Alan D. Taylor AND William S. Zwicker

SIMPLE GAMES

Desirability Relations, Trading, Pseudoweightings

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To our wives:

Carolyn A. Taylor

and

Catherine A. Brodie

and our children:

Gwendolyn H. Taylor

Harrison C. Taylor

and

Portia B. Zwicker

Contents

Preface

Few structures in mathematics arise in more contexts and lend themselves to more diverse interpretations than do hypergraphs or simple games. Much of our own intuition for what follows comes from the voting-theoretic context in which a single alternative, such as a bill or an amendment, is pitted against the status quo. In this setting, a collection of voters is said to be a winning coalition if passage of the issue at hand is guaranteed by yes votes from precisely those voters in the collection.

A simple model for a voting system such as this is a pair (P, W), where P is a finite set called the grand coalition and W is a collection of subsets of P. Elements of P are called players or voters, and sets in W are called winning coalitions. Subsets of P that are not in W are called losing coalitions.

In this generality, such a structure is usually called a hypergraph. A simple game is defined here to be a hypergraph that is monotonic: subsets of losing coalitions are again losing.

One natural way to construct a simple game is to assign a (positive) real number weight to each voter, and declare a coalition to be winning precisely when its total weight meets or exceeds some predetermined quota. Simple games that can be described in this way are said to be weighted.

Hypergraphs and simple games can also be thought of as logic gates or Boolean functions, with threshold functions playing the role of weighted simple games. Some of the deepest results in the present book are inspired by the remarkable body of mathematical work done in the area of threshold logic during the late 1950s and early 1960s by people such as Chow, Elgot, Gabelman, and Winder, as reported by Hu (1965) and Muroga (1971). Some of this work is only peripherally known to workers in simple games, although important aspects of it were anticipated by Isbell (1958).

It turns out, however, that the voting-theoretic context gives rise to some intuitions—most particularly, the idea of trading players among winning coalitions—that allow presentations that are more transparent, both conceptually and technically, than the original ones within threshold logic. Moreover, this trading notion—different though it is from that typically used by economists—is amenable to weakenings that reveal a number of important properties of simple games.

Thus, in terms of content, our starting point is the thesis that the class of weighted games is only one of a series of ever-widening collections of simple games with structural characterizations that are both natural and elegant. Although trading is our lingua franca, our focus is also on the complementary perspectives provided by two additional lines of inquiry.

The first line involves an investigation of some ordinal notions of power that allow one to compare the influence of individuals or coalitions. For example, in which games are there no cycles in the strict existential desirability relation on individuals given by $p \prec_I q$ iff X together with p is losing and X together with q is winning for some coalition X containing neither p nor q? A short argument shows that a game is acyclic in this sense iff two winning coalitions can never be rendered losing by a one-for-one trade of players. The problem of successfully extending $\prec_I$ to coalitions is a major theme of the latter half of the book.

The second line of inquiry involves a direct weakening of what it means for a simple game to be weighted. What is it, after all, that makes weightedness such an important property? The answer lies in the extent to which one has, in weighted games, a uniform explanation for many structural properties. For example, weighted games are certainly acyclic in the sense of the previous paragraph because any ordering induced by the real number weights cannot contain cycles.

But some nonweighted games are also acyclic. Is there a uniform structural explanation of acyclicity that applies in weighted *and* nonweighted games? The answer is yes. A "point-set additive pseudoweighting" is a real-valued function w, defined on coalitions in a simple game, such that $w(X) + w(\{p\}) < w(Y) + w(\{q\})$ whenever X together with p is losing and Y together with q is winning (assuming $p \notin X$ and $q \notin Y$). It turns out that a game is acyclic iff it bears such a point-set additive pseudoweighting. Thus here, as with several other desirability relations (on coalitions rather than individuals), pseudoweightings provide a uniform structural explanation of acyclicity.

One of our purposes, then, is to pull together material from a number of contexts and to present it as a coherent body of work. For the sake of simplicity and clarity, we write in one language—that of simple games—and adopt conventions appropriate to that context. Footnotes deal with issues such as nonmonotonic simple games that are more natural in areas such as threshold logic than they are in simple games. A second purpose is to present some contributions from our own research in hopes that others will take up where we have left off. For this reason, we have incorporated a number of open questions into the text and notes.

