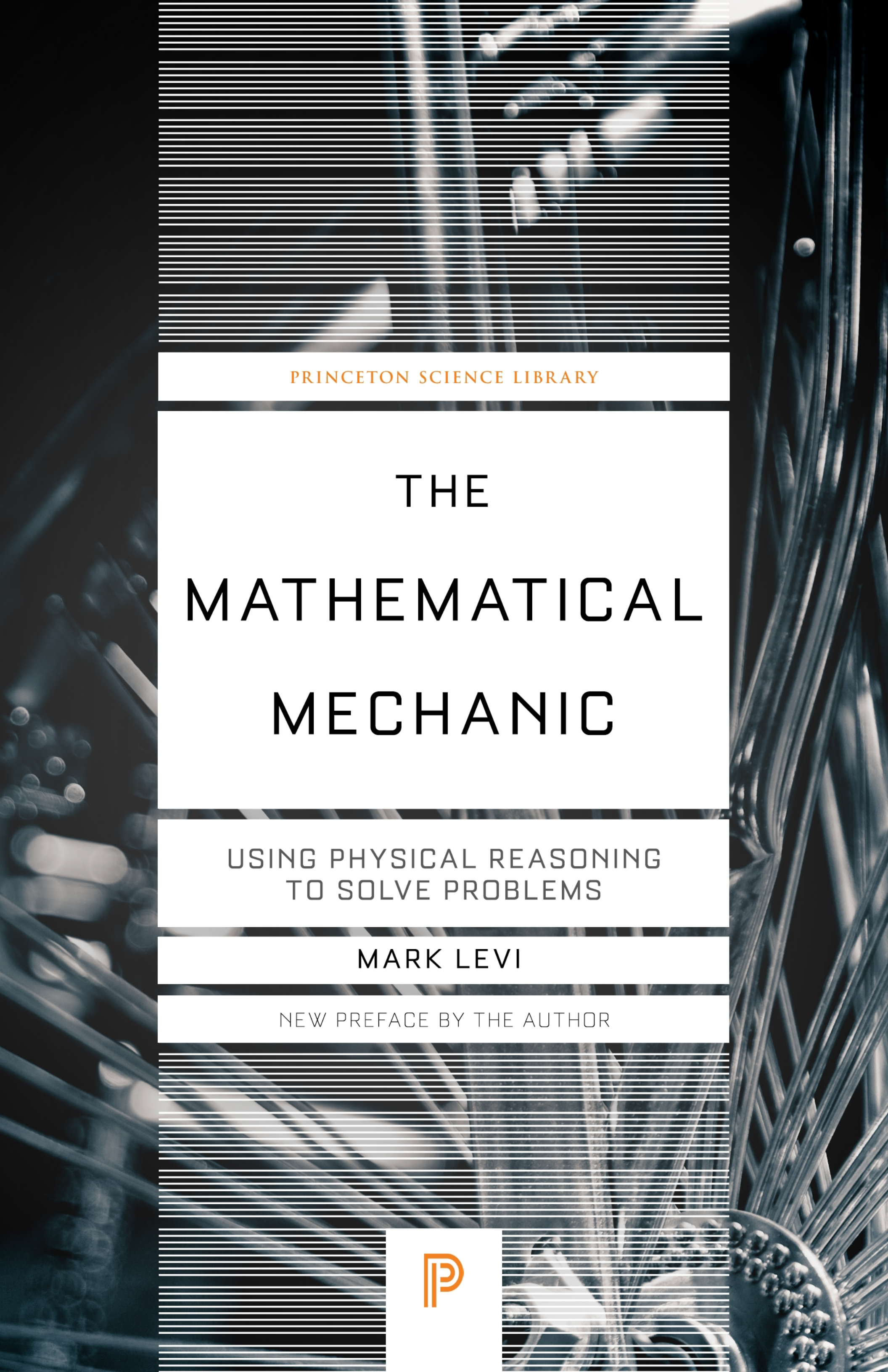

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THE MATHEMATICAL MECHANIC

USING PHYSICAL REASONING TO SOLVE PROBLEMS

MARK LEVI

NEW PREFACE BY THE AUTHOR

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Preface

Physical reasoning in mathematics goes back at least 2,400 years to Archimedes, who developed basic integral calculus using mechanical thought experiments. But how many calculus students are told nowadays that integral calculus started with Archimedes and that he computed definite integrals using imaginary balance scales? Typically all credit is ascribed to Newton, and perhaps Leibniz. Of course, calculus courses are not history of mathematics courses, and so this distortion of history is understandable—although still unfortunate.

From its early beginnings, Archimedean physical thinking has stayed in the shadow of the conventional mathematical reasoning in Euclidean geometry, algebra, trigonometry, and in the calculus of Newton and Leibniz, as well as in more advanced subjects. Still, although unadvertised as if to avoid embarrassment, the Archimedean approach has been responsible for some of the most consequential developments in mathematics. Three of the greatest mathematicians of all time—Hamilton, Riemann, and Poincaré—come to mind. In his work Hamilton used physics—more specifically, optics—as motivation, and Hamiltonian mechanics later became the central tool in quantum mechanics. Riemann's eponymous mapping theorem was certainly guided by physical reasoning. And Poincaré's treatment of what grew into the currently active and growing fields of symplectic geometry and dynamical systems was motivated by thinking of fluid motion.

When we consider a "Euclidean" approach versus an Archimedean approach, nothing is perfect. One could, of course, say that Euclidean geometry is perfect: in it things like points, lines, and planes are absolutely perfect objects devoid of all physical attributes. Lines have no thickness, or weight; they cannot be dirty, or hot, or conduct current. This act of perfecting real, palpable objects was an amazingly fruitful step that allowed pure logic to kick in, and geometry flourished into the greatest intellectual achievement of premodern times. However, in its perfection Euclidean geometry is a bit constraining. Indeed, in geometry we are forbidden to speak of forces (for example), which deprives us of using physical intuition and leaves an important part of the brain unused. This book is an attempt to advance the case for physical intuition in mathematics. Speaking more broadly, I am just advocating the general principle:

