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Singularities and Computer Algebra

Edited by
Christoph Lossen and Gerhard Pfister

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Singularities and Computer Algebra

Edited by

CHRISTOPH LOSSEN & GERHARD PFISTER

University of Kaiserslautern

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Dedicated to
Gert-Martin Greuel
on the Occasion of
His 60^{th} *Birthday*

Prof. Dr. Gert-Martin Greuel

Contents

Preface

From 18 to 20 October 2004, a conference *"Singularities and Computer Algebra"* was held at the University of Kaiserslautern on the occasion of Gert-Martin Greuel's 60th birthday. It was attended by 70 participants from Europe, Israel, Japan, Canada and the U.S.A. We were particularly happy that Greuel's teacher, Egbert Brieskorn, was among them.

Most of the participants have been influenced by Greuel's work on singularities and their computational aspects over the last 30 years. Among them, one could find colleagues and friends from the early years in Göttingen and Bonn, but also former and present diploma and Ph.D. students of Gert-Martin Greuel at Kaiserslautern. In particular, each of the invited speakers could look retrospectively at cooperating in one way or another with Greuel.

The papers of this volume concern ten of the invited lectures, supplemented by four articles which are written by participants of the conference and focus on computational aspects. Most of the contributions are intended to give an overview on a particular aspect of singularities. They describe the development of important areas of singularity theory over the past years and they discuss open questions.

In the lead text, we include a list of the invited lectures and a list of the participants as well as a picture of the septic with 99 nodes found by Oliver Labs and Duco van Straten, which has acted as a logo for the conference. Further, we include an article focussing on *Aspects of Gert-Martin Greuel's Mathematical Work*.

We would like to thank all the people who have contributed to the sucess of the conference and to this volume.

Christoph Lossen and Gerhard Pfister
(Organizers of the Conference)

Invited Lectures

Helmut A. HAMM: *Depth and De Rham Cohomology*

LÊ Dung Trang: *Singularity Invariants in Versal Deformations*

Eugenii SHUSTIN: *The Patchworking Construction and Applications to Tropical Enumerative Geometry*

Ragnar-Olaf BUCHWEITZ: *Free Divisors in the Representation Theory of Algebras*

Wolfgang EBELING: *Monodromy*

Yuri A. DROZD: *Derived Categories of Modules and Coherent Sheaves*

Antonio CAMPILLO: *Some Aspects and Applications of Singularities in Positive Characteristic*

Jonathan WAHL: *Topology, Geometry, and Equations of Normal Surface Singularities*

Ignacio LUENGO: *Superisolated Singularities*

Kyoji SAITO: *A Linearization Theorem of the Real Discriminants for Simple Singularities*

Charles T.C. WALL: *Transversality in families of mappings*

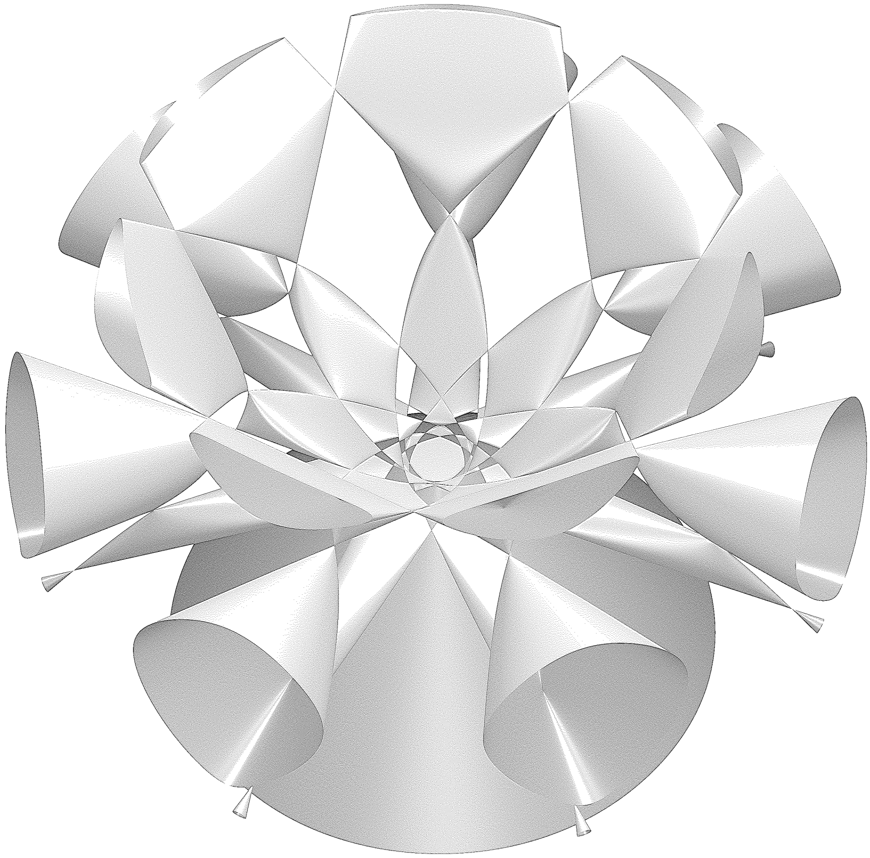
Joseph H.M. STEENBRINK: *Adjunction Conditions for 1-Forms on Surfaces in Projective Three-Space*

Duco VAN STRATEN: *Lagrangian Singularities*

Bernard TEISSIER: *On the Structure of the Newton Polyhedra of Certain Discriminants*

Wolfram DECKER: *SINGULAR and PLURAL*

Frank-Olaf SCHREYER: *An Experimental Approach to Numerical Godeaux Surfaces*



A septic surface S in $\mathbb{P}^3(\mathbb{C})$ with 99 real nodes. It was discovered in 2004 by O. Labs and D. van Straten using SINGULAR *experiments over small finite fields of prime order. If $\alpha \in \mathbb{C}$ satisfies $7\alpha^3 + 7\alpha + 1 = 0$, a defining equation for S over $\mathbb{Q}(\alpha)$ is the following:*

$$(z + a_5 w)\Big((z+w)(x^2+y^2) + a_1 z^3 + a_2 z^2 w + a_3 z w^2 + a_4 w^3\Big)^2$$
$$-x^7 + 21x^5y^2 - 35x^3y^4 + 7xy^6 - 7z\left(x^2+y^2\right)^3 + 56z^3\left(x^2+y^2\right)^2$$
$$-112z^5\left(x^2+y^2\right) + 64z^7\,,$$

where

$$a_1 := -\tfrac{12}{7}\alpha^2 - \tfrac{384}{49}\alpha - \tfrac{8}{7}, \qquad a_2 := -\tfrac{32}{7}\alpha^2 + \tfrac{24}{49}\alpha - 4,$$
$$a_3 := -4\alpha^2 + \tfrac{24}{49}\alpha - 4, \qquad a_4 := -\tfrac{8}{7}\alpha^2 + \tfrac{8}{49}\alpha - \tfrac{8}{7},$$
$$a_5 := 49\alpha^2 - 7\alpha + 50.$$

List of Participants

Klaus Altmann, Freie Universität Berlin (Germany)
Gottfried Barthel, Universität Konstanz (Germany)
Corina Baciu, TU Kaiserslautern (Germany)
Rocio Blanco, Univ. de Valladolid (Spain)
Lesya Bodnarchuk, TU Kaiserslautern (Germany)
Egbert Brieskorn, Bonn (Germany)
Steve Brüske, Universität Münster (Germany)
Ragnar-Olaf Buchweitz, University of Toronto (Canada)
Igor Burban, MPI Bonn (Germany)
Antonio Campillo, Univ. de Valladolid (Spain)
Jan A. Christophersen, Universitetet i Oslo (Norway)
Wolfram Decker, Universität des Saarlandes (Germany)
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Gert-Martin Greuel, TU Kaiserslautern (Germany)
Helmut A. Hamm, Universität Münster (Germany)
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Dmitry Kerner, Tel Aviv University (Israel)
Bernd Kreussler, University of Limerick (Ireland)
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Markus Perling, Universität Paderborn (Germany)
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Wilfred Pohl, TU Kaiserslautern (Germany)
Dorin Popescu, IMAR Bucharest (Romania)
Oswald Riemenschneider, Universität Hamburg (Germany)
Marko Roczen, Humboldt-Universität zu Berlin (Germany)
Diego Ruano, Univ. de Valladolid (Spain)
Kyoji Saito, Kyoto University (Japan)
Hans Schönemann, TU Kaiserslautern (Germany)
Frank-Olaf Schreyer, Universität des Saarlandes (Germany)
Mathias Schulze, Université d'Angers (France)
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Bernard Teissier, Université Paris VII (France)
Günther Trautmann, TU Kaiserslautern (Germany)
Ilya Tyomkin, Weizmann Institute, Rehovot (Israel)
Jonathan Wahl, University of N.C., Chapel Hill (U.S.A.)
Charles T.C. Wall, Liverpool (U.K.)
Anja Wenning, Universität Münster (Germany)
Eric Westenberger, TU Kaiserslautern (Germany)
Oliver Wienand, TU Kaiserslautern (Germany)

Aspects of Gert-Martin Greuel's Mathematical Work

Christoph Lossen Gerhard Pfister

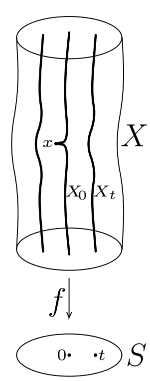
This article emanates from the opening speech of the conference *"Singularities and Computer Algebra"* which took place on October 18–20, 2004 on the occasion of Gert-Martin Greuel's 60th birthday. When preparing the speech, we realized soon that it is impossible to cover in such a speech Gert-Martin's complete work which is documented in more than eighty publications. Not to mention Gert-Martin's organisational work for the mathematical community.

