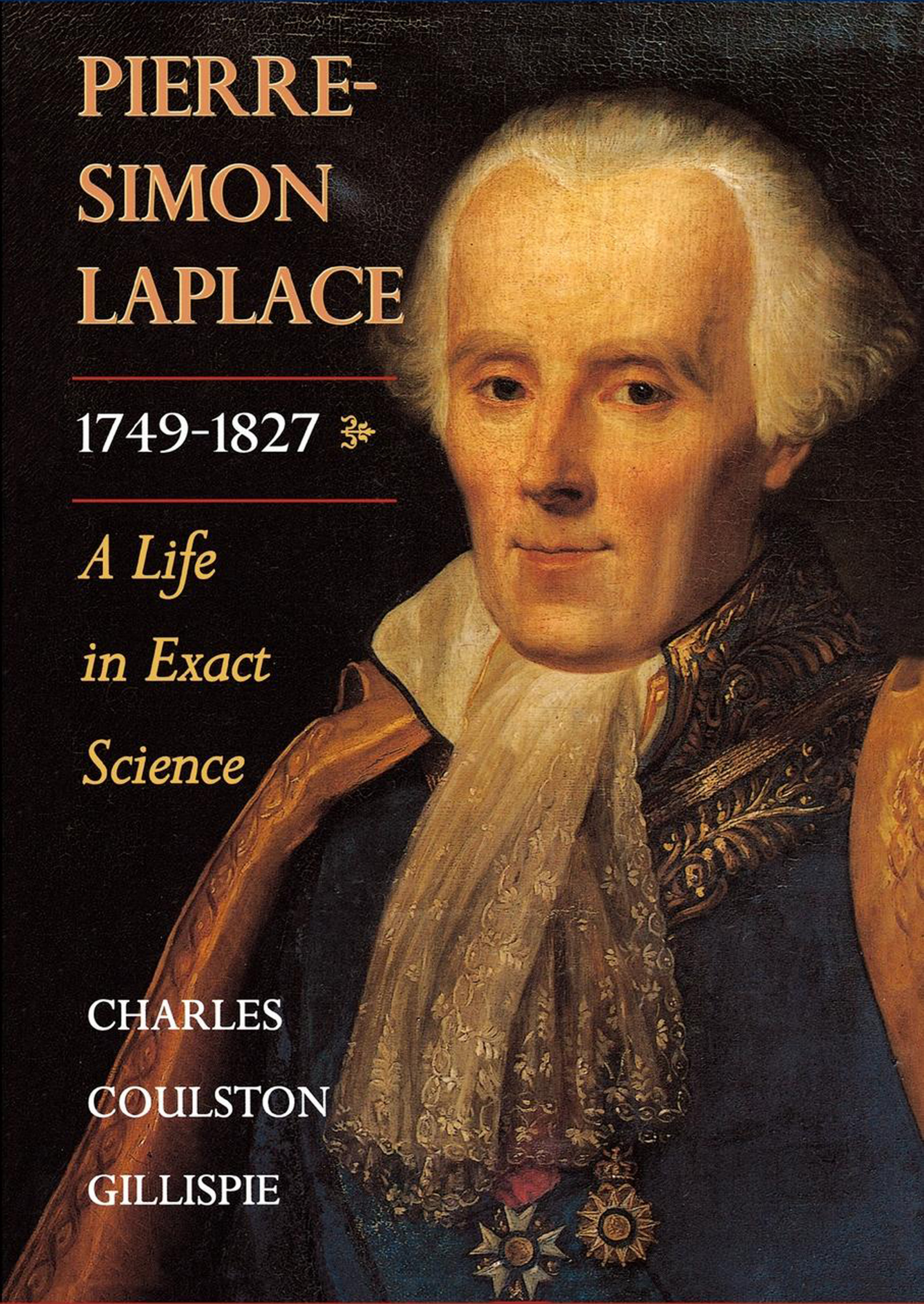

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1749-1827

A Life in Exact Science

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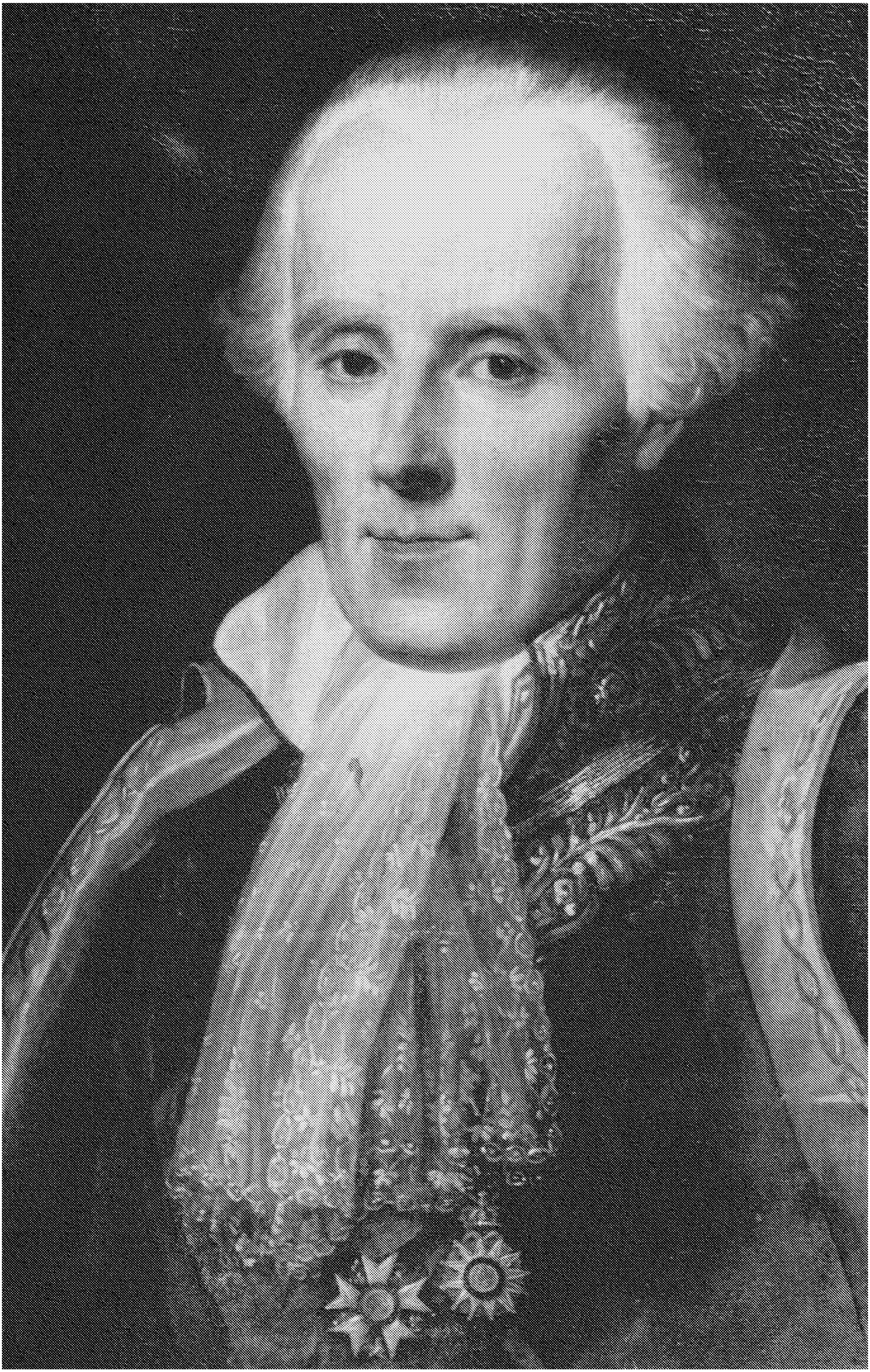

