Finite Strip Method

Y.K. Cheung
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Finite Strip Method

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CRC Press
Taylor & Francis Group
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CRC Press
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
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PREFACE

Since the publication of the first book on the finite strip method in 1976, tremendous developments of the method have been made. Interpolation functions have been generalized such that it permits the usage of functions other than the trigonometrical series for the series component of the shape functions. The availability of these shape functions has not only greatly widened the applications of this method in structural analysis but also extended the applications to fields beyond structural engineering. Recently, researchers have further carried out non-linear analyses using this method, and current publications have demonstrated the activities and interests in this area. Furthermore, finite strips formulated by carrying out appropriate Fourier or Hankel transformations have been widely adopted with particular reference to the geotechnical field. In view of the recent advancement, it was decided that an updated text on the finite strip method should be written such that "state of the art" reference on this subject could be made available to both researchers as well as practising engineers.

In this book, we attempt to provide a concise introduction to the theory of the finite strip method including the classical strip as well as the newly developed spline strip and computed shape function strip. The applications of various topics in structural engineering with special reference to practical structures such as bridges, box girder bridges, tall buildings, etc. are extensively discussed. References to the applications in the geotechnical field are also made. Recent development of the applications of the method in nonlinear analyses is also briefly described.

A detailed introduction on the theory of the finite strip method is made in Chapters 1 and 2. Chapters 3 to 5 cover the applications of the method in fundamental problems including plate bending, plane stress/strain as well as vibration and stability problems. The extension of the method to three-dimensional analysis is explored in Chapter 6. The domain transformation which has proved to be a valuable method for mapping complex geometry into regular domain is discussed in Chapter 7. The recent applications of finite strip method in various areas of engineering significance are fully covered in Chapters 8, 9 and 10. Development in nonlinear applications is reserved to Chapter 11, whereas the next chapter concentrates on the transformation approach of the method for semi-infinite domains.

The source code of a program for the static analysis of folded plate structures is listed in Appendix I. The program is written for the IBM SP2 but it can also easily be adapted for the micro-computer environment.

ACKNOWLEDGMENTS

The authors would like to express their sincere gratitude to Associate Professor J. C. Small of University of Sydney and Associate Professor S. C. Fan of Nanyang Technological University for reading parts of the manuscripts and making helpful suggestions, and to Ms. K. Wong for preparing the figures.

Permission from our former graduate students for us to use material from their theses are acknowledged :

Dr. F. T. K. Au,
Dr. C. R. C. Delcourt,
Associate Professor S. C. Fan,
Dr. D. J. Guo,
Dr. J. Kong,
Mr. H. Y. Szeto,
Associate Professor S. Swaddiwudhipong, and
Professor D. S. Zhu.

The contribution of the late Dr. W. Y. Li to the development of the spline finite strip method is also acknowledged. We wish to pay tribute to him by making extensive references to materials from his thesis.

Our sincere thanks are also extended to American Society of Civil Engineers, Elsevier Science, John Wiley & Sons Limited, Structural Engineering International and Thomas Telford Publishing for their permission to use material from their journals.

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CHAPTER ONE

INTRODUCTION

Rapid developments of various numerical techniques in engineering analyses have occurred in the last three decades as faster and bigger computers are made available in rapid succession. It is widely acknowledged that the finite element method[1] has dominated the field, but various other methods, such as the finite strip and the boundary element[2] methods, continue to have their own roles to play and have not been outclassed in their more specialized areas.

The finite element method, being the most versatile tool, requires discretisation in every dimension of the problems, and, therefore, will generally require more unknowns for approximation than the other methods. The advent of super-computers has provided engineers and scientists with the means to handle millions of unknowns, and problems which had once been considered to be intractable because of their complex nature and size have been solved successfully by the finite element method. Though the cost of the solutions has decreased considerably, it is still far from being considered cheap. The cost also multiplies by orders of magnitude when a more refined, or a higher dimensional analysis is required. Furthermore, the availability of such powerful machines is still limited to the privileged few, and this is particularly true in developing countries. In such countries, most computations have to be carried out either on less powerful machines or personal micro-computers. Very often the problem size of an accurate analysis might be so overwhelming as to overtax machines of these grades, so that the problem either will have to be solved roughly, or some additional lengthy, time-consuming subroutines have to be written to lower the core requirements. It is also well known that for many problems having regular geometric shapes and simple boundary conditions a full finite element analysis is very often both extravagant and unnecessary. Therefore, alternative methods that can reduce the computational effort and core requirements, while at the same time retaining to a great extent the versatility of the finite element analysis, are evidently desirable.

The finite strip method for two-dimensional problems, which has been developed since the late sixties, is one of the alternative methods. In the three-dimensional regime, the method is referred to as either the finite prism method or the finite layer method depending on the form of the functions chosen for describing the variables for the analyses.

The philosophy of the finite strip methods is similar to the Kantorovich[3] method which is used extensively for reducing partial differential equations to ordinary or partial differential equations of a lower order. In the finite strip method, the reduction is achieved either by assuming that the separation of variable approach can be applied in expressing the interpolation functions of the unknowns (*separation of variable approach*) or by carrying out suitable transformations (*transformation approach*).

Early formulations of the finite strips have been based on the former approach. The methods initially call for the use of simple polynomials in some directions and continuously differentiable smooth trigonometrical as well as hyperbolical series in the other directions. The general form of the displacement function is given as a product of polynomials and series. Broadly speaking, all series, which can satisfy *a priori* the boundary conditions at the ends of the strips (prisms or layers), can also be used. For example, exponential and decaying power series have been employed in the analyses of field problems and stress problems with one or two boundaries of infinite extent. Such an approach can be referred to as the *classical* finite strip method, and a brief comparison between this method and the finite element method is listed in Table 1.1.

The recently developed spline finite strip and computed shape function strip have further increased the flexibility in the choice of the interpolation functions by allowing polynomials to be used in all directions.

The spline finite strip, which has been devised to overcome the difficulties experienced by the classical finite strip method in dealing with multi-span or column-supported structures, has chosen the B-3 spline functions to replace the trigonometrical series and hyperbolic series in the interpolation functions for plates and shell analyses. Since the B-3 spline functions are, in fact, the solution of a beam under a point load, the spline strips also show considerable improvements in convergence in simulating the action of a point load. In conjunction with the domain transformation concept, the method can handle arbitrarily shaped structures easily. In addition, the localized property of the spline functions has allowed all characteristic matrices or vectors of the analyses to be formed section by section. Integration for the formation of these matrices or vectors can be carried out by explicit formulae (for regular domain) or numerical integration schemes (for arbitrarily shaped structures). Apart from being nearly as versatile as the standard finite element method, the method can also achieve a higher order of continuity (C^2 continuity) with a smaller number of degrees of freedom.

Two forms of computed shape function strips (transverse and longitudinal strips) were made available recently for structural analysis. The transverse (longitudinal) component of the shape function of a transverse (longitudinal) strip is obtained by analyzing a typical transverse (longitudinal) section. In the analysis, a unit load is placed sequentially on selected points (joints) of the section, and deflections of the joints due to such loads are computed by a simple frame stress analysis program. These deflections constitute the basis of the interpolation parameters along the transverse (longitudinal) direction. As

TABLE 1.1
Comparison between finite element and classical finite strip methods.

