a dictionary of PHILOSOPHICAL LOGIC

This dictionary introduces undergraduate and graduate students in philosophy, mathematics, and computer science to the main problems and positions in philosophical logic. Coverage includes not only key figures, positions, terminology, and debates within philosophical logic itself, but issues in related, overlapping disciplines such as set theory and the philosophy of mathematics as well.

Entries are extensively cross-referenced, so that each entry can be easily located within the context of wider debates, thereby providing a valuable reference both for tracking the connections between concepts within logic and for examining the manner in which these concepts are applied in other philosophical disciplines.

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ISBN 978 0 7486 2559 8

Edinburgh University Press
22 George Square
Edinburgh
EH8 9LF
www.euppublishing.com

Cover image: www.istockphoto.com
Cover design: www.paulsmithdesign.com

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Edinburgh

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A DICTIONARY OF PHILOSOPHICAL LOGIC

Dedicated to my mother,
Carol C. Cook,
who made sure that I got to learn all this stuff,
and to
George Schumm, Stewart Shapiro, and Neil Tennant,
who taught me much of it.

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Edinburgh University Press

Edinburgh University Press Ltd
22 George Square, Edinburgh

Typeset in Ehrhardt
by Norman Tilley Graphics Ltd, Northampton,
and printed and bound in Great Britain by
CPI Antony Rowe, Chippenham and Eastbourne

A CIP record for this book is available from the
British Library

ISBN 978 0 7486 2559 8 (hardback)

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Acknowledgements

I would like to thank the staff at Edinburgh University Press for making this volume possible, and for showing admirable patience in the face of the numerous extensions to the deadline that I requested. In addition, thanks are due to the University of Minnesota for providing me with research funds in order to hire a graduate student to assist with the final stages of preparing this manuscript, and to Joshua Kortbein for being that graduate student. A special debt is owed to the philosophy department staff at the University of Minnesota – Pamela Groscost, Judy Grandbois, and Anita Wallace – for doing all the important things involved in running a university department so that academics like myself have the time and energy to undertake tasks such as this. Finally, thank you Alice, for everything.

Introduction

The mathematical study of logic, and philosophical thought about logic, are two of the oldest and most important human undertakings. As a result, great advances have been made. The downside of this, of course, is that one needs to master a great deal of material, both technical and philosophical, before one is in a position to properly appreciate these advances.

This dictionary is meant to aid the reader in gaining such a mastery. It is not a textbook, and need not be read as one. Instead, it is intended as a reference, supplementing traditional study in the field – a place where the student of logic, of whatever level, can look up concepts and results that might be unfamiliar or have been forgotten.

The entries in the dictionary are extensively cross-referenced. Within each entry, the reader will notice that some terms are in bold face. These are terms that have their own entries elsewhere in the dictionary. Thus, if the reader, upon reading an entry, desires more information, these keywords provide a natural starting point. In addition, many entries are followed by a list of additional cross-references.

In writing the dictionary a number of choices had to be made. First was the selection of entries. In this dictionary I have tried to provide coverage, both broad and deep, of the major viewpoints, trends, and technical tools within philosophical logic. In doing so, however, I found it necessary to include quite a bit more. As a result, the reader will find many entries that do not seem to fall squarely under the heading "philosophical logic" or even "mathematical logic." In particular, a number of entries concern set theory, philosophy of mathematics, mereology, philosophy of language, and other fields connected to, but not identical with, current research in philosophical logic. The inclusion of these additional entries seemed natural, however, since a work intending to cover all aspects of philosophical logic should also cover those areas where the concerns of philosophical logic blur into the concerns of other subdisciplines of philosophy.

In choosing the entries, another issue arose: what to do about expressions that are used in more than one way in the literature. Three distinct sorts of cases arose along these lines.

The first is when the same exact sequence of letters is used in the literature to refer to two clearly distinct notions. An example is "Law of Non-Contradiction," which refers to both a theorem in classical propositional logic and a semantic principle occurring in the metatheory of classical logic. In this sort of case I created two entries, distinguished by subscripted numerals. So the dictionary contains, in the example at hand, entries for Law of Non-Contradiction$_1$ and Law of Non-Contradiction$_2$. The reader should remember that these subscripts are nothing more than a device for disambiguation.

The second case of this sort is when a term is used in two ways in the literature, but instead of there being two separate notions that unfortunately have the same name, there just seems to be terminological confusion. An example of this is "Turing computable," which is used in the literature to refer to both functions computable by Turing machines and to functions that are computable in the intuitive sense – i.e. those that are effectively computable. In this case, and others like it, I chose to provide the definition that seemed like the correct usage. So, in the present example, a Turing computable function is one that is computable by a Turing machine. Needless to say, such cases depend on my intuitions regarding what "correct usage" amounts to. I am optimistic that in most cases, however, my intuitions will square with my readers'.

Finally, there were cases where the confusion seemed so widespread that I could not form an opinion regarding what "correct usage" amounted to. An example is the pair of concepts "strong negation" and "weak negation" – each of these has, in numerous places, been used to refer to exclusion negation and to choice negation. In such cases I contented myself with merely noting the confusion.

Related to the question of what entries to include is the question of how to approach writing those entries. In particular, a decision needed to be made regarding how much formal notation to include. The unavoidable answer I arrived at is: quite a lot. While it would be nice to be able to explain all of the concepts and views in this volume purely in everyday, colloquial, natural language, the task proved impossible. As a result, many entries contain formulas in the notation of various formal languages. Nevertheless, in writing the entries I strove to provide informal glosses of these formulas whenever possible. In places where this was not possible, however, and readers are faced with a formula they do not understand, I can guarantee that an explanation of the various symbols contained in the formula is to be found elsewhere in this volume.

Regarding alphabetization, I have treated expressions beginning with, or containing, Greek or Hebrew letters as if these letters were their Latin equivalents. Thus, the Hebrew ℵ occurs in the "A" section of the book, while "κ-categorical" occurs in the "K" section. Also, numbers have been entered according to their spelling. Thus, "S4" is alphabetized as if it were "Sfour," and so occurs after "set theory" and before "sharpening."