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A Life in Exact Science

CHARLES COULSTON GILLISPIE

With the Collaboration of

ROBERT FOX *and* IVOR GRATTAN-GUINNESS

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Preface

PIERRE-SIMON LAPLACE (1749–1827) was among the most influential scientists in all history. His career was important for his technical contributions to exact science, for the philosophical point of view he developed in the presentation of his work, and for the part he took in forming the modern scientific disciplines. The main institutions in which he participated were the Académie Royale des Sciences, until its suppression in the Revolution, and then its replacement, the scientific division of the Institut de France, together with two other Republican foundations, the École Polytechnique and the Bureau des Longitudes. It will be convenient to consider the scientific life that he led therein as having transpired in four stages, the first two in the context of the Old Regime and the latter two in that of the French Revolution, the Napoleonic period, and the Restoration.

The boundaries must not be taken more categorically than biography allows, but in the first stage, 1768–1778, we may see Laplace rising on the horizon, composing memoirs on problems of the differential and integral calculus, mathematical astronomy, cosmology, theory of games of chance, and causality, pretty much in that order. During this formative period, he established his style, reputation, philosophical position, certain mathematical techniques, and a program of research in two areas, probability and celestial mechanics, in which he worked mathematically for the rest of his life.

In the second stage, 1778–1789, he moved into the ascendant, reaching in both those areas many of the major results for which he is famous and that he later incorporated into the great treatises *Traité de mécanique céleste* (1799–1825) and *Théorie analytique des probabilités* (1812 and later editions). They were informed in large part by the mathematical techniques that he introduced and developed, then or earlier, most notably generating functions, the transform since called by his name, the expansion also named for him in the theory of determinants, the variation of constants to achieve approximate solutions in the integration of astronomical expressions, and the generalized gravitational function that, through the intermediary of Poisson, later became the potential function of nineteenth-century electricity and magnetism. It was also during this period that Laplace entered on the third area of his mature interests, physics, in his collaboration with Lavoisier on the theory of heat, and that he became, partly in consequence of this association, one of the inner circle of influential members of the scientific community. In the 1780s he began serving on commissions important to the government and affecting the lives of others.

In the third stage, 1789–1805, the Revolutionary period and especially that of the Directory brought him to his zenith. The early 1790s saw the completion of the great series of memoirs on planetary astronomy and involved him centrally in the preparation of the metric system. More important, in the decade from 1795 to 1805 his influence was paramount for the exact sciences in the newly founded Institut de France; and his was a powerful position in the counsels of the École Polytechnique, where the first generation of mathematical physicists had their training. The educational mission attributed to all science in that period of intense civic consciousness changed the mode of scientific publication from academic memoir to general treatise. The first four volumes of *Mécanique céleste* (Laplace himself coined the term), generalizing the laws of mechanics for their application to the motions and figures of the heavenly bodies, appeared from 1799 through 1805. The last parts of the fourth volume and the fifth volume, really a separate work that appeared in installments from 1823 to 1825, contain important material (on physics) not already included in the sequence of Laplace's original memoirs published previously by the old Academy.

Laplace accompanied both *Mécanique céleste* and *Théorie analytique des probabilités* by verbal paraphrases addressed to the intelligent public in the French tradition of *haute vulgarisation*. The *Exposition du système du monde* preceded *Mécanique céleste* and initially appeared in 1796. The *Essai philosophique sur les probabilités*, first published in 1814 as an introduction to the second edition of *Théorie analytique* and printed separately earlier in the same year, originated in a course of lectures at the École Normale in 1795.

The work of the fourth stage, occupying the period from 1805 until 1827, exhibits elements of culmination and of decline. It was then that the mature—perhaps the aging—Laplace, in company with the chemist Claude-Louis Berthollet, surrounded himself with disciples in the informal Société d'Arcueil. But the science that he set out to shape was not astronomy. The center of their interest, following Volume IV of *Mécanique céleste*, was in physics—capillary action, the theory of heat, corpuscular optics, and the speed of sound. The Laplacian school of physics, as it has come to be called, has had a mixed scholarly press. But whatever else may be said about it, there can be no doubt about the encouragement that it gave to the mathematization of the science.

Beginning in 1810, Laplace turned his attention to probability again, moving back by way of error theory into the subject as a whole. Mathematically speaking, *Théorie analytique des probabilités* (1812) may be said to belong to the previous phase of drawing together and generalizing the researches on special topics of his younger years. There were important novelties in the application, however, notably in the treatment of least squares, in the extension of probability in later editions to analysis of the credibility of witnesses and the procedures of judicial panels and electoral bodies, and in the increasing sophistication of the statistical treatment of geodetic and meteorological data.

Before proceeding further, readers may wish to turn to the Bibliography and familiarize themselves with its several categories. Its organization reflects the complexity of Laplace's life work, and inevitably so. The central portion (Section I, pp. 284–306) constitutes a dual chronology of his writings, by order of composition and by order of publication. References to the former sequence (Section I, Part 1) are given in parentheses thus: (23). References to the latter sequence (Section I, Part 2) are given by bracketing the date, and the several memoirs published in any given year are distinguished by letters of the alphabet, thus: [1777a]. The great majority of Laplace's publications were reprinted in his *Oeuvres complètes* (see Section H, pp. 283–84). That edition is more accessible than the original papers and treatises. Where the footnotes indicate *OC*, the reference is to the location in those volumes of the passage cited. Other abbreviations are listed facing the Bibliography, p. 280. Secondary works are cited in the footnotes by the name of the author and the date of publication. The full reference will be found in the alphabetical section (O) of the Bibliography, pp. 309–17.

