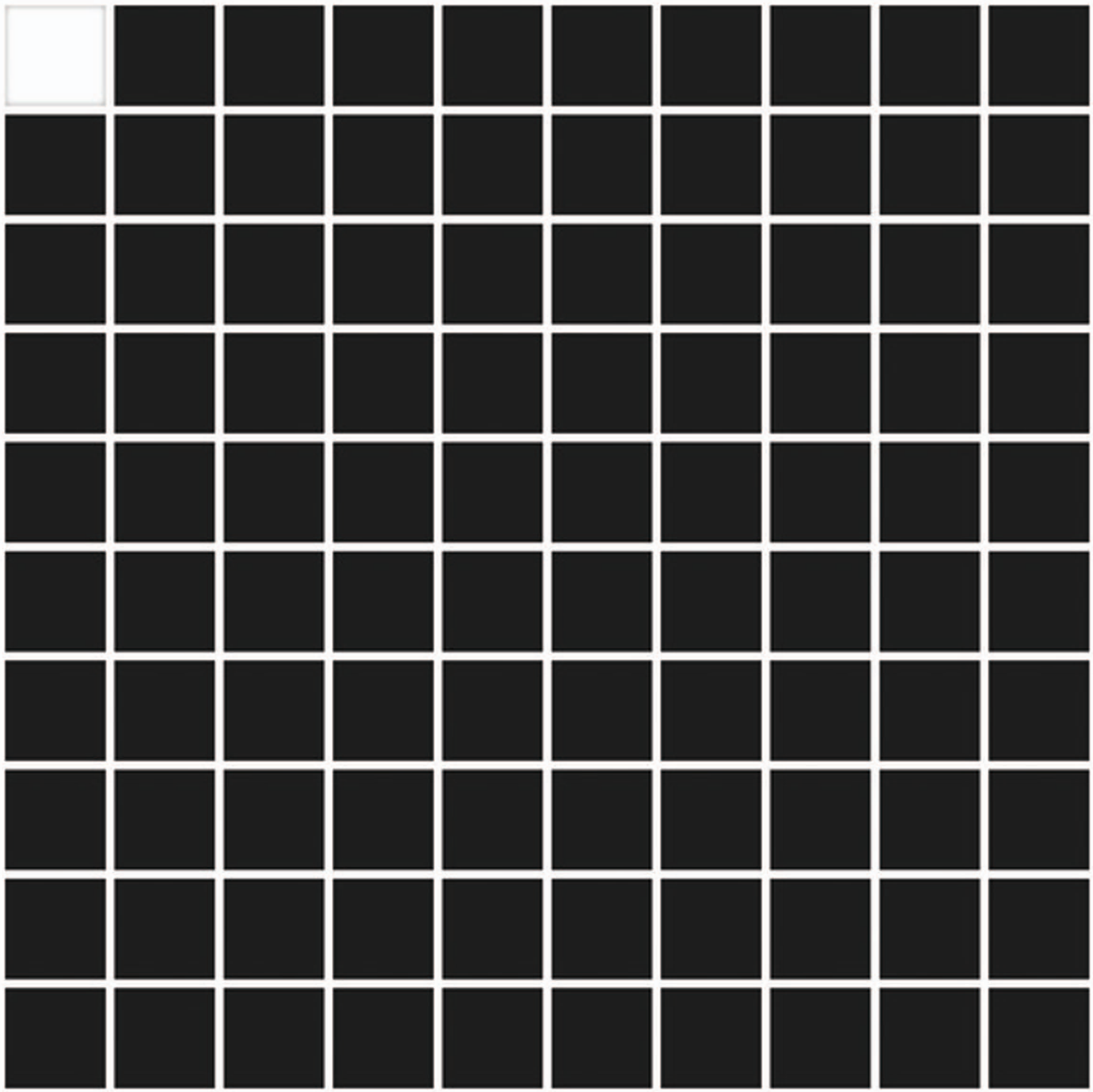

99 Variations on a Proof

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For Alexandra, who never said it couldn't be done.