In many cases there were concepts or views which have more than one name in the literature. In such cases I have attempted to place the definition under the name which is most common, cross-referencing other names to this entry. In a very few cases, however, where I felt there were good reasons for diverging from this practice, I placed the definition under the heading which I felt ought to be the common one. An example of such an instance is the entry for "Open Pair," which is more commonly called the "No-No paradox." In this case I think that the former terminology is far superior, so that is where I located the actual definition.

There are two things that the reader might expect from a work such as this that are missing. The first of these are bibliographical entries on famous or influential logicians. In preparing the manuscript I originally planned to include such entries, but found that length constraints forced these entries to be too short – in every case the corresponding entries on internet resources such as *The Stanford Encyclopedia of Philosophy*, the *Internet Encyclopedia of Philosophy*, or even Wikipedia ended up being far more informative. Thus, I discarded these entries in favor of including more entries on philosophical logic itself. The reader will find a list of important logicians in an appendix at the end of the volume, however.

Second, the reader might wonder why each entry does not have a suggestion for further reading. Again, space considerations played a major role here. With well over one thousand entries, such references would have taken up precious space that could be devoted to additional philosophical content. Instead, I have included an extensive bibliography, with references organized by major topics within philosophical logic.

A

A see **Abelian Logic**

ℵ The first letter of the Hebrew alphabet, **ℵ** denotes the **infinite cardinal numbers**. Subscripted **ordinal numbers** are used to distinguish and order the **ℵ**s (and thus the infinite cardinal numbers themselves). $\aleph_0$ is the first infinite cardinal number – that is, the cardinal number of any **countably infinite set**; $\aleph_1$ is the second infinite cardinal number; $\aleph_2$ is the third infinite cardinal number … $\aleph_\omega$ is the ω^{th} infinite cardinal number; $\aleph_{\omega+1}$ is the $\omega + 1^{th}$ infinite cardinal number … and so on.

See also: **ℶ**, **c**, **Cantor's Theorem**, **Continuum Hypothesis**, **Cumulative Hierarchy**, **Generalized Continuum Hypothesis**

ABACUS COMPUTABLE see **Register Computable**

ABACUS MACHINE see **Register Machine**

ABDUCTION An abduction (or **inference to the best explanation**, or **retroduction**) is an **inductive argument** whose **premise** (or premises) constitute the available evidence, and whose **conclusion** is a hypothesis regarding what best explains the evidence. Abduction often takes the same general form as the **fallacious deductive argument affirming the consequent**:

$$\frac{\begin{array}{l}A \rightarrow B\\ B\end{array}}{A}$$

where B is the evidence at hand, and A is the hypothesis regarding what brought about B.

See also: **Cogent Inductive Argument**, **Fallacy**, **Informal Fallacy**, **Strong Inductive Argument**

ABELIAN LOGIC Abelian logic (or **A**) is a **relevance logic**. Abelian logic is obtained by rejecting **contraction** and liberalizing the following **theorem** of **classical propositional logic**:

$$((A \rightarrow \bot) \rightarrow \bot) \rightarrow A$$

to:

$$((A \rightarrow B) \rightarrow B) \rightarrow A$$

The latter principle is the **axiom of relativity**.

Abelian logic is one of a very few **non-standard logics** which **extends** classical propositional logic. Abelian logic is not a **sub-logic** of classical logic; it contains **theorems** which are not theorems of classical logic and which result in **triviality** if added to classical logic.

See also: **Commutativity**

ABSOLUTE CONSISTENCY see **Post Consistency**

ABSOLUTE INCONSISTENCY see **Post Consistency**

ABSOLUTE INFINITE The absolute infinite is an **infinity** greater than the infinite **cardinal number** associated with any **set**. Thus, the **proper class** of all sets is an instance of the absolute infinite.

See also: **Indefinite Extensibility**, **Iterative Conception of Set**, **Limitation-of-Size Conception of Set**, **Universal Set**

ABSORBSION Given two **binary functions** f and g, absorbsion holds between f and g if and only if, for all a and b:

$$f(a, g(a, b)) = g(a, f(a, b)) = a$$

Within **Boolean algebra**, absorbsion holds between the **meet** and **join operators** – that is:

$$A \cap (A \cup B) = A$$
$$A \cup (A \cap B) = A$$

In **classical propositional logic**, absorbsion holds between the **truth functions** associated with **conjunction** and **disjunction** – that is:

$$A \wedge (A \vee B)$$

and:

$A \vee (A \wedge B)$

are **logically equivalent** to:

A

The principle of **contraction**:

$(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$

is also sometimes referred to as absorbsion.

See also: **Distributivity**, **Rule of Replacement**

ABSTRACT OBJECT An abstract object is any object that is not **part** of the physical or material world, or alternatively any object that is not causally efficacious. Typical examples of abstract objects include mathematical objects such as **numbers** and **sets**, as well as objects connected with **logic** such as **propositions**, **languages**, and **concepts**. An object that is not abstract is a **concrete object**.

See also: **Abstraction**, **Mathematical Abstractionism**, **Nominalism**, **Platonism**

ABSTRACTION$_1$ The process by which we come to understand universal representations of particular objects (that is, universals) by attending only to those things the objects have in common.

See also: **Abstract Object**, **Abstraction Principle**, **Concept**

ABSTRACTION$_2$ Abstraction is the process of obtaining knowledge of **abstract objects** through the stipulation of **abstraction principles**.

See also: **Abstraction Operator**, **Bad Company Objection**, **Basic Law V**, **Caesar Problem**, **Hume's Principle**, **Mathematical Abstractionism**

ABSTRACTION OPERATOR The **function implicitly defined** by an **abstraction principle** is an abstraction operator. For example, the abstraction operator **defined** by **Hume's Principle** is the function that maps **concepts** to their associated **cardinal number**, and the abstraction operator (intended to be) defined by the **inconsistent Basic Law V** is the function that maps each concept to the set (or **extension**) containing, as **members**, exactly the instances of the concept in question.