We decided to illuminate only some cornerstones of Gert-Martin's mathematical work: his Ph.D.-Thesis in 1973, his Habilitationsschrift in 1979, the SINGULAR project, the work on moduli spaces, and the work on equisingular families.

Ph.D.-Thesis (1973).

In his Diploma Thesis, titled *"Zur Picard-Lefschetz-Monodromie isolierter Singularitäten von vollständigen Durchschnitten"*, and his Ph.D.-Thesis, titled *"Der Gauß-Manin Zusammenhang isolierter Singularitäten von vollständigen Durchschnitten"*, G.-M. Greuel develops the theory of the Gauß-Manin connection for isolated complete intersection singularities: Let $f : (X, x) \to (S, 0)$ be a map of germs with the following properties:

- X is an m-dimensional complete intersection,
- S is a k-dimensional complex manifold,
- f is flat,
- $x \in X_0 = f^{-1}(0)$ is an isolated singular point,
- the critical set C of f is of dimension $k-1$.



Then (X_0, x) is an isolated complete intersection singularity of dimension $n := m - k$. We introduce $S' := S \setminus D_f$ and $X' := X \setminus f^{-1}(D_f)$, where $D_f = f(C) \subset S$ denotes the discriminant of f.

By a result of Hamm (extending a result of Milnor), we may assume that the restriction $f : X' \to S'$ is a locally trivial differentiable fibre bundle whose fibres are homotopy equivalent to a bouquet of n-spheres. The number of spheres in the bouquet, which equals $\dim_{\mathbb{C}} H^n(X_t, \mathbb{C})$, is called the *Milnor number* of the complete intersection germ (X_0, x) and denoted by $\mu(X_0, x)$.

The fibration $f : X' \to S'$ induces a vector bundle with fibre $H^n(X_t, \mathbb{C})$ over $t \in S'$. Its sheaf of holomorphic sections, $\mathcal{H}^n = R^n f_* \mathbb{C}_{X'} \otimes_{\mathbb{C}_{S'}} \mathcal{O}_{S'}$, has a canonical integrable connection $\nabla : \mathcal{H}^n \to \mathcal{H}^n \otimes_{\mathcal{O}_{S'}} \Omega^1_{S'}$, $\omega \otimes f \mapsto \omega \otimes df$. The monodromy of this connection is the Picard-Lefschetz monodromy of f in x, $\rho_{\mathbb{C}} : \pi_1(S', t) \to \mathrm{Aut}(H^n(X_t, \mathbb{C}))$, induced by the action of $\pi_1(S', t)$ on $H^n(X_t, \mathbb{Z})$. Greuel's main result is now the following theorem (extending Brieskorn's result for hypersurfaces):

Theorem. *The connection ∇ on S' can be extended to a (meromorphic) regular singular connection on S, the Gauß-Manin connection*

$$\nabla_{X/S} : \mathcal{H}^n_{DR}(X/S) \longrightarrow \mathcal{H}^n_{DR}(X/S) \otimes \Omega^1_S(D_f)$$

of coherent $\mathcal{O}_S$-modules, where $\mathcal{H}^n_{DR}(X/S)$ is the hypercohomology $\mathbb{R}^n f_* \Omega^\bullet_{X/S}$ of the complex of relative holomorphic differential forms.

This result is already contained in his diploma thesis. An important ingredient of the proof is a proof of the generalized de Rham lemma saying that, for each holomorphic map $h = (h_1, \ldots, h_t) : X \to \mathbb{C}^k$, the morphism

$$\Omega^p_{X/S} \Big/ \sum_{i=1}^{t} dh_i \wedge \Omega^{p-1}_{X/S} \longrightarrow \Omega^{p+t}_{X/S}, \quad [\omega] \longmapsto [dh_1 \wedge \ldots \wedge dh_t \wedge \omega]$$

is injective for $0 \leq p < \mathrm{codim}_X \mathrm{Sing}(f, h)$, where $(f, h) : X \to S \times \mathbb{C}^k$. Here, X does not need to have an isolated singularity. The de Rham lemma was later formulated by K. Saito in a more algebraic context, but the first proof is due to Greuel [1]. Indeed, Greuel proves a much more general statement, and the proof provides results which were recently used by Gusein-Zade and Ebeling to compute indices of vector fields.

In his Ph.D. thesis [2], published in [4], Greuel proves that $\mathcal{H}^n_{DR}(X/S)$ is locally free for $\dim S \leq 2$ and that a different extension $\mathcal{H}'''_{DR}$ (corresponding to Brieskorns $\mathcal{H}''$ is locally free for arbitrary S (of rank $\mu(X_0, x)$). Using these results and applying the index theorem of Malgrange to the Gauß-Manin connection, Greuel gets a purely algebraic formula for the Milnor number of an isolated complete intersection singularity $(X_0, x) \subset (X, x)$ as above:

Theorem. *The Milnor number $\mu(X_0, x)$ has the following properties:*

(1) $\mu(X_0, x) = \dim_{\mathbb{C}} \Omega^n_{X_0,x} / d\Omega^{n-1}_{X_0,x}$ *if* $n > 0$, *and* $\mu(X_0, x) = \dim_{\mathbb{C}} \mathcal{O}_{X_0,x} - 1$ *if* $n = 0$.

In particular, the Milnor number depends only on X_0 (and not on f).

(2) If $\dim S = 1$, then $\mu(X_0,x)+\mu(X,x) = \dim_{\mathbb{C}} \Omega^m_{X/S,x} = \dim_{\mathbb{C}} \mathcal{O}_{X,x}/\mathscr{C}$, where $\mathscr{C}$ denotes the ideal of $\mathcal{O}_{X,x}$ generated by the entries of the Jacobian matrix $\partial(g_1,\dots,g_r,f)/\partial \boldsymbol{x}$. Here $g_1,\dots,g_r$ are supposed to generate the ideal of X in $\mathbb{C}\{\boldsymbol{x}\} = \mathbb{C}\{x_1,\dots,x_m\}$.

In particular, we can compute $\mu(X_0,x)$ by recursion:

(3) If $X_i := V(f_1,\dots,f_{k-i}) \subset \mathbb{C}^m$ and $f_{k-i+1} : X_i \to S_i = \mathbb{C}$, then

$$\mu(X_0,0) = \sum_{i=1}^{k} (-1)^{k-i} \dim_{\mathbb{C}} \Omega^{n+i}_{X_i/S_i,0} = \sum_{i=1}^{k} (-1)^{k-i} \dim_{\mathbb{C}} \mathbb{C}\{\boldsymbol{x}\}\big/\mathscr{C}_i\,,$$

where $\mathscr{C}_i$ denotes the ideal of $\mathbb{C}\{\boldsymbol{x}\}$ generated by $f_1,\dots,f_{i-1}$ and the i-minors of the Jacobian matrix $\partial(f_1,\dots,f_i)/\partial\boldsymbol{x}$.

