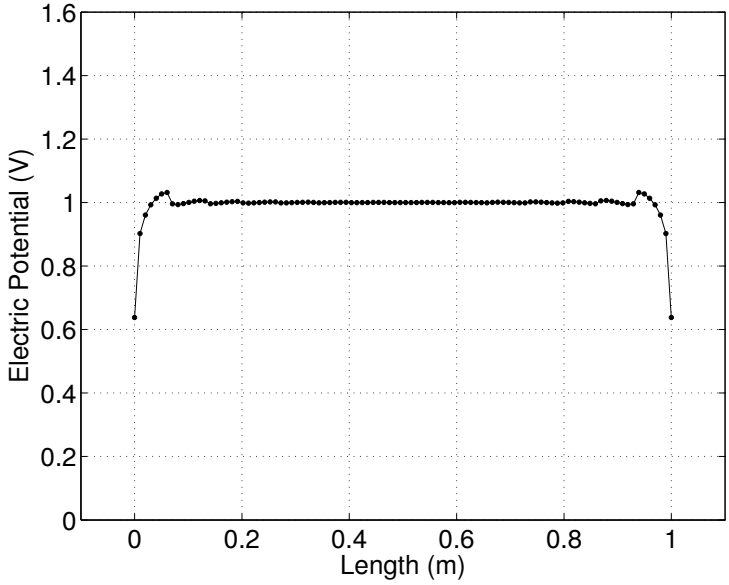


(a) 15 Segments

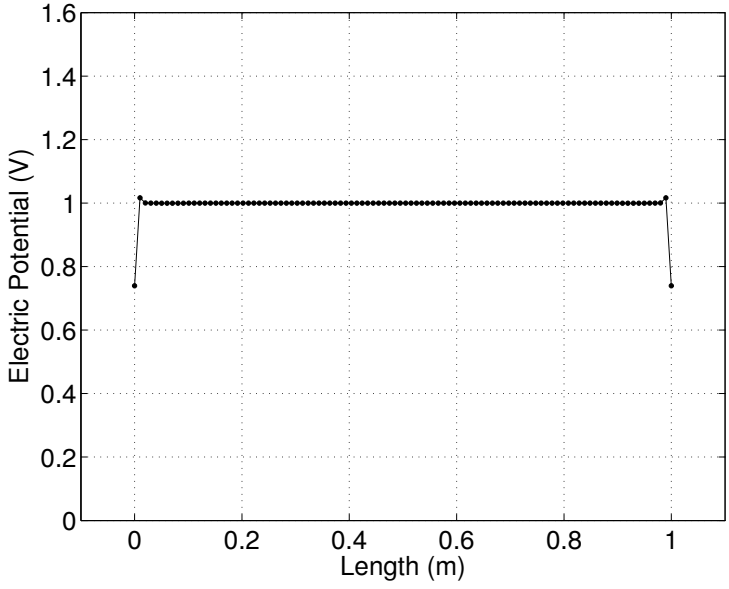


(b) 100 Segments

FIGURE 2.3: Straight Wire Charge Distribution



(a) 15 Segments



(b) 100 Segments

FIGURE 2.4: Straight Wire Potential

the center, and the computed voltage diverges from the true value. Realistic objects often have irregular surface features such as cracks, gaps, and corners that give rise to a more rapid variation in the solution at those points. To increase accuracy, it is advantageous to employ a denser level of discretization in areas where the most variation is expected.

2.1.2 Charged Plate

We next consider a similar problem involving a thin, charged conducting square plate of side length L, as shown in Figure 2.5. The potential on the plate can be written as

$$\phi_e(\mathbf{r}) = \int_{-\frac{L}{2}}^{\frac{L}{2}} \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{q_e(x', y')}{4\pi\epsilon|\mathbf{r} - \mathbf{r}'|}\, dx'\, dy' \tag{2.16}$$

and fixing the plate to a potential of 1V as before, (2.16) becomes

$$4\pi\epsilon = \int_{-\frac{L}{2}}^{\frac{L}{2}} \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{q_e(x', y')}{\sqrt{(x - x')^2 + (y - y')^2}}\, dx'\, dy' \,. \tag{2.17}$$

We now subdivide the plate into N square patches having an edge length of $2a$ and area $\Delta_s = 4a^2$, and we assume that the charge is constant within each patch. We choose N independent field points, each located at the center (x_m, y_m) of a patch. Doing so yields a matrix equation with elements Z_{mn} given by

$$Z_{mn} = \int_{S_n} \frac{1}{\sqrt{(x_m - x')^2 + (y_m - y')^2}}\, dx'\, dy' \tag{2.18}$$