We should note, however, that the text is not meant to be an encyclopedic treatment of simple games. Vast and important areas, such as homogeneous games (see, e.g., Ostmann, 1987; Peleg, 1966; Peleg and Rosenmüller, 1992; Rosenmüller, 1984; and Rosenmüller and Sudhölter, 1994), cardinal notions of power (see, e.g., Banzhaf, 1965; Felsenthal and Machover, 1996, 1999; Felsenthal, Machover, and Zwicker, 1998; Lucas, 1983; Penrose, 1946; Shapley and

Shubik, 1954; Straffin, 1982; and Taylor and Zwicker, 1996), and connections with social choice theory (see, e.g., Peleg, 1984), are not even mentioned. Indeed, a thorough treatment of any of these topics would require a monograph at least as long as what we have here.

The text is made up of five chapters. Chapter 1 treats material based on notions that are fundamental to the theory of simple games. It begins with a discussion of the algebra of simple games (sums, products, the dual game, etc.), but quickly moves on to more novel material: a focus on Boolean subgames in place of the more traditional reduced game or subgame, the fact that every simple game (even those that are neither strong nor proper) has a two-point constant-sum extension, the adaptation of the Rudin-Keisler ordering from ultrafilter theory to the context of simple games, the vector-weightedness of all simple games, and the exploration of a notion of dimension for simple games.

Chapter 2 begins with a formalization of the intuitive idea of trading via so-called trading matrices (and a discussion of what is gained by an "indexed" notion in place of the earlier approaches via asummability, etc.), and proceeds to a characterization of weighted games as those in which a sequence of winning coalitions can never be rendered losing by a series of trades. The proof we give is self-contained, combinatorial, and applicable to the more general context of pregames (in which some coalitions are winning, some are losing, and some are unclassified). This chapter also includes the related and more traditional results involving systems of linear inequalities and separating hyperplanes.

In Chapter 3, we focus on pairwise trading and two natural classes of simple games that arise in this context. The first of these classes consists of the games in which two winning coalitions can never be rendered losing by a one-for-one exchange of players. These are called linear games because they are precisely the ones wherein the desirability relation on individuals is a linear ordering of equivalence classes. Linear games are also characterized as those that carry certain kinds of pseudoweightings, one of which is the point-set additive pseudoweightings described earlier. If we consider many-for-many exchanges in place of one-for-one exchanges, we arrive at the second class of simple games treated in Chapter 3; we refer to games in this class as Winder games. It is well known that weighted games are either strong or proper—the Winder games turn out to be precisely the ones that have this property "hereditarily."

The central question in Chapter 4 is whether the desirability relation on individuals can be extended in a useful way to coalitions. Our starting point is an impossibility theorem asserting that no reasonable notion of coalitional equivalence can be transitive. This result notwithstanding, we study two notions of coalitional desirability—one introduced by the game theorist Eitan Lapidot and one by the threshold logician Robert Winder. The classes of games that naturally arise in this context are those in which the associated desirability relation on coalitions yields an acyclic ordering. We study these so-called weakly acyclic games and strongly acyclic games from the point of view of trading and pseudoweightings, and we use trading matrices to construct a constant-sum, strongly acyclic game that is, nevertheless, nonweighted.

If we consider trades among distinct coalitions only, then we arrive at a class of games slightly more general than the weighted ones. These are called Chow games, and their study occupies the first part of Chapter 5, where trading matrices are again used to construct an important example. We conclude Chapter 5 by coming full circle back to the class of weighted games, and showing that they can in fact be characterized by the acyclicity of certain desirability relations on coalitions. These are the desirability relations obtained by grafting the apparatus of trading onto the notions of Lapidot and Winder.