Acknowledgments

THE ORIGINAL edition of this book appeared in 1978 in Volume 15 of the *Dictionary of Scientific Biography*, published in sixteen volumes by Charles Scribner's Sons from 1970 to 1980. I had the primary editorial responsibility for that collection and also undertook the entry on Laplace. The purpose is to give an account of the sequence, range, and results of Laplace's scientific work, explained in terms of his own time and in his own notation. This book does not aspire to be a critical analysis of his mathematical achievements and limitations per se. For that, and for the entire mathematical movement of his generation and the succeeding one, the reader may and should turn to the magisterial study by my colleague and collaborator, Ivor Grattan-Guinness, *Convolutions in French Mathematics, 1800–1840* (1990).

Throughout I have benefited even more than normally from the advice and assistance of others. The preface to the *DSB* article specifies the help received from colleagues and students in preparing the earlier version. Dr. Grattan-Guinness composed chapter 29 on the history of the Laplace transform, while Robert Fox contributed chapters 22, 23, 24, and 27 on the Laplacian school of physics. Both collaborators have generously accepted the request to revise their contributions for the present edition, and both are assisting with suggestions for revision of the entire work. So also is Stephen M. Stigler, who gave invaluable guidance on Laplace's earliest memoirs, and on statistical aspects in general.

In undertaking this revision, I am incorporating material from a number of studies of various aspects of Laplace's life and work that have appeared since the initial publication in 1978, all of which are included in the Bibliography. Two of the authors, Bernard Bru and Curtis Wilson, have very kindly reviewed and suggested improvements in the sections that concern their specialties, those on probability and on the inequality of Jupiter and Saturn. A current graduate student, David Attis, is serving as a very capable research assistant. As always Emily Gillispie has read drafts and proof. Her eye for style and language has improved matters throughout. Finally, and also as always, I resolutely claim responsibility for errors and infelicities that have escaped the scrutiny of all these guardian angels. Unless otherwise indicated, all translations are my own.

A special word of thanks is due to Jack Repcheck, editor at Princeton University Press, whose idea it was to republish this monographic article as a scientific biography in its own right. The American Council of Learned Societies, which holds the copyright to the *Dictionary of Scientific Biography*, has granted the requisite permission.

Madame Christiane Demeulenaere-Douyère, Conservateur of the Archives de l'Académie des Sciences, Institut de France, unfailingly affords scholars and scientists the most cordial and expert guidance and assistance in consulting the invaluable materials under her care, and like all who work with these documents, I am correspondingly grateful to her and to her staff. On behalf of the Academy she has graciously authorized reproduction of the portrait that serves as frontispiece. Its date is 1842, and it seems highly probable that the painting would have been commissioned by Madame Laplace, who, with the very active assistance of Arago, was even then arranging for publication of the first collection of Laplace's *Oeuvres* (see Section G, p. 283).

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Part I

EARLY CAREER, 1768–1778

CHAPTER 1

Youth, Education, and Election to the Academy

GENEALOGICAL RECORDS of the Laplace family in lower Normandy go back to the middle of the seventeenth century.[1] Laplace's father, Pierre, was a syndic of the parish, probably in the cider business and certainly in comfortable circumstances. The family of his mother, Marie-Anne Sochon, were well-to-do farmers of Tourgéville. He had one elder sister, also called Marie-Anne, born in 1745. There is no record of intellectual distinction in the family beyond what was to be expected of the cultivated provincial bourgeoisie and the minor gentry. One paternal uncle, Louis, an abbé although not ordained, is said to have been a mathematician and was probably a teacher at the college (a secondary school) kept at Beaumont-en-Auge by the Benedictines. He died in 1759, when his nephew was ten. Laplace was enrolled there as a day student from the age of seven to sixteen. Pupils usually proceeded to the church or the army; Laplace's father intended him for an ecclesiastical vocation.

In 1766 he went up to the University of Caen and matriculated in the Faculty of Arts, still formally a cleric. During his two years there he must have discovered his mathematical gifts, for instead of continuing in the Faculty of Theology, he departed for Paris in 1768. Apparently, he never took a degree, although he may briefly have been a tutor in the family of the marquis d'Héricy and may also have taught at his former college. The members of the faculty at Caen who opened his eyes to mathematics, and their own to his talent, were Christophe Gadbled and Pierre Le Canu. All that is known about them is that they were points of light in the philosophic and scientific microcosm of Caen, professors with the sense to recognize and encourage a gifted pupil.

On Laplace's departure for Paris at the age of nineteen, Le Canu gave him a letter of recommendation to d'Alembert, who immediately set him a problem and told him to come back in a week. Tradition has it that Laplace solved it overnight. Thereupon d'Alembert proposed another, knottier puzzle, which Laplace resolved just as quickly.[2] The

[1] See Boncompagni (1883) and G. A. Simon, "Les origines de Laplace: sa généalogie, ses études," in Pearson (1929), pp. 202–16.

[2] Bigourdan (1931).

story may be apocryphal, but there is no doubt that d'Alembert was somehow impressed and took Laplace up, as he had other young men in the evening of his own career, although none of comparable merit mathematically. The next question was a livelihood, and d'Alembert himself answered to that necessity, securing his new protégé the appointment of professor of mathematics at the École Militaire. Imparting geometry, trigonometry, elementary analysis, and statics to adolescent cadets of good family, average attainment, and no commitment to the subjects afforded little stimulus, but the post did permit Laplace to stay in Paris. He taught there from 1769 to 1776.

It was expected of Laplace that he should concentrate his energies on making a mathematical reputation in order to win election to the Academy of Science. If its records are complete, he presented thirteen papers in just under three years, beginning in March 1770. The topics were extreme-value problems; adaptation of the integral calculus to the solution of difference equations; expansion of difference equations in a single variable in recurrent series and in more than one variable in recurro-recurrent series; application of these techniques to theory of games of chance; singular solutions for differential equations; and problems of mathematical astronomy, notably variation of the inclinations of the ecliptic and of planetary orbits, the lunar orbit, perturbations produced in the motion of the planets by the action of their satellites, and "the Newtonian theory of the motion of the planets" (9). Of these papers, four were published (1, 5, 8, 13). Laplace translated the first two into Latin and placed them in the *Nova acta eruditorum*, where the second was printed before the first.[3]

Laplace read the earlier paper, "Recherches sur les maxima et minima des lignes courbes," before the Academy on 28 March 1770, five days after his twenty-first birthday. After a review of extreme-value problems, he proposed several improvements in the development that Lagrange had given to Euler's *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes* (1744). One modification concerned Lagrange's finding in a 1761 paper that there was no need to follow Euler in assuming a constant difference.[4] If the assumption was justified, the number of equations might be reduced by at least one, and otherwise the problem was unsolvable. Laplace found the same result by a method that his commissioners called "less direct, less rigorous in appearance, but simpler and fairly elegant" (1). In cases where a

[3] [1774a], [1771a].