(4) If $X_0 = V(f_1,\dots,f_k) \subset \mathbb{C}^m$ is quasihomogeneous, then

$$\mu(X_0,0) = \dim_{\mathbb{C}} \mathbb{C}\{\boldsymbol{x}\}\big/\big(\mathscr{C}_k + \langle f_k\rangle\big)\,.$$

Related Publications 1971–1977

1. Zur Picard-Lefschetz-Monodromie isolierter Singularitäten von vollständigen Durchschnitten. *Diplomarbeit, Göttingen* (1971).
2. Der Gauß-Manin-Zusammenhang isolierter Singularitäten von vollständigen Durchschnitten. *Ph.D. Thesis, Göttingen* (1973).
3. Singularities of complete intersections. *In: A. Hattori: Manifolds, Tokyo 1973. Univ. of Tokyo Press, 123–129* (with E. Brieskorn, 1975).
4. Der Gauß-Manin-Zusammenhang isolierter Singularitäten von vollständigen Durchschnitten. *Math. Ann.* **214***, 235–266* (1975).
5. Spitzen, Doppelpunkte und vertikale Tangenten in der Diskriminante verseller Deformationen von vollständigen Durchschnitten. *Math. Ann.* **222***, 71–88* (with Lê Dung Tráng, 1976).
6. Cohomologie des singularités non isolés. *C.R. Acad. Sci. Paris Sér. A Math.* **284***, 321–322* (1977).
7. Die Zahl der Spitzen und die Jacobi-Algebra einer isolierten Hyperflächensingularität. *Manuscr. Math.* **21** *227–241* (1977).

Habilitationsschrift (1979).

Greuel's Habilitationsschrift, which has the title *"Kohomologische Methoden in der Theorie isolierter Singularitäten"*, consists of three parts:

I. *The Milnor number and deformations of complex curve singularities.*

II. *Deformation spezieller Kurvensingularitäten und eine Formel von Deligne.*

III. *Dualität in der lokalen Kohomologie isolierter Singularitäten.*

Large Parts of the Habilitationsschrift were written during a one year stay for research in France at IHES (Bures-sur Yvette) and at the mathematical institute of the Université de Nice.

In the first part, which has been based on a joint work with R.O. Buchweitz (see [11]), again a Milnor number $\mu(C,0)$ is the main object of investigation. This time for $(C,0) \subset (\mathbb{C}^n,0)$ being an arbitrary reduced complex curve singularity. Buchweitz and Greuel define this new invariant as

$$\mu(C,0) := \dim_{\mathbb{C}} \omega_{C,0}/d\mathcal{O}_{C,0},$$

where $\omega_{C,0} = \mathrm{Ext}^{n-1}_{\mathcal{O}_{\mathbb{C}^n,0}}\big(\mathcal{O}_{C,0}, \Omega^n_{\mathbb{C}^n,0}\big)$ is the dualizing module of Grothendieck, extending in this way the notion of the Milnor number of an isolated complete intersection curve singularity, $\mu(C,0) = \dim_{\mathbb{C}} \Omega^1_{C,0}/d\mathcal{O}_{C,0}$.

According to Greuel, the main results of Part I can be summarized by saying that also the general notion of a Milnor number reflects the topological nature of curve singularities:

> *"Obwohl μ für Kurvensingularitäten in Kodimension ≥ 2 keine topologische Invariante ist, spiegelt sie doch im Wesentlichen den topologischen Charakter der Singularität wider. Das ist der gemeinsame Nenner der Hauptresultate des ersten Teils."*

More precisely, Buchweitz and Greuel obtain the following results:

Theorem (Generalized Milnor formula). *If $(C,0)$ is a reduced complex curve singularity with r branches, then*

$$\mu(C,0) = 2\delta(C,0) - r + 1\,,$$

where $\delta(C,0) = \dim_{\mathbb{C}} \mathcal{O}_{\overline{C},\overline{0}}/\mathcal{O}_{C,0}$ for $(\overline{C},\overline{0}) \to (C,0)$ the normalization.

Theorem. *Let $f : \mathscr{C} \to D \subset \mathbb{C}$ be a good representative of a flat family of reduced curve singularities. Then, for all $t \in D$,*

(1) *The fibre $\mathscr{C}_t$ is connected.*

(2) $\mu(\mathscr{C}_0,0) - \sum_{x\in\mathscr{C}_t} \mu(\mathscr{C}_t,x) = \dim_{\mathbb{C}} H^1(C_t,\mathbb{C})\,.$

(3) $\mu(\mathscr{C}_0,0) - \sum_{x\in\mathscr{C}_t} \mu(\mathscr{C}_t,x) \geq \delta(\mathscr{C}_0,0) - \sum_{x\in\mathscr{C}_t} \delta(\mathscr{C}_t,x)\,.$

(4) $\mu_t := \sum_{x\in\mathscr{C}_t} \mu(\mathscr{C}_t, x)$ *is constant in* t *iff all fibres* $\mathscr{C}_t$ *are contractible.*

Statement (2) shows that the Milnor number is again a measure for the vanishing cohomology.

Theorem (μ-constant is equivalent to topological triviality).
Let $f : \mathscr{C} \to D \subset \mathbb{C}$ *be a good representative of a flat family of reduced curve singularities with section* $\sigma : D \to \mathscr{C}$ *such that* $\mathscr{C}_t \setminus \{\sigma(t)\}$ *is smooth for all* $t \in D$. *Then the following are equivalent:*

(a) $\mu(\mathscr{C}_t, \sigma(t))$ *is constant for* $t \in D$.

(b) $\delta(\mathscr{C}_t, \sigma(t))$ *and* $r(\mathscr{C}_t, \sigma(t))$ *are constant for* $t \in D$.

(c) $f : \mathscr{C} \to D$ *is topologically trivial.*

Theorem (Generalized Zariski discriminant criterion).
Let $f : \mathscr{C} \to D \subset \mathbb{C}$ *be a sufficiently small representative of a flat deformation of a reduced complete intersection curve singularity* $(C, 0)$. *Then the following are equivalent:*

(a) There exists a finite mapping $\pi = (\pi_1, f) : (\mathscr{C}, 0) \to (\mathbb{C}\times D, 0)$ *such that the multiplicity of the discriminant (with Fitting structure) along* $\{0\} \times D$ *is constant for* $t \in D$ *and equal to*

$$\sum_{x\in\mathscr{C}_t} \big(\mu(\mathscr{C}_t, x) + \mathrm{mult}(\mathscr{C}_t, x) - 1\big)\,, \quad t \neq 0\,.$$

(b) $f : \mathscr{C} \to D$ *admits a holomorphic section* $\sigma : D \to \mathscr{C}$ *such that* $\mathscr{C}_t$ *is smooth outside* $\{\sigma(t)\}$, *and* $\mu(\mathscr{C}_t, \sigma(t))$ *and* $\mathrm{mult}(\mathscr{C}_t, \sigma(t))$ *are constant in* t.

Note that condition (b) is stronger than equisingularity, since for a complete intersection in codimension ≥ 2 constant Milnor number does not imply constant multiplicity.

Part II of the Habilitationsschrift deals with smoothable singularities: Let $(C, 0) \subset (\mathbb{C}^n, 0)$ be a reduced complex curve singularity, and let

$$\begin{array}{ccc} (C,0) & \overset{i}{\hookrightarrow} & (\mathscr{C},0) \\ \downarrow & & \downarrow{\scriptstyle \phi\ \text{flat}} \\ \{0\} & \hookrightarrow & (S,0) \end{array}$$

be the semiuniversal deformation. Then $(C, 0)$ is called *smoothable* if there exists a component E of $(S, 0)$ such that the fibre $\mathscr{C}_t$ over a general point

$t \in E$ is smooth. The component $(E, 0)$ is then referred to as a *smoothing component for* $(C, 0)$.

Deligne's formula (1973) plays an important role in the investigation of smoothable singularities. It states that for a smoothable reduced complex curve singularity $(C, 0)$, each smoothing component has the dimension

$$e(C,0) := 3\delta(C,0) - \underbrace{\dim_{\mathbb{C}} \overline{\Theta}/\Theta}_{=:\, m_1(C,0)} ,$$

where $\Theta = \mathrm{Hom}_{\mathcal{O}_{C,0}}\big(\Omega^1_{C,0}, \mathcal{O}_{C,0}\big)$, and $\overline{\Theta} = \mathrm{Hom}_{\overline{\mathcal{O}}}\big(n_*\Omega^1_{\overline{C},\overline{0}}, \overline{\mathcal{O}}\big)$ for $\overline{\mathcal{O}} = n_*\mathcal{O}_{\overline{C},\overline{0}}$. And, for each reduced curve singularity $(C, 0)$, the codimension $m_1(C, 0)$ can be computed as $m_1(C,0) = r(C,0) + \dim_{\mathbb{C}} \overline{G}/G$, where $\overline{G} = \mathrm{Aut}\big(\overline{\mathcal{O}}/I\big)$ for some ideal $I \subset \overline{\mathcal{O}}$ contained in the conductor, and $G \subset \overline{G}$ the stabilisator of $\mathcal{O}_{C,0}/I$.

