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I am not interested in constructing a building, so much as in having a perspicuous view of the foundations of possible buildings.

Ludwig Wittgenstein, *Culture and Value*

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Preface

Jacques Tits's classification of thick irreducible spherical buildings of rank at least three (published in 1974 as Volume 386 of the Springer Lecture Notes [14]) is no doubt one of the great accomplishments of 20th century mathematics. At the heart of Tits's classification is the famous Theorem 4.1.2 of [14], which states that every 'local isomorphism' (in a suitable sense) from one thick irreducible spherical building to another extends to an isomorphism. Only with this result in hand does Tits invoke Coxeter's classification of finite reflection groups and begin a case-by-case analysis of buildings having a given irreducible spherical diagram.

Our main goal in this monograph is to give a detailed introduction to the theory of buildings culminating (in Chapter 10) in a proof of Theorem 4.1.2. Since the first four chapters of Tits's Lecture Notes accomplish this same goal quite beautifully, it behooves us to explain our purpose more carefully.

In 1981 Tits introduced his 'local approach' to buildings [16], central to which was the idea of viewing buildings as chamber systems. In the introduction to this paper, Tits wrote:

> So far, buildings have always been described as incidence geometries or simplicial complexes. The results of the present paper find their simplest expression in a somewhat more abstract framework, that of *chamber systems*.

In fact, a chamber system can be viewed as nothing more abstract than a certain class of edge-colored graphs, and the 'local approach' offers an indisputably more elementary way to learn about buildings than the 'complex' point of view taken in the Lecture Notes (but at a price: the important connection to the classical geometries is more difficult to describe in terms of chamber systems).

A basic aspect of the chamber system point of view is the notion of an isometry which can be found only implicitly in the Lecture Notes. The apartments of a building are the thin subbuildings having the same type as the building itself. In [16] Tits looked at an apartment as the image of an isometry from the corresponding Coxeter chamber system to the building. This suggested results

about extensions of isometries which turned out to be crucial. In [6] and [8], Bernhard Mühlherr and Mark Ronan developed these ideas in order to extend Theorem 4.1.2 to a large class of twin buildings. In particular, Ronan proved results in [8] which yielded an elegant simplified proof of Theorem 4.1.2 as a special case. (A spherical building can always be twinned with itself.) Ronan's proof follows the same basic lines as Tits's original proof, but his focus on extensions of isometries allowed him to circumvent a large portion of the technical difficulties encountered by Tits. Ronan's proof (with some minor simplifications of our own) forms the centerpiece of this book.

Another, more recent, development in the theory of buildings is the classification of Moufang polygons by Tits and the author [20]. In the addenda to his Lecture Notes, Tits introduced the Moufang property for spherical buildings, a certain symmetry property expressed in terms of apartments and roots. He also observed that, as a consequence of Theorem 4.1.2, all thick irreducible spherical buildings of rank at least three, as well as all the irreducible residues *of rank at least two* of such a building, have the Moufang property. This means, in particular, that the irreducible residues of rank two of such a building are Moufang polygons. Theorem 4.1.2 itself says that a thick irreducible spherical building of rank at least three is uniquely determined by the union (in an appropriate sense) of the residues of rank two containing a given chamber. These observations led Tits to suggest that a classification of Moufang polygons should make it possible to prove the classification of thick irreducible spherical buildings of higher rank in a simpler and more uniform fashion. This goal is also realized in [20].

In Chapter 11 of these notes, we introduce the Moufang property and show how it can be used to examine the structure of a spherical building and its automorphism group. In Chapter 12, we give an overview of the classification of thick irreducible spherical buildings as it is carried out in [20].

In [9] Ronan and Tits gave an existence criterion for buildings which applied, in particular, to all spherical buildings (thick, irreducible and of rank at least three). In the Lecture Notes, Tits proved existence by referring either to classical geometries (involving vector spaces, pseudo-quadratic forms, etc.) or to the theory of algebraic groups (with some additional effort for the buildings 'of mixed type'). The results of [9] allowed, by contrast, a completely elementary and uniform solution to the problem of existence. The way of looking at spherical buildings which we adopt in Chapters 11 and 12 reflects these ideas.

The first nine chapters of this book are heavily influenced by Ronan's beautiful monograph [7]. In Chapter 1 we introduce the language of chamber systems. In Chapters 2–6 we examine the structure of Coxeter chamber systems (whose

role in the theory of buildings is, in some geometrical sense, analogous to the role played by bases in the theory of vector spaces). Chapter 6 is devoted to a result of Mühlherr, which we use in Chapter 11 to link Theorem 4.1.2 to the Moufang property. Buildings and apartments are introduced in Chapters 7–8 and in Chapter 9 we specialize to spherical buildings.

We have tried to make this monograph accessible to as wide an audience as possible. In the theory of buildings there are often alternative definitions and different ways of reaching the same goal. This reflects the richness of the subject but is also a source of difficulty for the beginner. For the sake of simplicity we have, for the most part, refrained from giving alternatives. The only prerequisite for the first eleven chapters is a slight familiarity with the notion of a group defined by generators and relations; in fact, even this familiarity is needed only in the proof of two preliminary lemmas which the beginning reader would do just as well to accept as axioms. In the last chapter, we refer to results from [20], but it is of course hoped that the reader is keen to start looking at other sources by this point.

Virtually all the results in this book can be found in their original form in one of the sources in the bibliography. To be sure not to have overlooked anything, we have included [19], which contains a complete list of Tits's books and articles.

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