[4] "Essai d'une nouvelle méthode pour déterminer les maxima et les minima. des formules intégrales indéfinies," *OL*, 1, 334–62.

difference is not constant, the difficulty was shown to arise from a faulty statement of the problem. A variable was concealed that should have appeared in the function representing the curve to be optimized. When it was identified, the equations became determinate. If the solutions yielded maximum values, the equations involved double curvature. Laplace further gave a general analytic criterion for distinguishing a true maximum or minimum from instances in which two successive values happen to be equal, and he appeared to have regarded this as his chief contribution.

It was, however, with the other Leipzig paper, "Disquisitiones de calculo integrale," that Laplace made his debut in print [1771a]. The subject is a particular solution for one class of ordinary differential equations. The method he developed subsequently led to enunciation of a theorem, the statement of which he later annexed without proof to his first memoir on probability although it has nothing to do with that subject.[5] Reworking this material two years later, in 1773, Laplace repudiated this earliest publication, or very nearly so, apologizing for grave faults that he blamed on the printer.[6] That was the only reference he ever made to either of these youthful ventures into Latin.

Laplace won election to the Academy of Science on 31 March 1773, after five years in Paris. Six years his senior, Condorcet had become acting permanent secretary earlier that month. In that capacity he composed the preface to the volume containing the first of Laplace's memoirs to be published in Paris, one on recurro-recurrent series, the other on probability of causes.[7] The Academy had never, observed Condorcet, received from so young a candidate in such a short time so many important papers on varied and difficult topics as the sequence submitted by Laplace.[8] On two previous occasions his candidacy had been passed over: in 1771, in favor of Alexandre Vandermonde, fourteen years his elder, and the following year in favor of Jacques-Antoine-Joseph Cousin, ten years older and a professor at the Collège Royal de France.

Evidently, Laplace felt slighted, despite his youth. On 1 January 1773 d'Alembert wrote to Lagrange asking whether there was a possibility of obtaining a place in the Prussian Academy and a post at Berlin, since

[5] [1774c]. The equation was Equation (23), in chapter 6 below. For Laplace's statement of the theorem, see *OC*, **8**, pp. 62–63, and for the demonstration, see [1775a], Section VI, *OC*, **8**, pp. 335–46.

[6] [1774c], *OC*, **8**, p. 63, note, where Laplace refers the reader to [1777a], discussed in chapter 6.

[7] [1774b, 1774c].

[8] *SE*, **6**, (1774), "Histoire," p. 19.

the Paris Academy had just preferred a person of markedly inferior ability.[9] The approach lapsed three months later when Laplace was chosen an adjunct member in Paris.[10]

[9] *OL*, **13**, pp. 254–56.
[10] Bigourdan's statement that he was admitted directly to the second rank of *associé* is incorrect. (Bigourdan [1931] p. 384).

CHAPTER 2

Finite Differences, Recurrent Series, and Theory of Chance

AMONG LAPLACE'S early interests, it turned out to be a memoir on the solution of difference equations that marks the beginning of one of the main sequences of his lifework [1771b]. An unpublished sequel of the following year is entitled "Sur les suites récurrentes appliquées à la théorie des hasards" (10). We may reasonably surmise that the applicability of such series to problems of games of chance, and not any prior penchant for that subject matter, was mainly responsible for the appearance given by the record of his publications that probability attracted him more strongly than did celestial mechanics in this opening phase of his career.

The appearance is misleading, however, or at least ironic. For at the outset of the earlier paper, "Recherches sur le calcul intégral aux différences infiniment petites, et aux différences finies," Laplace observed that the equations he was studying turn up more frequently than any other type in applications of the calculus to nature. A general method for integrating them would be correspondingly advantageous to mechanics, and especially to "physical astronomy."[1] That science looms largest in his unpublished, as distinct from his published, record (1–13). Judging from a report in the archives of the Academy of Sciences, a paper of 27 November 1771, "Une théorie générale du mouvement des planètes," may even have been the germ of *Mécanique céleste* (9). The draft does not survive, but the referees, d'Alembert, Bezout, and Bossut, say that the subject was the Newtonian theory of the planets. Their report is full enough to permit the conclusion that Laplace expanded this memoir into his first general treatise on celestial mechanics, *Théorie du mouvement et de la figure elliptique des planètes* (1784).[2]

Since Laplace was still only knocking at the door of the Academy in 1771, he submitted "Recherches sur le calcul intégral aux différences infiniment petites, et aux différences finies" to the Royal Society of Turin for publication in its *Mélanges* [1771b]. It was almost surely the expansion of a paper on difference equations alone that he had read on

[1] [1771b], p. 273.
[2] [1784a], Chapter 15.

18 July 1770 in his second appearance before the Academy in Paris (2). Laplace now reserved difference equations for the second part of the memoir and proposed adapting to their solution—or as he said in the looser terminology of the time, to their "integration"—the method he had developed for infinitesimal expressions, presumably in a paper "Sur le calcul intégral des suites récurrentes," dated 13 February 1771 (6). He began by confirming in his own manner a theorem that Lagrange had recently proved concerning integration of equations of the following form:

$$X = y + H\frac{dy}{dx} + H'\frac{d^2y}{dx^2} + H''\frac{d^3y}{dx^3} + \cdots + H^{n-1}\frac{d(d^ny)}{dx^n}, \quad (1)$$

where X, H, H', H'', ..., are any functions of x. Lagrange had shown that such equations can always be integrated if integration is possible in the homogeneous case when $X = 0$. His proof was of a type classical in eighteenth-century analysis. It involved introducing a new independent variable, z; multiplying both sides of the equation by $z\,dt$; supposing integration of the resulting adjoint equation accomplished; and examining the steps needed to reduce the order one degree at a time until a solvable form should be reached. The approach worked for differential equations. The procedure presupposed the validity of infinitesimal methods in analysis, however, and therefore the operations were inapplicable to the solution of difference equations.

Laplace's approach appeared to be more cumbersome and turned out to be more general. Instead of introducing a multiplying factor and supposing the subsequent integration accomplished, Laplace employed integrating factors directly. The problem is to integrate equations of the form (1). To that end, he let

$$\omega\frac{dy}{dx} + y = T, \quad (2)$$

where T and ω are functions of x. Differentiating Equation (2) successively n times, he then multiplied the first of the resulting equations by ω', the second by ω'', the third by ω''', and so on. He next added all these equations to Equation (2), grouped the terms by orders of y, and compared the enormous resulting expression to Equation (1). Lengthy manipulation allowed him to determine the multipliers ω', ω'', ω''', ..., in terms of ω and H', H'', H''', ... Not only was Laplace able to prove Lagrange's theorem by this method, but he could also write equations equivalent to Equation (1) and evaluate them generally. In the second part of the memoir, and this was its motivation, Laplace adapted his approach to equations in finite differences by showing that all steps could be modified to conform to the rules of algebra.