The main goal in Part II of Greuel's Habilitationsschrift (published in [13]) is to extend Deligne's Formula to not necessarily smoothable singularities and to express it by means of invariants of $(C, 0)$ that are easier to compute:

Theorem. *(1) If $(C, 0)$ is a quasihomogeneous complex curve singularity,*

$$e(C,0) = \mu(C,0) + t(C,0) - 1 ,$$

where $t(C,0) = \dim_{\mathbb{C}}(\omega_{C,0}/\mathfrak{m}_{C,0}\omega_{C,0})$ is the Cohen-Macaulay type of $(C, 0)$.

(2) If $(C, 0)$ is Gorenstein and irreducible, then

$$e(C,0) \leq \mu(C,0) ,$$

and equality holds iff $(C, 0)$ is quasihomogeneous.

(3) For an arbitrary reduced complex curve singularity $(C, 0)$,

$$\begin{aligned} e(C,0) \;\; = \;\; & \mu(C,0) + t(C,0) - 1 \\ & + \dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{O}_{C,0}}\big(\Omega^1_{C,0}, \mathcal{O}_{C,0}\big) / \mathrm{Hom}_{\mathcal{O}_{C,0}}\big(\overline{\mathcal{O}}/\mathcal{O}_{C,0}\big) \\ & - \dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{O}_{C,0}}\big(\mathfrak{m}_{C,0}, \mathcal{O}_{C,0}\big) / \mathrm{Hom}_{\mathcal{O}_{C,0}}\big(\overline{\mathcal{O}}/\mathcal{O}_{C,0}\big) . \end{aligned}$$

The irreducibility assumption in (2) was later removed (see [17]). While Deligne's proof of the formula for the dimension of a smoothing component was global, a local proof was given as an application of the main result in [16].

Part III of the Habilitationsschrift (published in [12]) is devoted to the comparison of the Milnor and the Tjurina number of an isolated complete intersection singularity. The name "Tjurina number" for $\tau(X,0) := \dim_{\mathbb{C}} T^1_{X,0}$ (for an arbitrary singularity $(X,0)$) was coined by Greuel and introduced in that paper. If $(X,0)$ is unobstructed (e.g., a complete intersection), then $\tau(X,0)$ equals the dimension of the base space of the semiuniversal deformation. Part III contains the following result:

Theorem. *Let $(X,0)$ be an isolated complete intersection singularity.*

(1) If $(X,0)$ is quasihomogeneous, then $\mu(X,0) = \tau(X,0)$.

(2) If the neighbourhood boundary of $(X,0)$ is a rational homology sphere or if $\dim(X,0) = 1$, then $\mu(X,0) \geq \tau(X,0)$.

The last statement has been generalized by Looijenga and Steenbrink to arbitrary complete intersection singularities of dimension ≥ 2.

Related Publications 1978–1990

8. Invarianten quasihomogener vollständiger Durchschnitte. *Invent. Math.* **49**, *67–86* (with H.A. Hamm, 1978).
9. Le nombre de Milnor, équisingularité, et déformations de singularités des courbes réduites. *C.R. Acad. Sci. Paris Sér. A Math.* **288** *35–38 and Sém. sur les Singularités. Publ. Math. Univ. Paris 7, 13–30* (with R.-O. Buchweitz, 1979-80).
10. Kohomologische Methoden in der Theorie isolierter Singularitäten. *Habilitationsschrift, Bonn* (1979).
11. The Milnor number and deformations of complex curve singularities. *Invent. Math.* **58** *241-281* (with R.-O. Buchweitz, 1980).
12. Dualität in der lokalen Kohomologie isolierter Singularitäten. *Math. Ann.* **250** *157–173* (1980).
13. On deformation of curves and a formula of Deligne. *In: J.M. Aroca et al: Algebraic Geometry, La Rábida 1981. Springer LNM 961, 141-168* (1983).
14. On the topology of smoothable singularities. *In: P. Orlik: Singularities, Arcata 1981. Proc. Sympos. Pure Math.* **40**, *535–545* (with J.H.M. Steenbrink, 1983).
15. Einfache Kurvensingularitäten und torsionsfreie Moduln. *Math. Ann.* **270**, *417-425* (with H. Knörrer, 1985).
16. The dimension of smoothing components. *Duke Math. Journ.* **52**, *263–272* (with E. Looijenga, 1985).
17. Numerische Charakterisierung quasihomogener Gorenstein-Kurvensingularitäten. *Math. Nachr.* **124**, *123–131* (with B. Martin, G. Pfister, 1985).
18. Constant Milnor number implies constant multiplicity for quasihomogeneous singularities. *Manuscr. Math.* **56**, *159–166* (1986).
19. Torsion free modules and simple curve singularities. *Canad. Math. Soc. Conf. Proc.* **6**, *91–94* (1986).

20. Deformationen isolierter Kurvensingularitäten mit eingebetteten Komponenten. *Manuscr. Math.* **70**, *93–114* (with C. Brücker, 1990).
21. Simple Singularities in Positive Characteristic. *Math. Z.* **203**, *339–354* (with H. Kröning, 1990).

The SINGULAR Project.

The birth of the SINGULAR project can be dated back to about 1982, when G.-M. Greuel and the second author tried to generalize K. Saito's theorem which states that, *for a germ* $(X, 0)$ *of an isolated hypersurface singularity, the following conditions are equivalent:*

(a) $(X, 0)$ *is quasi-homogeneous (that is, has a good* $\mathbb{C}^*$*-action).*

(b) $\mu(X, 0) = \tau(X, 0)$.

(c) *The Poincaré complex of* $(X, 0)$ *is exact.*

Trying to extend this theorem to complete intersection curve singularities, they only succeeded in proving the equivalence of (a) and (b) (see [17]). They expected that (b) and (c) are, indeed, not equivalent for general complete intersection curve singularities. They succeeded in expressing the exactness of the Poincaré complex as an equality of dimensions of certain $\mathcal{O}_{X,0}$-modules. In those days, however, there was no computer algebra system available which could compute Milnor numbers, Tjurina numbers and the dimensions of the differential modules in the Poincaré complex. To be able to compute these numbers, such a system requires an implementation of T. Mora's tangent cone algorithm, a modification of Buchberger's Gröbner basis algorithm designed for computations over local rings.

Having implemented this algorithm, the expected counterexamples were found by H. Schönemann and the second author in C.T.C. Wall's list of unimodal complete intersection curve singularities: consider

$$\{xy + z^{\ell-1} = xz + yz^2 + y^{k-1} = 0\}$$

for $4 \leq \ell \leq k$, $5 \leq k$.

Motivated by this success, Greuel and the second author tried to attack Zariski's famous multiplicity conjecture by searching for a counterexample. Starting point was the following result of Greuel, confirming Zariski's conjecture for families of quasihomogeneous isolated singularities [18]:

Theorem. *Let* $f \in \mathbb{C}\{x_1, \ldots, x_n\}$ *be such that the ideal* $\langle f \rangle$ *can be generated by a semiquasihomogeneous polynomial, and let* $f_t \in \mathbb{C}\{x_1, \ldots, x_n, t\}$ *be a* μ*-constant deformation of* f*, then the multiplicity of* f_t *is constant (that is, independent of* t *for* $t \in \mathbb{C}$ *small).*

The method of proof suggested a way to look for possible counterexamples in the non-quasihomogeneous case. However, since Zariski's conjecture holds for curves and semiquasihomogeneous singularities, potential counterexamples have Milnor number > 1000. For these computations, the existing implementation of the tangent cone algorithm was not sufficient. Therefore, Greuel, the second author and H. Schönemann decided to set up such a computer algebra system with improved algorithms and extended functionality.

The result is nowadays known as SINGULAR which has grown to a major specialized computer algebra system used in mathematical research and teaching, and even in industrial applications (see the article on SINGULAR in this volume by H. Schönemann and the first author).

The computational complexity provided by the potential counterexamples to Zariski's conjecture was a big challenge and resulted in sophisticated strategies for the implementation of Buchberger's (resp. Mora's) algorithm. One can say that the hardness of the problem is one of the main reasons for SINGULAR to have one of the fastest implementations of a standard basis algorithm.

Although the search for a counterexample to Zariski's conjecture failed, it was not useless. The experiments with SINGULAR suggested a positive answer to Zariski's conjecture in another special case, proved in [24]. In general, Zariski's conjecture is still open.