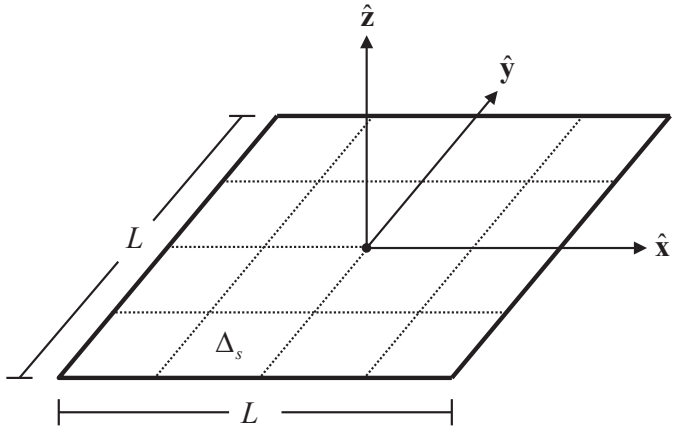


FIGURE 2.5: Thin Charged Plate Dimensions

where S_n comprises the extents of patch n. Right-hand side vector elements remain the same as in (2.13).

2.1.2.1 Matrix Element Evaluation

When the source and field patches are the same ($m = n$), the integrand has a singularity and the integral must be evaluated analytically. These matrix elements are called *self terms*, and represent the most dominant interactions between elements. We will devote significant attention to the evaluation of self-term matrix elements in this book. The self-term integral for the charged plate is

$$Z_{mm} = \int_{-a}^{a}\int_{-a}^{a} \frac{1}{\sqrt{(x')^2 + (y')^2}}\, dx'\, dy' \,. \tag{2.19}$$

Performing the innermost integration yields [1] (200.01)

$$Z_{mm} = \int_{-a}^{a} \log\left[\frac{\sqrt{a^2 + (y')^2} + a}{\sqrt{a^2 + (y')^2} - a}\right] dy' \tag{2.20}$$

and performing the outermost integration then yields [2]

$$Z_{mm} = 2a \log\left[y + \sqrt{a^2 + y^2}\right] + y \log\left[\frac{y^2 + 2a(a + \sqrt{a^2 + y^2})}{y^2}\right]\Bigg|_{-a}^{a} \tag{2.21}$$

which reduces to

$$Z_{mm} = \frac{2a}{\pi\epsilon} \log(1 + \sqrt{2}) \,. \tag{2.22}$$

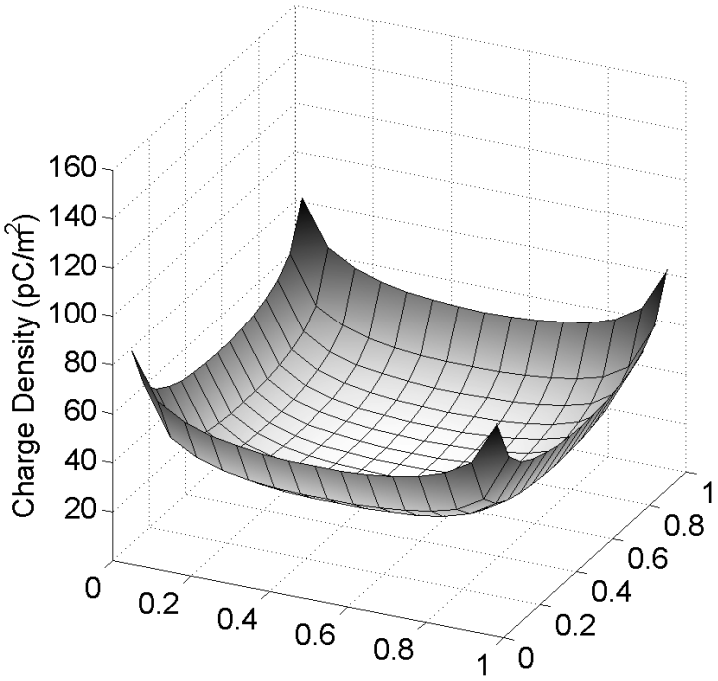
For patches that do not overlap ($m \neq n$) we will use a simple centroidal approximation to the integral, resulting in