Finite element	Finite strip
Applicable to any geometry, boundary conditions and material variation. Extremely versatile and powerful.	In static analysis, more often used for structures with two opposite simply supported ends and with or without intermediate elastic supports. In dynamic analysis it is used for structures with all boundary conditions and with discrete supports.
Usually large number of equations and matrix with comparatively large bandwidth. Can be very expensive and at times impossible to work out solution because of limitation in computing facilities.	Usually much smaller number of equations and matrix with narrow bandwidth, especially true for problems with an opposite pair of simply supported ends. Consequently much shorter computing time for solution of comparable accuracy.
Large quantities of input data and easier to make mistakes. Requires automatic mesh and load generation schemes.	Very small amount of input data because of the small number of mesh lines involved due to the reduction in dimensional analysis
Large quantities of output. As a rule all nodal displacements and element stresses are printed. Also many lower order elements will not yield correct stresses at the nodes and stress averaging or interpolation techniques must be used in the interpolation of results.	Easy to specify only those locations at which displacements and stresses are required and then output accordingly.
Requires a large amount of core and is more difficult to program. Very often, advanced techniques such as mass condensation, subspace iteration or Lanczos method have to be resorted to for eigenvalue problems in order to reduce core requirements.	Requires smaller amount of core and is easier to program. Because only the lowest few eigenvalues are required (for most cases anyway), the first two or three terms of the series will normally yield sufficiently accurate results for the classical finite strip method. Matrix can usually be solved by standard eigenvalue subroutines.

in the finite element method, simple polynomials in terms of the joint deflections are used to interpolate for the deflection between the joints. Standard polynomials are chosen for the other components of the shape functions.

Applications of Fourier and Hankel transforms in the analysis of geotechnical problems are examples of the transformation approach. The transformations reduce the governing differential equations for each strip (layer) into lower order equations for which solutions can be sought. The undetermined coefficients of the solutions can then be determined by conditions based on physical relations (e.g. flexibility relations).

Though the underlying principle is the same for the two approaches, the formulations and procedures are slightly different. For clarity, the transformation approach will be discussed in detail in Chapter 12 after allowing for a more systematic coverage on the separation of variable approach in the previous eleven chapters. Nevertheless, the following common procedures are adopted for both approaches:

(i) The continuum is divided into two- (strips) or three-dimensional (prisms, layers) subdomains in which one opposite pair of sides (2-dimensional) or one or more opposite pairs of faces (3-dimensional) of such a subdomain are in coincidence with the boundaries of the domain (Figure. 1.1). The geometry of the domain is preferably constant along one or two coordinate axes so that the width of a strip or the cross-section of a prism or layer will not change from one end to the other. Recent advances show that two-dimensional (three-dimensional) problems defined in irregular geometrical domain can be dealt with by transforming the domain into a unit square (cube) before the discretisation. For convenience, only the strip will be discussed in the general formulation, which is in any case also applicable to prisms and layers.

(ii) The strips are assumed to be connected to one another along a discrete number of nodal lines which coincide with the longitudinal boundaries of the strips. In some cases it is also possible to use internal nodal lines to arrive at a higher order strip.

The degrees of freedom (DOF) at each nodal line, called nodal parameters, are normally connected with the displacements and the partial derivatives of the displacements. They can also include non-displacement terms such as forces and strains (including direct strains, shear strain, bending and twisting curvatures) in stress analysis; pore pressure in consolidation and seepage analyses; as well as temperature in thermal analysis, etc.

Due to the use of continuous functions in the longitudinal direction, the DOF of the strip nodal line is usually less than that at the element node. For example, in plate bending, w and θ_x exist at each strip nodal line while w, θ_x, θ_y exist at each element node.

(iii) An interpolation function (or functions), in terms of the nodal parameters, is (are) chosen to represent the variables. Consequently, the stress/strain field (in the case of stress problems) and the velocity field (in the case of field problems) can be derived.

(iv) Based on the chosen functions, it is possible to obtain the respective characteristic matrices through variational or energy principles. In the case of static analysis of structural problems, the stiffness and load matrices which equilibrate the various external loadings acting on the strip are formed through minimum total potential energy principles. The principle states that

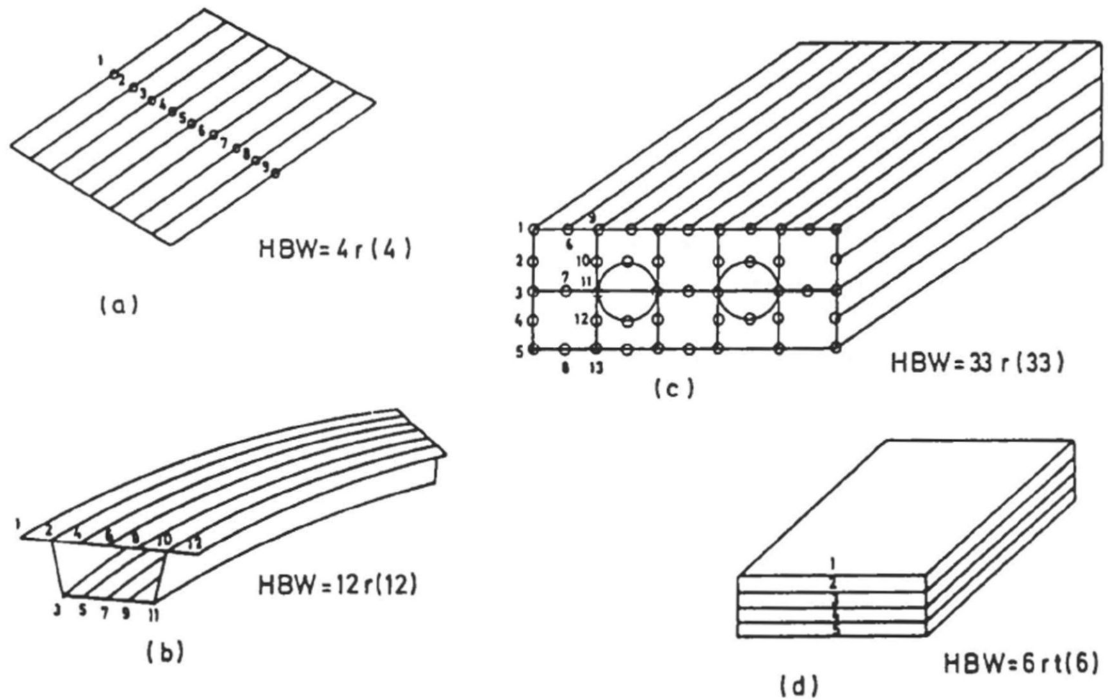

Figure 1.1 Some structures and typical mesh divisions: (a) Encastre slab (plate strips). (b) Curved box girder bridge (shell strips). (c) Voided slab bridge with circular holes (quadrilateral finite prisms). (d) Thick, multi-layer plate (finite layer).
HBW = half bandwidth; r,t = number of terms of series; ( ) =simply supported case.

of all compatible displacements satisfying the boundary conditions, those which satisfy the equilibrium conditions make the total potential energy assume a stationary value.

In mathematical form we have

$$\left[\frac{\partial \phi}{\partial \{\delta\}}\right] = 0 \tag{1.1}$$

in which the total potential energy ϕ is defined as the sum of the potential energy of external forces W and the strain energy U.

Other approaches may be used to formulate the solutions as well. For example, flexibility approach has been used to formulate the structural problems, and weighted residual method is widely used in the formulation for the field problems.

(v) The characteristic matrices of all strips are assembled to form a set of the overall equations for the whole domain under consideration. Since the bandwidth and the size of the matrix are usually small, the equations can be solved easily by any standard band matrix solution technique to yield the nodal parameters.

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CHAPTER TWO

INTERPOLATION SHAPE FUNCTIONS

2.1 CHOICE OF INTERPOLATION SHAPE FUNCTIONS

It has been pointed out in the previous chapter that the choice of suitable interpolation functions for a strip based on the separation of variable approach is the most crucial part of the analysis, and great care must be exercised at such a stage. A wrongly chosen interpolation function might not just produce obviously ridiculous answers but might even lead to results which converge to the wrong answer for successively refined meshes.[1]

A classical strip reduces a two-dimensional problem to a one-dimensional one with the shape functions for the interpolation variables in the analysis given as

$$\delta = \sum_{m=1}^{r} f_m(x)\, Y_m(y) \tag{2.1a}$$

Similarly, three-dimensional problems can be reduced to two-dimensional or one-dimensional ones in cases of finite prism and finite layer respectively. The shape functions for the finite prisms are

$$\delta = \sum_{m=1}^{r} f_m(x,z)\, Y_m(y) \tag{2.1b}$$

while those for the finite layers are

$$\delta = \sum_{n=1}^{t} \sum_{m=1}^{r} f_{mn}(x)\, Y_m(y)\, Z_n(z) \tag{2.1c}$$

In the above expressions, $f_m(x)$, $f_m(x,z)$ and $f_{mn}(x)$ are polynomial expressions with undetermined constants for the m-th and mn-th terms. $Y_m(y)$ and $Z_n(z)$ are series which satisfy the end conditions in the y and z directions respectively.