Look at both sides of the coin.

Here is an example of how fruitful this principle can be even in small things. In every trig course one is presented with the formula

$$\cos(\beta - \alpha) = \cos\beta\cos\alpha + \sin\beta\sin\alpha; \tag{1}$$

if students are lucky, they are also presented with a proof; and the luckiest ones are presented with a *good* proof. But remarkably and unexpectedly, (1) also follows from the *impossibility of the perpetual motion machine*, and with a snap of the fingers![1] As an added benefit, *each* term in (1) acquires an individual physical meaning. The beautiful but antiseptic (1) becomes almost palpable. And we get a glimpse of the unity of math and physics in a new way.

The most important place where Archimedean approach should be mentioned is our schools. Every student of geometry should have a chance to see how the Pythagorean theorem comes out of the impossibility of the perpetual motion machine, or how the theorem on the medians of a triangle comes out of balancing the triangle on a knife's edge, or how the inequality of arithmetic and geometric means comes out by connecting springs, or from connecting electrical resistors. I wish I had been told of these things as a student (I certainly wasn't, and neither was anyone I know). None of these require any physics background that cannot be explained in thirty minutes. And certainly explaining this minimal background will be time well spent.

After all, what is more important: (A) memorizing the quadratic formula, or (B) realizing that the Pythagorean theorem is a consequence of the impossibility of a perpetual motion machine? Let's

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[1] Further details on this can be found on the author's website at http://marklevimath.com.

see. On the one hand, (A) is a useful tool if you need to solve a quadratic equation. On the other hand, (B) connects the central theorem from mathematics with the central fact from physics; and (B) shows a striking unity of science (on this small example); and (B) broadens the mind by teaching one to disregard the boundaries between disciplines. And yet, every curriculum contains (A), and none mentions (B). We are missing an amazing opportunity to teach some math and some very basic physics inseparably, illustrating the unity of science. I hope a few gems like (B) will be included in our curricula—it requires only a tiny bit of space.