A number of the results we present are new in the strictest sense of the word—they have not appeared in print before. Those that are striking, yet quite easy, include the following:

- The construction, in Section 1.5, of a two-point constant-sum extension that handles simple games that are not initially strong or proper
- The construction of games of exponential dimension in Section 1.7
- The proof, in Section 3.4, that if there exists a counterexample to Chvátal's conjecture, then there is one that is strong and one that is proper
- The characterization of linear games in terms of local weightings in Section 3.7
- The characterization of Winder games (or "completely monotonic" games) as those that are hereditarily strong and proper in Section 3.10
- The impossibility results for notions of coalitional equivalence in Sections 4.2 and 4.3

Examples of new results that are more difficult include the following:

- The Chvátal-like proof of Snevily's theorem in Section 3.5
- The inductive construction of set-set additive pseudoweightings for weakly acyclic games in Section 4.6
- The pseudoweighting characterization of strong acyclicity in Section 4.8
- The characterization of weak and strong acyclicity in terms of sequential one-transfers in Section 4.9, and its application to the construction of a constant-sum game that is strongly acyclic but nonweighted in Section 4.10
- The construction, in Section 5.3, of a Chow game that is neither 4-trade robust nor strongly acyclic
- The material on multitrading, leading to the construction, in Section 5.5, of a new coalitional desirability relation $\prec_{TW}$ that extends both the original trading relation $\prec_{TL}$ and Winder's desirability relation $\prec_{W}$.

In addition to this new material, many of the results from threshold logic will be unfamiliar to some game theorists. These results were originally written with very different terminology or, in some cases, appeared only in unpublished Ph.D. dissertations.

The book also incorporates material from some of the authors' research papers of the last five years. Some of this material has been extensively reworked to smooth out technical difficulties, and it has all been reorganized and placed in a broader context. For example, although we have attempted in earlier papers to explain what it is about the idea of "trading" that imbues it with the descriptive power needed to characterize and compare many different classes of simple games, the discussion of trading matrices and indexed trading transforms in Sections 2.2 and 2.3 provides a clarity heretofore absent. A second example is the new matrix definition of the trading desirability relation $\prec_{\mathrm{TL}}$, which is more transparent than the original version, denoted $\prec_{\mathrm{T}}$ in Taylor and Zwicker (1996), and which explains the clear relationship between $\prec_{\mathrm{TL}}$ and the transitive closure of the original coalitional desirability relation $\prec_{\mathrm{L}}$ of game theory.

Thus, much of the material we present is new, and much more, we hope, will seem new. We have also taken several ideas that were not previously thought of as being central to the subject and cast them in the role of unifying themes. The list includes concepts, such as trading and pseudoweightings, that we first introduced ourselves, and yet another concept—the Rudin-Keisler ordering—that we imported from set theory, where its key role is widely acknowledged in the study of ultrafilters. But it also includes ideas known to the field, such as those of pregame and Boolean subgame, whose importance has not yet been fully exploited. Novel perspectives, we feel, are often as important as new results.

What is it that makes an area such as combinatorics a recognized field of mathematics, or a topic such as graph theory a recognized subfield of combinatorics? The answer, in our view, is twofold—they have coherency, and they have depth. The study of simple games has made progress on both fronts, and our hope is that both the perspectives and the results in this book will help sustain this effort.

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SIMPLE GAMES

1

Fundamentals

1.1 Introduction

Definition 1.1.1. *A* HYPERGRAPH *G is a pair (P, W) in which P is a finite set and W is a collection of subsets of P.*

In the context of hypergraphs, sets in W are called *edges*. A hypergraph in which all the edges contain exactly two elements of P is called a *graph*—an oft-studied and widely applicable mathematical notion, but one that takes us in quite a different direction from that suggested by the voting-theoretic context. Our primary concern is with the following.

Definition 1.1.2. *A* SIMPLE GAME *is a hypergraph $G = (P, W)$ that also satisfies* MONOTONICITY*: if $X \in W$ and $X \subseteq Y \subseteq P$, then $Y \in W$.*[1]

Simple games can be viewed as models of voting systems in which a single alternative, such as a bill or an amendment, is pitted against the status quo.[2] The asymmetry of outcomes allows the system to have a built-in bias in favor of the status quo or against it.[3] Any set of voters is called a *coalition*, and the set P is called the *grand coalition*. Members of P are called *players* or *voters*, and the subsets of P that are in W are called *winning coalitions*. The intuition here is that a set X is a winning coalition iff the bill or amendment passes when the players in X are precisely the ones who voted for it. A subset of P that is not in W is called a *losing coalition*, and the collection of losing coalitions is denoted by L.