See also: **Bad Company Objection**, **Caesar Problem**, **Mathematical Abstractionism**, **Singular Term**

ABSTRACTION PRINCIPLE An abstraction principle is any **formula** of the form:

$$(\forall\alpha)(\forall\beta)(\text{Abst}(\alpha) = \text{Abst}(\beta) \leftrightarrow \text{Equ}(\alpha, \beta))$$

where Abst is an **abstraction operator** mapping the type of entities ranged over by α and β (typically objects, **concepts**, **functions**, or **sequences** of these) to objects, and "Equ" is an **equivalence relation** on the type of entities ranged over by α and β.

According to **mathematical abstractionism**, abstraction principles are **implicit definitions** of the objects that fall in the **range** of the abstraction operator "Abst," and we gain knowledge of these objects merely through the stipulation of appropriate abstraction principles.

The most important abstraction principles are **Hume's Principle** and **Basic Law V**.

See also: **Bad Company Objection**, **Caesar Problem**

ABSTRACTIONISM see **Mathematical Abstractionism**

ABSURDITY RULE see **Ex Falso Quodlibet**

ACCESSIBILITY RELATION Within **formal semantics** for **modal logic**, an accessibility relation is a **relation** on the **set** of **possible worlds** in a **model** that indicates which worlds are accessible from which other worlds. The **validity** of different **modal axioms** is associated with different conditions on the accessibility relation. For example, the axiom **T**:

$$\Box A \rightarrow A$$

is valid if and only if the accessibility relation is **reflexive**.

See also: **Actual World**, **Kripke Semantics**, **Kripke Structure**, **Ternary Semantics**

ACKERMANN FUNCTION The Ackermann function (or **Ackermann-Péter function**) is a **binary recursive function defined** as:

$$A(m, n) = \begin{cases} n + 1 & \text{if } m = 0. \\ A(m - 1, 1) & \text{if } m > 0 \text{ and } n = 0. \\ A(m - 1, A(m, n - 1)) & \text{if } m > 0 \text{ and } n > 0. \end{cases}$$

The Ackermann function is a central example in **recursive function theory**, since it is recursive, but not **primitive recursive**. It is also an example of a function that grows rapidly – that is, the function outputs very large numbers for relatively small inputs.

See also: **Arithmetic**

ACKERMANN-PÉTER FUNCTION see **Ackermann Function**

ACTION TABLE An action table (or **transition function**) is the table of instructions governing the operation of a **Turing machine**.

See also: **Automaton**, **Deterministic Turing Machine**, **Non-Deterministic Turing Machine**, **Recursive Function Theory**, **Register Machine**

ACTUAL INFINITY see **Complete Infinity**

ACTUAL WORLD The actual world is the **possible world** we actually inhabit. It has been suggested that "actual" as used within **modal logic** (and thus the term "actual world") is an **indexical**. Thus, the actual world, for any reasoner in any possible world, is not the world we inhabit, but the one that they do.

See also: **Barcan Formula**, **Converse Barcan Formula**, **Counterpart Theory**, **Impossible World**, **Mere Possibilia**, **Trans-World Identity**

ACTUALISM see **Modal Actualism**

ACZEL SET THEORY see **Non-Well-Founded Set Theory**

ADDITION Addition (or **disjunction introduction**, or **or introduction**) is the **rule of inference** that allows one to **infer** a **disjunction** from either of the **disjuncts**. In symbols:

$$\frac{A}{A \vee B}$$

or:

$$\frac{B}{A \vee B}$$

See also: **Classical Dilemma**, **Constructive Dilemma**, **Destructive Dilemma**, **Disjunctive Syllogism**, **Introduction Rule**, **Vel**

AD HOMINEM Ad hominem (Latin, literally "to the man") is an **informal fallacy** which occurs when the reasoner, in attempting to demonstrate the inadequacy of another person's **argument**, attacks the character of the person presenting the argument instead of legitimately discrediting the **evidence** provided.

See also: **Straw Man**, **Tu Quoque**

ADICITY The adicity (or **arity**, or **degree**) of a **function** or **relation** is the number of inputs (or **arguments**) that it takes. Thus, a **unary function** is a function of adicity 1, and the adicity of a **binary relation** is 2.

See also: **Binary Function**, **N-ary Function**, **N-ary Relation**, **Ternary Function**, **Ternary Relation**, **Unary Relation**

AD IGNORANTIUM Latin for "to the point of ignorance," the phrase "ad ignorantium" is used to indicate an **informal fallacy** which occurs when the reasoner attempts to support a **conclusion** merely by pointing out that we have no **evidence** for the **negation** of the conclusion.

AD INFINITUM Latin for "to infinity," the phrase "ad infinitum" is used to indicate that a process is to be continued indefinitely, or that a particular **function** or operation is to be applied **infinitely** many times.

See also: **Complete Infinity**, **Cumulative Hierarchy**, **Hierarchy**, **Iteration**, **Iterative Conception of Set**, **Potential Infinity**

ADJUNCTION see **Conjunction Introduction**

ADMISSIBLE RULE A **rule of inference** is an admissible rule, relative to a particular **formal system**, if and only if its addition to the system does not allow one to **prove** any **theorems** or **demonstrate** the **validity** of any **arguments** that were not already provable using the original rules of the system.

An admissible rule is also a **derivable rule** if a **schema** can be provided which **demonstrates** how to obtain the **conclusion** of the derivable rule from the **premises** of the rule. Not every admissible rule is derivable, however.

See also: **Cut**, **Cut Elimination**, **Derivation**, **Sequent Calculus**

ADMISSIBLE SHARPENING see **Sharpening**

AFFINE LOGICS Affine logics are **substructural logics** within which the **structural rule contraction**:

$$\frac{\Delta, A, A \Rightarrow \Phi}{\Delta, A \Rightarrow \Phi}$$

fails.

