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Model Theory of Groups and Automorphism Groups

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David M. Evans
University of East Anglia

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Preface

The articles in this volume represent the invited lectures at the RESMOD Summer School on Model Theory of Groups and Automorphism Groups held in Blaubeuren, Germany, from 31 July to 5 August 1995. This was an EC-funded meeting directed at graduate students and researchers in Model Theory and Algebra and consisted mainly of invited lectures surveying various recent interactions between model theory and and other branches of mathematics, notably group theory.

RESMOD is the acronym for the European Human Capital and Mobility Network on Model Theory and Applications coordinated by the Équipe de Logique Mathématique at Université Paris 7. The programme committee for the meeting consisted of Wilfrid Hodges, Daniel Lascar and Dugald Macpherson. The meeting took place at the Heinrich Fabri Institut of the University of Tübingen, and the local organisers were Ulrich Felgner and Frieder Haug.

David M. Evans,
School of Mathematics,
University of East Anglia,
Norwich NR4 7TJ,
England.
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Introduction

The articles in this volume demonstrate the wide variety of interactions between algebra (particularly group theory) and current research in model theory. On the one hand, the analysis of direct questions about the first-order theories of classes of algebraic structures requires an interplay between model-theoretic and algebraic methods, and often such questions also evolve into ones which are interesting from a purely algebraic viewpoint. More indirectly, the model-theoretic analysis of classes of structures using some of the latest developments of model theory (particularly stability theory) has recently resulted in a wave of new applications of model theory to other parts of mathematics.

Alongside these developments there has been considerable interaction between model theory and the study of infinite permutation groups. Automorphism groups of model-theoretically interesting structures have provided a rich supply of examples and problems for the permutation group theorists, and the study of automorphism groups has been a crucial tool in certain model-theoretic questions.

Readers can judge for themselves the extent to which the articles in this volume fit into this pattern, but I shall give a brief sketch of them, emphasising the interactions between model theory and other parts of mathematics.

The article by Evans, Ivanov and Macpherson is a survey largely concerned with a question that originated in studying the fine detail of totally categorical structures, but which is now seen (and studied) as a problem about infinite permutation groups. The techniques in the papers by Lascar and Evans are model-theoretic in flavour, but the applications are to the study of the automorphism group of the field of complex numbers, and the papers are written without using model-theoretic terminology. The papers by Chatzidakis and Hodges form a survey of recent work on the model theory of pseudo-finite fields and in particular give surprising applications of these results (due to Hrushovski and Pillay) to the subgroup structure of Chevalley groups over prime fields.

Cameron's paper draws together strands from model theory, permutation groups and combinatorics. It studies a graded algebra which can be associated to any countable $\aleph_0$-categorical structure, and which is also significant in enumerative combinatorics.

The papers by Boffa, Oger and Chiswell are surveys of various aspects of the model theory of particular classes of groups. The questions considered start out as model-theoretic ones (equivalence of formulas, elementary equivalences at various levels of quantifier complexity *et cetera*), but also develop into questions which are interesting from a purely group-theoretic viewpoint. The techniques are a mixture of group theory and model theory (notably ultraproducts). The paper by Pfander follows on from Chiswell's article and gives new results on the finite presentability of groups with the same existential and universal theory as the non-abelian free groups.

The paper by Burke and Prest is a contribution to the theory of modules: an area where model-theoretic methods have had a significant impact on the algebraic theory. Finally, the paper by Kim is a survey of recent work (of Kim and Pillay) on Shelah's notion of *simplicity* of a first-order theory. In such theories a good notion of independence (*forking*) can be described and various unstable algebraic structures have recently been shown to have simple theories (for example, pseudo-finite fields).

In the remainder of this introduction I will give some background material and pointers to the literature which may be helpful to the non-specialist reader. This will be very brief, not least because there are already several excellent swift introductions to the area in print: for example, the opening sections of [1] and (more comprehensively) the article [6].

Section 2 below is based on notes of Dugald Macpherson originally prepared as an appendix to the 'Finite covers' paper in this volume.

1 Model theory

The books by Chang and Keisler [2] and Hodges [5] give a thorough treatment of model theory excluding stability theory. A good introduction to the latter can be found in the the book by Pillay [7], and [8] has many of the more recent developments.