A *minimal winning coalition* (MWC) is a winning coalition all of whose proper subsets are losing. Because of monotonicity, any simple game is completely determined by its set of minimal winning coalitions.

[1] According to Isbell (1958), hypergraphs satisfying monotonicity (with no other conditions imposed) were introduced and called "pseudogames" by D. B. Gilles in his 1953 Princeton thesis. Today, some game theorists are interested only in simple games that are proper or constant sum (to be defined later). With this in mind, we have provided constant-sum counterexamples whenever possible, while theorems are stated with appropriate generality. For further discussion of this issue, see the text following Proposition 1.3.7.

[2] As pointed out by Felsenthal and Machover (1998), such a model is often incomplete because many real-world voting systems allow abstentions.

[3] The constant-sum games referred to in note 1 serve to model those situations in which one must choose exactly one alternative from two *symmetric* options, with no such bias.

Our definition of a simple game differs in a rather trivial way from what most authors use in that we do not demand that the grand coalition be a winning coalition or that the null coalition be losing. In the presence of monotonicity, the only effect of omitting this requirement is to allow two somewhat uninteresting (although largely innocuous) hypergraphs as simple games: the one in which all coalitions are winning, and the one in which all coalitions are losing. The gain in our approach, albeit small, is in the naturality of the mathematics with regard to issues such as subgames.[4]

The assumption of monotonicity in the definition of a simple game is completely natural in the voting context, but less so in some other contexts. Thus, although our primary interest is in simple games, we have supplemented the text with footnotes wherever there is something interesting to say about the nonmonotonic case.

The study of simple games goes back at least to Dedekind's 1897 work, in which he determined the exact number of simple games with four or fewer players. Since that time, simple games have been investigated in a variety of different mathematical contexts: Boolean or switching functions, hypergraphs, coherent structures, Sperner systems, and clutters. In addition to hypergraphs (previously defined), we will make particular use of the notion of a Boolean function, central to the field of threshold logic.

The fundamental objects of study in threshold logic are the characteristic functions of hypergraphs. The intuition arises from thinking of such a function f as a complex circuit containing n on-off switches labeled with the integers $1, 2, \ldots, n$ corresponding to the players (or vertices) $p_1, p_2, \ldots, p_n$ in the grand coalition P of a hypergraph. A vector $\langle x_1, \ldots, x_n \rangle$ of zeros and ones thus codes a setting for each of the switches (1 for "on" and 0 for "off"). A *true vector* is one for which the output $f(\langle x_1, \ldots, x_n \rangle)$ is 1, meaning that the circuit as a whole passes current, acting as if it were a single switch set to "on", while a *false vector* is one for which $f(\langle x_1, \ldots, x_n \rangle)$ is 0, meaning that the circuit as a whole blocks current (see also note 16). Thus, a coalition X corresponds to its *characteristic vector* $\mathbf{x} = \langle x_1, \ldots, x_n \rangle$, in which $x_i = 1$ if $p_i \in X$ and $x_i = 0$ if $p_i \notin X$, winning coalitions correspond to true vectors, and losing coalitions correspond to false vectors.

The applications of simple games have been similarly diverse. For example, von Neumann himself did some early work in the area of reliability theory, wherein players are identified with components of a system, and winning coalitions correspond to collections of components sufficient for the operation of the system as a whole. Connections, both mathematical and historical, between

[4]For example, the winning coalitions of the reduced game of a simple game, defined in Section 1.4, may include the empty coalition, and those of the subgame need not contain its grand coalition.

the analysis of voting systems and the study of reliability theory are covered in Ramamurthy (1990).