Some Publications Related to SINGULAR 1996–2005

22. Standard bases, syzygies and their implementation in SINGULAR. In: *Beiträge zur angewandten Analysis und Informatik.* Shaker, Aachen, 69–96 (with H. Grassmann, B. Martin, W. Neumann, G. Pfister, W. Pohl, H. Schönemann, T. Siebert, 1994).
23. On an implementation of standard bases and syzygies in SINGULAR. *AAECC 7, 235–149* (with H. Grassmann, B. Martin, W. Neumann, G. Pfister, W. Pohl, H. Schönemann, T. Siebert, 1996).
24. Advances and improvements in the theory of standard bases and syzygies. *Arch. Math.* **66**, *163–176* (with G. Pfister, 1996).
25. Description of SINGULAR: A Computer Algebra System for Singularity Theory, Algebraic Geometry and Commutative Algebra. *Euromath Bulletin* **2**, *161–172* (1996).
26. The normalisation: a new algorithm, implementation and comparisons. *In: Proc. EUROCONFERENCE Computational Methods for Representations of Groups and Algebras (1.4.-5.4.1997). Birkhäuser* (with W. Decker, T. de Jong, G. Pfister, 1998).
27. Primary decomposition: algorithms and comparisons. *In: G.-M. Greuel, B.H. Matzat, G. Hiss: Algorithmic Algebra and Number Theory. Springer Verlag, Heidelberg, 187–220* (with W. Decker, G. Pfister, 1998).

28. Gröbner bases and algebraic geometry. *In: B. Buchberger and F. Winkler: Gröbner Bases and Applications. LNS 251 , CUP, 109–143* (with G. Pfister, 1998).
29. Applications of Computer Algebra to Algebraic Geometry, Singularity Theory and Symbolic-Numerical Solving. *In: European Congress of Mathematicians, Barcelona, July 10-14, 2000, Vol. II, 169–188* (2000).
30. Computer Algebra and Algebraic Geometry - Achievements and Perspectives. *Journ. Symb. Comp.* **30***, 253–290* (2000).
31. Three Algorithms in Algebraic Geometry, Coding Theory, and Singularity Theory. *In: C. Ciliberto et al: Application of Algebraic Geometry to Coding Theory, Physics and Computation, Proceedings. Kluwer, 161–194* (with C. Lossen, M. Schulze, 2001).
32. *A* SINGULAR *Introduction to Commutative Algebra*. Springer-Verlag, 605 pp. (with G. Pfister, and with contributions by O. Bachmann, C. Lossen and H. Schönemann, 2002).
33. Two-variable identities for finite solvable groups. *C.R. Acad. Sci. Paris, Ser. I* **337***, 581–586* (with T. Bandman, F. Grunewald, B. Kunyavskii, G. Pfister, E. Plotkin, 2003).
34. Engel-Like Identities Characterizing Finite Solvable Groups. To appear in Compos. Math. (with T. Bandman, F. Grunewald, B. Kunyavskii, G. Pfister, E. Plotkin, 2005).

Applying Computer Algebra Methods in Mathematical Research.

The publications [33,34] are related to a problem from group theory. The solution to this problem may serve as a model for how computer algebra methods may be used for establishing conjectures and for proving theorems in other fields of mathematics.

The was problem addressed to G.-M. Greuel and the second author by B. Kunyavskii. It can be stated as follows:

> *Characterize the class of solvable finite groups G by explicit two-variable identities.*

To explain this problem, note that a group G is Abelian iff the two-variable identity $xy = yx$ is satisfied for all $x, y \in G$. Moreover, Zorn (1930) proved that, setting

$$v_1(x,y) := \big[x,y\big] := xyx^{-1}y^{-1}\,, \qquad v_{k+1}(x,y) := \big[v_k, y\big]\,,$$

a finite group G is nilpotent iff there exists some $n \geq 1$ such that the two-variable identity $v_n(x,y) = 1$ holds for all $x, y \in G$. The identity $v_n(x,y) = 1$ is referred to as an *Engel Identity*.

The existence of two-variable (but non-explicit) identities for finite solvable groups has been proved by R. Brandl and J.S. Wilson (1981,1988). B. Plotkin suggested that there should be an explicit definition for such a two-variable identity $U_n(x,y) = 1$, using the recursion $U_{k+1} = [xU_k x^{-1}, yU_k y^{-1}]$. A key point has been to find an appropriate candidate for $U_1(x,y)$. Indeed, experimenting with SINGULAR such a candidate was found (see [33,34]):

Theorem. *Define U_k inductively by*

$$U_1(x,y) := x^{-2}y^{-1}x, \qquad U_{k+1}(x,y) := \left[xU_k(x,y)x^{-1}, yU_k(x,y)y^{-1}\right].$$

Then a finite group G is solvable iff there exist some n such that the two-variable identity $U_n(x,y) = 1$ holds for all $x, y \in G$.

That solvable groups satisfy the identity above is clear by the definition of a solvable group. Thus, it remains to show that for a (minimal) non-solvable finite group no such equality holds. Fortunately, the minimal non-solvable finite groups have been classified by Thompson (1968): his list consists of

1. $\mathrm{PSL}(2, \mathbb{F}_p)$, $p \geq 5$ prime,
2. $\mathrm{PSL}(2, \mathbb{F}_{2^p})$, p prime,
3. $\mathrm{PSL}(2, \mathbb{F}_{3^p})$, p prime,
4. $\mathrm{PSL}(3, \mathbb{F}_3)$,
5. the Suzuki groups $\mathrm{Sz}(2^p)$, p prime.

The key observation that allows one to translate B. Plotkins suggestion to a problem of algebraic geometry is the following[1]: *if $x, y \in G$ satisfy $1 \neq U_1(x,y) = U_2(x,y)$, then $U_n(x,y) \neq 1$ for all $n \in \mathbb{Z}$.*

It thus remains to show that for each group in Thompson's list, there are elements $x, y \in G$ such that $1 \neq U_1(x,y) = U_2(x,y)$.

It is quite instructive to show how such a problem from group theory can be translated to a problem in algebraic geometry and how to solve it with the help of computer algebra. Let us consider here the family of groups $G = \mathrm{PSL}(2, \mathbb{F}_p)$, $p \geq 5$ a prime. The next two cases of Thomson's list can be handeled similarly, the fourth case is treated by giving an explicit example. The last case, however, has turned out to be much more difficult, with a surprising complexity and involving in addition deep theorems from arithmetic geometry.

We represent two elements x and y of G by two matrices of the following types:

$$x = \begin{pmatrix} t & 1 \\ -1 & 0 \end{pmatrix}, \qquad y = \begin{pmatrix} 1 & b \\ c & 1+bc \end{pmatrix},$$

[1] This observation is independent of the choice of U_1.

with $b, c, t \in \mathbb{F}_p$.

Clearly we have $y \neq x^{-1}$ for all $(b, c, t) \in \mathbb{F}_p^3$, thus $U_1(x, y) \neq 1$. It remains to show that for each choice of p the equation $U_1(x, y) = U_2(x, y)$ has a solution $(b, c, t) \in \mathbb{F}_p^3$.

The ideal $I \subset \mathbb{Z}[b, c, t]$ spanned by the entries of $U_1(x, y) - U_2(x, y)$ is generated by four polynomials of degree at most 8. For a fixed prime number p, it defines a curve in the three-dimensional space over $\mathbb{F}_p$. To prove that there are $\mathbb{F}_p$-rational points on the curve we use the the Hasse-Weil-Theorem as generalized by Aubry and Perret for singular curves: *If $C \subseteq \overline{\mathbb{F}_q}^n$ is an irreducible affine curve, defined over $\mathbb{F}_q$, $q = p^m$, and if $\overline{C} \subset \mathbb{P}^n$ is its projective closure, then*

$$\#C(\mathbb{F}_q) \geq q + 1 - 2p_a\big(\overline{C}\big)\sqrt{q} - \deg\big(\overline{C}\big)\,.$$

To be able to apply the theorem to our situation, we have to show that the image of the ideal I in $\mathbb{F}_p[b, c, t]$ defines an irreducible curve C over the algebraic closure $\overline{\mathbb{F}_p}$ of $\mathbb{F}_p$. In algebraic terms, we have to show that the image of I generates a prime ideal of $\overline{\mathbb{F}_p}[b, c, t]$.

If this is the case, we may compute the degree and the arithmetic genus of the projective curve $\overline{C} \subset \mathbb{P}^3$ via the Hilbert-polynomial which equals $H(t) = 10t - 11$. Hence, $\deg\big(\overline{C}\big) = 10$ and $p_a\big(\overline{C}\big) = 11 + 1 = 12$, and the Hasse-Weil formula gives $\#C(\mathbb{F}_p) > 0$ for all primes $p > 593$.