Recently developed spline and computed shape function strips have allowed the replacement of the series component by polynomials. The interpolation variables are then written as

$$\delta = f(x)\ \phi_m(y) \tag{2.1d}$$

where m represents the node number and mode number for the spline and computed shape function strips respectively.

To ensure convergence to correct results the following simple requirements have to be satisfied:

(i) The series part (Y_m and Z_n) or ϕ_m (after modifications) of the interpolation function should satisfy *a priori* the end conditions (for vibration problems only the displacement conditions have to be satisfied). For example, for a simply supported plate strip in bending, the displacement function should be able to satisfy the conditions of both deflection w and normal curvature $\partial^2 w/\partial y^2$ being equal to zero at the two ends.

(ii) The polynomial part of the interpolation function ($f_m(x)$, $f_m(x,z)$ and $f_{mn}(x)$) must be able to represent a state of constant 'strain' in the transverse (x) direction or x-z plane. If this is not obeyed, then there is no guarantee of convergence as the mesh is further and further refined. The methods for the checking of the constant 'strain' conditions are discussed in detail in Appendix 2.A.

(iii) The interpolation function must satisfy the compatibility requirement along the boundaries with the neighbouring strips, and this might include the continuity of the partial derivatives of the interpolated variables as well as the variables themselves.
The above statement in the stress analysis context can also be rephrased as:
"*the displacement function should be chosen in such a way that the strains which are required in the energy formulations should remain finite at the interface between the strips*".
Thus for in-plane and three-dimensional elasticity the strains involved are the first partial derivatives, and only the displacements need to be continuous. On the other hand, for bending problems the strains involved are the second partial derivatives of the deflection (curvatures), and therefore both the deflection and its first derivatives have to be continuous along the interface.
If such conditions are complied with, then there will be no infinite strains at the interface and therefore no contribution to the energy formulation from the interface, which can be considered as a narrow strip of area converging to zero. Only in this way can we be assured that a simple summation of the total potential energy of all strips will in fact be equal to the potential energy of the elastic body in question. The total potential energy of such a finite strip representation will always provide an approximate energy greater than the true one and, therefore, give a bound to the absolute total potential energy of the elastic system. A detailed mathematical discussion of this condition was presented in a paper by Tong and Pian.(2)
Similar argument for field problems, which are described by second order differential equations, leads to the fact that only the variables, such as

excess pore water pressure and temperature, have to be continuous along the boundaries.

2.2 AVAILABLE INTERPOLATION FUNCTIONS FOR CLASSICAL FINITE STRIPS

Since an interpolation function is always made up of two parts [see Eqs. 2.1a to 2.1c], it would be more convenient for us to discuss each part separately. The polynomial component (shape function) is governed by the shape of the cross-section (e.g. line, triangle, etc.) together with the nodal arrangement within the cross-section, and the series part is determined by the end conditions.

2.2.1 SERIES PART OF INTERPOLATION FUNCTIONS

Depending on the boundary conditions as well as the nature of the problems, different series have been sought. Some of the choices are listed as follows:

(1) Vibration eigenfunctions
(2) Buckling eigenfunctions
(3) Exponential functions
(4) Decaying power series

2.2.1.1 Vibration eigenfunctions

The most commonly used series for stress analysis are the basic functions (or eigenfunctions) which are derived from the solution of the beam vibration differential equation

$$\frac{\partial^4 Y}{\partial y^4} = \frac{\mu^4}{l^4} Y \tag{2.2}$$

where l is the length of the strip and μ is a parameter.

The general solution of Eq. 2.2 is

$$Y(y) = C_1 \sin(\frac{\mu y}{l}) + C_2 \cos(\frac{\mu y}{l}) + C_3 \sinh(\frac{\mu y}{l}) + C_4 \cosh(\frac{\mu y}{l}) \tag{2.3}$$

with the coefficients C_1, etc., to be determined by the end conditions. These have been worked out explicitly in the literature[3] for the various end conditions and are listed below:

(a) Both ends simply supported ($Y(0) = Y''(0) = 0$, and $Y(l) = Y''(l) = 0$)

$$Y_m(y) = \sin(\frac{\mu_m y}{l}) \qquad (\mu_m = \pi, 2\pi, 3\pi, \ldots m\pi) \tag{2.4}$$

(b) Both ends clamped ($Y(0) = Y'(0) = 0$, and $Y(l) = Y'(l) = 0$)

$$Y_m(y) = \sin(\frac{\mu_m y}{l}) - \sinh(\frac{\mu_m y}{l}) - \alpha_m[\cos(\frac{\mu_m y}{l}) - \cosh(\frac{\mu_m y}{l})] \tag{2.5}$$

$$\alpha_m = \frac{\sin\mu_m - \sinh\mu_m}{\cos\mu_m - \cosh\mu_m} \qquad (\mu_m = 4.7300,\ 7.8532,\ 10.9960,..\ \frac{2m+1}{2}\pi\ )$$

(c) One end simply supported and the other end clamped (Y(0) = Y"(0) = 0, and Y(l) = Y'(l) = 0)

$$Y_m(y) = \sin(\frac{\mu_m y}{l}) - \alpha_m \sinh(\frac{\mu_m y}{l}) \tag{2.6}$$

$$\alpha_m = \frac{\sin\mu_m}{\sinh\mu_m} \qquad (\mu_m = 3.9266,\ 7.0685,\ 10.2102,...\ \frac{4m+1}{4}\pi\ )$$

(d) Both ends free (Y"(0) = Y"'(0) = 0, and Y"(l) = Y"'(l) = 0)

$$Y_1(y) = 1, \qquad \mu_1 = 0 \tag{2.7a}$$

$$Y_2(y) = 1 - \frac{2y}{l} \qquad \mu_2 = 1 \tag{2.7b}$$

$$Y_m(y) = \sin(\frac{\mu_m y}{l}) + \sinh(\frac{\mu_m y}{l}) - \alpha_m[\cos(\frac{\mu_m y}{l}) + \cosh(\frac{\mu_m y}{l})] \tag{2.7c}$$

$$\alpha_m = \frac{\sin\mu_m - \sinh\mu_m}{\cos\mu_m - \cosh\mu_m} \qquad (\mu_m = 4.7300,\ 7.8532,\ ..\ \frac{2m-3}{2}\pi\ ,\ m=3,4,\ ....)$$

(e) One end clamped and the other end free (Y(0) = Y'(0) = 0, and Y"(l) = Y"'(l) = 0)

$$Y_m(y) = \sin(\frac{\mu_m y}{l}) - \sinh(\frac{\mu_m y}{l}) - \alpha_m[\cos(\frac{\mu_m y}{l}) - \cosh(\frac{\mu_m y}{l})] \tag{2.8}$$

$$\alpha_m = \frac{\sin\mu_m + \sinh\mu_m}{\cos\mu_m + \cosh\mu_m} \qquad (\mu_m = 1.875,\ 4.694,\ ....\ \frac{2m-1}{2}\pi\ )$$

(f) One end simply supported and the other end free (Y(0) = Y"(0) = 0, and Y"(l) = Y"'(l) = 0)

$$Y_1(y) = \frac{y}{l}, \qquad \mu_1 = 1 \tag{2.9a}$$

$$Y_m(y) = \sin(\frac{\mu_m y}{l}) + \alpha_m \sinh(\frac{\mu_m y}{l})] \tag{2.9b}$$

$$\alpha_m = \frac{\sin\mu_m}{\sinh\mu_m} \qquad (\mu_m = 3.9266,\ 7.0685,\ 10.2102,...\ \frac{4m-3}{4}\pi\ )$$

(g) Continuous beam[4]

To derive the series solution for a continuous beam consisting of S spans (Figure 2.1), the eigenfunctions can be computed by using a general stiffness approach. The vibration of the s-th span of such a beam, with reference to the local coordinate system, is defined by:

$$\frac{d^4 Y_s}{dy^4} - \frac{m_s \omega^2}{E_s I_s} Y_s = 0 \tag{2.10}$$

Y_s is the amplitude of the mode and ω is the natural frequency of the beam. I_s, m_s, and E_s are respectively the second moment area of the section, the mass per unit length and Young's modulus.