Contents

Preface

On April 19, 1610, upon receiving an advance copy of Galileo's *Starry Messenger*, Johannes Kepler composed a fan letter. "I may perhaps seem rash in accepting your claims so readily with no support of my own experience," Kepler wrote to Galileo. "But why should I not believe a most learned mathematician, whose very style attests the soundness of his judgement?"[1] We are not accustomed today to thinking of a mathematician's work in terms of style. A proof is a form of argument, but the truth of the theorem it proves would hardly seem to depend on any rhetorical, let alone stylistic, features of its proof. The received wisdom is that mathematics, the universal language of science, has one style—*the mathematical style*—characterized by symbolic notation, abstraction, and logical rigor.[2]

This book aims to challenge that conception of mathematics. While a belief in the universality and unity of *ars mathematica* is not without reason, a moment's reflection gives rise to some basic questions. Where did "the" mathematical style come from? How has it developed with the growth of mathematical knowledge? What opportunities does it open or foreclose? How has its potential evolved with changes in the forms of writing and, therefore, ways of reading, mathematics? What are its expressive, cognitive, and imaginative possibilities?

These questions, at heart, concern the *literature* of mathematics. To survey this literature—a vast body of material ranging in subject from algebra to geometry, number theory to physics, logic to statistics, and dating from Babylonian tablets of the Bronze Age to the peer-reviewed journals and electronic preprints of today—is clearly beyond the scope of a book this size. Instead, I will describe a cross section of mathematics using a method inspired by Raymond Queneau's *Exercises in Style*. This literary work from 1947 takes the same simple story—that of a peculiar individual who is first seen in a dispute on a bus, and then later in conversation with a friend about the position of a coat button—and transforms it in ninety-nine different ways. Queneau's stylistic exercises exemplify various forms of prose, poetry, and speech, as well as more striking contortions, such as "Onomatopoeia," "Dog Latin," and "Permutations by Groups of 2, 3, 4, and 5 Letters." Queneau, in addition to being an author and poet, was also an amateur mathematician, and together with the mathematical historian François Le Lionnais he cofounded the experimental writing group known as the Oulipo. The name of the mostly French writing group is an acronym for *Ouvroir de Litérature Potentielle* (Workshop for Potential Literature), and its membership includes writers, artists, and mathematicians such as Georges Perec, Italo Calvino, Marcel Duchamp, Jacques Roubaud, Claude Berge, and Michèle Audin. The stated purpose of the group is to explore the possibilities for literature derived from mathematically inspired rules or constraints.[3] As soon as I learned about the Oulipo and Queneau's book, I wanted to see what effect constrained writing strategies would have on a mathematical narrative—a proof.

The theme I chose for *99 Variations on a Proof* is an algebraic equation known as a cubic equation, and every chapter proves the same minor—some might say trivial—theorem about its solutions. Many proofs, from 16 Ancient to 61 Modern, emerge from the mathematical literature on cubics. In some cases this happened quite directly, the extreme example being 7 Found, which I discovered, ready-made, on a page of the most famous Renaissance treatise on algebra. More often than not, however, variations required considerable interpretation and invention. Sometimes this was because the style originated in a subject area peripheral to cubics, as in 6 Axiomatic or the physics-based proof 96 Electrostatic. Still more distant translations were needed to convey styles outside of mathematics entirely, such as the musical score 26 Auditory and the architectural 62 Axonometric.

Some proofs aim to satisfy a particular standard of rigor, some fall short of today's standard, and some have other aims entirely.

Each variation, with relatively few exceptions, appears on a single page with a brief discussion on the reverse side of the page. The secondary text includes explanatory details, source information, and my comments on the nature and significance of each style. Cross references to related variations invite readers to deviate from the idiosyncratic order that I've given to the chapters and find their own paths through the book.

This is not a mathematical treatise on cubic equations, and my choice of the particular cubic here was made almost arbitrarily. Despite the historical threads implied by the chapter titles, this is not a book of mathematical history; while the ontological status of content and style is a matter of some debate, this is also not a work of philosophy. It is a book *about* mathematics, its attitudes, norms, perspectives, and practices—in short, its culture.[4]

Other comparative studies of mathematical proof have addressed the relationship between content and form in different ways. In 1938, one H. Pétard published "A Contribution to the Mathematical Theory of Big Game Hunting," which offers thirty-eight applications of modern mathematics and physics to the problem of catching a lion.[5] During the writing of *99 Variations on a Proof*, two other mathematical renderings of Queneau's *Exercises* appeared: *Rationnel mon Q* by Ludmila Duchêne and Agnès Leblanc and *Exercises in (Mathematical) Style* by John McCleary. While there is necessarily some overlap between these books, it is surprising that studies of style could themselves vary in style so much. This in and of itself further confirms the potential of the basic premise of Queneau's original.