1.1 First-order languages and structures

In a first-order language one has an alphabet of symbols and certain finite sequences of these symbol (the formulas of the language) are the objects of interest. The symbols are connectives $\wedge$ (*and*), $\vee$ (*or*), $\neg$ (*not*); quantifiers $\forall$ and $\exists$; punctuation (parentheses and commas); variables; and constant, relation and function symbols, with each of the last two coming equipped with a finite 'arity' specifying how many arguments it has. The number of these constant, relation and function symbols (together with their arities) is referred to as the *signature* of the language.

The *terms* of the language are built inductively. Any variable or constant symbol is a term and if f is an n-ary function symbol and $t_1, \ldots, t_n$ are terms, then $f(t_1, \ldots, t_n)$ is also a term (all terms are built in this way).

Now we can build the *formulas* of the language. Again, this is done inductively. If R is an n-ary relation symbol in the language and $t_1, \ldots, t_n$ are terms then $R(t_1, \ldots, t_n)$ is a formula (an *atomic* formula). If ϕ, ψ are formulas and x a variable, then $(\phi) \wedge (\psi)$, $(\phi) \vee (\psi)$, $\neg(\phi)$, $\forall x(\phi)$, $\exists x(\phi)$ are formulas (of higher 'complexity'). A formula not involving any quantifiers is called *quantifier free* or *open*. There is a natural notion of a *free variable* in a formula, and when we write a formula as $\phi(x_1, \ldots, x_m)$ we mean that its free variables are amongst the variables $x_1, \ldots, x_m$. A formula with no free variables is called a *sentence*. For more details the reader could consult ([2], Section 1.3) or ([5], Section 2.1).

If L is a first-order language then an *L-structure* consists of a set M equipped with a constant (that is, a distinguished element of M), n-ary relation (that is, a subset of M^n), and n-ary function $M^n \to M$ for each constant symbol and

n-ary relation and function symbol in L. If $\phi(x_1, \ldots, x_m)$ is an L-formula and $a_1, \ldots, a_m \in M$ then one can 'read' $\phi(a_1, \ldots, a_m)$ as a statement about the behaviour of $a_1, \ldots, a_m$ and these constants, relations and functions (interpreting each constant, relation or function symbol as the corresponding constant, relation or function of M), which is either true or false. If it is true, then we write

$$M \models \phi(a_1, \ldots, a_m).$$

All of this can of course be made completely precise (defined inductively on the complexity of ϕ): see ([2], Section 1.3) and ([5], Section 2.1) again. We shall always have $=$ as a binary relation symbol in L and interpret it as true equality in any L-structure.

If Φ is a set of L-sentences and M an L-structure we say that M is a *model* of Φ (and write $M \models \Phi$) if every sentence in Φ is true in M. If there is a model of Φ we say that Φ is *consistent*. The set of L-sentences true in M is called the *theory* of M. Two L-structures M_1 and M_2 are *elementarily equivalent* if they have the same theory. This is written as $M_1 \equiv M_2$. Thus in this case the structures M_1 and M_2 cannot be distinguished using the language L. The following basic result of model theory shows that one should not expect first-order languages to be able to completely describe infinite structures.

Theorem 1.1 (Löwenheim-Skolem) *Let L be a first-order language with signature of cardinality λ. Let μ, ν be cardinals with $\mu, \nu \geq \max(\lambda, \aleph_0)$, and suppose M_1 is an L-structure with cardinality μ. Then there exists an L-structure M_2 elementarily equivalent to M_1 and of cardinality ν.*

The 'upward' part of this result (where $\nu \geq \mu$) follows easily from the fundamental theorem of model theory:

Theorem 1.2 (The Compactness Theorem) *Let L be a first-order language and Φ a set of L-sentences. If every finite subset of Φ is consistent, then Φ is consistent.*

The original version of this is due to Gödel (1931). Proofs (using a method due to Henkin (1949)) can be found in ([2], Theorem 3.2.2) and ([5], Theorem 6.1.1). Algebraists may prefer the proof using ultraproducts ([2], Corollary 4.1.11) and the theorem of Los ([2], Theorem 4.1.9, or [5], Theorem 9.5.1).

If M, N are L-structures with $M \subseteq N$ and the distinguished relations, functions (and constants) of N extend those of M, then we say that M is a *substructure* of N. If also for every L-formula $\phi(x_1, \ldots, x_m)$ and $a_1, \ldots, a_m \in M$ we have

$$M \models \phi(a_1, \ldots, a_m) \Leftrightarrow N \models \phi(a_1, \ldots, a_m)$$

then we say that M is an *elementary* substructures of N (and that N is an elementary extension of M) and write $M \preceq N$. A stronger version of the Löwenheim-Skolem Theorem (1.1) is true: the smaller of M_1, M_2 may be taken to be an elementary substructure of the larger. Proofs can be found in ([2], Theorems 3.1.5 and 3.1.6) and ([5], Corollaries 3.1.5 and 6.1.4).