Although the memoir had opened with mention of mechanics and astronomy, Laplace introduced its raison d'être, the solution of difference equations, with a reminder that their calculus was the foundation of the entire theory of series.[3] Coherently enough, therefore, he continued the discussion with a determination of the general term of series of the important class[4]

$$y^x = A\phi^x y^{x-1} + {}'A\phi^x\phi^{x-1}y^{x-2} + {}''A\phi^x\phi^{x-1}\phi^{x-2}y^{x-3} + \cdots, \quad (3)$$

where A, $'A$, $''A$ are constants, and ϕ is a function of x. In the simplest case, in which $\phi = 1$, Equation (3) reduces to the recurrent form,

$$y^x = Ay^{x-1} + {}'Ay^{x-2} + {}''Ay^{x-3} + \cdots + {}^{n-1}Ay^{x-n} \quad (4)$$

The memoir ends with an application of the calculus of finite differences to a solution of this equation, and Laplace gives a method for determining the constants.

The episode is largely typical of the relation of Laplace's point of departure to the work of elders and near contemporaries. There is the not quite ritual obeisance to a principle or practice, in this case the formulation of problems in terms of differential equations in general, attributed to d'Alembert, the patron. There is the tactful nod to a result found quite differently by Condorcet, the well-placed official. There is the pioneering analytical breakthrough achieved by Euler, although in restricted form. There is the formal mathematical theorem stated by Lagrange, emphasizing analyticity. There is, finally, the adaptation imagined and executed by Laplace, his motivation being the widest applicability to problems in the real world.

His finding that equations of the form (3) are always integrable became the starting point of the next memoir, which deals with what Laplace called "recurro-recurrent" series and their application to the theory of chance [1774b]. Still not a member of the Academy, he submitted it for their judgment on 5 February 1772, and they placed it in the *Savants étrangers* collection. Recurrent series of the familiar form (3) were restricted to a single variable index, the definition being that "every term is equal to any number of preceding terms, each multiplied by a function of x taken at will.[5] It had been while investigating certain

[3] [1771b], p. 299.
[4] Ibid., p. 330.
[5] [1774b], *OC*, **8**, p. 5.

problems in the theory of chance, so Laplace said a little later, that he had come upon equations in finite differences of another, novel type.[6] They are the analogues in finite analysis of partial differential equations and give rise to a complex set of series, the general term of which has two or more variable indices. In such series as Equation (4), if ϕ is a function of x and n rather than of x alone, and if the integers 1, 2, 3,..., are substituted for x and n, then for each value of n, a series results in which ${}^{n}y^{x}$ designates the term corresponding to the numbers x and n. The definition of a recurro-recurrent series is that ${}^{n}y^{x}$ is equal to any number of preceding terms, taken in rank or in order, in any number of such series, each multiplied by some function of x. Here is the example that Laplace displayed:

$$
\begin{aligned}
{}^{1}y^{x} + A'\,y^{x-1} + B'\,y^{x-2} + \cdots + N &= 0,\\
{}^{2}y^{x} + A''\,{}^{2}y^{x-1} + B''\,{}^{2}y^{x-2} + \cdots + N'' &= H''^{1}\,y^{x} + M''\,{}^{1}y^{x-1}\\
&\quad + P''\,{}^{1}y^{x-2} + \cdots,\\
\cdots\cdots\cdots\cdots\cdots\cdots&\cdots\cdots,\\
{}^{n}y^{x} + A^{n}\,{}^{n}y^{x-1} + B^{b}\,{}^{n}y^{x-2} + \cdots + N^{n} &= H^{n}\,{}^{n-1}y^{x}\\
&\quad + M^{n}\,{}^{n-1}y^{x-1} + \cdots. \qquad (5)
\end{aligned}
$$

The value of ${}^{n}y^{x}$ must be determined when $A^{n}, B^{n}, \ldots, N^{n}, H$ are any functions of n; when also A'', B'', ..., A''', B''',... are what those functions become on substituting 1, 2, 3,... for n; and when, finally, $A, B, \ldots, N$ are any constants.

The solution consists in showing that Equation (5) can always be transformed as follows:

$$
{}^{n}y^{x} = a^{n}\,{}^{n}y^{x-1} + b^{n}\,{}^{n}y^{x-1} + c^{n}\,{}^{n}y^{x-3} + \cdots + u^{n}, \qquad (6)
$$

where a^{n}, $b^{n}, \ldots, u^{n}$ are functions of n and of constants that can be determined.[7] This Equation (6), in turn, is an expression for a recurrent series precisely of the type (3) that he had shown to be integrable in his Turin memoir.[8] A further problem of a third-degree equation in finite differences is then shown to be reducible successively to the forms (5) and (6), and Laplace went on to propose a general procedure for reducing an equation of any degree r to a lower degree, the requirement being that by the assumption made for the value of n, the equation of degree r become one of degree $r - 1$.

[6] [1776a, 1°], *OC*, **8**, p. 71.
[7] *OC*, **8**, p. 7.
[8] [1771b].

A passage included many years later in the *Essai philosophique sur les probabilités* makes clearer to the layman how he was visualizing these series.[9] A recurrent series is the solution of a difference equation with a single variable index. Its degree is the difference in rank between its two extreme terms. The terms may be determined by means of the equation, provided the number of known terms equals its degree. These terms are in effect the arbitrary constants of the expression for the general term or (which comes to the same thing) of the solution of the difference equation. The reader is next to imagine a second series of terms arranged horizontally above the terms of the first series, a third series above the second, and so on to make a display infinite upward and to the right. It is supposed that there is a general equation between the terms that are consecutive both horizontally and vertically and the numbers that indicate their rank in both directions. This will be an equation in finite partial differences, or recurro-recurrent.

The reader is finally to imagine that on top of the plane containing this pattern of series there is another containing a similar pattern, and so on to infinity, and that a general equation relates the terms that are consecutive in the three dimensions with the numbers indicating their rank. That would be an equation in finite partial differences with three indices. Generally, and independently of the spatial model, such equations may govern a system of magnitudes with any number of indices. Some eight years later, Laplace replaced the use of recurro-recurrent series with the more efficient tool of generating functions for solving problems in finite differences, which he encountered mainly in the calculus of probability.[10]

It is important biographically to notice his passing remark that investigations in the theory of chance had led him to the formulation of recurro-recurrent series. The latter part of the memoir illustrates how they might be applied to the solution of several problems concerning games of chance. In the first such example, two contestants, A and B, play a game in which the loser at each turn forfeits a crown to his opponent. Their relative skills are as a to b. At the outset A has m crowns, and B has n. What is the probability that the game will not end with x or fewer turns? Laplace found the answer by substituting values given by the conditions of the problem in a series of equations of the form (5), first for the case in which $a = b$, $m = n$, and n is even, and then for all possible suppositions about the parameters.

