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The Descriptive Set Theory of Polish Group Actions

Howard Becker
& Alexander S. Kechris



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PREFACE

This book contains the results of the research of the authors in the foundations of the theory of definable actions of Polish groups and the associated orbit equivalence relations. It is addressed to graduate students and researchers in this area as well as to mathematicians in other fields, whose work comes in contact with this subject.

The reader will also find some new theorems which are attributed to someone other than the authors. Many people have helped in one way or another to improve this book: by allowing us to include these new theorems; by simplifying some of our proofs; by bringing to our attention some work we were unaware of; or by reading earlier versions of this book – or parts of it – and suggesting improvements in the exposition. We thank all of them: G. Cherlin, W. Comfort, R. Dougherty, G. Hjorth, A. Louveau, M. Megrelishvili, M. G. Nadkarni, R. Sami, S. Solecki, S. Thomas, and an anonymous referee.

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November 1995

INTRODUCTION

A *Polish space (group)* is a separable, completely metrizable topological space (group). This book is about actions of Polish groups, in connection with – or from the point of view of – the subject of descriptive set theory. *Descriptive set theory* is the study of definable sets and functions in Polish spaces. The basic classes of definable sets are the classes of Borel, analytic and co-analytic sets, and these certainly constitute the main topic of the book, but at times we also consider other classes of definable sets.

The structure of Borel actions of Polish *locally compact*, i.e., second countable locally compact, topological groups has long been studied in ergodic theory, operator algebras and group representation theory. See, for example, Auslander–Moore [66], Feldman–Hahn–Moore [78], Glimm [61], Kechris [92a], Mackey [57, 62, 89], Moore [82], Ramsay [82, 85], Sinai [89], Varadarajan [63], Vershik–Fedorov [87], Zimmer [84] for a sample of this work. This is closely related to the subject matter of this book. More recently, there has been increasing interest in an extension of the above: studying the structure of Borel actions of *arbitrary*, not necessarily locally compact, Polish groups. Such groups are ubiquitous in mathematics, most often as groups of symmetries of mathematical structures, e.g., the symmetric group S_∞ on a countably infinite set, the group of homeomorphisms $H(X)$ of a compact metrizable space X , the unitary group $U(H)$ of a separable infinite dimensional Hilbert space H , groups of automorphisms of standard measure spaces, diffeomorphism groups, etc. The nonexistence of Haar measure for such groups makes their study quite challenging and requires the development of new methods, one of which is the use of Baire Category techniques.

One area where Borel actions of nonlocally compact groups occur, particularly of the symmetric group S_∞ and its closed subgroups, is in mathematical logic. Of particular importance is the *Vaught Conjecture*, a well-known open problem in logic, and the *Topological Vaught Conjecture* (see Becker [94], Burgess–Miller [75], Lascar [85], Miller [77], Sami [94], Steel [78], Vaught [74]). This falls within the more general framework of *invariant descriptive set theory*, introduced by Vaught [74] and motivated, to a large extent, by mathematical logic. If the term “invariant descriptive set theory” is interpreted in a broad enough fashion, then this is a book on invariant descriptive set theory. However, our perspective is somewhat

different from that of traditional invariant descriptive set theory.

Another place where such actions arise is in the ergodic theory and unitary group representation theory of nonlocally compact groups (see, e.g., Olshanski [82], Vershik [74]). Finally, such actions occur in the theory of operator algebras; see, in particular, Effros [65], [81], two papers motivated by some basic problems in that field which have also played an important role in our work as well.

This book contains a number of results about actions of general Polish groups, some of which are new even in the locally compact case. We will not attempt to describe in detail the contents of the book in this introduction. We do, however, wish to summarize the major themes that we are studying here and the main results related to each of them.

One theme is that of changing the topology on the space being acted upon. One of the main results here is that for a Borel action of a Polish group G on a Polish space X , there is a new Polish topology on X having the same Borel structure such that the action becomes now continuous in the new topology. Equivalently, every Borel action on a Polish space is Borel isomorphic to a continuous one. This answers a question of Ramsay [85], who raised it in the case of locally compact G , and Miller [77]. It was known classically for countable (discrete) groups and was proved recently for $G = \mathbb{R}$ in Wagh [88].

Another result along these lines is that for any continuous action of a Polish group G on a Polish space X and any invariant Borel set $B \subseteq X$, there is a new finer Polish topology on X (which therefore has the same Borel structure) in which B is now clopen and the action remains continuous. This result was known for countable groups and for S_∞ .

A second theme is that of *universal actions*. We establish the existence of universal actions for any given Polish group, and, moreover, we show that the universal action can be taken to be a continuous action on a compact Polish space. This extends a result of Mackey [62] and Varadarajan [63] for the locally compact case.

These first two themes appear in §5 and §2, respectively. There are several applications of these results to various topics. A classical theorem of Tarski asserts that for any action of a group G on a set X , there is a G -invariant finitely additive probability measure on X iff there is no G -paradoxical decomposition of X . The problem has been raised (see, e.g., Wagon [93] for the history of this problem) whether there is an analog of Tarski's Theorem for countably additive probability measures. We establish in §4, using also some recent work of Nadkarni [90], the following analog: for a Borel action of a Polish group on a Polish space, there is a G -invariant

probability Borel measure iff there is no countable G -paradoxical Borel decomposition of X . In view of known counterexamples, this appears to be an optimal analog of Tarski's Theorem.

