London Mathematical Society Lecture Note Series 189

Locally Presentable and Accessible Categories

Jiří Adámek & Jiří Rosický

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We dedicate this book to the memory of our excellent colleague and very dear friend Jan Reiterman

Contents

Preface

The basic theme of our monograph is the syntax and semantics of a categorical theory of mathematical structures (as used in algebra, model theory, computer science, etc.). The semantics part is the study of properties of the categories of structures. We concentrate on two kinds of categories: locally presentable categories, which are rather close to quasivarieties of algebras, and the broader (and less "pleasant") accessible categories, which are close to classes of structures axiomatizable in first-order logic. The syntax part describes categories of structures by means of sketches: a sketch is a small category with specified limits and colimits, and a model of the sketch is a set-valued functor preserving the specified limits and colimits. Locally presentable categories are precisely the categories of models of limit-sketches (i.e., no colimit is specified), and accessible categories are precisely the categories of models of sketches. A different approach to syntax is by means of first-order logic: we characterize theories needed to axiomatize locally presentable and accessible categories.

The fundamentals of the theory of locally presentable categories are exhibited in Chapter 1, and those of accessible categories in Chapter 2. The rest of our monograph is devoted to some related topics: algebraic categories, injectivity, categories of models, and Vopěnka's large-cardinal principle. The book is completely self-contained: we expect the reader to be familiar with the basic category-theoretical concepts (such as adjoint, limit, etc.) which we mention briefly in the Preliminaries, but all the more advanced concepts are carefully explained in the text. Facts about large cardinal numbers, used in the last chapter, are presented in the Appendix.

Organization

Every chapter is concluded by a set of easy exercises which illustrate some of the features of the text. References appear in the historical remarks. Open problems are listed at the end of the book.

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April 1993 J. A. & J. R.

Introduction

Locally presentable categories

The concept of a locally presentable category is one of the most fruitful concepts of category theory. The definition, generalizing the concept of an algebraic lattice, is natural and simple. The scope is very broad: varieties and quasivarieties of (many-sorted) algebras, Horn classes of relational structures, and functor-categories are all locally presentable. Furthermore, locally presentable categories enjoy a number of important properties: they are complete and cocomplete, wellpowered and co-wellpowered, and they have a strong generator.

The definition of a locally presentable category is due to P. Gabriel and F. Ulmer. Their lecture notes [Gabriel, Ulmer 1971], by now classical, are a profound, but by no means easily readable, treatise on the topic. One of the aims of our monograph is to make the fundamentals of the theory of locally presentable categories more accessible to readers who work in category theory, computer science, and related areas. We have collected these fundamentals in Chapter 1, where the basic properties of locally presentable categories are proved and several equivalent ways of introducing these categories are exhibited. For example, we show that locally presentable categories are precisely the categories sketchable by a limit sketch (i.e., the categories of all set-valued functors preserving specified limits).

Accessible categories

These generalize locally presentable categories by weakening cocompleteness to the existence of some directed colimits. The collection of all categories obtained by this generalization is much broader than that of all locally presentable categories, and it includes categories such as

fields and homomorphisms,

Hilbert spaces and linear contractions,

linearly ordered sets and order-preserving functions,

sets and one-to-one functions.

An important special case: for each sketch in the sense of C. Ehresmann the category of set-valued models of the sketch (i.e., set-valued functors preserving specified limits and specified colimits) is accessible. It turns out that this is actually no special case: each accessible category is sketchable, i.e., equivalent to the category of all models of some sketch. This fundamental relationship between accessible and sketchable categories was discovered by [Lair 1981], who called accessible categories "catégories modelables". The name "accessible" is due to [Makkai, Paré 1989], whose book is a comprehensive treatise devoted to accessible categories. Our prime aim in presenting (in Chapter 2) the fundamentals of the theory of accessible categories is to make this theory easy to grasp. Thus, for example, our proof of the equivalence of accessible and sketchable categories is conceptually simpler than any previously published proof, being based on the concept of a pure morphism (a concept "borrowed" from model theory). Unlike M. Makkai and R. Paré, we do not stress the 2-categorical aspects of the theory, although some of the basic results (e.g., on limits of accessible categories) are addressed.

Algebraic categories

Locally presentable categories are closely related to varieties and quasivarieties of many-sorted algebras. We devote Chapter 3 to this interrelationship. We present J. R. Isbell's characterization of quasivarieties (i.e., implicational classes) of algebras as precisely the locally presentable categories with a regularly projective regular generator. Varieties are precisely the quasivarieties with effective equivalence relations. We also introduce the Lawvere–Linton concept of algebraic theory, and prove that varieties are precisely the categories of models of algebraic theories (= product sketches). We finally study the concept of essentially algebraic theory due to P. Freyd, which is an equational theory of partial operations in which the domain of definition of each operation is determined by equations in the "preceding" operations. We prove that locally presentable categories are precisely the categories of models of essentially algebraic theories—this is a folklore result whose proof has not been published before.