What distinguished the style of that most learned mathematician, Galileo? "For him, good thinking means quickness, agility in reasoning, economy in argument, but also the use of imaginative examples,"[6] according to Italo Calvino. This Oulipian finds the clearest statement of Galilean style in the following passage of *The Assayer* from 1623: while criticizing an adversary's reliance on authority to carry an argument, Galileo asserts, "but discoursing is like coursing, not like carrying, and one Barbary courser can go faster than a hundred Frieslands."[7] Calvino calls this Galileo's "declaration of faith—style as a method of thought and as literary taste."[8] This is a faith that I have tried to keep.

My motivation for this project, from beginning to end, has been to try to conceptualize mathematics as a literary or aesthetic medium. There is no shortage of evidence that professional mathematicians describe their work in aesthetic terms, but the terms they use, at least publicly, are very limited. The oft-repeated "beauty" and "elegance" may be important components of mathematical taste, but they fail to convey its range or subtlety or how it relates to literary and aesthetic experiences beyond mathematics.[9] The ninety-nine (or, if you admit an omitted proof the same status as the others, one hundred) proofs serve to highlight the material differences in logic, diction, imagery, and even typesetting that give tone and flavor to mathematics.[10] I hope that readers with little or no predisposition to the subject matter will begin to perceive these stylistic differences by merely paging through examples, stopping to look more closely at proofs that reflect—or offend—their sensibility and moving lightheartedly forward from any that do not. The reader inclined to delve deeper might recognize that the book itself is a mathematical game. In any case, if mathematics is made more vivid as a result of it passing through the reader's hands, the book will have served its intended purpose.

Theorem. *If* $x^3 - 6x^2 + 11x - 6 = 2x - 2$, *then* $x = 1$ *or* $x = 4$.

Proof. Omitted. □

0

Omitted

Omitted

Why bother with proof at all? Proofs are explicitly omitted by authors for a variety of reasons, one being aesthetic. In a standard textbook of undergraduate level abstract algebra, one finds: "The proof of this proposition is notationally unpleasant without having any interesting features, so we omit it."[11] Readers often forgive such omissions in expository work; still, one has to be careful taking mathematics on faith.

It is worth noting a few peculiarities of the proposition before embarking on its proof, notationally pleasant or not.

We are given an algebraic equation in terms of some numbers, an unknown variable x, its square x^2, and its cube x^3. This makes it a *degree-three polynomial* equation or, simply, a *cubic* equation. A more standard form for the equation would organize all the terms to one side of the equals sign as $x^3-6x^2+9x-4=0$. That's the first peculiarity, and several proofs will take this normalizing step as a point of departure.

Actually, that's the second peculiarity—the *first* peculiarity of this theorem is that it doesn't state what *x is*. Some mathematical readers will tolerate a skipped proof, but the introduction of a variable without specifying its domain is a universally condemned sin of omission. Why? Because it invites ambiguity. For the purposes of our exploration of style, however, it will function as a highly productive ambiguity, to borrow a term from the American philosopher and poet Emily Grosholz.[12]

A final, more mathematically interesting, feature is that the cubic equation has just two solutions. If you haven't forgotten the quadratic formula and its $\pm$ sign, you may recall that degree-two equations have two roots. Though I have yet to meet anyone who has committed it to memory, there is a *cubic formula* (see 30 Formulaic), and it extracts *three* roots from any degree-three equation. Lacking a distinct third root, our equation is what's known technically as a *degenerate case*.

1

One-Line

Theorem. *Let x be real. If* $x^3 - 6x^2 + 11x - 6 = 2x - 2$*, then* $x = 1$ *or* $x = 4$.

Proof. By subtraction, $x^3 - 6x^2 + 9x - 4 = 0$, which factors as $(x-1)^2(x-4) = 0$. □

Like poets, mathematicians often strive for economy, and the one-line proof is a sort of monostich. Even the abbreviation QED for *quod erat demonstrandum* ("which was to be shown"), which traditionally marks the end of a proof, is deemed too prolix by modern standards. Instead we find the tombstone □, sometimes called the halmos, after the Hungarian American mathematician Paul Halmos who first incorporated it into mathematical writing. Economy is an ideal that extends from proofs to larger scale works. Once, in *The Bulletin of the American Mathematical Society*, there appeared a research article written by two number theorists that, in its entirety, consisted of only two sentences.[13] I guess the authors couldn't agree on one.