This problem, like many others that Laplace adduced, appears in the writings of De Moivre, who seems to have furnished his first reading in

[9] *OC*, **8**, pp. xxvi–xxvii.
[10] [1782a], Chapter 11.

the subject, but whose solutions were less direct.[11] A further example was suggested to Laplace by a bet made on a lottery at the École Militaire. What is the probability that all the numbers 1, 2, 3, . . . , n will be taken after x draws? That, too, he found by formulating the problem in a recurro-recurrent series in two variable indices, observing that the approach could clearly have wide applicability in the theory of chance, where the most difficult problems often concern the duration of events.

Laplace chose precisely this juncture for defining probability. The statement occurs immediately after his development of the method of recurro-recurrent series and just before its application to the foregoing examples:

> The probability of an event is equal to the sum of each favorable case multiplied by its probability, divided by the sum of the products of each possible case multiplied by its probability, and if each case is equally probable, the probability of the event is equal to the number of favorable cases divided by the number of all possible cases.[12]

The definition is noteworthy, not for its content, which was standard, but for its location in the development of his work and for its phrasing. The wording should serve to temper the criticism often made of Laplace, particularly in respect to inverse probability (which, to be sure, he had not yet started), that he gratuitously assumed equal a priori probabilities or possibilities. It is true that he often did—although not, as will appear, when he had any data. It is also interesting that in the passage immediately preceding the definition, Laplace should have written "duration of events" rather than "duration of play." This was the first remark he ever printed about the subject as a whole, and already he was thinking about its applicability to the world, and not merely to games of chance.

[11] De Moivre (1756). The problem in question is LVIII, p. 191.

[12] [1774b], *OC*, **8**, pp. 10–11.

CHAPTER 3

Probability of Events and of Their Causes: The Origin of Statistical Inference

HISTORICALLY and biographically, the instinctive choice of words is often more indicative than the deliberate. There is much anachronism in the literature, and it may be well, therefore, to take this, the juncture at which Laplace was entering upon one of the central preoccupations of his life, as the occasion to venture an observation about the early history of probability. More substance has sometimes been attributed to it than the actual content warrants. It is true that adepts of the subject were much given to celebrating its applicability, but prior to Laplace what they were praising was more prospect than actual accomplishment. Jakob Bernoulli's *Ars conjectandi* (1713) is justly famous mathematically, although it is seldom mentioned that part IV, headed "Usum et applicationem praecedentis doctrinae in civilibus, moralibus, & oeconomicis," contained simply the law of large numbers and otherwise remained uncompleted.

In the Dutch and English insurance industry in the eighteenth century, risks were estimated empirically, and the tables on which actuarial transactions depended were numerically insecure. De Moivre and others did employ probability in calculation of annuities, and Daniel Bernoulli undertook a theoretical analysis of the risks attending inoculation for smallpox.[1] These were particular problems, however, and the only extended field of application open to probabilistic analysis was the theory of games of chance. Even there, most of the experiments were thought experiments.

The statement that games of chance provided the principal subject matter in which theorems could be demonstrated and problems solved mathematically will be confirmed by close attention to contemporary usage, which refers to *théorie des hasards*, or "theory of chance." The word "probability" was not used to designate the subject. That word appears in two ways, one more restricted and the other vaguer than in the post-Laplacian science. In the mathematical "theory of chance,"

[1] Daniel Bernoulli, "Essai d'une nouvelle analyse de la mortalité causée par la petite vérole et des avantages de l'inoculation pour la prévenir," *MARS* (1760/1766), part 2, pp. 1–79.

probability was a quantity, its basic quantity, that which Laplace defines in the passage quoted in the preceding chapter. "*Calcul des probabilités*" refers to calculation of its amount for certain outcomes in given situations. The phrase "*théorie des probabilités*" rarely, if ever, occurs.

The word "probability" had its second and larger sense, one that would have befitted a theory if any had existed, rather in the philosophic tradition started by Pascal's wager on the existence of God.[2] The changes rung on that idea belong to theology, epistemology, and moral philosophy, some pertaining to what is now called decision making and others to political economy. Such is the discourse in the article "Probabilité" in the *Encyclopédie* (1765).[3] It was largely skepticism about the mathematical prospects for that sort of thing that inspired d'Alembert's overly deprecated hostility to the subject, and optimism about it that inspired Condorcet's overly celebrated enthusiasm.[4] Laplace himself was clear about the difference, although in other terms. Indeed, he insisted upon it in the distinction between mathematical and moral expectation in one of the pair of papers to be discussed next, which between them did begin to join mathematical theory of games with philosophic probability and scientific methodology.

The bibliographical circumstances need to be discussed before the significance of these two papers can be fully appreciated. Both were composed before Laplace became a member of the Academy. They were thus printed in the *Savants étrangers* series, in successive volumes. The more famous was entitled "Mémoire sur la probabilité des causes par les événements" [1774c]. The second, and lengthier, was delayed for two years and then combined with an astronomical memoir under the title "Recherches, 1°, sur l'intégration des équations différentielles aux différences finies, et sur leur usage dans la théorie des hasards. 2°, sur le principe de la gravitation universelle, et sur les inégalités séculaires des planètes qui en dépendent" [1776a, 1° and 2°].

As we shall see, this early coupling of probability with astronomy was no mere marriage of convenience. Laplace spent his entire professional life faithful to the pattern that it started. Before discussing that, however, we shall need to consider the way in which the first part of the dual memoir completed his earlier application of recurro-recurrent series to solving problems in the theory of chance and at the same time complemented his new departure into the determination of cause. He

[2] For the historiography of probability, see Bibliography, pp. 308–9.

[3] Long thought to have been written by Diderot himself, the article is now known to have been composed for the most part by the Swiss contributor, Gabriel Cramer. See Bru and Crépel (1994), p. 225.

[4] For d'Alembert's reservations, see Daston (1988), chapter 2, section 3, and for Condorcet's way around them, Brian (1994), part I, chapters 4–5.

submitted this resumption of his work on difference equations to the Academy on 10 March 1773.[5]

These were the writings in which Laplace began broadening probability from the mathematics of actual games and hypothetical urns into the basis for statistical inference, philosophic causality, estimation of scientific error, and quantification of the credibility of evidence, to use terms not then coined. The preamble of the paper on cause declares, and cross-references in both essays confirm, that they were conceived as companion pieces on the subject that Laplace was now beginning to call probability. The one breaks new ground in what was later called its inverse aspect. The other extends and systematizes a direct approach. In preferring the word "probability" to suggest the wider scope that he was giving the subject itself as a branch of mathematics, Laplace may well have been following the precept of Condorcet, newly the acting permanent secretary of the Academy, who, in prefatory remarks to the volume in which the memoir on causes appeared, praised it for its approach to predicting the probability of future events. "It is obvious," wrote Condorcet, "that this question comprises all the applications that can be made of the doctrine of chance to the uses of ordinary life, and of that whole science; it is the only useful part, the only one worthy of the serious attention of philosophers."[6]

The memoir on causes opens with a preamble, most of which might more appropriately have belonged to the concurrent piece on the solution of difference equations. Laplace referred readers to the latter memoir after reviewing what De Moivre and Lagrange had contributed to these problems and the way in which they had then involved him in the theory of chance. The present memoir had a different object, the determination of the probability of causes, given knowledge of events. Uncertainty, the reader is told, concerns both events and their causes (notice that at the outset Laplace took probability to be an instrument for repairing defects in knowledge). When it is given that an urn contains a set number of black and white slips in some definite ratio, and the probability is required of drawing a white one, then we know the cause and are uncertain about the event. But if the ratio is not given, and after a white slip is drawn the probability is required that it be as p is to q, then we know the event and are uncertain about the cause. All problems of theory of chance could be reduced to one or the other of these classes.