Collecting mathematical problems amenable to physical reasoning has been my hobby for many years, and I hoped to get this obsession out of my system by writing this book. The "getting it out my system" hope didn't quite pan out: after this book came out in 2009, I came up with more examples and learned of some others; here are some theorems/identities admitting simple physical proofs that are not included in this book:

- More min-max problems
- Altitudes in a triangle are concurrent (i.e., meet at one point)
- Euler's polyhedral formula (communicated to me by Peter Lax)
- Physical proof of the coordinate expression of the dot product
- The Cauchy-Schwarz inequality
- Hölder's inequality
- Orthogonality of eigenvectors of symmetric matrices via work
- Polar decomposition of matrices
- Theorem of cosines
- Moser's theorem on Jacobians via diffusion
- Uniformization theorem by heat conduction
- Conformal modulus via electrical resistance

The differences in people's personality types spill over into the way they learn mathematics. We all differ in our balances between intuitive/visual versus logical/combinatorial approaches. Our educational system disadvantages the former types by emphasizing recipes and rules and by deemphasizing intuition, whether geometrical or physical, for example. Perhaps small curriculum changes such as including some of the Archimedean ideas I mentioned earlier might help.

Some observations described in this book were so revealing and so enjoyable to me that I wanted to share them with others. And I hope that the Archimedean approach will make its way into our school curricula. All students deserve to see the palpable side of mathematics that they almost certainly never saw before.

THE
MATHEMATICAL MECHANIC

1

INTRODUCTION

> IT SO HAPPENS THAT ONE OF THE GREATEST MATHEMATICAL DISCOVERIES OF ALL TIMES WAS GUIDED BY PHYSICAL INTUITION.
>
> —GEORGE POLYA, ON ARCHIMEDES' DISCOVERY OF INTEGRAL CALCULUS

1.1 Math versus Physics

Back in the Soviet Union in the early 1970s, our undergraduate class—about forty mathematics and physics sophomores—was drafted for a summer job in the countryside. Our job included mixing concrete and constructing silos on one of the collective farms. My friend Anatole and I were detailed to shovel gravel. The finals were just behind us and we felt free (as free as one could feel in the circumstances). Anatole's major was physics; mine was mathematics. Like the fans of two rival teams, each of us tried to convince the other that his field was superior. Anatole said bluntly that mathematics is a servant of physics. I countered that mathematics can exist without physics and not the other way around. Theorems, I added, are permanent. Physical theories come and go. Although I did not volunteer this information to Anatole, my own reason for majoring in mathematics was to learn the main *tool* of physics—the field which I had planned to eventually pursue. In fact, the summer between high school and college I had bumped into my high school physics teacher, who asked me about my plans for the Fall. "Starting on my math major," I said. "What? Mathematics? You are nuts!" was his reply. I took it as a compliment (perhaps proving his point).

1.2 What This Book Is About

This is not "one of those big, fat paperbacks, intended to while away a monsoon or two, which, if thrown with a good overarm action, will bring a water buffalo to its knees" (Nancy Banks-Smith, a British television critic). With its small weight this book will not bring people to their knees, at least not by its *physical* impact. However, the book does exact revenge—or maybe just administers a pinprick—against the view that mathematics is a servant of physics. In this book physics is put to work for mathematics, proving to be a very efficient servant (with apologies to physicists). Physical ideas can be real eye-openers and can suggest a strikingly simplified solution to a mathematical problem. The two subjects are so intimately intertwined that both suffer if separated. An occasional role reversal can be very fruitful, as this book illustrates. It may be argued that the separation of the two subjects is artificial.[1]

Some history. The physical approach to mathematics goes back at least to Archimedes (c. 287 BC – c. 212 BC), who proved his famous integral calculus theorem on the volumes of the cylinder, a sphere, and a cone using an imagined balancing scale. The sketch of this theorem was engraved on his tombstone. Archimedes' approach can be found in [P]. For Newton, the two subjects were one. The books [U] and [BB] present very nice physical solutions of mathematical problems. Many of fundamental mathematical discoveries (Hamilton, Riemann, Lagrange, Jacobi, Möbius, Grassmann, Poincaré) were guided by physical considerations.