See also: **Abelian Logic**, **Sequent Calculus**

AFFIRMATIVE PROPOSITION The **quality** of a **categorical proposition** is affirmative – that is, the categorical proposition is an affirmative proposition (or **positive proposition**) – if and only if it asserts that (some or all) **members** of the **class denoted** by the **subject term** are also **members** of the class denoted by the **predicate term**. **A-propositions** and **I-propositions** are affirmative, while **E-propositions** and **O-propositions** are not. Categorical propositions that are not affirmative are **negative**.

See also: **Particular Proposition**, **Quantity**, **Square of Opposition**, **Universal Proposition**

AFFIRMING THE ANTECEDENT see **Modus Ponens**

AFFIRMING THE CONSEQUENT Affirming the consequent is the **formal fallacy** that occurs when one moves from a **conditional**, and the **consequent** of that conditional, to the **antecedent** of that conditional. In symbols:

$$\begin{array}{l} P \rightarrow Q \\ \underline{Q \qquad} \\ P \end{array}$$

See also: **Abduction**, **Conditional Proof**, **Denying the Antecedent**, **Material Conditional**, **Modus Ponens**, **Modus Tollens**

ALETHIC MODAL LOGIC Alethic modal logic is the branch of **modal logic** that deals with the modal **operators** "it is **necessary** that Φ" and "it is **possible** that Φ," typically symbolized as "□ Φ" and "◇ Φ" or "L Φ" and "M Φ," respectively. Any modal logic dealing with modal operators other than these, such as **deontic modal logic**, **doxastic modal logic**, **epistemic modal logic**, and **temporal modal logic**, are **non-alethic modal logics** or **analethic modal logics**.

See also: **Contingency**, **Impossibility**, **Kripke Semantics**, **Kripke Structure**, **Normal Modal Logic**, **Possibility**

ALGEBRA An algebra is a **set** of objects and one or more **functions** or **relations** on that set. Within **logic**, important algebras include the **natural numbers**, the **real numbers**, **Boolean algebras**, **lattices**, and orderings of various types. One fruitful way to view a **formal system** is as an algebra where the set in question contains all **well-formed formulas** and the operations are the functions defined by the **formation rules** (e.g. **conjunction** is associated with the **binary** function that takes two **formulas** as inputs and gives their conjunction as output).

See also: **Algebraic Logic**, **Induction on Well-formed Formulas**, **Partial Ordering**

ALGEBRAIC LOGIC The branch of **mathematical logic** that studies the algebraic structures – that is, **algebras** – associated with particular **formal systems**. Algebraic logic is especially useful when studying **many-valued logics**, since one can compare the algebras generated by these systems to the **Boolean algebras** generated by **classical logics**.

See also: **Lattice**, **Partial Ordering**

ALGORITHM see **Effective Procedure**

ALTERNATE DENIAL see **Sheffer Stroke**

ALTERNATIVE LOGIC see **Non-Standard Logic**

AMBIGUITY An expression displays ambiguity if it has more than one legitimate meaning or interpretation in a given context.

See also: **Amphiboly**, **Equivocation**, **Informal Fallacy**, **Punctuation**

AMPHIBOLY A type of **ambiguity**, amphiboly occurs when a complex expression has more than one legitimate interpretation, and the ambiguity in question is not due to any single word having more than one meaning. In cases of amphiboly, the multiple interpretations are due instead to a structural, **logical**, or grammatical defect in the construction of the expression.

See also: **Equivocation**, **Informal Fallacy**, **Punctuation**

ANALETHIC LOGIC Analethic logic is a **three-valued logic** where the third **truth value** is the **truth value gap** "neither **true** nor **false**" (typically denoted "N"), and the **designated values** are "true" and "neither true nor false." **Compound sentences** are assigned truth values based on the **truth tables** for the **strong Kleene connectives**. Analethic logic has the same **proof-theoretic** behavior of the **logic of paradox**, without requiring the acceptance of a **truth value glut**.

See also: **Contradiction**, **Designated Value**, **Dialetheism**, **Dialethic Logic**, **Ex Falso Quodlibet**, **Paraconsistent Logic**

ANALETHIC MODAL LOGIC see **Alethic Modal Logic**

ANALYSIS Analysis is either the **first-order theory** of the **real numbers** or the **second-order theory** of the **natural numbers** (that is, **second-order arithmetic**). There is no **ambiguity** here, since the two theories are equivalent in **proof-theoretic** strength.

See also: **Intuitionistic Mathematics**, **Non-standard Analysis**

ANALYTIC A **statement** is analytic if and only if it is **true** in virtue of the meanings of the expressions contained in it. If a statement is not analytic, then it is **synthetic**.

ANAPHORA Anaphora occurs when the **referent** of an expression depends on the referent of another expression occurring in the same **statement** or in another appropriately connected statement. For example, in:

> Bobby was tired. He said he was suffering from lack of sleep.

"He" occurs anaphorically. Often (but not always) anaphoric terms are pronouns such as "it," "she," "there," etc.

See also: **Demonstrative**, **Indexical**

ANCESTRAL The ancestral of a **relation** R is the relation R* that holds between x and y if and only if there is a chain of objects z_1, z_2, … z_n such that Rxz_1, Rz_1z_2, … Rz_ny. Within **second-order logic** the ancestral is **defined** as follows. First, a **concept** F is **hereditary** relative to a relation R if and only if:

$$\text{Hered}(F, R) \quad = \quad (\forall x)(\forall y)(\, Rxy \rightarrow (Fx \rightarrow Fy))$$

Loosely, F is hereditary relative to R if and only if everything R-related to an F is an F. We can now define the ancestral of R:

$$R^*(x, y) \quad = \quad (\forall F)(((\forall z)(Rxz \rightarrow Fz) \wedge \text{Her}(F, R)) \rightarrow Fy)$$

See also: **Frege's Theorem**, **Transitive Closure**

AND see **Conjunction**

AND ELIMINATION see **Conjunction Elimination**

AND INTRODUCTION see **Conjunction Introduction**

ANTECEDENT The antecedent of a **conditional** is the **subformula** of the conditional occurring between the "if" and the "then," or, if the conditional is not in strict "If … then …" form, then the antecedent is the subformula occurring between "if" and "then" in the "if … then …" statement **logically equivalent** to the original conditional.