In its cryptic way, the one line here does at least give the reader something to do. Go ahead, combine like terms with like—and look, divide out these common factors.

2

Two-Column

Hypothesis: $x^3-6x^2+11x-6=2x-2$, where x is a real number.
To prove: $x=1$ or 4.

	STATEMENT	REASON
1.	$x^3-6x^2+11x-6=2x-2$	Given.
2.	$x^3-6x^2+11x-6+2=2x-2+2$	Addition property of equations.
3.	$x^3-6x^2+11x-4=2x$	Addition.
4.	$x^3-6x^2+11x-4-2x=2x-2x$	Subtraction property of equations.
5.	$x^3-6x^2+9x-4=0$	Subtraction.
6.	$x^3-(1+5)x^2+(5+4)x-4=0$	Addition.
7.	$x^3-x^2-5x^2+5x+4x-4=0$	Distributive property.
8.	$x^2(x-1)-5x(x-1)+4(x-1)=0$	Factoring.
9.	$(x^2-5x+4)(x-1)=0$	Factoring.
10.	$[x^2-(1+4)x+4](x-1)=0$	Addition.
11.	$(x^2-x-4x+4)(x-1)=0$	Distributive property.
12.	$[x(x-1)-4(x-1)](x-1)=0$	Factoring.
13.	$[(x-4)(x-1)](x-1)=0$	Factoring.
14.	$x-1=0$ or $x-4=0$	Zero product property.
15.	$x-1+1=1$ or $x-4+4=4$	Addition property of equations.
16.	$x=1$ or $x=4$	Addition.
		QED

The form is familiar to American high school geometry students, who may not be wrong in guessing that it was devised for ease of grading rather than as a model of understanding. The addition of fifteen lines over the last proof does not occasion a corresponding addition in insight, no matter how much reassurance we get from that vertical line separating the action of the proof on the left from its justification on the right.

Moreover, the gains in logical transparency come at a rhetorical cost. The two-column method absolves the student from having to bother with style, not to mention grammar. According to Patricio Herbst, Professor of Educational Studies and Mathematics at the University of Michigan, this may have been one of the intentions of the two-column proof. In "Establishing a Custom of Proving in American School Geometry: Evolution of the Two-Column Proof in the Early Twentieth Century," he writes,

> as students had thus far been used to memorizing the demonstrations of a geometry text, the mental discipline that geometry made possible was being lost. Instructional changes were needed in order to enable geometry to do its job. . . .[The two-column format] gave students an "objective" representation that enabled smoother recognition of the similarities between such different activities as proving fundamental propositions and solving proof exercises.[14]

See 18 Indented for an elaboration on the two-column proof.