Other applications include neurons in living organisms,[5] and logical computing devices. With reference to the latter, Muroga (1971, p. vi) has pointed out that a switching gate, called a "parametron" (from "parametrically excited oscillation"), was developed independently by E. Goto and J. von Neumann in 1954, with von Neumann filing for a patent three months before Goto. These were, according to Muroga (1971, p. 13), the precursor of threshold logic, and by 1970, more than four thousand parametron computers had been built for commercial use in Japan.[6]

Of fundamental importance to us is the class of weighted simple games.

Definition 1.1.3. *A simple game* $G = (P, W)$ *is said to be* WEIGHTED *if there exists a "weight function"* $w\colon P \to \mathbf{R}$ *(the set of real numbers) and a real number "quota"* $q \in \mathbf{R}$ *such that a coalition* X *is winning precisely when the sum of the weights of the players in* X *meets or exceeds quota.*

If the players in P are indexed as $p_1, \ldots, p_n$, then $\langle w(p_1), \ldots, w(p_n)\rangle$ is called the *associated weight vector.* Any specific example of such a weight function $w\colon P \to \mathbf{R}$ and quota q as in Definition 1.1.3 are said to *realize* G as a weighted game. If w is a weight function and X is a coalition, we will use $w(X)$ to denote $\Sigma\{w(p)\colon p \in X\}$.

Note that a simple game G is weighted iff there is a weight function $w\colon P \to \mathbf{R}$ such that $w(X) < w(Y)$ whenever X is losing in G and Y is winning in G. Without specifying a quota, however, G is not uniquely determined by w, and this lack of uniqueness prevents one from starting with a weight function w (alone) and saying, "Let G be the simple game determined by w." On the other hand, if we start with a weighted game G, then it makes perfect sense, in light of this equivalent definition of weightedness, to say, "Choose a weighting w that realizes G."

Let us illustrate a piece of notation with an example. Suppose G is the weighted game with grand coalition $P = \{a, b, c, d, e\}$, respective weights $1, 1, 2, 2, 3$, and quota $q = 5$. Then we will have occasion to denote G by $[5; 1_a, 1_b, 2_c, 2_d, 3_e]$. This is a slight variant of the standard notation in which the subscripts are deleted and one simply writes $[5; 1, 1, 2, 2, 3]$.

[5]The original mathematical model (see McCulloch and Pitts, 1943) of a neuron is nothing other than a weighted simple game, so one might expect significant overlap between the study of simple games and that of neural nets and artificial intelligence. This does not seem to be the case, however, in part because these fields ask such different questions. A second factor may be that the neural nets that are complex enough to be useful as learning machines are typically too complex, at this time, to admit a thoroughgoing analysis (see, e.g., Benítez, Castro, and Requena, 1997).

[6]Current interest in parametron switches appears to have no direct connection with the ability of such switches to implement threshold logic. Rather, the concern is with other matters, such as adiabatic computing (see, e.g., Merkle, 1993).

Hypergraphs or Boolean functions that meet the corresponding condition of weightedness are known as *threshold hypergraphs* or *threshold functions*, respectively. Threshold graphs are an important special case, and are discussed in Section 1.6 and Section 2.5 (a good general reference is Mahadev and Peled, 1995).

An important advantage of translating a subject from one mathematical arena to another is that the new venue suggests approaches that were somewhat less obvious in the old. An important example is provided by the following proposition, whose easy proof we leave for the reader.

Proposition 1.1.4. *The weight $w(X)$ of a coalition X under a given weight function w is equal to the dot product $\mathbf{w} \bullet \mathbf{x}$ of the weight vector $\mathbf{w}$ with the characteristic vector $\mathbf{x}$ of X.*

Weights and quota are typically nonunique, in part because sufficiently small variations in weight can be compensated for with a corresponding adjustment in the quota. Similar considerations show that one could use the phrase "exceeds quota" in place of "meets or exceeds quota" in Definition 1.1.3 and have an equivalent notion.