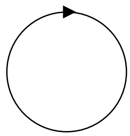
See also: **Affirming the Consequent**, **Consequent**, **Denying the Antecedent**, **Modus Ponens**, **Modus Tollens**

ANTI-EXTENSION The anti-extension of a **predicate** is the **set** of objects that fail to **satisfy** the predicate. Thus, the anti-extension of "is red" is the set of things that fail to be red. More generally, the anti-extension of an n-ary **relation** is the set of **n-tuples** that fail to satisfy the relation.

Typically, the anti-extension of a predicate is the **complement** of the **extension** of the predicate. Some **non-standard logics**, however, such as **supervaluational semantics**, allow there to be objects that are in neither the extension nor the anti-extension of a predicate.

See also: **Disjoint**, **Exclusive**, **Exhaustive**, **Partition**, **Sharpening**

ANTI-FOUNDATION AXIOM The anti-foundation axiom is the **axiom** that replaces the **axiom of foundation** within **non-well-founded set theory**, and which allows for **sets** with **non-well-founded membership relations**. The axiom states that, given any directed graph, there is a **function** f from the universe of sets **V** onto the nodes of that graph such that, for any two sets A and B, A is a member of B if and only if there is an edge in the graph leading from the node f(A) to the node f(B). For example, the graph:



represents the non-well-founded set Ω where Ω = {Ω}.

See also: **Iterative Conception of Set**, **Non-Well-Founded Set Theory**

ANTILOGISM An antilogism (or **inconsistent triad**) is any triple of **statements** such that the **truth** of any two of them guarantees the **falsity** of the third. Antilogisms were used as a tool for testing the **validity** of **categorical syllogisms**, since a categorical syllogism will be valid if and only if the triple containing the two **premises** and the **contradictory** of the **conclusion** is an antilogism.

See also: **Term Logic**, **Venn Diagram**

ANTINOMY An antinomy occurs when two laws, or two **conclusions** of apparently acceptable **arguments**, are incompatible with each other. The term "antinomy" is also sometimes used more loosely as a synonym for "**paradox**."

See also: **Insolubilia**, **Sophism**, **Sophisma**

ANTIREALISM see **Logical Antirealism**

ANTISYMMETRY A **relation** R is antisymmetric if and only if, for any a and b, if:

Rab

and:

Rba

then:

a = b.

See also: **Asymmetry**, **Linear Ordering**, **Partial Ordering**, **Strict Ordering**, **Symmetry**, **Well-Ordering**

A POSTERIORI see **A Priori**

A PRIORI A **statement** is a priori if and only if it can be known to be **true** independent of any empirical experience (other than those experiences that might be necessary in order to understand the statement). A statement that is not a priori is **a posteriori**.

A-PROPOSITION An A-proposition is a **categorical proposition** asserting that all objects which are **members** of the **class designated** by the **subject term** are members of the class designated by the **predicate term**. In other words, an A-proposition is a categorical proposition whose **logical form** is:

All P are Q.

The **quality** of an A-proposition is **affirmative** and its **quantity** is **universal**. An A-proposition **distributes** its subject term, but not its predicate term.

See also: **E-Proposition**, **I-Proposition**, **O-Proposition**, **Square of Opposition**

ARGUMENT$_1$ An argument is a **sequence** of **statements** where all but one of the statements (the **premises**) are intended to provide evidence, or support, for the remaining statement (the **conclusion**).

Sometimes, in technical contexts such as the **sequent calculus**, an argument can have more than one **conclusion**.

See also: **Conditionalization**, **Deductive Argument**, **Formal Fallacy**, **Inductive Argument**, **Inference**, **Informal Fallacy**

ARGUMENT$_2$ An argument of a **function** or **relation** is any value that can be input into the function or relation.

See also: **Domain**, **Field**, **Range**

ARISTOTELIAN COMPREHENSION SCHEMA The Aristotelian comprehension schema is the following **formula** in **second-order logic** (for any formula Φ not containing Y **free**):

$$(\exists x)\Phi \rightarrow (\exists Y)(\forall x)(Yx \leftrightarrow \Phi)$$

The Aristotelian comprehension schema guarantees there is a **concept** holding of exactly the objects **satisfying** Φ, as long as at least one object satisfies Φ. Unlike the standard **comprehension schema**, the Aristotelian comprehension schema does not guarantee the existence of an **empty concept**.

See also: **Aristotelian Second-order Logic**, **Empty Set**, **Schema**

ARISTOTELIAN LOGIC see **Categorical Logic**

ARISTOTELIAN SECOND-ORDER LOGIC Aristotelian second-order logic is a variant of **second-order logic** where the **comprehension schema** is replaced by the weaker **Aristotelian comprehension schema**. The main difference between standard second-order logic and Aristotelian second-order logic is that in Aristotelian second-order logic there is no guarantee that the **empty concept** exists.

See also: **Empty Set**, **Schema**

ARISTOTLE'S SEA BATTLE Aristotle's sea battle example is meant to challenge what we now call **classical logic**. Aristotle has us consider two **statements**:

(1) There will be a sea battle tomorrow.

(2) There will not be a sea battle tomorrow.

According to classical reasoning, one of these is **true** and the other **false**. But if that is the case, then we have no control over whether there will be a sea battle tomorrow or not – the facts of the matter have already been determined. Since the **argument** generalizes to any statement, we are left with an uncomfortable determinism regarding the future.

See also: **Bivalence**, **Law of Excluded Middle**, **Non-Standard Logic**

ARISTOTLE'S THESIS Aristotle's thesis is the following **formula** on **propositional logic**:

$\sim (\sim A \rightarrow A)$

This formula is a theorem in **connexive logic**, yet it is not a theorem within **classical logic** – in the classical context Aristotle's thesis is equivalent to $\sim A$.

See also: **Boethius' Theses**

ARITHMETIC Any **theory** regarding the **natural numbers** is an arithmetic. Within **logic**, there are a number of important arithmetic theories, including **Robinson arithmetic**, **Peano arithmetic**, and **non-standard arithmetic**.