It can be shown readily that the general solution (basis function) for Eq. 2.10 is

$$Y_{sm} = A_{sm}\sin G_s\mu_m y_s + B_{sm}\cos G_s\mu_m y_s + C_{sm}\sinh G_s\mu_m y_s + D_{sm}\cosh G_s\mu_m y_s \quad (2.11)$$

where $\mu_m = \left[\dfrac{m\omega_m^2}{EI}\right]^{1/4}$ and $G_s = \left[\dfrac{m_s}{E_sI_s}\right]^{1/4}\left[\dfrac{m}{EI}\right]^{-1/4}$. E, I and m are the reference values of the corresponding variables and are usually taken as the parameters of the first span.

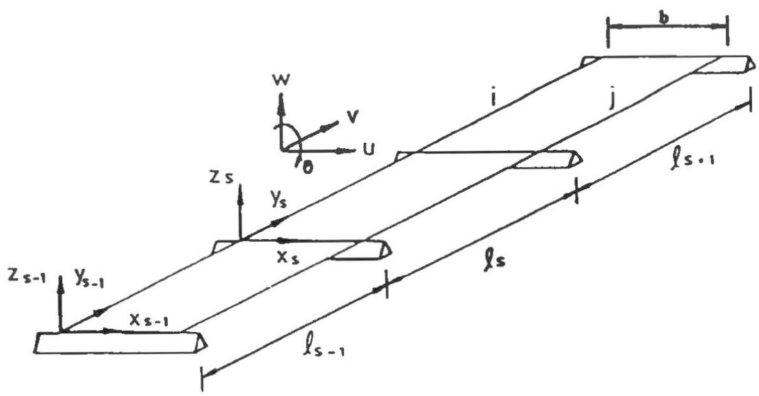


Figure 2.1 A typical continuous finite strip.

The undetermined coefficients, μ_m, A_{sm}, B_{sm}, C_{sm}, D_{sm} are determined by considering the boundary conditions at both end supports and at all intermediate supports. They can be determined by the procedures described as follows.

Inserting the coordinates of the two end supports of the s-th span, the end deflections and rotations can be written as

$$\begin{Bmatrix} Y_{sm}(0) \\ \theta_{sm}(0) \\ Y_{sm}(l_s) \\ \theta_{sm}(l_s) \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ -\beta & 0 & -\beta & 0 \\ s_1 & c_1 & s_2 & c_2 \\ -\beta c_1 & \beta s_1 & -\beta c_2 & \beta s_s \end{bmatrix} \begin{Bmatrix} A_{sm} \\ B_{sm} \\ C_{sm} \\ D_{sm} \end{Bmatrix} \quad (2.12)$$

The two end moments for this particular span can also be written in terms of the undetermined coefficients as:

$$\begin{bmatrix} M_{sm}(0) \\ M_{sm}(l_s) \end{bmatrix} = E_s I_s \beta^2 \begin{bmatrix} 0 & -1 & 0 & 1 \\ s_1 & c_1 & s_2 & c_s \end{bmatrix} \begin{Bmatrix} A_{sm} \\ B_{sm} \\ C_{sm} \\ D_{sm} \end{Bmatrix} \tag{2.13}$$

In the above equations (Eqs. 2.12 and 2.13) the following definitions are adopted:

$$\theta_{sm} = -\frac{dY_{sm}}{dy_s}, \quad M_{sm} = -E_s I_s \frac{d^2 Y_{sm}}{d^2 y_s}, \quad \beta = G_s \mu_m$$

$s_1 = \sin\beta l_s$, $s_2 = \sinh\beta l_s$, $c_1 = \cos\beta l_s$, $c_2 = \cosh\beta l_s$

Taking into account the fact that the intermediate supports are rigid, end moments and slopes for each span (except the first and last spans) are related as:

$$\begin{Bmatrix} M_{sm}(0) \\ M_{sm}(l_s) \end{Bmatrix} = \begin{bmatrix} \alpha(l_s) & \gamma(l_s) \\ \gamma(l_s) & \alpha(l_s) \end{bmatrix} \begin{Bmatrix} \theta_{sm}(0) \\ \theta_{sm}(l_s) \end{Bmatrix} \tag{2.14}$$

in which $\alpha(l_s) = E_s\, I_s\, \beta(s_1\, c_2 - s_2\, c_1)/(1 - c_1\, c_2)$, and
$\gamma(l_s) = E_s\, I_s\, \beta(s_2 - s_1)/(1 - c_1\, c_2)$.

Having established the moment equilibrium and compatibility conditions at all intermediate supports, the following matrix equation can be obtained:

$$\begin{bmatrix} \gamma(l_1) & \alpha(l_1)+\alpha(l_2) & \gamma(l_2) & 0 & . & 0 & 0 & 0 & 0 \\ & \gamma(l_2) & \alpha(l_2)+\alpha(l_3) & \gamma(l_3) & . & 0 & 0 & 0 & 0 \\ & & & & . & . & . & . & . \\ & & & & & . & . & . & . \\ & & & & & & \gamma(l_{s-1}) & 0 & 0 \\ & & & & & \gamma(l_{s-1}) & \alpha(l_{s-1})+\alpha(l_s) & 0 & 0 \end{bmatrix} \begin{Bmatrix} \theta_{1m}(0) \\ \theta_{2m}(0) \\ \theta_{3m}(0) \\ . \\ . \\ \theta_{sm}(0) \\ \theta_{sm}(l_s) \\ Y_{1m}(0) \\ Y_{sm}(l_s) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ . \\ . \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{2.15}$$

Incorporating the boundary conditions at the end supports (two at each end), this equation is reduced to a square matrix. As examples, the reduced matrix equations for a three span beam with simply supported, free and clamped ends respectively are given as follows:[4]

(a) simply supported ends

$$\begin{bmatrix} \alpha(l_1) & \gamma(l_1) & 0 & 0 \\ \gamma(l_1) & \alpha(l_1)+\alpha(l_2) & \gamma(l_2) & 0 \\ 0 & \gamma(l_2) & \alpha(l_2)+\alpha(l_3) & \gamma(l_3) \\ 0 & 0 & \gamma(l_3) & \alpha(l_3) \end{bmatrix} \begin{Bmatrix} \theta_{1m}(0) \\ \theta_{2m}(0) \\ \theta_{3m}(0) \\ \theta_{3m}(l_3) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{2.16}$$