$$Z_{mn} = \frac{\Delta_s}{\sqrt{(x_m - x_n)^2 + (y_m - y_n)^2}} \tag{2.23}$$

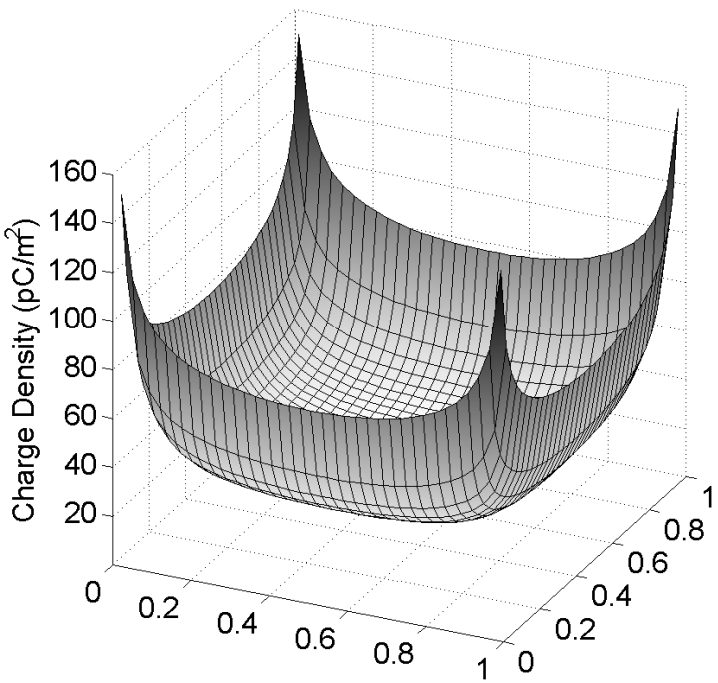
where x_n and y_n are at the center of the source patch. This approximation is not very accurate for elements that are near to one other, though it serves to illustrate solving the problem. In such cases, an analytic or adaptive numerical integration should be used instead. Matrix elements involving source and field points that do not overlap but are still sufficiently close are called *near terms*, and are also considered in greater detail in this book.

2.1.2.2 Solution

In Figures 2.6a and 2.6b is shown the computed surface charge density obtained using 15 and 35 patches in the x and y directions, which comprises 225 and 1225 unknowns, respectively. In Figures 2.7a and 2.7b is shown the surface charge density on the patches along the diagonal of the plate. As with the