Notice that the definition of a weighted game does not require the weights to be nonnegative. However, monotonicity guarantees that if w is a weight function that realizes G as a weighted game, then replacing every negative weight with zero weight results in a nonnegative weight function that, with the same quota, also realizes G. Hence weights can be taken to be nonnegative. In fact, weights can be taken to be nonnegative integers by first using a small adjustment in each weight (as in the previous paragraph) to make them rational, and then multiplying everything by a common denominator.[7]

There are a few pieces of terminology pertaining to players in a simple game that we will want to have at hand.

Definition 1.1.5. *Suppose that $G = (P, W)$ is a simple game and that $p \in P$. Then we shall say that*

p is a DUMMY *iff $\forall X \subseteq P$, if $X \in W$ then $X - \{p\} \in W$;*
p is a PASSER *iff $\forall X \subseteq P$, if $p \in X$ then $X \in W$; and*
p is a VETOER *iff $\forall X \subseteq P$, if $p \notin X$ then $X \notin W$;*
p is a DICTATOR *iff $\forall X \subseteq P$, $X \in W$ iff $p \in X$.*

[7]The definition of weightedness given in Definition 1.1.3 can be used for nonmonotonic simple games, but in this broader setting, negative weights are often necessary. The characterizations of weighted simple games given in Chapter 2 apply to the nonmonotonic case as well. In theoretical models for neurons (see note 5), inputs with positive weights are called *excitors* and those with negative weights are called *inhibitors* (Muroga, 1971, p. 12).

Notice that a dummy corresponds to a player whose vote can never affect the outcome of an election,[8] and that p is a dictator iff p is both a vetoer and a passer. Sometimes, "master player" is used in place of "passer."

Our set-theoretic notation is reasonably standard. We use $\wp(X)$ to denote the power set of X and $\varnothing$ to denote the empty set. Set-theoretic difference is denoted by a minus sign; thus $X - Y$ is the set of things in X and not in Y. The symbol $\dot{\cup}$ is used to denote disjoint union. Thus, the use of $X \dot{\cup} Y$ carries with it the assertion that X and Y are disjoint. We often use X^c to denote the complement of X, especially when it is understood that $X \subseteq P$. The number of elements in the set X is denoted by $|X|$, and $[X]^k$ denotes the collection of subsets of X of size exactly k. If f is a function, then $f^{-1}[X]$ denotes the preimage of the set X, and $f|X$ denotes the restriction of the function f to the set X. For a real number x, we use $[x]$ for the greatest integer less than or equal to x.

The remainder of Chapter 1 is organized as follows. In Section 1.2, we present four examples of simple games. These examples are based on real-world voting systems, which we use to illustrate (or at least foreshadow) several classes of simple games, discussed later.

In Section 1.3, we introduce the dual game, with sums and products being presented in the beginning of Section 1.4. This much is standard fare. The rest of Section 1.4 is less standard, although the notion of substructure we introduce there dates back at least to Isbell (1958), and was used by workers in threshold logic as well. It yields the more widely known notions of subgame and reduced game as special cases. We also introduce there an ordering of simple games, borrowed from the theory of ultrafilters and ideals, known as the Rudin-Keisler ordering. It is related to the notion of voting blocs, and these again were known to Isbell (1958).

In Section 1.5, we show that every simple game—whether it is strong, proper, or neither—has a two point constant sum extension (where "extension" corresponds to the more general notion of substructure introduced in Section 1.4). If the game happens to be either strong or proper, this construction essentially reduces to the traditional one involving the addition of a single point, reduced games, and subgames.

It turns out to be useful to consider structures that are like simple games in the sense of having winning coalitions and losing coalitions, but also have coalitions that are unclassified. These are called pregames, with graphs providing the important special case in which all coalitions of size other than two are unclassified. Pregames and graphs are discussed in Section 1.6.

Section 1.7 contains material on vector-weighted games, and builds on this to provide a notion of dimension for simple games. We prove, for example,

[8] In the original European Economic Community, discussed in Section 1.2, Luxembourg was a dummy in exactly this sense (see also note 10).

that for every n, there is a simple game of exact dimension n, and that there are simple games with "exponential dimension."