1.2 Definable sets; types

Suppose L is a first-order language and M an L-structure. Let $n \in \mathbb{N}$. A subset A of M^n is called (parameter) *definable* if there exist $b_1, \ldots, b_m \in M$ and an L-formula $\phi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ with

$$A = \{\bar{a} \in M^n : M \models \phi(\bar{a}, \bar{b})\}.$$

If the parameters $\bar{b}$ can be taken from the subset $X \subseteq M$ then A is said to be X-definable. The union of the finite X-definable subsets of M is called the *algebraic closure* of X, denoted by $\mathrm{acl}(X)$, and the union of the X-definable singleton subsets of M is the *definable closure* of X, denoted by $\mathrm{dcl}(X)$. It is not hard to check that both of these are indeed closure operations on M.

So the definable subsets of M^n are the ones which can be described using L-formulas (and parameters). Conversely one could take a particular n-tuple $\bar{a} \in M^n$ and a set of parameters $A \subseteq M$ and ask what the language L can say about $\bar{a}$ (in terms of A and M). This gives the notion of the *type* of $\bar{a}$ over A, which by definition is

$$\mathrm{tp}_M(\bar{a}/A) = \{\phi(x_1, \ldots, x_n, b_1, \ldots, b_m) : b_1, \ldots, b_m \in A,\ M \models \phi(\bar{a}, \bar{b})\}$$

(the subscript M is dropped if this is clear from the context). It is sometimes useful to consider the type of $\bar{a}$ (over A) using only certain L-formulas. For example, for the *quantifier free type* of $\bar{a}$ over A one takes only quantifier free ϕ in the above definition. It is also possible to define the type of an infinite sequence of elements of M. The reader can consult ([5], Section 6.3) or ([2], Section 2.3) for further details here.

More generally, a (complete) n-type over A is a set of L-formulas with parameters from A equal to $\mathrm{tp}_N(\bar{a}/A)$ for some elementary extension N of M and some $\bar{a} \in M^n$. There is no reason to suppose, for arbitrary M and A, that this type should be *realised* in M, that is, there exists $\bar{a}' \in M^n$ with $\mathrm{tp}_M(\bar{a}'/A) = \mathrm{tp}_N(\bar{a}/A)$. For example, this would clearly be impossible if $A = M$ and $\bar{a} \notin M^n$. However, it can happen that for some infinite cardinal κ if $|A| < \kappa$ then every complete n-type over A is realised in M: in this case M is called *κ-saturated,* and if $\kappa = |M|$ then M is *saturated.* The reader should consult ([2], Section 2.3) and then ([2], Chapter 5) and ([5], Chapter 10) for more on this subject as the need arises.

1.3 Interpreted structures; imaginary elements

Some structures can be built out of others in a definable way. The classical example is the construction of the field of rational numbers from the ring of integers. Another example is algebraic groups over a particular field.

Formalising this leads to the notion of an *interpretation* of one structure in another. Suppose K and L are first-order languages, M a K-structure and N an L-structure. We say that N is *interpretable* in M if for some $n \in \mathbb{N}$ there exist:

1. a $\emptyset$-definable subset D of M^n;

2. a $\emptyset$-definable equivalence relation E on D;

3. a bijection $\gamma : N \to D/E$

such that for every $\emptyset$-definable subset R of N^m the subset of M^{mn} given by

$$\hat{R} = \{(\bar{a}_1, \ldots, \bar{a}_m) \in (M^n)^m : (\gamma^{-1}(\bar{a}_1/E), \ldots, \gamma^{-1}(\bar{a}_m/E)) \in R\}$$

is $\emptyset$-definable in M.

Thus the set N can be identified with a $\emptyset$-definable subset of M^n factored by a $\emptyset$-definable equivalence relation, and with this identification all of the L-structure on N can be derived from the K-definable structure on M. There is a considerable amount of redundancy in the definition: it is only necessary to have $\emptyset$-definability of $\hat{R}$ when R is a distinguished constant or relation, or the graph of a distinguished function.