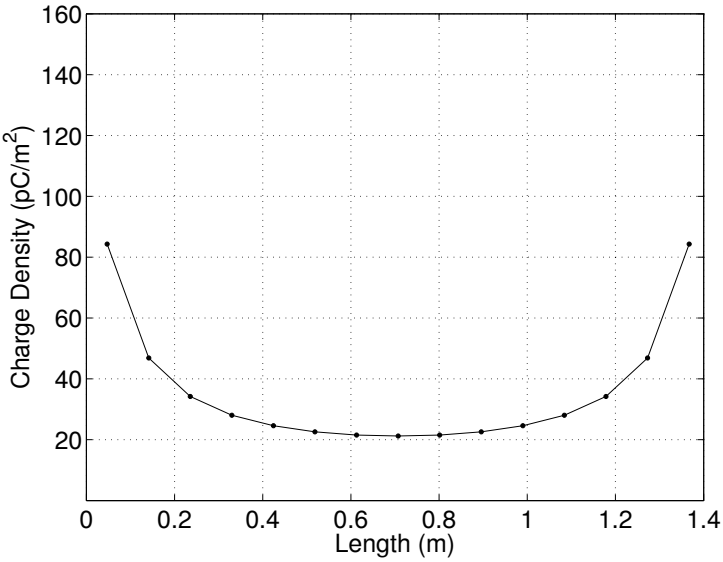


(a) 225 Patches

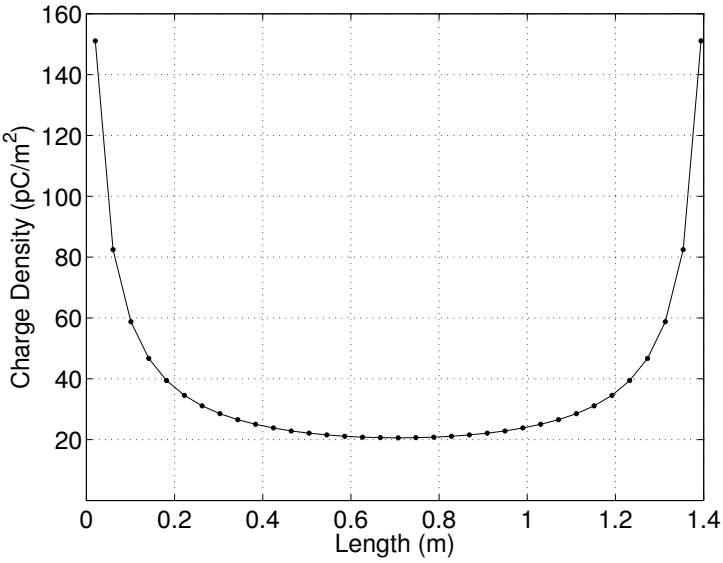


(b) 1225 Patches

FIGURE 2.6: Charge Distribution on Square Plate



(a) 225 Patches



(b) 1225 Patches

FIGURE 2.7: Potential on Square Plate Diagonal

thin wire, the charge accumulates near the corners and the edges of the plate, and our solution would likely benefit from additional discretization density in those areas.

2.2 The Method of Moments

In the previous section, we considered the problem of computing an unknown charge distribution on a thin wire or plate at a known potential. Our basic approach was to expand the unknown quantity using a set of known functions with unknown coefficients. We then converted the resulting equation into a linear system of equations by enforcing the boundary conditions, in this case the electric potential, at a number of field points on the object. The resulting linear system was then solved numerically for the unknown coefficients. Let us formalize this process by introducing a method of weighted residuals known as the Method of Moments. Consider the generalized problem

$$L(f) = g \tag{2.24}$$

where L is a linear operator, g is a known forcing function, and f is unknown [3]. In electromagnetic problems, L is typically an integro-differential operator such as $\mathcal{L}$ (3.77) and $\mathcal{K}$ (3.78), f is the unknown function (current) and g is a known driving function (incident field). Let us now expand f into a sum of N weighted basis functions

$$f = \sum_{n=1}^{N} a_n f_n \tag{2.25}$$

where a_n are unknown coefficients. Because L is linear, substitution of the above into (2.24) yields

$$\sum_{n=1}^{N} a_n L(f_n) \approx g \tag{2.26}$$

where the residual is

$$R = g - \sum_{n=1}^{N} a_n L(f_n)\ . \tag{2.27}$$

The basis functions f_n are chosen so they model the expected behavior of the unknown function throughout its domain, and they may be scalars or vectors depending on the problem. If the basis functions have local support in the domain, they are called *local* or *subsectional* basis functions. If their support spans the entire problem domain, they are called *global* or *entire-domain* basis functions. In this book we focus almost exclusively on local basis functions.

Let us now generalize the method by which the boundary conditions were previously enforced. We define an inner product or *moment* between a basis function $f_n(\mathbf{r}')$ and a *testing* or *weighting* function $f_m(\mathbf{r})$

$$\langle f_m, f_n \rangle = \int_{f_m} f_m(\mathbf{r}) \cdot \int_{f_n} f_n(\mathbf{r}') \, d\mathbf{r}' d\mathbf{r} \tag{2.28}$$

where the integrals can be line, surface, or volume integrals depending on the support of the basis and testing functions. Requiring the inner product of each testing function with the residual function to be zero yields

$$\sum_{n=1}^{N} a_n \langle f_m, L(f_n) \rangle = \langle f_m, g \rangle \tag{2.29}$$

which results in the $N \times N$ matrix equation $\mathbf{Za} = \mathbf{b}$ with matrix elements

$$Z_{mn} = \langle f_m, L(f_n) \rangle \tag{2.30}$$

and right-hand side vector elements

$$b_m = \langle f_m, g \rangle \, . \tag{2.31}$$

In electromagnetic problems, basis functions interact with all others via the Green's function and the resulting system matrix is full. This can be compared to other algorithms such as the Finite Element Method, where the matrix is typically sparse, symmetric, and banded, and many elements of each matrix row are zero [4].

2.2.1 Point Matching

In Sections 2.1.1 and 2.1.2, we enforced the boundary conditions by testing the integral equation at a set of discrete points on the object. This is equivalent to using a delta function as the testing function in (2.28), resulting in

$$f_m(\mathbf{r}) = \delta(\mathbf{r}) \, . \tag{2.32}$$

This method, referred to as *point matching* or *point collocation*, offers advantages as well as disadvantages. One benefit is that in evaluating the matrix elements, no integral is required over the range of the testing function, only that of the source function, which may make evaluation of the matrix elements easier. The primary disadvantage is that the boundary conditions are matched only at discrete locations throughout the solution domain. Regardless, the results are often still reasonable, and we will use this method for a few problems in this book for comparative purposes.