[5] The date appears in the procès-verbaux of the Academy (13). A marginal note in the printed text dating it 10 February must be erroneous.

[6] *SE*, **6** (1774), "Histoire," p. 18. On the interplay between Condorcet and Laplace here, see Gillispie, (1972) and Brian (1994), part I, chapter 5.

Laplace here proposed to investigate problems of the second type. He began on the basis of a theorem that, like the definition of probability in the previous memoir, he enunciated verbally:

> If an event can be produced by a number *n* of different causes, the probabilities of the existence of these causes, given the event (*prises de l'événement*), are to each other as the probabilities of the event, given the causes: and the probability of each cause is equal to the probability of the event, given that cause, divided by the sum of all the probabilities of the event, given each of the causes.[7]

In substance, this theorem is the same as that published posthumously by Thomas Bayes in 1763, eleven years previously. Not only is it now named "Bayes's theorem" or "Bayes's rule," but in the twentieth century the entire approach to probability and statistics depending on it has come to be called Bayesian. That usage derives ultimately from the early-nineteenth-century vindication by Augustus De Morgan and George Boole of their obscure countryman's priority, which was also recognized by Poisson in his *Recherches sur la probabilité des jugements* (1837).[8] Laplace nowhere mentioned Bayes in the memoir on probability of causes. Later he did refer to Bayes in one sentence in the *Essai philosophique sur les probabilités.*[9] In 1774 he had almost certainly not read Bayes's paper in the *Philosophical Transactions*. In this period continental mathematicians seldom read or referred to their British contemporaries, and the statement and approach in Bayes's piece are very different.

On the other hand, Laplace may well have heard of Bayes. His paper was submitted to the Royal Society after his death by his friend and fellow minister, the theologian and liberal moral philosopher Richard Price, who accompanied it with a covering note and an appendix.[10] Price was known on the Continent, especially among political theorists, including Condorcet, and Condorcet did mention Bayes in a comment of 1781.[11] Moreover, in introducing the analysis of cause, Laplace did not claim that it was an altogether new subject. He said it was novel "in many respects." More important, he echoed Condorcet's prefatory

[7] [1774c], *OC*, **8**, p. 29.

[8] Bayes's paper, "An essay towards solving a problem in the doctrine of chances," is reprinted in Pearson and Kendall (1970), pp. 134–53. For its importance to Laplace, see Stigler (1978) and (1982).

[9] *OC*, **7**, p. cxlviii.

[10] For a discussion of the motivation and significance of Bayes's paper, and of the relations between Bayes and Price, see Gillies (1987).

[11] In the prefatory comment to Laplace's later "Mémoire sur les probabilités," [1781a], *HARS*, (1778–1781), pp. 43–46.

remarks to the effect that the approach "the more merits being developed in that it is mainly from that point of view that the science of chance can be useful in civil life." It should also be said, finally, that in the sequel the analysis of inverse probability derived from Laplace's memoir and from his further work. Bayes remains one of those pioneers remembered only after the subject they intrinsically might have started had long been flourishing thanks to work of others that did have consequence.

However that may be, Laplace proceeded from the statement of the theorem to an example. From an urn containing an infinite number of white and black slips in unknown ratio, $p + q$ slips are drawn, of which p are white and q black. What is the probability that the next slip will be white? The above theorem gives the following formula for the probability that x is the true ratio[12]

$$\frac{x^p(1-x)^q\,dx}{\int x^p(1-x)^q\,dx}, \tag{7}$$

where the integral is taken from $x = 0$ to $x = 1$. Laplace calculated that the required probability of drawing a white slip on the next try is

$$\frac{p+1}{p+q+2}. \tag{8}$$

A second example from the same urn leads to the application that was the point of this analysis. After pulling p white and q black slips in $p + q$ draws, what is the probability of taking m white and n black slips in the next $m + n$ draws? For that probability, Laplace obtained the expression

$$\frac{\int x^{p-m}(1-x)^{q-n}\,dx}{\int x^p(1-x)^q\,dx}, \tag{9}$$

where the limits are again 0 to 1, and went on to ask a more significant question. How large would $(p + q)$ need to be, and how small would $(m + n)$ need to remain, in order to permit calculation of the probability of the ratio of m to n given the ratio of p to q? Clearly, the solution of that problem would permit calculating the probability of a future event from past experience, which is to say for statistical inference (although Laplace never used that phrase). He did not try to solve the problem in general, pleading the lengthiness of the calculation. Instead,

[12] [1774c] *OC*, **8**, p. 30.

he proceeded to the demonstration of a limit theorem. In effect, he showed, in what he called an interesting (*curieux*) proof, that the numbers $p + q$ can be supposed large enough to bring as close to certainty as one pleases the probability that the ratio of white slips to the total lies between $p/(p + q) + \omega$ and $p/(p + q) - \omega$, where ω is less than any given magnitude.

Laplace had two changes to ring on one of the classic problems, the division of stakes after a game is interrupted. First, he referred the reader to the companion memoir for the method of employing recurro-recurrent series in deducing the canonical solution for the standard case, in which the relative skills of the two players are given. He further promised a general solution in the case of three or more players. So far as he knew, that problem had never been solved. (Never reticent about claiming credit, Laplace was mistaken in this instance, for De Moivre had already solved the problem.)[13] Second, Laplace addressed the case in which the relative skill of the players is unknown, pointing out that it pertained to the probability of future events. Laplace solved it for a two-person game in this inverse example, without attempting the generality of three or more players.

Altogether more significant is the next topic: first, for its subject, which was the determination of the mean value among a series of observations; second, for the area from which Laplace took it, which was astronomy; and third, for the application, which was to theory of error. In view of the whole development of Laplace's later career, this article may be considered highly indicative, not least in his manner of introducing it. Two years previously, he wrote, he had worked out a solution for taking the mean value among several observations of the same phenomenon. Thinking it would be of little use, he had deleted it from the memoir on recurro-recurrent series, where he had originally intended it as a postscript. He had since learned from an astronomical journal that Daniel Bernoulli and Lagrange had both investigated the same problem.[14] Their memoirs remained unpublished, and he had never seen them.