If E is simply equality on D then we say that N is *definable* in M. If also $D = M$ then we say that N is a *reduct* of M (so N just consists of M with some of its definable structure forgotten). It is also possible to formulate a notion of interpretation using parameters. The reader should consult ([5], Section 5.3) for further information on interpretations.

Equivalence classes in D/E as above are referred to as *imaginary* elements of M. Taking the set of all imaginary elements (as D and E range over all $\emptyset$-definable sets and equivalence relations) gives us the set M^{eq}. We wish to regard this as a first-order structure, so we extend the language K of M in a canonical way (to a first-order language K^{eq}), and part of the K^{eq}-theory of M^{eq} describes how the imaginary elements correspond in a $\emptyset$-definable way to the original K-structure M. The reader can consult ([5], Section 4.3) for the precise details of how to do all of this. Once we have this concept, it makes sense to extend notions such as 'parameter definable', 'types', 'algebraic closure' etc. to subsets of M^{eq}. Again we refer the reader to ([5]) for further details if the need arises.

2 Permutation groups

Most of what is said here can be found in more detail in ([5], Section 4.1), ([1], Chapters 1 and 2) and ([6]). A general reference on permutation groups which usefully contains material on automorphism groups of infinite structures is [3].

2.1 Actions and orbits

Let X be any set. The group of all permutations of X is called the *symmetric group* on X and is denoted by $\mathrm{Sym}(X)$. A *permutation group* on X is a subgroup of this. The image of the element $x \in X$ under the permutation $g \in \mathrm{Sym}(X)$ is denoted by gx. More generally, an *action* of a group G on X is a function $\alpha : G \times X \to X$ such that for all $x \in X$ and $g, h \in G$ we have $\alpha(1, x) = x$ and $\alpha(g, \alpha(h, x)) = \alpha(gh, x)$. It is easy to see that this is equivalent to the map $g \mapsto \alpha(g, -)$ being a homomorphism from G into $\mathrm{Sym}(X)$. Thus each element of

G induces a permutation of X and a product of elements in the group induces the corresponding product of permutations. Henceforth, we shall also denote $\alpha(g, x)$ by gx if the action is clear from the context. (It should be noted that some people write their actions on the right, and so would write xg instead of gx, with corresponding changes needed for other pieces of notation. This rarely causes confusion.)

Given an action of a group G on a set X the *orbits* are the equivalence classes under the equivalence relation $\sim$ on X, where $x_1 \sim x_2$ if there is $g \in G$ with $gx_1 = x_2$. We say that the action is *transitive* on X if there is a unique orbit. One way of manufacturing transitive actions of a group G is via *coset spaces.* Let H be a subgroup of G and let $Y = \{gH : g \in G\}$ be the set of left cosets of H in G. Define an action of G on Y by setting $\alpha(g_1, g_2H) = g_1g_2H$. Clearly this is a transitive action. However, in a strong sense this gives us all transitive actions of G. Suppose G acts transitively on a set X. Let $x \in X$ and let $H = \{g \in G : gx = x\}$ (the *stabiliser* of x in G, usually denoted by G_x). Then the map $\theta : Y \to X$ given by $\theta(gH) = gx$ is a well-defined bijection and for all $y \in Y$ and $g \in G$ we have $\theta(gy) = g\theta(y)$. Thus the actions of G on X and Y are *equivalent.* This is the *Orbit-Stabiliser Theorem.*

Out of any given action of a group G on a set X we can produce various other actions of G. For example, if $Y \subseteq X$ is a union of G-orbits, then one can simply restrict the action to Y. Also, suppose there is a G-invariant equivalence relation on X. Then one can consider the action of G on the set of equivalence classes. Next, suppose k is a positive integer. Then there is an induced action of G on X^k (given by $g(x_1, \ldots, x_k) = (gx_1, \ldots, gx_k)$), and an equally natural action on $X^{\{k\}}$, the set of k-sets from X. We say that G is *k-transitive* if, in the first action, all k-tuples of distinct elements lie in the same orbit. We say G is *k-homogeneous* if it is transitive on $X^{\{k\}}$. The original action is *highly* transitive (or homogeneous) if it is k-transitive (or k-homogeneous) for all $k \in \mathbb{N}$.