As the remaining finitely many cases can be checked directly with a computer, it remains to prove that for any prime $p \geq 5$, the ideal $I \cdot \overline{\mathbb{F}_p}[b, c, t]$ is, indeed, a prime ideal.

We have $I \cdot \overline{\mathbb{F}_p}[b, c, t] = \big(I \cdot \overline{\mathbb{F}_p}(t)[b, c]\big) \cap \overline{\mathbb{F}_p}[b, c, t]$, and $I \cdot \overline{\mathbb{F}_p}(t)[b, c]$ is generated by two polynomials $f_1 \in \mathbb{F}_p[b, t]$, $f_2 \in \mathbb{F}_p[b, c, t]$, as obtained using Singular and verified by hand later on. Thus, it is enough to prove that $I \cdot \overline{\mathbb{F}_p}(t)[b, c]$ is a prime ideal, which is equivalent to showing that f_1 is irreducible in $\overline{\mathbb{F}_q}[t, b]$. As the polynomial f_1 has a small degree (namely 4) in x, this could be proved by making an Ansatz and showing that the resulting systems of polynomial equations have no solution over the algebraic closure (which was done first by the computer, then by hand).

Work on Cohen-Macaulay Modules and Moduli Spaces.

In the joint paper [15] with H. Knörrer, G.-M. Greuel showed that a reduced plane curve singularity is of finite CM-representation type, that is, its analytic local ring has only finitely many isomorphism classes of indecomposable Cohen-Macaulay modules iff it is a simple (ADE-) singularity.

This result has been extended later by Knörrer and Greuel in a joint paper with R.-O. Buchweitz and F.-O. Schreyer to arbitrary isolated hypersurface singularities [41]. It attracted interest by mathematicians working in representation theory of finite dimensional algebras, and the question came up whether the so-called tame–wild dichotomy for finite dimensional algebras (proved by Y. Drozd) also holds for curve singularities w.r.t. Cohen-Macaulay modules. During a workshop in Bielefeld in 1990 organized by C.M. Ringel, Greuel proposed this as a conjecture when he gave a talk about the construction of moduli spaces of CM modules over a fixed local ring of a reduced curve singularity. Y. Drozd, who was in the audience, immediately realized that the sandwiched construction used for the construction of moduli spaces could be used to reduce the question to a matrix problem.

The tame–wild dichotomy for CM-modules over curve singularities was finally proved by Greuel and Drozd in a joint paper. Several other joint papers of Greuel and Drozd were devoted to the classification of tame curve and surface [47] singularities and their CM-modules. Moreover, in [45], the tame–wild dichotomy was shown to hold also for singular projective curves with a particular nice geometric description of the tame curves for which a classification of all indecomposable vector bundles resp. torsion free sheaves was achieved.

In the remaining part of this section, we focus on the general approach to constructing moduli spaces for singularities and related objects developed by G.-M. Greuel and the second author in the 1980s. This approach basically consists of the following steps:

1. Fix some rough invariants.
2. Find the worst object among them you want to classify.
3. Consider the versal deformation $X \to T$ of the worst object with fixed invariants.
4. Prove that this family contains all objects you want to classify.
5. Compute the kernel $\mathcal{L}$ of the Kodaira-Spencer map of the family.
6. Compute a stratification $\{T_\alpha\}$ of T, by fixing suitable invariants such that the geometric quotient $T_\alpha/\mathcal{L}$ exists.
7. Modulo the action of a finite group, we obtain coarse moduli spaces.

To illustrate this general idea, let us consider an example:

> *Classify all* $R = \mathbb{C}[[t^c, t^{c+1}, \ldots]]$*-modules of rank one with set of values* $\Gamma = \{\gamma_0, \ldots, \gamma_k, c, c{+}1, \ldots\}, \quad 0 = \gamma_0 < \gamma_1 < \ldots < \gamma_k < c\,.$

Following our philosophy, we determine the *worst object*:

$$M_0 = \sum_{i=1}^{k} t^{\gamma_i} + t^c\mathbb{C}[[t]]\,.$$

Its *versal deformation* is given by

$$\mathscr{M}_\Gamma = \sum_{i=1}^{k} m_i \cdot \mathbb{C}[\boldsymbol{\lambda}][[t^c, t^{c+1}, \dots]] + t^c\mathbb{C}[\boldsymbol{\lambda}][[t]],$$

where $\boldsymbol{\lambda} = \{\lambda_{i,j}\}_{j+\gamma_i \notin \Gamma}$, and $m_i = t^{\gamma_i} + \sum_{j+\gamma_i\notin\Gamma} \lambda_{i,j} t^{j+\gamma_i}$. The *Kodaira-Spencer map* is a mapping

$$\rho : \mathrm{Der}_{\mathbb{C}}\, \mathbb{C}[\boldsymbol{\lambda}] \longrightarrow \mathrm{Ext}^1_{\mathbb{C}[\boldsymbol{\lambda}][[t^c,\dots]]}(\mathscr{M}_\Gamma, \mathscr{M}_\Gamma)\,,$$

and $t, t' \in T = \mathrm{Spec}(\mathbb{C}[\boldsymbol{\lambda}])$ define isomorphic modules iff they are in the same integral manifold of the kernel $\mathcal{L}$ of ρ. This kernel is of the form $\sum_\ell \mathbb{C}[\boldsymbol{\lambda}]\delta_\ell$, with $L = \sum_\ell \mathbb{C}\delta_\ell$ an Abelian Lie algebra.

The computation of the moduli spaces as geometric quotients is based on the following theorem:

Theorem. *Let A be a K-algebra, $\mathcal{L} \subset \mathrm{Der}^{\mathrm{nil}}_K(A)$ a Lie algebra, $\delta_1, \dots, \delta_n \in \mathcal{L}$ such that $\mathcal{L} \subset \sum_{\ell=1}^n A\delta_\ell$, and let $x_1, \dots, x_n$ be elements of A such that $\det\big(\delta_\ell(x_j)\big)$ is a unit in A and such that for each k-minor M of the first k columns of $\big(\delta_\ell(x_j)\big)$ we have $\boldsymbol{\delta}(M) \in \sum_{j<k} A\boldsymbol{\delta}(x_j)$. Then the following holds:*

(1) $A^{\mathcal{L}}[x_1, \dots, x_n] = A$ and $x_1, \dots, x_n$ are algebraically independent over $A^{\mathcal{L}}$. In particular, $\mathrm{Spec}(A) \to \mathrm{Spec}\big(A^{\mathcal{L}}\big)$ is a (trivial) geometric quotient.

(2) If, additionally, $\mathcal{L} = L$ is a finite dimensional nilpotent Lie algebra of dimension n, then $H^1(\mathcal{L}, A) = 0$.

This theorem has as consequence the following corollary which is the basis for the applications:

Corollary. *Let A be a Noetherian K-algebra, $L \subset \mathrm{Der}^{\mathrm{nil}}_K(A)$ a finite dimensional, nilpotent Lie algebra, and let $d : A \to \mathrm{Hom}_K(L, A)$ be the differential, $da(\delta) = \delta(a)$. Assume that the following holds:*

- *$0 = Z_{k+1}(L) \subset Z_k(L) \subset \dots \subset Z_0(L) = L$ is a finite filtration of L satisfying $[L, Z_j(L)] \subset Z_{j+1}(L)$.*
- *$0 = F^{-1}(A) \subset F^0(A) \subset F^1(A) \subset \dots$ is a filtration of the K-algebra A such that $\delta(F^i(A)) \subset F^{i-1}(A)$ for all i and all $\delta \in L$.*
- *$\mathrm{Spec}(A) = \bigcup_\alpha U_\alpha$ is the flattening stratification of the modules*

  $$\mathrm{Hom}_K(L, A)\big/A \cdot d(F^i(A))\,, \qquad \mathrm{Hom}_K(Z_j(L), A)\big/\pi_j(A \cdot d(A))\,,$$

  where $\pi_j : \mathrm{Hom}_K(L, A) \to \mathrm{Hom}_K(Z_{j+1}(L))$ denotes the canonical projection.

Then U_α is L-invariant and admits a locally trivial geometrical quotient with respect to the action of L.

We illustrate the use of this corollary by continuing the example treated above: in this case, L is Abelian, therefore no Z-filtration is needed.