Is there a general recipe to the physical approach? As with any tool—physical[2] or intellectual—this one sometimes works and sometimes does not. The main difficulty is to come up with a

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[1] "Mathematics is the branch of theoretical physics where the experiments are cheap" (V. Arnold [ARN]). Not only are the experiments in this book cheap—they are even free, being the thought experiments (see, for instance, problems 2.2, 3.3, 3.13, and, in fact, most of the problems in this book).

[2] With apologies for the pun.

physical incarnation of the problem.[3] Some problems are well suited for this treatment, and some are not (naturally, this book includes only the former kind). Finding a physical interpretation of a particular problem is sometimes easy, and sometimes not; readers can form their own opinions by skimming through these pages.

One lesson a student can take from this book is that looking for a physical meaning in mathematics can pay off.

Mathematical rigor. Our physical arguments are not rigorous, as they stand. Rather, these arguments are sketches of rigorous proofs, expressed in physical terms. I translated these physical "proofs" into mathematical proofs only for a few selected problems. Doing so systematically would have turned this book into a "big, fat ...". I hope that the reader will see the pattern and, if interested, will be able to treat the cases I did not treat. Having made this disclaimer I feel less guilty about using the word "proof" throughout the text without quotation marks.

The main point here is that the physical argument can be a tool of discovery and of intuitive insight—the two steps preceding rigor. As Archimedes wrote, "For of course it is easier to establish a proof if one has in this way previously obtained a conception of the question, than for him to seek it without such a preliminary notion" ([ARC], p. 8).

An axiomatic approach. Instead of translating each physical "proof" into a rigorous proof, an interesting project would entail systematically developing "physical axioms"—a set of axioms equivalent to Euclidean geometry/calculus—and then repeating the proofs given here in the new setting.

One can imagine an extraterrestrial civilization that first developed mechanics as a rigorous and pure axiomatic subject. In this dual world, someone would have written a book on using geometry to prove mechanical theorems.

Perhaps the real lesson is that one should not focus solely on one or the other approach, but rather look at both sides of the coin. This

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[3] It is a contrarian approach: normally one starts with a physical problem, and abstracts it to a mathematical one; here we go in the opposite direction.

book is a reaction to the prevalent neglect of the physical aspect of mathematics.

Some psychology. Physical solutions from this book can be translated into mathematical language. However, something would be lost in this translation. Mechanical intuition is a basic attribute of our intellect, as basic as our geometrical imagination, and not to use it is to neglect a powerful tool we possess. Mechanics is geometry with the emphasis on motion and touch. In the latter two respects, mechanics gives us an extra dimension of perception. It is this that allows us to view mathematics from a different angle, as described in this book.

There is a sad Darwinian principle at work. Physical reasoning was responsible for some fundamental mathematical discoveries, from Archimedes, to Riemann, to Poincaré, and up to the present day. As a subject develops, however, this heuristic reasoning becomes forgotten. As a result, students are often unaware of the intuitive foundations of subjects they study.

The intended audience. If you are interested in mathematics and physics you will, I hope, not toss this book away.

This book may interest anyone who thinks it is fascinating that

- The Pythagorean theorem can be explained by the law of conservation of energy.
- Flipping a switch in a simple circuit proves the inequality $\sqrt{ab} \le \frac{1}{2}(a+b)$.
- Some difficult calculus problems can be solved easily with no calculus.
- Examining the motion of a bike wheel proves the Gauss-Bonnet formula (no prior exposure is assumed; all the background is provided).
- Both the Cauchy integral formula and the Riemann mapping theorem (both explained in the appropriate section) become intuitively obvious by observing fluid motion.