2.2 Automorphism groups and topological groups

Suppose L is a first-order language and M an L-structure. By an *automorphism* of M we mean a permutation of M which preserves each of the distinguished constants, relations and functions of M. The set of these forms a subgroup of $\mathrm{Sym}(M)$, called the *automorphism group* of M, and is denoted by $\mathrm{Aut}(M)$. It is clear that if $A \subseteq M$, then $\mathrm{Aut}(M/A) = \{g \in \mathrm{Aut}(M) : ga = a \;\forall a \in A\}$ is a subgroup of $\mathrm{Aut}(M)$ which stabilises any A-definable subset of M^k. Furthermore, if $\bar{b} \in M^k$ and $g \in \mathrm{Aut}(M/A)$ then $\mathrm{tp}_M(\bar{b}/A) = \mathrm{tp}_M(g\bar{b}/A)$. More subtly, if M is saturated then the converse is also true: if $\bar{b}$ and $\bar{b}'$ have the same type over A and $|A| < |M|$ then $\bar{b}$ and $\bar{b}'$ are in the same $\mathrm{Aut}(M/A)$-orbit (for example, see ([5], Corollary 10.4.12), or the proof of ([2], Theorem 2.3.9) if M is countable). Note that any element of $\mathrm{Aut}(M)$ induces an automorphism of M^{eq}.

Conversely, if G is a permutation group on a set X then there is a natural first-order structure with domain X, on which G acts as a group of automorphisms (with, for each $n \in \mathbb{N}$, the same orbits on n-tuples as the full automorphism group). For each orbit Ω of G on X^n (as n ranges through $\mathbb{N}$) introduce an n-ary relation symbol R_Ω, interpreted on X by the orbit Ω. The corresponding language is known as the *canonical language*, and the structure on X as the *canonical structure.*

Suppose that X is any set. Then there is a natural topology on $\mathrm{Sym}(X)$ which makes it into a topological group (so multiplication and inversion are continuous maps). The open sets are unions of cosets of pointwise stabilisers of finite subsets of X. We then make any permutation group G on X into a topological group by giving it the relative topology. To put this another way, if $g \in G$ then the cosets $gG_{(F)}$ as F ranges over the finite subsets of X form a basis of open neighbourhoods of g in G, where $G_{(F)} = \{h \in G : hx = x \ \forall x \in F\}$. Clearly this topology is Hausdorff. In fact, as any open coset is closed, the topology is totally disconnected. It it separable if and only if X is countable and discrete if and only if $G_{(F)} = \{1\}$ for some finite $F \subseteq X$. It is not hard to show that a closed subgroup G of $\mathrm{Sym}(X)$ is compact if and only if all of its orbits on X are finite.

For us, the most important fact about this topology will be that a subgroup of $\mathrm{Sym}(X)$ is closed if and only if it is the full group of automorphisms of a first-order structure with domain X. In fact, if $G \leq \mathrm{Sym}(X)$ then the automorphism group of the canonical structure of G on X is the closure of G in $\mathrm{Sym}(X)$.

If X is countable the topology is metrisable: enumerate X as $(x_n : n \in \mathbb{N})$, and define a metric d on $\mathrm{Sym}(X)$ by putting, for distinct $g, h \in \mathrm{Sym}(X)$, the distance $d(g, h)$ to be $1/m$ where m is as large as possible such that g agrees with h, and g^{-1} with h^{-1}, on x_l for all $l < m$. Thus, $\mathrm{Sym}(X)$ becomes a complete metric space with a countable basis of open sets (a *Polish space*).

2.3 $\aleph_0$-categoricity

For saturated structures M there is a strong connection between what is definable in a first-order way and the automorphism group: over small subsets of M orbits equate to types. For countable $\aleph_0$-categorical structures the connection is even stronger, and automorphism groups of $\aleph_0$-categorical structures are probably the most widely studied class of infinite permutation groups.

If M an L-structure and κ an infinite cardinal we say that M is *κ-categorical* if its theory has a model of size κ and all such are isomorphic. The case $\kappa = \aleph_0$ (that is, countably infinite κ) has a group-theoretic formulation. Say that a permutation group G on an infinite set X is *oligomorphic* if it has finitely many orbits on X^k for all positive integers k. Then the theorem of Engeler, Ryll-Nardzewski, and Svenonius asserts that, for a countably infinite structure M, the following are equivalent:

1. M is $\aleph_0$-categorical;
2. $\mathrm{Aut}(M)$ acts oligomorphically on M;
3. for every $n \in \mathbb{N}$ there are only finitely many n-types over $\emptyset$ (realised in elementary extensions of M).