3

Illustrated

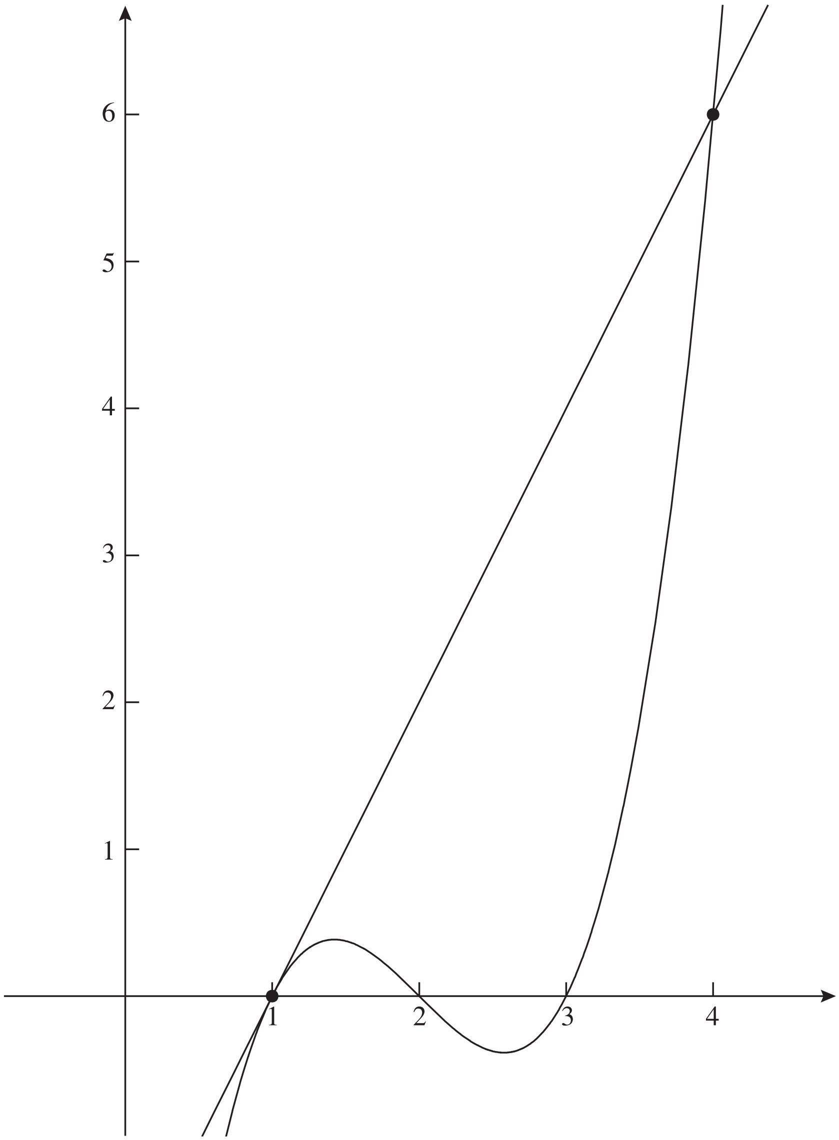

FIGURE. The two points of intersection of the cubic $y = x^3 - 6x^2 + 11x - 6$ and the line $y = 2x - 2$ occur at $(1,0)$ and $(4,6)$.

When I showed a draft of a few dozen proofs to a physicist colleague, he declared this one *the* proof. In a sense he was right. I originally conceived of the proposition as a claim about the intersection of two graphs.

But this example of “proof by inspection” wouldn’t qualify as proof according to most mathematicians’ standards. How can we be sure that the curves really do meet under those two beauty spots? What’s going on outside the region plotted? I’m reminded of a story told by the French geometer Étienne Ghys. After giving a talk in the Bourbaki seminar (see the comments in 6 Axiomatic) on a geometric construction that included illustrations, he was approached by the eminent French mathematician and Fields Medalist Jean-Pierre Serre, who remarked, “That was interesting what you said. I have a question. Would you consider this to be a theorem?”[15] Nevertheless, computer plots like this as well as 12 Ruler and Compass diagrams and 21 Blackboard sketches are effective tools for discovering and communicating mathematics.

4

Elementary

Proposition. *If x is a real number and $x^3-6x^2+11x-6=2x-2$, then $x=1$ or $x=4$.*

Proof. The degree three equation admits the standard form $x^3-6x^2+9x-4=0$. Expand the linear term $9x$ as the sum $5x+4x$ so that a common factor is apparent on the left hand side.

$$(x^3-6x^2+5x)+(4x-4)=(x^2-5x)(x-1)+4(x-1)$$

Factoring out $x-1$ leaves a quadratic, which also factors easily.

$$(x^2-5x+4)(x-1)=(x-4)(x-1)(x-1)$$

Since the right side of the equation is zero, one of the factors $(x-1)$ or $(x-4)$ must be zero. Thus $x=1$ or $x=4$.

Elementary

Economy of means can be as valuable as economy of form. I understand a proof to be *elementary* if its argument only relies on techniques that are basic in the field to which the proposition belongs, judging from its terminology. In this sense, an elementary proof is a naturally occurring constraint for mathematical *reasoning*. Other forms of reasoning or methods include 13 Reductio ad Absurdum and 22 Substitution.

Authors of textbooks and mathematics curricula often try to organize their material so that the student only sees problems that are elementary in this sense. This is likely responsible for the widely held assumption that mathematics itself develops according to the same logical pattern. An example to the contrary is the prime number theorem, which describes how the average gap between primes grows as primes increase. This result belongs to the field of number theory, but it took half a century for mathematicians to find a number theoretic proof for the theorem, which was first proved by methods using complex analysis.