This book should appeal to anyone curious about geometry or mechanics, or to many people who are not interested in mathematics because they find it dry or boring.

Uses in courses. Besides its entertainment value, this book can be used as a supplement in courses in calculus, geometry, and teacher education. Professors of mathematics and physics may find some problems and observations to be useful in their teaching.

Required background. Most of the book (chapters 2–5) requires only precalculus and some basic geometry, and the level of difficulty stays roughly flat throughout those chapters, with a few crests and valleys. Chapters 6 and 7 require only an acquaintance with the derivative and the integral. At the end of chapter 7 I mention the divergence, but in a way that requires no prior exposure. This chapter should be accessible to anyone familiar with precalculus.

The second part (chapters 6–11) uses on rare occasions a few concepts from multivariable calculus, but I tried to avoid the jargon as much as possible, hoping that intuition will help the reader jump over some technical gaps.

Everything one needs from physics is described in the appendix; no prior background is assumed.

This book can be read one section or problem at a time; if you get stuck, it only takes turning a page to gain traction. A few exceptions to this topic-per-page structure occur, mostly in the later chapters.

Sources. Many, but not all *solutions* in this book are, to my knowledge, original. These include solutions to problems 2.6, 2.9, 2.10, 2.11, 2.13, 3.3, 3.7, 3.8, 3.9, 3.10, 3.11, 3.12, 3.17, 3.18, 3.19, 3.20, 3.21, 5.2, 5.3, 6.1, 6.2, 6.3, 6.4, 6.5, 7.1, and 7.2. The interpretations in chapter 8 and in sections 9.3, 9.8 and 11.8 appear to be new.

There is not much literature on the topic of this book. When I was in high school, an example from Uspenski's book [U] struck me so much that the topic became a hobby.[4] More problems of the

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[4] This is the first example of this book, in section 2.2. Tokieda's article [TO] contains, together with this example, some very nice additional ones.

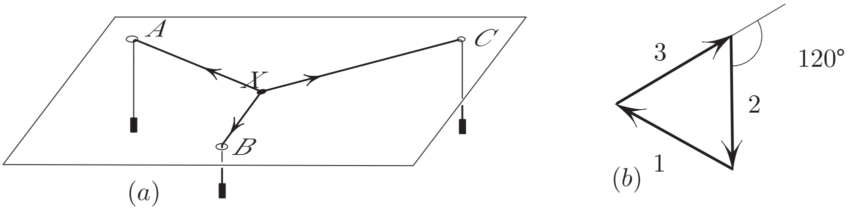

Figure 1.1. If X minimizes total distance $XA + XB + XC$, then the angles at X are 120°.

kind described here are in the small book by Kogan [K] and Balk and Boltyanskii [BB], and in chapter 9 of Polya's book [P]. And the main source of such problems and solutions is the 24-centuries-old work by Archimedes [ARC].

1.3 A Physical versus a Mathematical Solution: An Example

Problem. *Given three points A, B, and C in the plane, find the point X for which the sum of distances $XA + XB + XC$ is minimal.*

Physical approach. We start by drilling three holes at A, B, and C in a tabletop (this is cheaper to do as a thought experiment or at a friend's home). Having tied the three strings together, calling the common point X, I slip each string through a different hole and hang equal weights under the table, as shown in figure 1.1. Let us make each weight equal to 1; the potential energy of the first string is then AX: indeed, to drag X from the hole A to its current position X we have to raise the unit weight by distance AX. We endowed the sum of distances $XA + XB + XC$ with the physical meaning of potential energy. Now, if this length/energy is minimal, then the system is in equilibrium. If each angle in ΔABC is less than 120°, then the equilibrium position of X is not at A, B, C. The three forces of tension acting on X then add up to zero and hence they form a triangle (rather than an open path) if placed head-to-tail, as shown in