Proofs can be found in ([2], Theorem 2.3.13) or ([5], Theorem 7.3.1). There is a translation between the group-theoretic and model-theoretic terminology in this case. The countable model M is saturated so (realisations in M of) n-types over a finite subset A of M are exactly $\mathrm{Aut}(M/A)$-orbits on M^n. But as there are only finitely many of these, each of them is actually A-definable. So a subset of M^n is A-definable if and only if it is invariant under $\mathrm{Aut}(M/A)$. It then follows from the Orbit-Stabiliser Theorem that $a \in M$ is in the algebraic closure of A if and only if $\mathrm{Aut}(M/A \cup \{a\})$ is of finite index in $\mathrm{Aut}(M/A)$. The same is true in M^{eq}. Moreover stabilisers of elements of M^{eq} are exactly the open subgroups of $\mathrm{Aut}(M)$ (use ([1], 1.2, Exercise 4)).

Obvious examples of countable $\aleph_0$-categorical structures include a pure set, the set of unordered pairs from a pure set (with a natural induced graph structure, two 2-sets adjacent if they intersect in a singleton), the rationals as an ordered set and the countable atomless boolean algebra. The paper [4] is a survey of various ways of constructing $\aleph_0$-categorical structures and classification results relating to them. The book [1] contains a large amount of information about automorphism groups of these structures.

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Finite Covers

David M. Evans, Dugald Macpherson, Alexandre A. Ivanov

Contents

0 Outline of the Notes

These notes examine a technique for building new structures from simpler ones. The original motivation for this construction is Zil'ber's 'ladder theorem' (Theorem 1.6.4 here), which describes how totally categorical structures are built from strictly minimal sets by a sequence of covers. Similar results exist for several other classes of structures, such as $\aleph_1$-categorical structures, $\aleph_0$-categorical ω-stable structures, and smoothly approximated structures.

We will concentrate on finite covers of countable $\aleph_0$-categorical structures, and we often describe structures entirely by their automorphism groups, without reference to any particular language (in the $\aleph_0$-categorical case this is justified by the Ryll-Nardzewski theorem). The following terminology is convenient.

If Ω is a set then we regard $\mathrm{Sym}(\Omega)$, the symmetric group on Ω, as a topological group with a base of open sets being given by cosets of pointwise stabilisers of finite subsets of Ω. Then a *permutation structure* is a pair $\langle W; G\rangle$ where W is a non-empty set (the *domain*), and G is a closed subgroup of $\mathrm{Sym}(W)$ (the group of *automorphisms*). We shall usually write $G = \mathrm{Aut}(W)$ and refer simply to 'the permutation structure W.' If A is a subset of W and B a subset of W (or more generally of some set on which $\mathrm{Aut}(W)$ is acting in an obvious way), then $\mathrm{Aut}(A/B)$ denotes the permutations of A which extend to elements of $\mathrm{Aut}(W)$ fixing every element of B. We shall write permutations on the left of the elements of W.

Permutation structures are all obtained by taking automorphism groups of first-order structures on W, and we often regard a first-order structure as a permutation structure without explicitly saying so. When we do this, the group of automorphisms for the permutation structure is, of course, just the automorphism group of the first-order structure. We can now define a finite cover (a model-theoretic definition is given in 1.1.2).

Definition 0.0.1 If C, W are permutation structures, then a finite-to-one surjection $\pi : C \to W$ is a *finite cover* if its fibres form an $\mathrm{Aut}(C)$-invariant partition of C, and the induced map $\mu : \mathrm{Aut}(C) \to \mathrm{Sym}(W)$ given by $\mu(g)w = \pi(g\pi^{-1}(w))$ for $g \in \mathrm{Aut}(C)$ and $w \in W$ has image $\mathrm{Aut}(W)$. We refer to μ as the *restriction* map. The *kernel* of the finite cover is $\ker\mu = \mathrm{Aut}(C/W)$.

The main problem which concerns us is:

> **The Cover Problem:** For a given $\aleph_0$-categorical structure W, describe its finite covers.

An overview of how the material in this paper relates to this problem can be found in Section 1.7, after we have given the basic definitions, examples and results. For the rest of this section, we simply describe the structure of these notes and highlight some of the principal results in each section.

Section 1 first gives the basic definitions and some 'naturally occuring' examples of covers. We discuss notions closely related to finite covers, notably symmetric extensions, and give some of the basic theory, sometimes in this wider context. Finally we review some of the model-theoretic background to the cover problem.