Let a be the multiplicity of the maximal semigroup $\Gamma_0 \subset \Gamma$ acting on Γ. Then we define $F^i(\mathbb{C}[\boldsymbol{\lambda}])$ to be the $\mathbb{C}$-vector space generated by all quasihomogeneous polynomials in $\mathbb{C}[\boldsymbol{\lambda}]$ of degree less than $(i+1)\cdot a$. Here, we assign the degree j to λ_{ij}, which makes the vector fields δ_ℓ homogeneous of degree $-\ell$.

Then the assumptions of the corollary are satisfied for the nilpotent Lie algebra $L^{(0)} := \sum_{\ell \geq a} \mathbb{C}\delta_\ell \subset L$. Hence, if $\operatorname{Spec}(\mathbb{C}[\boldsymbol{\lambda}]) = \bigcup_\alpha U_\alpha$ is the flattening stratification of the modules $\operatorname{Hom}_{\mathbb{C}}(L^{(0)}, \mathbb{C}[\boldsymbol{\lambda}])/\mathbb{C}[\boldsymbol{\lambda}] \cdot d(F^i(\mathbb{C}[\boldsymbol{\lambda}]))$, then $U_\alpha \to U_\alpha/L^{(0)}$ is a geometric quotient. Using an H^1-vanishing argument, one can show that $U_\alpha \to U_\alpha/L$ is a geometric quotient, too (see [37] for details). This quotient turns out to be the moduli space of all modules with a certain Hilbert function fixed.

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Work on Equisingular Families

The short paper [48], actually an appendix to a paper of A. Tannenbaum in Compos. Math. 1984, was, in a sense, the initial point for G.-M. Greuel to start his own research on equisingular families. Tannenbaum observed that Segre's analysis of families with prescribed singularities can be rigorously justified for curves with at most ordinary nodes and cusps as singularities: Segre associated his characteristic linear series to $H^0\big(C, n_*\widetilde{\mathcal{N}}\big)$, where $\widetilde{\mathcal{N}}$ is a certain locally free sheaf on the normalization. *Correct* would have been $H^0\big(C, \mathcal{I}_{Z^{ea}(C)}(C)\big)$, where $Z^{ea}(C)$ is the zero-dimensional scheme locally defined by the Tjurina ideal. Indeed, Tannenbaum proved the existence of an exact sequence

$$0 \longrightarrow \mathcal{I}_{Z^{ea}(C)}(C) \longrightarrow n_*\widetilde{\mathcal{N}} \longrightarrow \mathscr{T} \longrightarrow 0\,,$$

where $\mathscr{T}$ is a torsion sheaf supported at the singular locus of C, with stalk $\mathscr{T}_x = 0$ if (C,x) is a node or a cusp.

As an addendum, Greuel computed the dimension of $\mathscr{T}_x$ as

$$\dim_{\mathbb{C}} \mathscr{T}_x = \tau(C,x) + r(C,x) - \operatorname{mult}(C,x) - \delta(C,x)\,.$$

In particular, at a singular point x of C, $\mathscr{T}_x$ is nonzero unless this singular point is either a node or a cusp.

Greuel's interest in equisingular families of curves with arbitrary singularities (not just nodes and cusps) was stimulated, and more questions came up. To describe some of these, we restrict ourselves to the case of plane curves, using the following definition:

Definition. Let $S_1,\dots,S_r$ be (analytic or topological) types of plane curve singularities. Then we set

$$V_d(S_1,\dots,S_r) := \left\{ C \subset \mathbb{P}^2 \;\middle|\; \begin{array}{l} C \text{ is a reduced curve of degree } d, \\ \text{having exactly } r \text{ singular points of} \\ \text{types } S_1,\dots,S_r \end{array} \right\},$$

respectively $V_d^{irr}(S_1,\dots,S_r)$ where it is additionally assumed that C is irreducible.

In a joint work first with U. Karras [49], Greuel showed that the set $V_d(S_1,\dots,S_r)$ carries a natural structure as a complex space, even for analytic types of arbitrary isolated singularities, given by deformation theory. Moreover, with U. Karras and then with E. Shustin and the first author, G.-M. Greuel looked for general numerical criteria answering the following questions (for $V = V_d(S_1,\dots,S_r)$, resp. $V_d^{irr}(S_1,\dots,S_r)$):

- Is V non-empty?
- Is V smooth (that is, is the characteristic linear series complete)?
- Is V *T-smooth* (that is, smooth and of the expected dimension)?
- Is V irreducible?

The general method for answering these questions is based on a translation to a statement about the cohomology of ideal sheaves of zero-dimensional schemes.

For instance, the T-smoothness property for analytic types translates as

$$V \text{ is T-smooth at } C \iff H^1\big(\mathcal{J}_{Z^{ea}(C)}(d)\big) = 0\,,$$

for $Z^{ea}(C) \subset \mathbb{P}^2$ the zero-dimensional scheme which is locally given by the Tjurina ideal for a local equation of C. For topological types, $Z^{ea}(C)$ has to be replaced by the zero-dimensional scheme $Z^{es}(C)$ which is locally given by the equisingularity ideal in the sense of J. Wahl.

The first criteria obtained have been based on the following vanishing theorem of Riemann-Roch-type (see [49,50]):

Theorem. *Let S be a smooth projective surface, $C \subset S$ a reduced curve. Let $\mathcal{F}$ be a torsion-free, coherent $\mathcal{O}_C$–module having rank 1 on each irreducible component C_i of C, $i = 1,\dots,s$. Then $H^1\big(C,\mathcal{F}\big)$ vanishes if*

$$\chi(\omega_C \otimes \mathcal{O}_{C_i}) - \mathrm{isod}_{C_i}(\mathcal{F},\mathcal{O}_C) \;<\; \chi(\mathcal{F}_{C_i})$$

for all $i = 1,\dots,s$. Here, $\mathcal{F}_{C_i} = \mathcal{F} \otimes \mathcal{O}_{C_i}$ (mod torsion). Moreover,

$$\mathrm{isod}_{C_i,x}(\mathcal{F},\mathcal{G}) := \min\big(\dim_{\mathbb{C}} \mathrm{coker}(\varphi_{C_i} : \mathcal{F}_{C_i,x} \hookrightarrow \mathcal{G}_{C_i,x})\big)\,,$$

where the minimum is taken over all φ_{C_i}, which are induced by homomorphisms $\varphi : \mathcal{F}_x \to \mathcal{G}_x$.

As a consequence, it was shown in [49,50] that V is T-smooth at C if the total (equisingular) Tjurina number of C is bounded by a linear function in the degree of d. The resulting criteria are usually referred to as the $3d$-criterion and the $4d$-criterion.

A slightly weaker $4d$-criterion was found before by E. Shustin by different methods. G.-M. Greuel met E. Shustin at the ICM 1990 in Kyoto (Japan) where they discussed the different approaches and realized that joining efforts could result in a major progress in this area.

A major breakthrough in the study of equisingular families was the first asymptotically proper general sufficient condition for the existence of plane curves with prescribed (topological types of) singularities obtained in [53]:

Theorem. *If $S_1, \ldots, S_r$ are topological types of singularities, and if*

$$\sum_{i=1}^{r} \mu(S_i) \leq \frac{1}{392}(d+2)^2,$$

then $V_d^{irr}(S_1, \ldots, S_r)$ is non-empty.

This criterion is referred to as being asymptotically proper, since (from the asymptotical point of view) it differs from the necessary criterion

$$\sum_{i=1}^{r} \mu(S_i) \leq (d-1)^2$$

for non-emptyness only by a constant factor. Later, this factor $\frac{1}{392}$ has been improved to $\frac{1}{9}$, and the statement has been extended to analytic types by E. Shustin.

For the T-smoothness problem, the linear right-hand side (as in the $3d$- and $4d$-criterion) could be replaced by a quadratic function in d, too. So far, the best known general sufficient criterion has been obtained in [55,56]:

Theorem. *Let $d \geq 6$. Then $V_d^{irr}(S_1, \ldots, S_r)$ is T-smooth at C if*

$$\sum_{i=1}^{r} \gamma^{ea}(C, z_i) \leq (d+3)^2, \qquad \left(\textit{resp.} \;\; \sum_{i=1}^{r} \gamma^{es}(C, z_i) \leq (d+3)^2 \right),$$

for new invariants $\gamma^{ea} \leq (\tau+1)^2$, $\gamma^{es} \leq (\tau^{es}+1)^2$.