See also: **Finitary Arithmetic**, **Gödel's First Incompleteness Theorem, Gödel's Second Incompleteness Theorem, Hume's Principle, Inconsistent Arithmetic**, **Intuitionistic Arithmetic**

ARITHMETIC HIERARCHY The arithmetic hierarchy (or **Kleene hierarchy**) is a classification of the **formulas** of **first-order arithmetic** based on their **complexity**. A formula is designated a Π_0 (or Σ_0) formula if it is, or is **equivalent** to, a formula containing only **bounded quantifiers**. Π_n and Σ_n formulas, for any **natural number** greater than 0, are **defined recursively** as follows:

> Φ is Π_{n+1} if and only if Φ is **logically equivalent** to some formula of the form:
> $(\forall x_1)(\forall x_2) \ldots (\forall x_m)\Psi$ where Ψ is a Σ_n formula.
>
> Φ is Σ_{n+1} if and only if Φ is logically equivalent to some formula of the form:
> $(\exists x_1)(\exists x_2) \ldots (\exists x_m)\Psi$ where Ψ is a Π_n formula.

Every formula of first-order arithmetic is **equivalent** to a formula in **prenex normal form**, guaranteeing that this definition assigns every formula of arithmetic a rank in the arithmetic hierarchy.

See also: **Hierarchy**, **Π-Formula**, **Π-Sentence**, **Σ-Formula**, **Σ-Sentence**, **Skolem Normal Form**

ARITHMETIC PREDECESSOR see **Arithmetic Successor**

ARITHMETIC SUCCESSOR The arithmetic successor of a **natural number** is the next natural number. In other words, the arithmetic successor of n is n + 1. If n is the arithmetic successor of m, then m is the **arithmetic predecessor** of n.

See also: **Axiom of Infinity**, **Cardinal Successor**, **Inductive Set**, **Ordinal Successor**, **Successor Function**

ARITHMETIZATION Arithmetization is the method by which **numerals** in formalized **arithmetic** are assigned to symbols, **formulas**, and **sequences** of formulas within that system of arithmetic. Various claims about the **syntax**, **proof theory**, etc. of the arithmetical theory can be formulated and studied within that same theory by using the numerals assigned to expressions by the arithmetization process as proxies for the expressions themselves. **Gödel's first incompleteness theorem** and **Gödel's second incompleteness theorem** are the paradigm instances of using arithmetization in order to study characteristics of **formal systems**.

See also: **Diagonalization**, **Diagonalization Lemma**, **Gödel Numbering**, **Gödel Sentence**, **Peano Arithmetic**

ARITY see **Adicity**

ASSERTION Assertion (or **pseudo modus ponens**) is the following principle of **propositional logic**:

$$(A \wedge (A \rightarrow B)) \rightarrow B$$

Assertion is the **conditionalization** of the valid argument form **modus ponens**.

ASSOCIATIVE LAW see **Associativity**

ASSOCIATIVITY$_1$ A **function** f is associative if and only if the following holds for any a, b, and c:

$$f(a, f(b, c)) = f(f(a, b), c)$$

Any function that **satisfies** the above **formula** is said to satisfy the **associative law**.

See also: **Absorbsion**, **Boolean Algebra**, **Join**, **Lattice**, **Meet**

ASSOCIATIVITY$_2$ Within **propositional logic**, associativity is the **rule of replacement** that allows one to replace a **formula** of the form:

$(A \wedge (B \wedge C))$

with:

$((A \wedge B) \wedge C)$,

or to replace a formula of the form:

$(A \vee (B \vee C))$

with:

$((A \vee B) \vee C)$

Multiple applications of associativity allow one to rearrange the **parentheses** in long **sequences** of **conjunctions** or in long sequences of **disjunctions**.

See also: **Commutativity, Distributivity**

ASYMMETRY A **relation** R is asymmetric if and only if it is not **symmetric** – that is, if there exist an a and b such that:

Rab

but not:

Rba.

In some contexts asymmetry is understood more strictly, however, so that a relation R is asymmetric if and only if it is nowhere symmetric – that is, if for any x and y, if:

Rxy

then:

$\sim Ryx$

See also: **Antisymmetry**, **Strict Ordering**

ATOM$_1$ Within **mereology**, an atom is any object that has no **proper parts**, that is, no **parts** other than itself. Formally, we can **define** this notion as (where P is the part **relation**):

$$Ax \quad = \quad \sim (\exists y)(Pyx \wedge y \neq x)$$

See also: **Gunk, Mereological Nihilism**

ATOM$_2$ see **Atomic Formula, Atomic Sentence, Propositional Letter**

ATOMIC FORMULA An atomic formula (or **atom**, or **simple formula**) is a **formula** that consists of a single **n-ary predicate** followed by n **singular terms**. Note that the terms might be simple or **complex**, and might be, or contain, either **constants** or **variables**.

See also: **Atomic Sentence**, **Compound Formula**, **Compound Statement**, **Formation Rules**, **Propositional Letter**, **Singular Proposition**

ATOMIC LETTER see **Propositional Letter**

ATOMIC SENTENCE Within first-order logic, an atomic sentence (or **atom**, or **simple sentence**) is a **formula** that consists of a single **n-ary predicate** followed by n **singular terms** where none of the terms contains any **variables**. In other words, an atomic sentence is an **atomic formula** where all the terms are **constants**, or are **complex terms** containing only constants.

See also: **Compound Formula**, **Compound Statement**, **Formation Rules**, **Propositional Letter**, **Singular Proposition**, **Well-Formed Formula**

ATOMLESS GUNK see **Gunk**

ATTRIBUTE see **Concept**

AUSSUNDERONG see **Axiom(s) of Separation**

AUSSONDERONG AXIOM see **Axiom(s) of Separation**

AUTOLOGICAL A **predicate** is autological if and only if it applies to itself. For example, “polysyllabic” is autological, since “polysyllabic” is polysyllabic, but “unpronounceable” is not autological, since “unpronounceable” is pronounceable. A predicate that is not autological is **heterological**. The **Grelling paradox** arises when one considers whether “heterological” is heterological.