Injectivity and generalizations of locally presentable categories

Some natural generalizations of locally presentable categories are studied in Chapter 4: weakly locally presentable categories, i.e., accessible categories with weak colimits (equivalently: with products) and locally multipresentable categories, i.e., accessible categories with multicolimits (equivalently: with connected limits). These concepts are closely related to those of orthogonality or injectivity w.r.t. a morphism, or w.r.t. a cone. Weakly locally presentable categories are precisely the full subcategories of locally presentable categories which are specified by injectivity w.r.t. a set of morphisms. And locally multipresentable categories are precisely the full subcategories of locally presentable categories specified by orthogonality w.r.t. a set of cones.

Categories of models

Chapter 5 deals with first-order logic. We call categories of models of first-order theories axiomatizable. In the finitary, many-sorted first-order logic L_ω we prove that categories axiomatizable by so-called limit theories are precisely the locally finitely presentable categories. This is a result in [Coste 1979]. More generally, in the λ-ary logic L_λ categories axiomatizable by limit theories are precisely the locally λ-presentable categories.

We also exhibit a full characterization of accessible categories by means of (more general) basic theories: a category is accessible iff it is axiomatizable by a basic theory in some of the logics L_λ. A crucial difference between the above two characterization results is that in the case of accessible categories we cannot restrict to a given λ: we show an example (1) of a finitely accessible category which cannot be axiomatized in L_ω and (2) of a basic theory in L_ω whose category is not finitely accessible.

Vopěnka's principle

Some results on locally presentable and accessible categories depend on the existence of certain large cardinal numbers. Notably, the large-cardinal Vopěnka's principle turns out to be equivalent to important properties of locally presentable categories. Vopěnka's principle implies e.g. that

(∗) a category is locally presentable iff it is cocomplete and has a colimit-dense set of objects.

This is quite surprising because it is thus possible to define "locally presentable" without a reference to the concept of presentable object. (The explanation is that, under Vopěnka's principle, in each category with a colimit-dense set all objects are presentable.) Conversely, the statement (∗) implies Vopěnka's principle. Thus, large cardinal numbers turn out to have a close link to locally presentable categories.

We devote Chapter 6 to the role of Vopěnka's principle in the theory of locally presentable and accessible categories. All concepts concerning large cardinal numbers which are needed in that chapter are explained in the Appendix.

0. Preliminaries

This monograph is devoted to a theory of locally presentable categories and accessible categories. We assume that the reader has basic knowledge of categories, functors, and adjoints, but we are careful to explain all the necessary concepts of model theory, logic, and set theory, as well as all the more advanced categorical notions in the text. We have concentrated all the required facts concerning cardinal numbers in the Appendix. We now recall some conventions and facts of category theory necessary for avoiding later misunderstandings. The proofs of the (standard) statements presented here can be found e.g. in [Adámek, Herrlich, Strecker 1990]*.

0.1 Set Theory. We distinguish, as in the Bernays–Gödel set theory, between *sets* and *classes*. Until Chapter 6 this is all that need be said—in other words, we just use naive set theory with a distinction between "small" and "large". But we use transfinite induction frequently; thus, the axiom of choice (for classes) is assumed without mention.

The first infinite cardinal is denoted by ω or $\aleph_0$, the next one by ω_1 or $\aleph_1$. Categories $\mathcal{K}$ are understood to be locally small, i.e., objects and morphisms form classes $\mathcal{K}^{\mathrm{obj}}$ and $\mathcal{K}^{\mathrm{mor}}$, respectively, whereas $\hom(A, B)$ is a set (for any pair A, B of objects). A class of objects of a category is called *essentially small* if it has a set of representatives w.r.t. isomorphism.

0.2 Composition is written from right to left, that is, if $f\colon A \to B$ and $g\colon B \to C$ are morphisms, then $g \cdot f$ [or gf] is their composite.

0.3 Comma-categories. For each object K of a category $\mathcal{K}$ we form the comma-category $K \downarrow \mathcal{K}$ of all arrows with the domain K, whose morphisms from $K \xrightarrow{a} A$ to $K \xrightarrow{b} B$ are the $\mathcal{K}$-morphisms $f\colon A \to B$ with $b = fa$. Dually, $\mathcal{K} \downarrow K$ denotes the comma-category of all arrows with the codomain K.

*References to the literature listed at the end of our book are denoted by square brackets.

0.4. By a **diagram** in a category $\mathcal{K}$ is meant a functor $D\colon \mathcal{D} \to \mathcal{K}$ from a small category $\mathcal{D}$ (called the scheme of the diagram D). The diagram D is said to be *finite* if $\mathcal{D}$ has finitely many morphisms. A category is called *(co)complete* provided that every diagram in it has a (co)limit.