(b) free ends

$$\begin{bmatrix} \alpha(l_1) & -\beta\eta_1 & \gamma(l_1) & 0 & 0 & 0 \\ -\beta\eta_1 & \beta^2\xi_1 & -\beta\zeta_1 & 0 & 0 & 0 \\ \gamma(l_1) & -\beta\zeta_1 & \alpha(l_1)+\alpha(l_2) & \gamma(l_2) & 0 & 0 \\ 0 & 0 & \gamma(l_2) & \alpha(l_2)+\alpha(l_3) & \gamma(l_3) & \beta\zeta_3 \\ 0 & 0 & 0 & \gamma(l_3) & \alpha(l_3) & \beta\eta_3 \\ 0 & 0 & 0 & \beta\zeta_3 & \beta\eta_3 & \beta^2\xi_3 \end{bmatrix} \begin{Bmatrix} \theta_{1m}(0) \\ Y_{1m}(0) \\ \theta_{2m}(0) \\ \theta_{3m}(0) \\ \theta_{3m}(l_3) \\ Y_{3m}(l_3) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{2.17}$$

where $\zeta_s = E_s I_s \beta\,(c_2 - c_1)/(1 - c_1c_2)$; $\eta_s = E_s I_s \beta\, s_2 s_1/(1 - c_1c_2)$
$\xi_s = E_s I_s \beta\,(s_1c_2 + c_1s_2)/(1 - c_1c_2)$

(c) clamped ends

$$\begin{bmatrix} \alpha(l_1)+\alpha(l_2) & \gamma(l_2) \\ \gamma(l_2) & \alpha(l_2)+\alpha(l_3) \end{bmatrix} \begin{Bmatrix} \theta_{2m}(0) \\ \theta_{3m}(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{2.18}$$

In the next step, the non-trivial solution of μ_m is obtained by equating the determinant of this matrix to zero. The solution can be obtained either by the Newton-Raphson method[4] or 'modified regula falsi' method.[4] Attention should be drawn here to the fact that for the case of a beam clamped at both ends, Eq. 2.18 will miss out on the roots corresponding to the situation in which the whole displacement vector is equal to zero, that is, the continuous structure degenerates into a series of single span substructures with both end clamped. The eigenvalues of such beams are, of course, common knowledge and they can be inserted into the correct positions in the ascending array of μ_m. The position of missing roots can also be determined by using a method proposed by Wittrick and Williams[6] in which a 'sign count' of the matrix is made.

Having μ_m determined, the eigenfunction coefficients (A_{sm} to D_{sm}) can be determined by utilizing the moment equilibrium, the compatibility condition and the zero deflection conditions at intermediate and end supports,

a) for all spans

$$Y_s(0) = 0$$

$$Y_{s-1}(l_{s-1}) = 0$$

b) for all intermediate supports

$$Y'_{s-1}(l_{s-1}) = Y'_s(l_s)$$

$$EI_{s-1}Y''_{s-1}(l_{s-1}) = EI_s Y''_s(l_s)$$

c) for simple end support

$$Y''_1(0) = 0$$

d) for clamped end support

$$Y'_1(0) = 0$$

If A_{1m} is taken as a unit reference constant, the other coefficients can be determined accordingly. Referring to the simply-supported three span example once again, the independent unknowns can be chosen to be[4]

$$[A_{1m}, B_{1m}, C_{1m}, A_{2m}, B_{2m}, C_{2m}, A_{3m}, B_{3m}, C_{3m}]$$

and they can be determined by solving the following matrix equation:

$$\begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = [\mathbf{ABC}] \begin{Bmatrix} A_{1m} \\ B_{1m} \\ C_{1m} \\ A_{2m} \\ B_{2m} \\ C_{2m} \\ A_{3m} \\ B_{3m} \\ C_{3m} \end{Bmatrix} \tag{2.19}$$

The matrix [ABC] is given explicitly in Table 2.1.

The above functions [Cases (a) to (g)] are primarily used for bending strips. For two- and three-dimensional elasticity problems, which will be discussed in more detail in Chapter 4, both Y_m and Y_m' will be used for the u, v (and w) displacements. Only one other function has been employed successfully in plane analysis and will be discussed later.

Some of these basic functions also possess the valuable properties of orthogonality, i.e. for $m \neq n$

$$\int_0^l Y_m Y_n \, dy = 0 \tag{2.20a}$$

$$\int_0^l Y''_m Y''_n \, dy = 0 \tag{2.20b}$$

It will be observed that these integrals appear in all subsequent formulations, and the utilization of these properties results in a significant saving in computational effort.

TABLE 2.1
Matrix [ABC] (simply supported three span beam)[4].

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sin\mu_m l_1 & \cos\mu_m l_1 - \cosh\mu_m l_1 & \sinh\mu_m l_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \sin\mu_m l_2 & \cos\mu_m l_2 - \cosh\mu_m l_2 & \sinh\mu_m l_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \sin\mu_m l_3 & \cos\mu_m l_3 - \cosh\mu_m l_3 & \sinh\mu_m l_3 \\
\cos\mu_m l_1 & -\sin\mu_m l_1 - \sinh\mu_m l_1 & \cosh\mu_m l_1 & -1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & \cos\mu_m l_2 & -\sin\mu_m l_2 - \sinh\mu_m l_2 & \cosh\mu_m l_2 & -1 & 0 & -1 \\
-\sin\mu_m l_1 & -\cos\mu_m l_1 - \cosh\mu_m l_1 & \sinh\mu_m l_1 & 0 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\sin\mu_m l_2 & -\cos\mu_m l_2 - \cosh\mu_m l_2 & \sinh\mu_m l_2 & 0 & -2 & 0 \\
0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

The existence of so many functions should in no way be interpreted as a need for a matching number of computer subroutines. In actual practice all the relevant functions are stored in the same subroutine, and integrals involving different combinations of the functions are integrated numerically. Consequently problems with many different types of boundary conditions can be solved by the same program.

2.2.1.2 Buckling eigenfunctions

In stability analysis, more accurate results can be obtained if the buckling mode shape is used for the interpolation series. For example, the functions for a structure clamped at the base and free at the tip can be chosen to be:[7]

$$Y_m = 1 - \cos\frac{(2m-1)\pi y}{2} \tag{2.21}$$

2.2.1.3 Exponential functions

Attempts have been made to model the behaviour of layered medium of infinite extent in the horizontal plane by exponential functions,[8]

a) rectangular coordinate system

$$Y_m(x) = \exp(-\frac{mx}{L}) \tag{2.22}$$

b) circular coordinate system

$$Y_m(r) = \exp(-\frac{mr}{R}) \tag{2.23}$$

where L and R are length constants and are sufficiently large in magnitude so that the infinite extent of the medium can be modelled approximately. In most cases, only one term is necessary for the analysis.

Exponential series in the time variable have also been used in the solution of the parabolic differential equation.[9] For example, the temperature field is defined as:

a) one-dimensional problem

$$T(x,\tau) = \sum_{m=1}^{r} f_m(x)[1 - \exp(-\frac{m^2\pi^2\tau}{L_x^2})] \tag{2.24a}$$

b) two-dimensional problem

$$T(x,y,\tau) = \sum_{m=1}^{r}\sum_{n=1}^{r} f_{mn}(x,y)[1 - \exp(-\frac{m^2\pi^2\tau}{L_x^2} - \frac{n^2\pi^2\tau}{L_y^2})] \tag{2.24b}$$

c) three-dimensional problem

$$T(x,y,z,\tau) = \sum_{m=1}^{r}\sum_{n=1}^{r}\sum_{k=1}^{r} f_{mnk}(x,y,z)[1 - \exp(-\frac{m^2\pi^2\tau}{L_x^2} - \frac{n^2\pi^2\tau}{L_y^2} - \frac{k^2\pi^2\tau}{L_z^2})] \tag{2.24c}$$

where τ is the temporal variable; L_x , L_y and L_z are characteristic lengths in the corresponding directions.

2.2.1.4 Decaying power series

Instead of exponential series, decaying power series may also be chosen to model a medium of infinite extent in two directions,[10] such as layered soils. In order to reflect the difference in behaviour between the near and far fields, the series should consist of two parts:

a) near field functions ($r < R$): series of r^m

$$Y_1^1(r) = 1 \tag{2.25a}$$

$$Y_2^1(r) = \frac{r}{R} \tag{2.25b}$$

$$Y_3^1(r) = 3(\frac{r}{R}) - 4(\frac{r}{R})^2 \tag{2.25c}$$

$$Y_m^1(r) = \sum_{k=1}^{m-1} (-1)^{k-1} \frac{(m+k-1)!}{(m-k-1)!(k+1)!(k-1)!} (\frac{r}{R})^k \tag{2.25d}$$

b) far field function ($r > R$): series of $1/r^m$

$$Y_m^2(r) = (\frac{R}{r})^m \tag{2.25e}$$

in which R is distance of the interface of the far and near field from the centre.