Three general constructions of finite covers are described in Section 2: free covers, digraph coverings and coverings of two-graphs. We show that free covers are uniquely determined by choice of fibre and binding groups, and so we have a satisfactory classification of these. The material on digraph and two-graph coverings is suggested by ideas from topology (covering spaces) and finite combinatorics. Both constructions provide examples of finite covers with finite kernels.

In Section 3 we give some preliminary results on finite covers. We then give various ways of dividing up the general cover problem and make various reductions which show that we should focus on some special types of covers (minimal, superlinked and abelian kernel). Any finite cover is an expansion of a free finite cover with the same fibre and binding groups, and we aim for classification up to conjugacy within the automorphism group of this free cover. On the other hand, any finite cover is a reduct of a minimal cover. We show that the kernel of a minimal cover of an $\aleph_0$-categorical structure is nilpotent, and thereby reduce certain problems to consideration of finite covers with abelian kernels.

Finite covers whose kernels are finite are analysed in Section 4. We show that in some cases these can all be described in terms of digraph coverings. In some other cases not covered by these results, a careful analysis of the example of a vector space covering its projective space provides a different answer. The techniques and notions in Section 4 parallel very clearly some ideas from stability theory (strong types, stationarity, and distinguished extensions of types). In Section 5 we amplify further on this, and consider the results of Section 4 from this viewpoint.

In Section 6 we consider finite covers with abelian kernels. Following the approach of Ahlbrandt and Ziegler we divide the problem into two parts: describe the possibilities for the kernels, then work out what the possible covers can be with each particular kernel. For the first part we outline how Pontriagin duality can sometimes be useful. For the second part, we describe the construction of the cohomology group H^1_c which parametrises the covers with a given kernel. We use these ideas to show how results on the cohomology and representation theory of finite groups can be used in our context.

Section 7 contains further results which can be used to calculate cohomology groups. These are all standard results from cohomology of discrete groups, adapted to our purposes. We show how these can be used to prove finiteness of H^1_c, given additional constraints on W.

Section 8 contains some open problems and questions which occurred to us during the writing of this paper.

1 Introduction to covers

1.1 Definitions

We give the basic definitions associated with permutation structures and finite covers. We suggest that the reader skims over them quickly and refers back when necessary.

1.1.1 Permutation structures

If W_1 and W_2 are sets of the same cardinality then any bijection $\phi : W_1 \to W_2$ induces an isomorphism $f_\phi : \mathrm{Sym}(W_1) \to \mathrm{Sym}(W_2)$. We say that permutation structures $\langle W_1; G_1\rangle$ and $\langle W_2; G_2\rangle$ are *isomorphic* if for some bijection ϕ we have $f_\phi(G_1) = G_2$. (As pointed out to us by Martin Ziegler, this produces a slight conflict in terminology: the group of isomorphisms from a permutation structure $\langle W; G\rangle$ to itself is actually the normaliser in $\mathrm{Sym}(W)$ of G, so it might be more correct to refer to *this* as the 'automorphism group of the permutation structure,' rather than G.)

Two permutation structures are *bi-interpretable* if their automorphism groups are isomorphic as topological groups. If the permutation structures arise from countable $\aleph_0$-categorical structures there is a model-theoretic interpretation of this notion due to G. Ahlbrandt and M. Ziegler ([2]: see also Section 7 of [42]). The following useful observation is due to E. Hrushovski ([36]).

Lemma 1.1.1 *A permutation structure $\langle W; G\rangle$ such that G has finitely many orbits on W is bi-interpretable with a transitive permutation structure $\langle W_1; G_1\rangle$.*

Proof. Let $\bar{x}$ be a finite tuple of elements from W containing (at least) one element from each G-orbit. Let W_1 be the orbit under G of $\bar{x}$. We get a natural continuous, injective homomorphism $G \to \mathrm{Sym}(W_1)$, and it is easy to see that the image G_1 of this is closed in $\mathrm{Sym}(W_1)$. The inverse map $G_1 \to G$ is also continuous, and so we have the result. □

Related to this construction is the notion of a *Grassmannian* of a transitive permutation structure W. First recall that if W has the property that $\mathrm{Aut}(W/X)$ has finitely many finite orbits for all finite subsets X of W then we define the *algebraic closure* $\mathrm{acl}(X)$ of X to be the union of the finite $\mathrm{Aut}(W/X)$-orbits. This is a closure operation on the finite subsets of W. If A is a finite algebraically closed subset of W then the Grassmannian $\mathrm{Gr}(W; A)$ is the permutation structure having domain $W_A = \{gA : g \in \mathrm{Aut}(W)\}$ and automorphism group those permutations induced on this set by $\mathrm{Aut}(W)$. To see that this is a closed subgroup of $\mathrm{Sym}(W_A)$ observe that, as in Lemma 1.1.1, the group of permutations induced by $\mathrm{Aut}(W)$ on the orbit of an enumeration of A is closed, and there is an invariant finite-to-one map from this orbit to W_A, so what we want follows from Lemma 1.4.2. If $\mathrm{Aut}(W)$ acts faithfully on W_A then $\mathrm{Gr}(W; A)$ is bi-interpretable with W.