See also: **Liar Paradox**, **Liar Sentence**, **Russell Paradox**, **Russell Set**

AUTOMATON An automaton is a **finitely** describable abstract machine or computing device. The study of automata is central to

computability theory. Examples of automata include **Turing machines** and **register machines**.

See also: **Church-Turing Thesis**, **Deterministic Turing Machine**, **Non-Deterministic Turing Machine**, **Recursive Function Theory**, **Turing Test**

AUTOMORPHISM An automorphism is an **isomorphism** between a **structure** and itself.

See also: **Endomorphism**, **Epimorphism**, **Homomorphism**, **Monomorphism**

AXIOLOGICAL LOGIC Axiological logic is the **logic** of "good," "bad," and "better than." Typically, axiological logics contain a **binary relation** P where "Pxy" represents "x is preferred to y" or "x is better than y." This relation is usually assumed to be **asymmetric** and **transitive**.

See also: **Modal Logic**, **Partial Ordering**

AXIOM An axiom is a **formula** used as a starting assumption and from which other **statements** – **theorems** – are **derived**. Thus, many statements are **proved** using axioms, but axioms need not, and given their **definition** cannot, be proved. In the past, axioms were meant to be self-evident and thus in need of no additional support or evidence. Now, however, an axiom is any principle that is assumed without proof.

See also: **Axiom Schema**, **Axiomatized Theory**, **Finitely Axiomatizable**, **Recursively Axiomatizable Theory**

AXIOM OF AUSSONDERONG see **Axiom(s) of Separation**

AXIOM OF CHOICE The axiom of choice (or **multiplicative axiom**) asserts that, given a **set** containing one or more **pairwise disjoint sets**, there exists a second set containing exactly one **member** of each of the sets contained in the original set – in other words, given a set of non-overlapping sets, the axiom of choice tells us that we can "select" one member from each of the non-overlapping sets and form these into a "new" set. In **first-order logic** supplemented with the membership symbol "$\in$," this can be formulated as:

$(\forall x)(((\forall y)(y \in x \rightarrow (\exists z)(z \in y)) \wedge (\forall y)(\forall z)((y \in x \wedge z \in x) \rightarrow \sim (\exists w)(w \in y \wedge w \in z))) \rightarrow (\exists y)(\forall z)(z \in x \rightarrow (\exists! t)(t \in z \wedge t \in y)))$

The axiom of choice is equivalent to **Zorn's lemma**, the **well-ordering principle**, and the **trichotomy law**.

See also: **Axiom of Countable Choice**, **Axiom of Dependent Choice**, **Choice Function**, **Choice Set**, **Global Choice**, **Zermelo Fraenkel Set Theory**

AXIOM OF CONSTRUCTIBILITY The axiom of constructibility is a **set-theoretic** principle that states that the universe of sets (**V**) is identical to the **constructible sets** (**L**). Thus, the axiom can be succinctly stated as:

$$V = L$$

The axiom of constructibility is **independent** of **Zermelo Fraenkel set theory**, and Kurt Gödel proved that both the **axiom of choice** and the **continuum hypothesis** are **consistent** with Zermelo Fraenkel set theory by showing that both follow from the axiom of constructibility (which is itself consistent with Zermelo Fraenkel set theory).

See also: **Independence Result**, **Inner Model**

AXIOM OF COUNTABLE CHOICE The axiom of countable choice (or **axiom of denumerable choice**) is a weak version of the **axiom of choice**. It states that, given a **countable set** containing one or more **pairwise disjoint sets**, there exists a set containing exactly one **member** of each of the sets contained in the original set.

The axiom of countable choice is implied by both the full axiom of choice and the **axiom of dependent choice**.

See also: **Choice Function**, **Choice Set**, **Global Choice**, **Trichotomy Law**, **Well-Ordering Principle**, **Zorn's Lemma**

AXIOM OF DENUMERABLE CHOICE see **Axiom of Countable Choice**

AXIOM OF DEPENDENT CHOICE The axiom of dependent choice is a weak version of the **axiom of choice**. It states that, given:

(1) Any non-empty **set** X.

and:

> (2) Any **relation** R on X such that, for any member a of X, there is a member b of X such that Rab (that is, for any **serial relation** R on X).

there is a **sequence** $x_1, x_2, \ldots$ such that, for all n, x_n is in X, and Rx_nx_{n+1}.

The axiom of dependent choice is **implied** by the full axiom of choice and implies the **axiom of countable choice**.

See also: **Choice Function**, **Choice Set**, **Global Choice**, **Trichotomy Law**, **Well-Ordering Principle**, **Zorn's Lemma**

AXIOM OF DETERMINATENESS see **Axiom of Extensionality**

AXIOM OF EMPTY SET The axiom of empty set (or **axiom of null set**, or **empty set axiom**, or **null set axiom**) asserts that there exists a **set** containing no **members**. In **first-order logic** supplemented with the membership symbol "∈," this can be formulated as:

$$(\exists x)(\forall y)(y \notin x)$$

or:

$$(\exists x)(\forall y)(y \in x \leftrightarrow y \neq y)$$

The **empty set**, the set whose existence is asserted by this **axiom**, is typically **denoted** by "∅."

See also: **Axiom of Infinity**, **Zermelo Fraenkel Set Theory**

AXIOM OF EXTENSIONALITY The axiom of extensionality (or **axiom of determinateness**, or **extensionality axiom**) asserts that two **sets** are **identical** if and only if they have exactly the same **members**. In **first-order logic** supplemented with the membership symbol "∈," this can be formulated as:

$$(\forall x)(\forall y)(x = y \leftrightarrow (\forall z)(z \in x \leftrightarrow z \in y))$$

Satisfaction of the axiom of extensionality is often thought to be constitutive of the concept of set – in other words, something cannot be a set unless it satisfies this axiom, regardless of which other set-theoretic axioms are **true**.