Enforcing the continuity along the interface, it is possible to relate the two sets of interpolation parameters. For example, if L terms are chosen for the far field functions, we will have

$$\Sigma Y_m^1(R)\,\delta_m^1 = \Sigma Y_m^2(R)\,\delta_m^2 \tag{2.26a}$$

$$\Sigma \frac{d}{dr} Y_m^1(R)\,\delta_m^1 = \Sigma \frac{d}{dr} Y_m^2(R)\,\delta_m^2 \tag{2.26b}$$

$$\Sigma \frac{d^{L-1}}{dr^{L-1}} Y_m^1(R)\,\delta_m^1 = \Sigma \frac{d^{L-1}}{dr^{L-1}} Y_m^2(R)\,\delta_m^2 \tag{2.26c}$$

where δ_m^1 and δ_m^2 are the interpolation parameters for the near field and far fields respectively.

In matrix form,

$$\mathbf{A}\,\Delta^1 = \mathbf{B}\,\Delta^2 \tag{2.27}$$

where Δ^1 and Δ^2 are the vectors containing δ_m^1 and δ_m^2.

The elements ($\mathbf{a}_{ij}$ and $\mathbf{b}_{ij}$) of $\mathbf{A}$ and $\mathbf{B}$ are defined as follows:

$$\mathbf{a}_{ij} = \mathbf{r}_{ik}\,\mathbf{a}_{kl}^*\,\mathbf{c}_{jl} \text{ and } \mathbf{b}_{ij} = \mathbf{r}_{ik}\,\mathbf{b}_{kl}^*$$

$$\mathbf{r}_{ij} = \begin{cases} (\frac{1}{R})^{i-1} & i = j \\ 0 & i \neq j \end{cases} \quad ; \quad \mathbf{a}_{ij}^* = \begin{cases} \frac{(i-1)!}{(j-1)!} & i \geq j \\ 0 & i < j \end{cases}$$

$$c_{ij} = \begin{cases} 1 & i=j=1 \\ 0 & i=1\neq j \text{ or } i\neq j=1 \text{ or } i<j \\ (-1)^{j-2}\dfrac{(i+j-2)!}{(i-j)!(j-2)!j!} & i,j>1 \text{ and } i\geq j \end{cases}$$

$$b_{ij}^{*} = (-1)^{i-1}\frac{(i+j-2)!}{(i-1)!}$$

On simplification, the n-th term of the series can be written as

$$Y_m = \Gamma_{r_0 R}\, Y_m^1 + \Gamma_{R\infty} \sum_{l=1}^{L} e_{lm}\, Y_m^2 \tag{2.28}$$

where $e_{lm} = \sum_{k=1}^{L} b_{lk} a_{km}$

$$\Gamma_{pq} = \begin{cases} 1 & p \leq r \leq q \\ 0 & \text{elsewhere} \end{cases}$$

2.2.2 SHAPE FUNCTION PART OF THE INTERPOLATION FUNCTION

A polynomial shape function is a polynomial associated with a nodal interpolation parameter, and it describes the corresponding interpolation field within the cross-section of a strip when the nodal interpolation parameter in question is given a unit value. Such shape functions are, in fact, derived by specifying a normalized unit value of the relevant interpolation component at its own node, and a value of zero for the same interpolation component at all other nodes.

For example, the shape functions for a strip with three nodes (Eq. 2.1a) can be written as

$$w = \sum_{m=1}^{r} [C_1 \quad C_2 \quad C_3] \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{Bmatrix}_m Y_m = \sum_{m=1}^{r} Y_m \sum_{k=1}^{3} [C_k]\{\delta_k\}_m \tag{2.29}$$

in which $\{\delta_k\}_m$ is a vector representing the m-th term nodal displacement parameters (deflection) at nodes 1, 2, and 3, and C_1 , C_2 , C_3 are the shape functions associated with δ_1, δ_2, δ_3 respectively.

In order to satisfy the stated criterion, it can be shown readily that

at x=0	$[C_1\, C_2\, C_3] = [1\ 0\ 0]$
at x=b/2	$[C_1\, C_2\, C_3] = [0\ 1\ 0]$
and at x=b	$[C_1\, C_2\, C_3] = [0\ 0\ 1]$

These three relations suggest that the shape functions should be second degree polynomials. The methods for the derivation of such shape functions can be found in most standard finite element texts.

The main purpose of using the shape function directly instead of a simple polynomial with the undetermined constants is twofold: to avoid the lengthy process of relating the undetermined constants to the nodal displacement parameters, and to make sure that there is compatibility of displacement along the adjoining strips (prisms). The first point is rather obvious and requires no discussion. The second point can be best explained by noting that the displacements along any interface of adjoining strips (prisms) are uniquely determined by the displacement parameters at the node (or nodes) common to the adjoining strips (prisms), since by definition the shape function for the displacement parameters of any other node will take up zero value at the said interface. Many shape functions are available, and some of the more common ones are listed below.

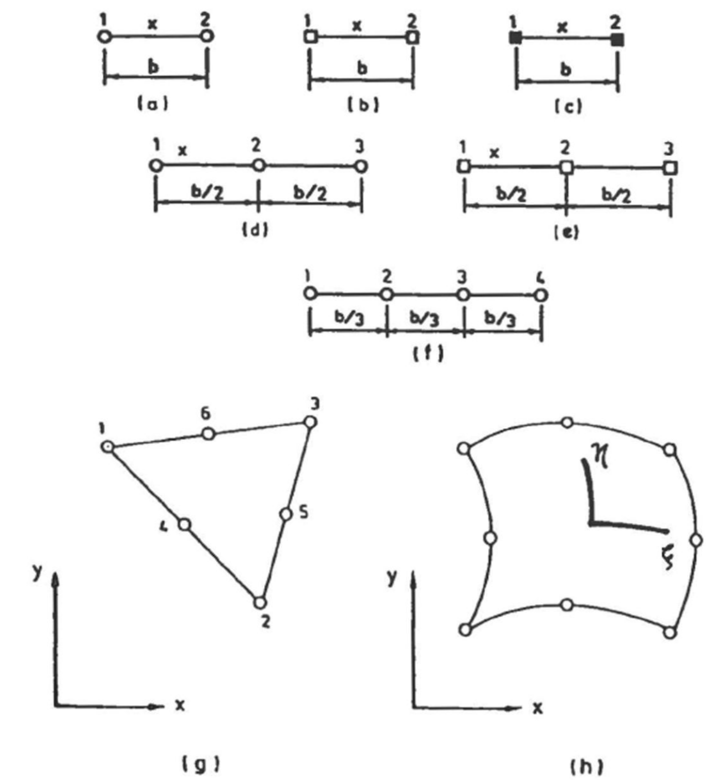


Figure 2.2 Strip and prism cross-sections.