This announcement, together with what he now calls the utility of the matter, had led him to set out his own ideas. It would appear that he must have given them some further development, for the approach turned on treating the true value as the unknown cause of three observed values, taken for effects. Afterward, the mean value giving the minimum probability of error would be determined. It is evident from

[13] Todhunter (1865) p. 468.

[14] [1774c], *OC*, **8**, pp. 41–42. He says *Journal astronomique*, but the name in fact was *Recueil des astronomes*. On this matter, see Stigler (1978), 247–48.

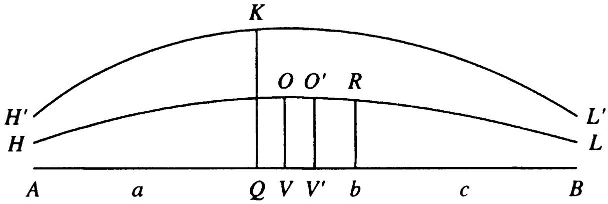

FIGURE 1

the outset that the observations Laplace had in mind were astronomical, for the point to be fixed indicated the time at which the event occurred. Laplace now constructed a graph and a posterior distribution curve, *HOL* in Figure 1. The line *AB* represents time. Three observations of the phenomenon occur at points a, b, and c. The interval between a and b is p seconds, and between b and c it is q seconds. The problem is to find the point V at which to fix the mean between the three readings a, b, and c.

Laplace begins by considering the error distribution in general. It is more likely that an observation deviates from the true value by two seconds than by three, by three seconds than by four, and so on, but the law relating probability of error inversely to its degree is unknown. The probability that an observation differs from the true value V by the amounts Vp and Vp' may be represented along the curve RMM' in Figure 2. If x stands for the abscissa Vp and y for the corresponding ordinate pM, the equation of the curve may be written $y = \phi(x)$. The curve is symmetrical around VR since it is equally probable that the deviation from the true value is to the left as to the right.

Next Laplace returns to Figure 1, where the point a is at distance x from the true value V. The probability that the three observations deviate by the distances Va, Vb, and Vc will be $\phi(x) \cdot \phi(p - x) \cdot \phi(p + q - x)$. If it be supposed that the true instant is at V' and that $aV' = x'$, this probability will be equal to $\phi(x') \cdot \phi(p - x') \cdot \phi(p + q - x')$. Then by the fundamental principle of inverse probability stated at the outset (i.e., Bayes's rule), the probabilities that the true instant of

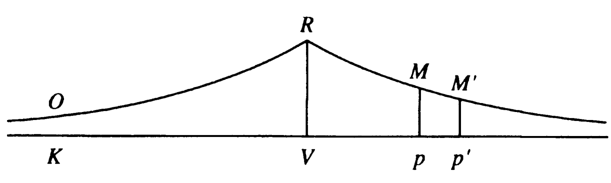

FIGURE 2

the phenomenon is at V or V' will be to each other in the proportion of these two expressions. The equation of the curve HOL is $y = \phi(x) \cdot \phi(p - x) \cdot \phi(p + q - x)$, and its ordinates represent the probabilities of the corresponding points on the abscissa.

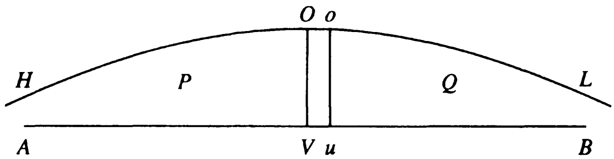

FIGURE 3

In speaking of the mean to be taken among several such observations, either of two things may be intended. One is the instant at which it is equally probable that the true time of the phenomenon occurs before or after it. That may be called the "probabilistic mean." The other is the instant at which the sum of errors to be incurred ("feared" in Laplace's terminology) multiplied by their probabilities is a minimum. Laplace named it the "mean of error" or "astronomical mean," the latter because it is the one that astronomers should prefer.

Laplace now shows by a nice piece of geometrical reasoning that the probabilistic and astronomical means come to the same value, which may be found by bisecting the area under the curve HOL in Figure 3. The probabilistic mean may be found by determining the ordinate OV, which divides the area under the curve in two equal parts, since it is then as probable that the true instant falls to the left as to the right of the midpoint V. For the probabilistic mean, the point V has to be located where the sum of the ordinates of the curve HOL, multiplied by their distance from the point V, is a minimum. To prove that there is no difference between the two, Laplace introduces the ordinate ou infinitely close to OV and lets $VU = dx$ and $OV = y$. He also lets Q be the center of gravity of the part uol of the curve, M be the area under this part, and z be the distance from the point Q to the ordinate OV. If the point V is taken as the mean, the sum of the ordinates multiplied by their distances from V will be

$$Mz + Nz' + \frac{1}{2} y\, dx^2,$$

whereas if u is taken as the mean, then the sum of the ordinates multiplied by their distance from u will be

$$M(z - dx) + N(z' + dx) + \frac{1}{2} y\, dx^2.$$

The difference between these two quantities is $Ndx - Mdx$, and that must be equal to zero in the case of a minimum. In that case, $M = N$, so that the ordinate OV bisects the area under the curve HOL. Thus, the astronomical and probabilistic means are one and the same.

In an illuminating commentary accompanying a translation of the essay, Stephen Stigler considers that this argument characterizes, in the terminology of modern statistics, a posterior median (the probabilistic mean) as optimal for a certain loss function (the astronomical mean), and he takes it to be one of the earliest results that pertains to mathematical statistics as distinct from theory of probability.[15]

In order to choose a mean value in practice, Laplace needed to specify the form of the probability curve $\phi(x)$. It is reasonable to suppose, he argued, that the ratio of two consecutive infinitesimal differences is the same as that of the two corresponding ordinates, so that $d\phi/\phi$ is constant. If so, the equation relating the ordinates to their infinitesimal differences is

$$\frac{d\phi(x + dx)}{d\phi(x)} = \frac{\phi(x + dx)}{\phi(x)}, \tag{10}$$

whence

$$\frac{d\phi(x)}{dx} = -m\phi(x); \qquad \text{thus } \phi(x) = \beta e^{-mx}. \tag{11}$$

The parameter m is constant, and since the area under the curve ORM is supposed to equal unity, and the curve is symmetrical, $\beta = m/2$, and

$$\phi(x) = \frac{m}{2} e^{-mx}. \tag{12}$$

Thus Laplace did not here arrive at the curve that would lead to the famous least-squares rule, first published by Legendre in 1805 (chapter 21). Laplace's purpose was to convince astronomers that their normal practice of taking an arithmetical average was erroneous, although he had to acknowledge that his method was difficult to use. A further application of the principle of inverse probability calculates the probabilities that the different values of m are to each other as the

[15] Stigler (1986b), p. 360.