Is there an elementary style of writing mathematics? If so, I think it's best captured by the eminent Hungarian mathematician and educator Georg Pólya:

> Rules of style: The first rule of style is to have something to say. The second rule of style is to control yourself when, by chance, you have two things to say; say first one, then the other, not both at the same time.[16]

5

Puzzle

Suppose that among four consecutive numbers, the product of the first three equals twice the third. What's the fourth number?

Answer: 4

Maybe the first question is, "Why bother with an equation?" There are worse things to worry about. In an article entitled "What is Mathematics For?", American mathematician Underwood Dudley concludes: "In mathematics problems can be solved, using reason, and the solutions can be checked and shown to be correct.... That is what mathematics education is for and what it has always been for: to teach reasoning, usually through the medium of silly problems."[17]

The 68 Word Problem or story problem is probably the most recognizable genre of silly problems in mathematics education.

To see how the answer to this brain-teaser solves our equation (or doesn't), call the fourth number in question x. Then the three consecutive numbers prior to x (inferring that they are integers) are, in numerical order, $x-3$, $x-2$, $x-1$. If their product equals twice the third, then $(x-3)(x-2)(x-1)=2(x-1)$. Multiply these numbers and the result is a big mess, but a mess that can be tidied into the form of the equation in 0 Omitted.

6

Axiomatic

Notations

Zero and *one* are numbers, and they are denoted by 0 and 1 respectively. The *sum* of numbers x and y is the result of adding x and y, and it is denoted by $x+y$; their *product* is the result of multiplying x and y, and it is denoted by $x\times y$ or xy. Numbers x and y are *equal* if they are identical, a fact which is denoted by the equation $x=y$.

Definitions

1. The numbers 2 through 11 are defined by the sums $2=1+1$, $3=2+1$, $\ldots$, $11=10+1$.
2. The *additive inverse* of a number x is the number $-x$ such that $x+(-x)=0$.
3. The *difference* of two numbers x and y is denoted $x-y$, and it is defined as the sum $x+(-y)$.
4. The *square* of a number x is the the product of x with itself, and it is denoted by x^2.
5. The *cube* of a number x is the product of x with its square, and it is denoted by x^3.

Axioms

6. Given any proposition P, if P or P, then P.
7. For all numbers x and y, if $x=y$, then $y=x$.
8. For all numbers x, y, z, if x is equal to y and y is equal to z, then x is equal to z.
9. For all numbers x and y, and any equation E, if x is equal to y, then y may be substituted for any occurrence of x in E without changing the truth value of E.
10. If x and y are numbers, then the sum $x+y$ and product $x\times y$ are numbers also.
11. For all numbers x, y, z, if x is equal to y, then the sums $x+z$ and $y+z$ are equal, as are the products $x\times z$ and $y\times z$.
12. For all numbers x, y, the transposed sums $x+y$ and $y+x$ are equal, as are the transposed products $x\times y$ and $y\times x$.
13. For all numbers x, y, z, the triple sums $(x+y)+z$ and $x+(y+z)$ are equal, as are the triple products $(x\times y)\times z$ and $x\times(y\times z)$.
14. If x, y, z are numbers, then the product of x with the sum $y+z$ is equal to the sum of the products $x\times y+x\times z$.
15. The number 1 is not equal to the number 0.
16. For any number x, the sum $0+x$ is equal to x.
17. For any number x, the product $1\times x$ is equal to x.
18. For any number x, there exists a unique additive inverse $-x$.
19. For any numbers x, y, if $x\times y=0$, then $x=0$ or $y=0$.

Theorems

20. For all numbers x, y, z, if $x = y$, then $x - z = y - z$.
21. For any number x, $x - x = 0$.
22. For any number x, $0 \times x = 0$.
23. For any numbers x, y, $(-x)y = -(xy) = x(-y)$
24. For any number x, $-(-x) = x$
25. For any numbers x, y, z, $x(y-z) = xy - xz = (y-z)x$
26. For any numbers x, y, z, w, $(x-y)(z-w) = xz - xw - yz + yw$
27. For any number x, $x + x = 2x$
28. For any numbers x, y, $-(x+y) = -x - y$
29. $-2 + (-4) = -6$
30. $1 + 4 \times 2 = 9$
31. For any number x, $(x-1)^2 = x^2 - 2x + 1$.
32. For any number x, $(x-1)^2(x-4) = x^3 - 6x^2 + 9x - 4$.
33. For any number x, $x^3 - 6x^2 + 9x - 4 = (x^3 - 6x^2 + 11x - 6) - (2x - 2)$.
34. For any number x, if $x^3 - 6x^2 + 11x - 6 = 2x - 2$, then $x = 1$ or $x = 4$.