Nodal DOF : ○ = f; □ = f, $\frac{\partial f}{\partial x}$; ■ = f, $\frac{\partial f}{\partial x}$; $\frac{\partial^2 f}{\partial x^2}$

(a) Straight line with two nodes (Figure 2.2a) and with variables (zero derivatives) as nodal parameters:

$$C_1 = (1-\bar{x}) \quad , \quad C_2 = \bar{x} \tag{2.30}$$

where $\bar{x} = x/b$

(b) Straight line with two nodes (Figure 2.2b); variables and first derivatives:

$$[C_1] = [(1-3\bar{x}^2+2\bar{x}^3) \, , \, x(1-2\bar{x}+\bar{x}^2)\,] \tag{2.31a}$$
$$[C_2] = [(3\bar{x}^2-2\bar{x}^3) \, , \, x(\bar{x}^2-\bar{x})\,] \tag{2.31b}$$

(c) Straight line with two nodes (Figure 2.2c); variables, first and second derivatives:

$$\begin{aligned}[C_1] = [&(1-10\bar{x}^3+15\bar{x}^4-6\bar{x}^5) \, , \, x(1-6\bar{x}^2+8\bar{x}^3-3\bar{x}^4), \\ &x^2(0.5-1.5\bar{x}+1.5\bar{x}^2-0.5\bar{x}^3)]\end{aligned} \tag{2.32a}$$
$$\begin{aligned}[C_2] = [&(10\bar{x}^3-15\bar{x}^4+6\bar{x}^5) \, , \, x(-4\bar{x}^2+7\bar{x}^3-3\bar{x}^4), \\ &x^2(0.5\bar{x}-\bar{x}^2+0.5\bar{x}^3)]\end{aligned} \tag{2.32b}$$

(d) Straight line with three nodes (Figure 2.2d); variables only:

$$C_1 = (1-3\bar{x}+2\bar{x}^2) \quad , \quad C_2 = (4\bar{x}-4\bar{x}^2) \quad , \quad C_3 = (-\bar{x}+2\bar{x}^2) \tag{2.33}$$

(e) Straight line with three nodes (Figure 2.2e); variables and first derivatives:

$$[C_1] = [(1-23\bar{x}^2+66\bar{x}^3-68\bar{x}^4+24\bar{x}^5) \, , \, x(1-6\bar{x}+13\bar{x}^2-12\bar{x}^3+4\bar{x}^4)] \tag{2.34a}$$
$$[C_2] = [(16\bar{x}^2-32\bar{x}^3+16\bar{x}^4) \, , \, x(-8\bar{x}+32\bar{x}^2-40\bar{x}^3+16\bar{x}^4)] \tag{2.34b}$$
$$[C_3] = [(7\bar{x}-34\bar{x}^3+52\bar{x}^4-24\bar{x}^5) \, , \, x(-\bar{x}+5\bar{x}^2-8\bar{x}^3+4\bar{x}^4)] \tag{2.34c}$$

(f) Straight line with four nodes (Figure 2.2f); variables only

$$C_1 = -\frac{9}{2}\,(\bar{x}-\frac{1}{3})(\bar{x}-\frac{2}{3})(\bar{x}-1) \tag{2.35a}$$
$$C_2 = \frac{27}{2}\,\bar{x}(\bar{x}-\frac{2}{3})(\bar{x}-1) \tag{2.35b}$$
$$C_3 = -\frac{27}{2}\,\bar{x}(\bar{x}-\frac{1}{3})(\bar{x}-1) \tag{2.35c}$$
$$C_4 = \frac{9}{2}\,\bar{x}(\bar{x}-\frac{1}{3})(\bar{x}-\frac{2}{3}) \tag{2.35d}$$

(g) Triangle with six nodes (Figure 2.2g); variables only:

(1) Corner nodes: $C_i = (2L_i-1)L_i \qquad (i=1,2,3)$ (2.36a)

(2) Midside nodes: $C_4 = 4L_1L_2$, $C_5 = 4L_2L_3$, $C_6 = 4L_3L_1$, (2.36b)

in which L_i, etc., are area coordinates.

A useful formula for integrating area coordinate quantities over the area of a triangle is

$$\iint L_1^a L_2^b L_3^c \, dxdy = \frac{a!b!c!}{(a+b+c+2)!} 2\Delta \tag{2.37}$$

in which Δ refers to the area of the triangle.

(h) Serendipity and Lagrange elements: variable only. For example 8-noded Serendipity elements (in natural coordinates) (Figure 2.2h):

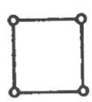
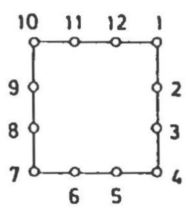
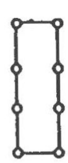
(1) Corner nodes:

$$C_i = \frac{1}{4}(1+\xi_o)(1+\eta_o)(\xi_o+\eta_o-1) \tag{2.38a}$$

(2) Midside nodes ($\xi_i = 0$)

$$C_i = \frac{1}{2}(1-\xi^2)(1+\eta_o) \tag{2.38b}$$

TABLE 2.2
Family of Serendipity and Lagrange elements ($\xi_o = \xi\xi_i$, $\eta_o = \eta\eta_i$).

	$\frac{1}{4}(1+\xi_o)(1+\eta_o)$
	$\frac{1}{32}(1+\xi_o)(1+\eta_o)[9(\xi^2+\eta^2)-10]$ (Corner nodes) $\frac{9}{32}(1+\xi_o)(1-\eta^2)(1+9\eta_o)$ (Nodes : 2,3,8,9) $\frac{9}{32}(1+9\xi_o)(1-\xi^2)(1+\eta_o)$ (Nodes : 5,6,11,12)
	$\frac{1}{32}(1+\xi_o)(1+\eta_o)(9\eta^2-1)$ (Corner nodes) $\frac{9}{32}(1+\xi_o)(1-\eta^2)(1+9\eta_o)$ (Side Nodes)

3) Midside nodes ($\eta_i = 0$)

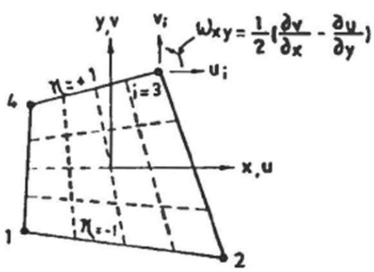
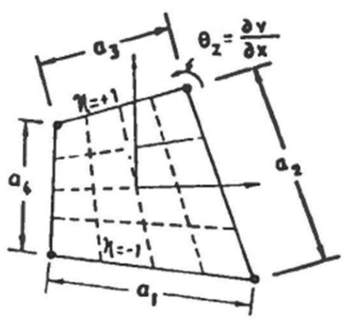
$$C_i = \frac{1}{2}\ (1+\xi_o)(1-\eta^2) \tag{2.38c}$$

with $\xi_o = \xi\xi_i$, $\eta_o = \eta\eta_i$, and ξ_i and η_i being the ξ and η coordinates of the i-th node.

Table 2.2 lists a number of other members of the element families that have been called up for use in the finite strip analysis. In addition, such elements can also be used for domain (geometric) transformation[11].

(i) Higher order quadrilateral elements;[12] interpolation parameters include the variables as well as the derivatives of the variables. Some examples of such elements are given in Table 2.3.

TABLE 2.3
Higher order quadrilateral elements ($\xi_o = \xi\xi_i$, $\eta_o = \eta\eta_i$).