In another direction, we show in §6 that various forms of the Topological Vaught Conjecture, originally proposed by Miller (see Rogers *et al.* [80], p. 484) turn out to be equivalent. One form is that in a continuous action of a Polish group on a Polish space there are either countably many or perfectly many orbits. Other forms assert the same conclusion for Borel actions or continuous actions restricted to Borel invariant sets. We also show in §6 that the Topological Vaught Conjecture for the symmetric group S_∞ is equivalent to the usual model theoretic Vaught Conjecture for the language $L_{\omega_1\omega}$ (which asserts that any sentence in this language has either countably many or perfectly many countable models, up to isomorphism.) Both the Vaught Conjecture (even for first-order theories) and the Topological Vaught Conjecture are of course still open.

A third theme is that of the *logic actions*. These are particular continuous actions by the group S_∞ on *codes* for countable L -structures, where the orbit equivalence relation is isomorphism. The logic actions serve three purposes: as a means of applying the study of group actions to questions in model theory; as motivation for generalizing concepts and theorems from model theory (that is, from logic actions) to other actions; and, as we show in §2, as concrete examples of the universal S_∞ -action.

The fourth theme is that of *dichotomies* for the orbit space, that is, theorems to the effect that the orbit space is either “small” or else contains a copy of some specific “large” set. The Vaught Conjecture asserts such a dichotomy, and, indeed, that conjecture is what motivated quite a bit of the research in this book. Other important dichotomies, discussed in §3, are the Silver dichotomy and the Glimm–Effros type dichotomies.

The fifth theme is that of *descriptive complexity* for the orbit equivalence relation induced by a Borel action of a Polish group on a Polish space. In general such an equivalence relation is analytic but not Borel. We show though in §7 that the space acted upon can be canonically decomposed into an \aleph_1 sequence of invariant Borel sets on which the orbit equivalence is Borel and this sequence is cofinal among such invariant Borel sets. Finally, we characterize the Borelness of the orbit equivalence relation in terms of the complexity of the stabilizer function (from points in the space to closed subgroups of the Polish group): It is Borel iff the stabilizer function is Borel.

The sixth and final theme is that of the “*definable cardinality*” of the orbit space of an action. This is related to some of the earlier themes, as for example that of dichotomies, and is developed in more detail in §8.

Some of the results contained here first appeared, without complete proofs, in the research announcement Becker–Kechris [93]. (In our research announcement we stated that we did not know whether the universal action can always be taken to be a continuous action on a *compact* space; as pointed out above, this has now been solved.) Some of them have also been previously announced in Becker [94] and Kechris [94].

This book contains a fair amount of background material – definitions and known theorems, sometimes with proofs, sometimes without. These preliminaries are introduced when needed, and so are scattered throughout the book. In particular, most of §1 and of the first halves of §2, 3 survey the basic facts about Polish groups, actions of Polish groups, and the induced equivalence relations, respectively, that are needed in the sequel.

0. DESCRIPTIVE SET THEORY

In this book we will assume that the reader is familiar with the basic notions and results of descriptive set theory, which can be found in Moschovakis [80] and Kechris [95].

In particular, a *Polish space* is a separable completely metrizable topological space. The *Borel sets* of a topological space are those generated by the open sets by the operations of complementation and countable union. A *standard Borel space* is a measurable space, i.e., a set X equipped with a σ -algebra \mathcal{S} , such that there exists a Polish topology on X with \mathcal{S} its σ -algebra of Borel sets. A *countably separated measurable space* is a measurable space admitting a countable family of measurable sets which separate points.

We will often make use of the *Effros Borel space* on the class $\mathcal{F}(X)$ of all closed subsets of a Polish space X . This is the measurable space generated by the sets

$$\{F \in \mathcal{F}(X) : F \cap U \neq \emptyset\}$$

for $U \subseteq X$ open. It is a standard Borel space.

The class of Borel sets in a separable metrizable space ramifies into a transfinite hierarchy of Borel classes $\Sigma_\alpha^0, \Pi_\alpha^0, \Delta_\alpha^0, 1 \leq \alpha < \omega_1$, defined as follows:

$$\begin{aligned} \Sigma_1^0 &= \text{open}, \quad \Pi_1^0 = \text{closed}, \\ \Sigma_\alpha^0 &= \left\{ \bigcup_{n \in \mathbb{N}} A_n : A_n \in \Pi_{\beta_n}^0 \text{ for some } \beta_n < \alpha \right\}, \\ \Pi_\alpha^0 &= \text{the class of complements of } \Sigma_\alpha^0 \text{ sets.} \end{aligned}$$

Also let

$$\Delta_\alpha^0 = \Pi_\alpha^0 \cap \Sigma_\alpha^0$$

be the class of *ambiguous* sets at level α . We have $\Sigma_\alpha^0 \cup \Pi_\alpha^0 \subseteq \Delta_\beta^0$ for $\alpha < \beta$ and $\bigcup_{\alpha < \omega_1} \Sigma_\alpha^0$ is the class of Borel sets. We call a function between separable metrizable spaces *Borel-measurable* if the inverse image of a Borel set is Borel. This notion clearly makes sense for standard Borel spaces as well.

We denote by Σ_1^1 the class of *analytic* sets in a Polish space, i.e., the continuous images of Borel sets. Let also Π_1^1 be the class of *co-analytic*