In particular, for families of curves with n nodes and k cusps (resp. for families of curves with ordinary m_i-fold points) the sufficient condition reads

$$4n + 9k \;\leq\; (d+3)^2 \qquad \left(\text{resp.} \;\; 4 \cdot \#(\text{nodes}) + \sum_{m_i > 2} 2m_i^2 \;\leq\; (d+3)^2 \right).$$

For the irreducibility problem, the best known general sufficient criterion is:

Theorem. *If* $\max_{i=1..r} \tau'(S_i) \leq (2/5) \cdot d - 1$ *and*

$$\frac{25}{2} \cdot \#(nodes) + 18 \cdot \#(cusps) + \frac{10}{9} \cdot \sum_{\tau'(S_i) \geq 3} \left(\tau'(S_i) + 2\right)^2 \; < \; d^2,$$

then $V_d^{irr}(S_1, \ldots, S_r)$ *is non-empty and irreducible.*

Here τ' refers to the Tjurina number, resp. to the equisingular Tjurina number, that is, the codimension of the equisingularity ideal.

In contrast to the conditions for non-emptiness and T-smoothness, this condition seems not to be asymptotically proper. Indeed, for instance, for plane curves with r ordinary m-fold points, the known examples of reducible families (see [56]) satisfy $d^2 \sim r \cdot m^2$ while the left-hand side of our criterion is of type $r \cdot \frac{m^4}{4}$.

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Exterior Algebra Methods for the Construction of Rational Surfaces in the Projective Fourspace

Hirotachi Abo Frank-Olaf Schreyer

Abstract

The aim of this paper is to present a construction of smooth rational surfaces in projective fourspace with degree 12 and sectional genus 13. The construction is based on exterior algebra methods, finite field searches and standard deformation theory.

Introduction

This paper is dedicated to Gert-Martin Greuel on the occasion of his sixtieth birthday. The use of computer algebra systems is essential for the proof of the main result of this paper. It will become clear that without computer algebra systems like Singular and Plural developed in Kaiserslautern we could not obtain the main result of this paper at all. We thank the group in Kaiserslautern for their excellent program.

Hartshone conjectured that only finitely many components of the Hilbert scheme of surfaces in $\mathbb{P}^4$ correspond to smooth rational surfaces. In 1989, this conjecture was positively solved by Ellingsrud and Peskine [6]. The exact bound for the degree is, however, still open. This motivates our search for smooth rational surfaces in $\mathbb{P}^4$. Examples of smooth rational surfaces in $\mathbb{P}^4$ prior to this paper were known up to degree 11, see [4]. Our main result is the proof of existence of the following example.

Theorem 0.1. *There exists a family of smooth rational surfaces in $\mathbb{P}^4$ over $\mathbb{C}$ with $d = 12$, $\pi = 13$ and hyperplane class*

$$H \equiv 12L - \sum_{i_1=1}^{2} 4E_i - \sum_{i_2=3}^{11} 3E_i - \sum_{i_3=12}^{14} 2E_i - \sum_{i_4=15}^{21} E_i.$$

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in terms of a plane model.

Abstractly, these surfaces arise as the blow up of $\mathbb{P}^2$ in 21 points. L and E_i in the Theorem denote the class of a general line and the exceptional divisors.

The 21 points lie in special position due to the fact that we need $\mathrm{h}^0(X, \mathcal{O}(H) = 5$ and $\mathrm{h}^1(X, \mathcal{O}(H)) = 4$. Indeed, it will turn out that the component of the Hilbert scheme corresponding to these surfaces has dimension 38, hence up to projectivities this is a $38 - 24 = 14$ dimensional family of abstract surfaces. This fits with the fact that the 21 points have to satisfy a condition of codimension $\leq 20 = 4 \cdot 5$, which leaves us with a family of collections of points in $\mathbb{P}^2$ of dimension $\geq 2 \cdot 21 - 20 = 22$. Up to automorphism of $\mathbb{P}^2$ this leads to a family of dimension $\geq 22 - 8 = 14$, and hence equality holds. The great difficulty to find points in $\mathbb{P}^2$ in very special positions was one of the sources, which led Hartshorne to his conjecture.

We construct these surfaces via their "Beilinson monad": Let V be an $n+1$-dimensional vector space over a field K and let W be its dual space. The basic idea behind a Beilinson monad is to represent a given coherent sheaf on $\mathbb{P}^n = \mathbb{P}(W)$ as a homology of a finite complex of vector bundles, which are direct sums of exterior powers of the tautological rank n subbundle $U = \ker\big(W \otimes \mathcal{O}_{\mathbb{P}(W)} \to \mathcal{O}_{\mathbb{P}(W)}(1)\big)$ on $\mathbb{P}(W)$. (Thus $U \simeq \Omega^1(1)$ is the twisted sheaf of 1-forms. As Beilinson, we will use the notation $\Omega^p(p)$ for the exterior powers of U.)

The differentials in the monad are given by homogeneous matrices over an exterior algebra $E = \bigwedge V$. To construct a Beilinson monad for a given coherent sheaf, we typically take the following steps: Determine the type of the Beilinson monad, that is, determine the vector bundles of the complex, and then find differentials in the monad.

Let X be a smooth rational surface in $\mathbb{P}^4 = \mathbb{P}(W)$ with degree 12 and sectional genus 13. The type of a Beilinson monad for the (suitably twisted) ideal sheaf of X can be derived from the knowledge of its cohomology groups. Such information is partially determined from general results such as the Riemann-Roch formula and the Kodaira vanishing theorem. It is, however, hard to determine the dimensions of all cohomology groups needed to determine the type of the Beilinson monad. For this reason, we assume that the ideal sheaf of X has the so-called "natural cohomology" in some range of twists. In particular, we assume that in each twist $-1 \leq n \leq 6$ at most one of the cohomology groups $\mathrm{H}^i(\mathbb{P}^4, \mathcal{I}_X(n)$ for $i = 0 \ldots 4$ is non-zero. This is an open condition for surfaces in a given component of the Hilbert scheme. Under this assumption the Beilinson monad for the twisted ideal sheaf $\mathcal{I}_X(4)$ of X has the following form:

$$4\Omega^3(3) \xrightarrow{A} 2\Omega^2(2) \oplus 2\Omega^1(1) \xrightarrow{B} 3\mathcal{O}. \qquad (1)$$

To detect differentials in (1), we use the following techniques developed recently: (1) the first technique is an exterior algebra method due to Eisenbud, Fløystad and Schreyer [5] and (2), the other one is the method using small finite fields and random trials due to Schreyer [9].

(1) Eisenbud, Fløystad and Schreyer presented an explicit version of the Bernstein-Gel'fand-Gel'fand correspondence. This correspondence is an isomorphism between the derived category of bounded complexes of finitely generated S-graded modules and the derived category of certain "Tate resolutions" of E-modules, where $S = \mathrm{Sym}_K(W)$. As an application, they constructed the Beilinson monad from the Tate resolution explicitly. This enables us to describe the conditions, that the differentials in the Beilinson monad must satisfy in an exterior algebra context.

(2) Let $\mathbb{M}$ be a parameter space for objects in algebraic geometry such as the Hilbert scheme or a moduli space. Suppose that $\mathbb{M}$ is a subvariety of a rational variety $\mathbb{G}$ of codimension c. Then the probability for a point p in $\mathbb{G}(\mathbb{F}_q)$ to lie in $\mathbb{M}(\mathbb{F}_q)$ is about $(1 : q^c)$. This approach will be successful if the codimension c is small and the time required to check $p \notin \mathbb{M}(\mathbb{F}_q)$ is sufficiently small as compared with q^c. This technique was applied first by Schreyer [9] to find four different families of smooth surfaces in $\mathbb{P}^4$ with degree 11 and sectional genus 11 over $\mathbb{F}_3$ by a random search, and he provided a method to establish the existence of lifting these surfaces to characteristic 0. This technique has been successfully applied to solve various problems in constructive algebraic geometry (see [10], [12] and [1]).

The Singular or Macaulay2 scripts used to construct and to analyse these surfaces are available at `http://www.math.uni-sb.de/~ag-schreyer` and `http://www.math.colostate.edu/~abo/programs.html`.

1 The Exterior Algebra Method

Our construction of the rational surfaces uses the "Beilinson monad". A Beilinson monad represents a given coherent sheaf in terms of direct sums of (suitably twisted) bundles of differentials and homomorphisms between these bundles, which are given by homogeneous matrices over an exterior algebra E. Recently, Eisenbud, Fløystad and Schreyer [5] showed that for a given sheaf, one can get the Beilinson monad from its "Tate resolution", that is a free resolution over E, by a simple functor. This enables us to discuss the Beilinson monad in an exterior algebra context. In this section, we take a quick look at the exterior algebra method developed by Eisenbud, Fløystad and Schreyer.