<table>
<tr>
<td>
</td>
<td>u displacement : $f_{1i}\, u_i + f_{2i}\, \omega_{xyi}$
v displacement : $f_{1i}\, v_i$
w displacement : $f_{1i}\, w_i + f_{3i}\, \omega_{xyi}$
where
$f_{1i} = \frac{1}{4}\ (1+\xi_o)(1+\eta_o)$
$f_{2i} = \frac{1}{16}\eta_i[(z_3-z_2)(1+\xi_i)+(z_4-z_1)(1-\xi_i)]$
$(1-\eta^2)(2+\xi_o-\xi^2)f_{1i}$
$f_{3i} = -\frac{1}{16}\xi_i[(x_3-x_4)(1+\eta_i)+(x_2-x_1)(1-\eta_i)]$
$(1-\xi^2)(2+\eta_o-\eta^2)f_{1i}$</td>
</tr>
<tr>
<td>
</td>
<td>u displacement : $f_{1i}\, u_i$
v displacement : $f_{1i}\, v_i$
w displacement : $f_{1i}\, w_i + f_{3i}\, \theta_{zi}$
where
$f_{1i} = \frac{1}{4}\ (1+\xi_o)(1+\eta_o)$
$f_{2i} = \frac{1}{16}\ (1+\xi_o)[2(1+\eta_o)(2+\xi_o-\xi^2)$
$-\alpha_i\xi_i\eta_o(1-\xi^2)]$
$f_{3i} = -\frac{1}{4}\beta_i\xi_i\,(1-\xi^2)\, f_{1i}$
$\alpha_1 = \alpha_4 = (a_2-a_4)/a_4$; $\alpha_2 = \alpha_3 = (a_2-a_4)/a_4$
$\beta_1 = (3a_1a_4+a_1a_2+a_3a_4-a_2a_3)/4a_4$
$\beta_2 = (3a_1a_2+a_2a_3+a_1a_4-a_3a_4)/4a_2$
$\beta_3 = (3a_3a_2+a_3a_4+a_2a_1-a_4a_1)/4a_2$
$\beta_4 = (3a_3a_4+a_4a_1+a_3a_2-a_1a_2)/4a_2$</td>
</tr>
</table>

2.3 LONGITUDINAL FUNCTIONS FOR SPLINE FINITE STRIPS[13]

The spline finite strip was initially developed for the analysis of plates and shells. In such a model, a strip is divided into a number of sections by knots (nodes) (Figure 2.3), and variables are expressed in terms of the knot values.

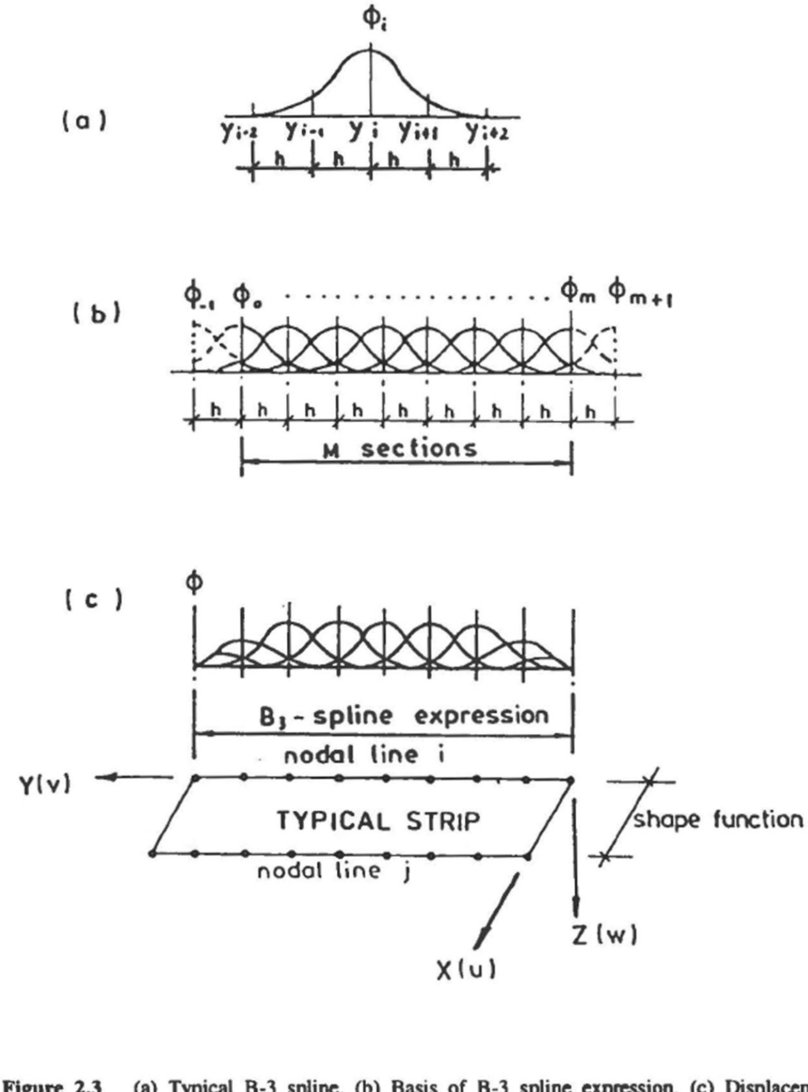


Figure 2.3 (a) Typical B-3 spline. (b) Basis of B-3 spline expression. (c) Displacement functions for typical strip.[13]

The shape functions, which consist of two parts, are written as the product of spline functions and suitable piecewise continuous polynomials (see Section 2.2.2). A variety of spline functions[14] are available, but it has been demonstrated that the B-3 spline function is the most appropriate choice for most plate and shell problems.

Each local spline function has non-zero values over four consecutive sections with the section knot $y=y_i$ as the centre. The functions with equal spacings are

$$\phi_i = \frac{1}{6h^3}\begin{cases} 0 & y < y_{i-2} \\ (y-y_{i-2})^3 & y_{i-2} \le y \le y_{i-1} \\ h^3 + 3h^2(y-y_{i-1}) + 3h(y-y_{i-1})^2 - 3(y-y_{i-1})^3 & y_{i-1} \le y \le y_i \\ h^3 + 3h^2(y_{i+1}-y) + 3h(y_{i+1}-y)^2 - 3(y_{i+1}-y)^3 & y_i \le y \le y_{i+1} \\ (y_{i+2}-y)^3 & y_{i+1} \le y \le y_{i+2} \\ 0 & y < y_{i+2} \end{cases} \tag{2.39a}$$

Spline functions of unequal spacings were also adopted in analyses, and the explicit forms of the local B-3 splines in the normalized variable η $(-1 \le \eta \le 1)$ are:

$$\phi_i = \begin{cases} 0 & \eta < \eta_{i-2} \\ A_i(\eta-\eta_{i-2})^3 & \eta_{i-2} \le \eta \le \eta_{i-1} \\ A_i(\eta-\eta_{i-2})^3 + C_i(\eta-\eta_{i-1})^3 & \eta_{i-1} \le \eta \le \eta_i \\ B_i(\eta_{i+2}-\eta)^3 + D_i(\eta_{i+1}-\eta)^3 & \eta_i \le \eta \le \eta_{i+1} \\ B_i(\eta_{i+2}-\eta)^3 & \eta_{i+1} \le \eta \le \eta_{i+2} \\ 0 & \eta < \eta_{i+2} \end{cases} \tag{2.39b}$$

where $A_i = [h_{i-1}(h_{i-1}+h_i)(h_{i-1}+h_i+h_{i+1})]^{-1}$
$B_i = [h_{i+2}(h_{i+2}+h_{i+1})(h_{i+2}+h_{i+1}+h_i)]^{-1}$
$C_i = -(h_{i-1}+h_i+h_{i+1}+h_{i+2})[h_{i-1}h_i(h_{i+1}+h_i)(h_{i+2}+h_{i+1}+h_i)]^{-1}$
$D_i = -(h_{i-1}+h_i+h_{i+1}+h_{i+2})[h_{i+2}h_{i+1}(h_{i+1}+h_i)(h_{i+1}+h_i+h_{i-1})]^{-1}$
$h_i = \eta_i - \eta_{i-1}$.

Therefore, the displacement function for the deflection of a lower order plate strip (Figure 2.3) is:

$$w = [[C_1]\ [C_2]]\begin{bmatrix} [\phi] & & & \\ & [\phi] & & \\ & & [\phi] & \\ & & & [\phi] \end{bmatrix}\{\delta\} \tag{2.40}$$

where C_1 and C_2 are given by Eq. 2.31,
$\{\delta\} = [(\alpha_w)_j^T\ (\alpha_{wr})_j^T\ (\alpha_w)_{j+1}^T\ (\alpha_{wr})_{j+1}^T]^T$
$[\bar{\phi}] = [\phi_{-1}\ \phi_0\ \ldots..\phi_M\ \phi_{M+1}]$