We shall frequently employ the following terminology. Suppose C_0 and C are permutation structures with the same domain, and $\mathrm{Aut}(C) \leq \mathrm{Aut}(C_0)$. Then we say that C_0 is a *reduct* of C, or C is an *expansion* of C_0. We use the adjective *proper* to indicate that $\mathrm{Aut}(C) < \mathrm{Aut}(C_0)$.

1.1.2 Finite covers

We first give the model-theoretic definition of *finite cover.* In practice, however, we will use the group-theoretic translation of this given in the opening remarks

(0.0.1).

Definition 1.1.2 Let C and W be first-order structures. A finite-to-one surjection $\pi : C \to W$ is a *finite cover* of W if there is a 0-definable equivalence relation E on C whose classes are the fibres of π, and any relation on W^n (respectively, C^n) which is 0-definable in the 2-sorted structure (C, W, π) is already 0-definable in W (respectively, C).

Observe that a finite cover $\pi : C \to W$ induces a homomorphism

$$\mu : \mathrm{Aut}(C) \to \mathrm{Aut}(W),$$

given by putting $\mu(g)(w) = \pi(g\pi^{-1}(w))$ for all $g \in \mathrm{Aut}(C)$ and $w \in W$. In fact, if W is countable $\aleph_0$-categorical, then the above definition of a finite cover is equivalent to saying that the fibres of π are the classes of an $\mathrm{Aut}(C)$-invariant equivalence relation on C, and the map $\mathrm{Aut}(C) \to \mathrm{Aut}(W)$ induced by π has image $\mathrm{Aut}(W)$ (Lemma 1.4.2 below ensures that Definition 1.1.2 implies the surjectivity), and so this agrees with what was given as Definition 0.0.1. We refer to μ as the *restriction* homomorphism.

Suppose that $\pi : C \to W$ is a finite cover. Then $\mathrm{Aut}(C)$ has a normal subgroup K, the *kernel* of the cover, defined by

$$K := \{g \in \mathrm{Aut}(C) : \pi(x) = \pi(gx) \text{ for all } x \in C\},$$

(so also the kernel of the restriction homomorphism $\mathrm{Aut}(C) \to \mathrm{Aut}(W)$). We have a short exact sequence

$$1 \to K \to \mathrm{Aut}(C) \xrightarrow{\mu} \mathrm{Aut}(W) \to 1.$$

The cover *splits* if K has a closed complement in $\mathrm{Aut}(C)$, that is, there is a closed subgroup H of $\mathrm{Aut}(C)$ such that $KH = \mathrm{Aut}(C)$ and $K \cap H = 1$. Equivalently, C is a reduct of a cover of W with trivial kernel (namely, a structure with automorphism group H).

For each $a \in W$ let $C(a)$ denote the fibre above a, that is $\{x \in C : \pi(x) = a\}$. We also define, for any $a \in W$, the *fibre group* of the cover at a as the permutation group induced by $\mathrm{Aut}(C)$ on $C(a)$. The *binding group* at a is a normal subgroup of the fibre group, and is the permutation group induced on a fibre $C(a)$ by the kernel K. Clearly, if $\mathrm{Aut}(W)$ acts transitively on W then all of the fibre groups are isomorphic as permutation groups, as are the binding groups. We refer to these as the fibre and binding groups of the cover. If these are unequal, we say that the cover is *twisted.*

We mention some special kinds of covers. We say that $\pi : C \to W$ is *free* if $\mathrm{Aut}(C/W) = \Pi_{w \in W} \mathrm{Aut}(C(w)/W)$, that is, the kernel is the full direct product of the binding groups (so as big as possible). At the other extreme, the cover is *trivial* if its kernel $\mathrm{Aut}(C/W)$ is the trivial group (this differs from the terminology in [3] and [4] where 'trivial' means 'split'). A *principal* cover $\pi : C \to W$ is a free finite