2.2.2 Galerkin's Method

For testing, we are free to use whatever functions we wish. However, for most problems the choice of testing function is crucial to obtaining a good solution. One of the most commonly used is the *Method of Galerkin*, where the basis functions are used as the testing functions. This has the advantage of enforcing the boundary conditions throughout the solution domain, instead of at discrete points as with point matching. However, this comes at the expense of increased complexity and computation time required to evaluate the matrix elements. Most of the MoM problems encountered in the literature use Galerkin testing as the standard method (with its use often assumed or implied), and we will do so as well for most of the problems in this book.

2.3 Common One-Dimensional Basis Functions

The most important characteristic of a basis function is that it can reasonably represent the behavior of the unknown function throughout its domain. If the solution has a high level of variation in a particular region, pulse basis functions may not be as good a choice as a linear or higher-order function. The choice of basis function also determines the level of difficulty in evaluating the MoM matrix elements, which in some cases may be quite high. We will now briefly consider some one-dimensional local basis functions commonly used in Moment Method problems, as well as entire-domain functions.

2.3.1 Pulse Functions

A set of pulse basis functions is depicted in Figure 2.8, where the domain has been divided into N points with $N-1$ subsegments/pulses. In our figure

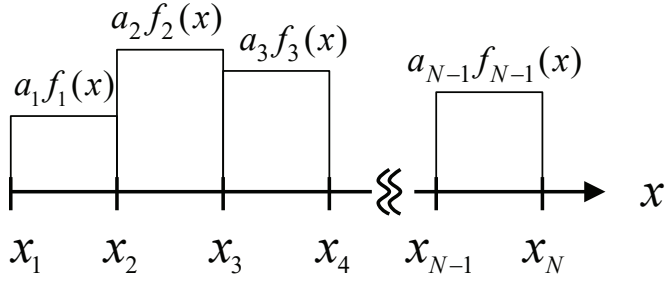


FIGURE 2.8: Pulse Functions

the segments all have equal lengths, However, this is not required. The pulse function is defined as

$$f_n(x) = 1 \qquad x_n \le x \le x_{n+1} \tag{2.33}$$

and zero otherwise. Pulse functions comprise a simple and crude approximation to the solution over each segment, but can greatly simplify the evaluation of MoM matrix elements. Note that since the derivative of pulse functions is impulsive, they cannot be used when the integral operator contains a derivative with respect to x [5].

2.3.2 Piecewise Triangular Functions

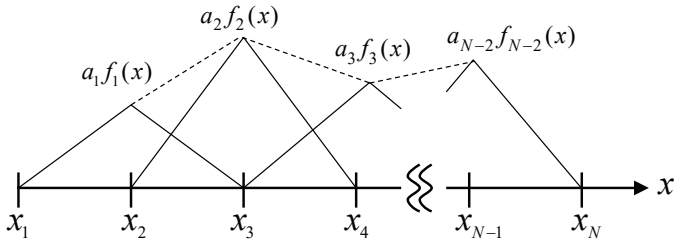


FIGURE 2.9: Triangle Functions (End Condition 1)

Where pulse functions are constant on a single segment, a triangle function spans two segments and varies from zero at the outer points to unity at the center. A set of triangle functions is shown in Figure 2.9. The domain has been divided into N points and $N-1$ subsegments, resulting in $N-2$ basis functions. Though we have again shown the segments as having equal length, this is not required. Since adjacent functions overlap by one segment, triangles provide a piecewise linear variation of the solution between segments. The triangle function is defined as

$$f_n(x) = \frac{x - x_{n-1}}{x_n - x_{n-1}} \qquad x_{n-1} \le x \le x_n \tag{2.34}$$

and

$$f_n(x) = \frac{x_{n+1} - x}{x_{n+1} - x_n} \qquad x_n \le x \le x_{n+1}\ . \tag{2.35}$$

Triangle functions can be used when the integral operator contains a derivative with respect to x, which is useful in redistributing vector derivatives.

Note that the configuration in Figure 2.9 forces the solution to zero at x_1 and x_N. This may be desirable when the value of the solution at the ends of the domain is known to be zero *a priori*, however it should not be used when