    PROOF. Suppose x is a number.

Theorem 33	$x^3 - 6x^2 + 9x - 4 = (x^3 - 6x^2 + 11x - 6) - (2x - 2)$	(1)
Hypothesis	$x^3 - 6x^2 + 11x - 6 = 2x - 2$	(2)
Axiom 10	$2x - 2$ is a number	(3)
Axiom 9, (1), (2), (3)	$x^3 - 6x^2 + 9x - 4 = (2x - 2) - (2x - 2)$	(4)
Theorem 21, (3)	$(2x-2) - (2x-2) = 0$	(5)
Axiom 8, (4), (5)	$x^3 - 6x^2 + 9x - 4 = 0$	(6)
Theorem 32	$(x-1)^2(x-4) = x^3 - 6x^2 + 9x - 4$	(7)
Axiom 8, (7), (6)	$(x-1)^2(x-4) = 0$	(8)
Axiom 19, (8)	$(x-1)^2 = 0$ or $x - 4 = 0$	(9)
Definition 4, (9)	$(x-1)(x-1) = 0$ or $x - 4 = 0$	(10)
Axiom 19, (10)	$x - 1 = 0$ or $x - 1 = 0$ or $x - 4 = 0$	(11)
Axiom 6, (11)	$x - 1 = 0$ or $x - 4 = 0$	(12)
Definition 3, (12)	$x + (-1) = 0$ or $x + (-4) = 0$	(13)
Axiom 11, (13)	$x + (-1) + 1 = 0 + 1$ or $x + (-4) + 4 = 0 + 4$	(14)
Definition 2, (14)	$x + 0 = 0 + 1$ or $x + 0 = 0 + 4$	(15)
Axiom 16, (15)	$x = 1$ or $x = 4$	(Theorem)

The influential German mathematician David Hilbert offers one of the most succinct descriptions of the axiomatic approach: "If we consider a particular theory more closely, we always see that a few distinguished propositions of the field of knowledge underlie the construction of the framework of concepts, and these propositions then suffice by themselves for the construction, in accordance with logical principles, of the entire framework."[18] This proof is based on Italian mathematician Giuseppe Peano's *The principles of arithmetic, presented by a new method* from 1889.[19] The theorem is proved at the end of a sequence of theorems, each of which relies on one or more of the axioms, definitions, and theorems that precede it. Terms that are deemed too primitive to warrant precise definition are considered mere notations. It may seem absurd to consider a simple equation like $1+4\cdot 2=9$ to be a theorem, but it is a result that can be logically deduced from the axioms stated.[20]

The axiomatic method of organizing knowledge into a logical hierarchy dates back at least as far as Euclid (see 52 Antiquity), but axiomatic systems took on a new character and prominence in the modern era. The group of young mathematicians writing collectively under the pseudonym Nicolas Bourbaki sought to organize vast tracts of the discipline along the lines of the modern axiomatic style inspired by Hilbert and the celebrated algebraist Emmy Noether.[21] Bourbaki's 1948 manifesto, "The Architecture of Mathematics," explains their perspective (emphasis added):

> From the axiomatic point of view, mathematics appears thus as a storehouse of abstract forms—the mathematical structures.... Of course, it can not be denied that most of these forms had originally a very definite intuitive content; but, it is exactly by *deliberately throwing out this content*, that it has been possible to give these forms all the power which they were capable of displaying and to prepare them for new interpretations.[22]

Setting aside the brazenness of these lines, it's not difficult to imagine that some might have a hard time going along with such a formalist program. Without questioning the benefits afforded by the Bourbaki approach, mathematician and philosopher Gian-Carlo Rota observed at the end of the century:

> The axiomatic method of presentation of mathematics has reached a pinnacle of fanaticism in our time.... Clarity has been sacrificed to such shibboleths as consistency of notation, brevity of argument and the contrived linearity of inferential reasoning. Some mathematicians will go as far as to pretend that mathematics *is* the axiomatic method, neither more nor less. This pretense of "identifying" mathematics with a style of exposition is having a corrosive effect on the way mathematics is viewed by scientists in other disciplines.[23]

Other mathematicians will find that mathematics itself pays a price for our over reliance on formalism; see the comments following 33 Calculus.