See also: **Zermelo Fraenkel Set Theory**

AXIOM OF FOUNDATION The axiom of foundation (or the **axiom of regularity**, or the **axiom of restriction**, or **foundation axiom**, or **regularity axiom**, or **restriction axiom**), asserts that, given any non-empty **set** A, there is a **member** of A that is **disjoint** from A – in other words, any set that has any members at all has a member that shares no members with the original set. In **first-order logic** supplemented with the membership symbol "∈," this can be formulated as:

$$(\forall x)((\exists y)(y \in x) \rightarrow (\exists z)(z \in x \wedge \sim(\exists w)(w \in z \wedge w \in x)))$$

Although the exact import of the axiom of foundation is difficult to summarize, its main role is to rule out the existence of **non-well-founded sets**.

See also: **Anti-Foundation Axiom**, **Non-Well-Founded Set Theory**, **Zermelo Fraenkel Set Theory**

AXIOM OF INFINITY The axiom of infinity asserts that there exists a **set** A such that (1) the **empty set** is a **member** of A, and (2) for any set that is a member of A, its **ordinal successor** is also a member of A. In **first-order logic** supplemented with the membership symbol "∈" and standard abbreviations, this can be formulated as:

$$(\exists x)(\varnothing \in x \wedge (\forall y)(y \in x \rightarrow y \cup \{y\} \in x))$$

The set whose existence is asserted by this axiom can easily be shown to contain **infinitely** many members.

The set-theoretic axiom of infinity should be clearly distinguished from the **infinity axiom**, which merely states that infinitely many objects exist.

See also: **Axiom of Zermelo Infinity**, ω, **Singleton**, **Successor**, **Union**, **Zermelo Fraenkel Set Theory**

AXIOM OF NULL SET see **Axiom of Empty Set**

AXIOM OF PAIRING The axiom of pairing (or **pairing axiom**) asserts that, for any two objects A and B, there is a **set** whose **members** are exactly A and B – in other words, for any two objects, the **unordered pair** containing just those two objects as members exists. In **first-order logic** supplemented with the membership symbol "∈," this can be formulated as:

$$(\forall x)(\forall y)(\exists z)(\forall w)(w \in z \leftrightarrow (w = x \vee w = y))$$

The axiom of pairing implies that the **singleton** of every set exists, and it also provides the resources to construct **ordered pairs** and, more generally, **ordered n-tuples**.

See also: **Pairing Function**, **Zermelo Fraenkel Set Theory**

AXIOM OF POWERSET The axiom of powerset (or **powerset axiom**) asserts that, given any **set** A, there exists a second set B such that the **members** of B are exactly the **subsets** of A (including the **empty set**, and A itself). B is the **powerset** of A. In **first-order logic** supplemented with the membership symbol "∈," this can be formulated as:

$$(\forall x)(\exists y)(\forall z)(z \in y \leftrightarrow (\forall w)(w \in z \rightarrow w \in x))$$

See also: **Cantor's Theorem**, **Continuum Hypothesis**, **Generalized Continuum Hypothesis**, **Zermelo Fraenkel Set Theory**

AXIOM OF REDUCIBILITY In **ramified type theories**, the axiom of reducibility states that, for any **concept** (of any **type**) of **order** n, there is a concept of order 0 (of the same type) that has the same **extension** – that is, that holds of exactly the same entities. The axiom of reducibility is often **formalized** as:

$$(\forall X_n)(\exists Y_0)(\forall z)(X_n z \leftrightarrow Y_0 z)$$

The axiom of reducibility in effect reduces the ramified theory of types to the **simple theory of types**.

AXIOM OF REGULARITY see **Axiom of Foundation**

AXIOM OF RELATIVITY The axiom of relativity is the following **theorem** of **Abelian logic**:

$$((A \rightarrow B) \rightarrow B) \rightarrow A$$

This axiom is notable since its addition to most **logics** renders the logic **trivial**. One can see this quite simply in the case of **classical logic**, since **consequential mirabilis**:

$$(\sim A \rightarrow A) \rightarrow A$$

is a theorem of classical logic.

See also: **Paraconsistent Logic**, **Relevance Logic**

AXIOM OF REPLACEMENT see **Axiom(s) of Replacement**

AXIOM OF RESTRICTION see **Axiom of Foundation**

AXIOM OF SEPARATION see **Axiom(s) of Separation**

AXIOM OF SUMSET see **Axiom of Union**

AXIOM OF TRICHOTOMY see **Trichotomy Law**

AXIOM OF UNION The axiom of union (or **axiom of sumset**, or **sumset axiom**, or **union axiom**) asserts that, given any **set** of sets A, there exists a set B such that B contains all of the **members** of the members of A. B is the **union** of A. In **first-order logic** supplemented with the membership symbol "∈," this can be formulated as:

$$(\forall x)(\exists y)(\forall z)(z \in y \leftrightarrow (\exists w)(z \in w \wedge w \in x))$$

The **union** of a set A is typically **denoted** by "$\cup$A."

See also: **Intersection**, **Zermelo Fraenkel Set Theory**

AXIOM OF ZERMELO INFINITY The axiom of Zermelo infinity asserts that there exists a set A such that (1) the empty set is a **member** of A, and (2) for any set that is a member of A, its **singleton** is also a member of A. In **first-order logic** supplemented with the **membership** symbol "∈" and standard abbreviations, this can be formulated as:

$$(\exists x)(\varnothing \in x \wedge (\forall y)(y \in x \rightarrow \{y\} \in x))$$

This axiom should be clearly distinguished from both the set-theoretic **axiom of infinity** and the **infinity axiom**.

See also: **Ordinal Successor**, **Zermelo Fraenkel Set Theory**

AXIOM SCHEMA An axiom schema is a **formula** in the **metatheory** within which one or more **metalinguistic schematic variables** occur. Given an axiom schema, one obtains an **axiom** by systematically replacing each schematic variable with an **object language** formula of the appropriate type. Since there are usually **infinitely** many different object language formulas of the type in question, an axiom schema provides a **finite** formulation of an infinite list of axioms that are similar in structure.

See also: **Axiom(s) of Replacement**, **Axiom(s) of Separation**,