In Section 1.8 we show that the general class of simple games can be arrived at by starting with a small natural class of simple games (the symmetric, or k-out-of-n, games) and closing out under three natural operations: the formation of voting blocs, isomorphism, and a special case of product called the bicameral meet (all defined later). Finally, Section 1.9 contains a brief discussion of the strategic form of a simple game.

1.2 Examples

Before proceeding, we collect together a few examples of simple games. We have several purposes in mind, in addition to the traditional one of helping the reader decode later abstractions. For example, we have tried to preview some of the key mathematical concepts that we will use later by applying them in the context of these examples. While such early treatments are perforce informal, the reader should be reassured that precise definitions appear later in the book, and we provide the references for these later definitions as we go.

These examples additionally serve to introduce many of the techniques for constructing simple games. A side benefit may be that the resulting variety dispels any thought that simple games need be "simple." In particular, not every yes-no voting system can be realized as a weighted voting system.[9]

We will consider, for each of the four real-world examples presented here, the class of simple games to which the example gives rise, and a few special instances of the class that we will make use of later in the book. In fact, with the exception of the somewhat complicated games constructed in Sections 2.7, 4.10, and 5.3, we will need little more in the way of specific examples than the dozen or so that we present here.

In some of the real-world examples presented, we have made simplifying assumptions to make them fit the context of simple games. For example, in the U.S. federal system, we consider a presidential veto to be the same as a "no vote" by the president, and we ignore the possibility of abstentions or absences.

Grants of Certiorari and the Class of Symmetric Games

Example 1.2.1. Grant of certiorari. One way to have a case heard by the U.S. Supreme Court is to have this further consideration approved of by at least four of the nine Justices. Such approval is called a grant of certiorari. This game is

[9]This is an important particular. The literature is scattered with passages that leave the impression that all yes-no voting systems are weighted. If this seems surprising, keep in mind that an elementary trap is easy to avoid only if one suspects that it might exist.

an example of what is called a symmetric game, or a k-out-of-n game, and the detailed study of these games dates back at least to Bott (1953).

Symmetric games are special cases of the weighted games introduced in the last section (and returned to in the next example); in fact, the phrase "k-out-of-n" refers to a description of the game in which each of n players is given a weight of one, and the quota is set at k. Notice that it takes six Supreme Court Justices to block (that is, to guarantee, by their opposition, that passage will not happen) a grant of certiorari, which is more than it takes to approve.

Symmetric games are the simplest venue within which to explore the relative ease of passing versus blocking, and the following symmetric games will be used for this purpose in Sections 3 and 5:

(i) The game [4; 1, 1, 1, 1, 1], in which every winning coalition is a blocking coalition, but not vice versa
(ii) The game [2; 1, 1, 1, 1, 1], in which every blocking coalition is a winning coalition, but not vice versa
(iii) The game [3; 1, 1, 1, 1, 1], in which the blocking coalitions are precisely the winning coalitions

The European Economic Community and the Class of Weighted Games

Example 1.2.2. The European Economic Community. The European Economic Community, as set up by the Treaty of Rome in 1958, consisted of six countries: France, Germany, Italy, Belgium, the Netherlands, and Luxembourg. Because of the different populations of the countries, votes were not distributed equally; France, Germany, and Italy were given four votes each, Belgium and the Netherlands two votes each, and Luxembourg just one. The winning coalitions were those with at least twelve of the seventeen possible votes.[10]

That both the European Economic Community and the k-out-of-n games fall within the important class of weighted games is completely clear. There are also simple games that may not, at first blush, appear to be weighted, but turn out to be. The best-known example is the UN Security Council, wherein passage requires approval of at least nine of the fifteen countries, subject to a veto by any one of the five so-called permanent members (we ignore the possibility of an abstention). A moment's reflection will show that an assignment of weight 1 to each nonpermanent member, an assignment of weight 7 to each permanent member, and a quota of 39 prove this to be a weighted system.

It is hardly an original observation that for a weighted game, the choice of weights is not unique, yet the extent of this "wiggle room," as well as the

[10] Actually, the EEC used several different voting rules, depending on the kind of issue in question. The rule we are describing was known as qualified majority voting.