Definition. Let $\mathcal{A}$ be a small, full subcategory of a category $\mathcal{K}$. For each object K in $\mathcal{K}$ the *canonical diagram* of K (w.r.t. $\mathcal{A}$) is the diagram of all arrows $A \to K$ where A lies in $\mathcal{A}$; more precisely, the canonical diagram is the natural forgetful functor $D\colon \mathcal{A} \downarrow K \to \mathcal{K}$. We say that K is a *canonical colimit of $\mathcal{A}$-objects* provided that the canonical diagram has a colimit with the colimit-object K and the colimit maps $D(A \xrightarrow{a} K) \xrightarrow{a} K$.

0.5 Hierarchy of Monomorphisms. A monomorphism $m\colon A \to B$ is called

(1) *regular* if m is an equalizer of some pair $f_1, f_2\colon B \to C$;

(2) *strong* if each commuting square

$$\begin{array}{ccc} P & \xrightarrow{\;e\;} & Q \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[\;m\;]{} & B \end{array}$$

such that e is an epimorphism has a diagonal fill-in (i.e., a morphism $d\colon Q \to A$ with $f = d \cdot e$ and $g = m \cdot d$);

(3) *extremal* if every epimorphism $e\colon A \to A'$ through which m factorizes is an isomorphism.

Every regular monomorphism is strong, and the converse is true in each category with (epi, regular mono)-factorizations of morphisms. Every strong monomorphism is extremal, and the converse is true in each category with pushouts.

Every complete, wellpowered category has (epi, extremal mono)-factorizations as well as (extremal epi, mono)-factorizations of morphisms.

0.6 Generators. A set $\mathcal{G}$ of objects of a category is called a *generator* provided that for each pair $f_1, f_2\colon K \to K'$ of distinct morphisms there exists an object $G \in \mathcal{G}$ and a morphism $g\colon G \to K$ with $f_1 \cdot g \neq f_2 \cdot g$. The dual concept is cogenerator.

A generator $\mathcal{G}$ is called *strong* provided that for each object K and each proper subobject of K there exists a morphism $G \to K$ with $G \in \mathcal{G}$ which does not factorize through that subobject. A shorter definition is possible in a cocomplete category: $\mathcal{G}$ is a strong generator if every object is an extremal quotient of a coproduct of $\mathcal{G}$-objects. (It would be more reasonable, but unfortunately less standard, to call $\mathcal{G}$ an extremal generator.) Every category $\mathcal{K}$ with a strong generator $\mathcal{G}$ is *wellpowered*, i.e., each object has only a set of subobjects.

0.7 Adjoint Functors. A functor $F\colon \mathcal{K} \to \mathcal{L}$ is *right adjoint* to a functor $G\colon \mathcal{L} \to \mathcal{K}$ provided that there exists a natural isomorphism

$$\hom(G-,-) \cong \hom(-,F-).$$

Notation: $G \dashv F$. We often use *Freyd's adjoint functor theorem*: if $\mathcal{K}$ is a complete category, then a functor $F\colon \mathcal{K} \to \mathcal{L}$ is a right adjoint iff F preserves limits and satisfies the *solution-set condition* (which says that for each object L in $\mathcal{L}$ there exists a set of arrows $L \xrightarrow{f_i} FK_i$ in $\mathcal{L}$ such that every arrow $L \xrightarrow{f} FK$ factorizes as $f = Fk \cdot f_i$ for some i). We also have *Freyd's special adjoint functor theorem*: if $\mathcal{K}$ is a complete, wellpowered category with a cogenerator, then a functor $F\colon \mathcal{K} \to \mathcal{L}$ is a right adjoint iff F preserves limits.

0.8 Reflective Subcategories. A subcategory $\mathcal{A}$ of a category $\mathcal{K}$ is called

(1) *isomorphism-closed* provided that for each isomorphism $i\colon A \to A'$ in $\mathcal{K}$ with A in $\mathcal{A}$ we have A' and i in $\mathcal{A}$ too,

(2) *closed under limits* if every limit cone in $\mathcal{K}$ of a diagram in $\mathcal{A}$ lies in $\mathcal{A}$,

(3) *reflective* provided that the inclusion functor $\mathcal{A} \hookrightarrow \mathcal{K}$ is right adjoint.

The latter means that each object K of $\mathcal{K}$ has a *reflection map* $r_K\colon K \to A$, $A \in \mathcal{A}$, with the universal property that each morphism from K into an $\mathcal{A}$-object uniquely factorizes through r_K by an $\mathcal{A}$-morphism. If each r_K is an epimorphism, $\mathcal{A}$ is said to be *epireflective* in $\mathcal{K}$.

For complete, wellpowered, and co-wellpowered categories $\mathcal{K}$ the following holds: a full, isomorphism-closed subcategory of $\mathcal{K}$ is epireflective iff it is closed in $\mathcal{K}$ under products and extremal subobjects.

0.9 The Yoneda Lemma. For each small category $\mathcal{K}$ the *Yoneda embedding* is the functor $Y\colon \mathcal{K} \to \mathbf{Set}^{\mathcal{K}^{\mathrm{op}}}$ assigning to each object K of $\mathcal{K}$ the contravariant hom-functor $\hom(-,K)\colon \mathcal{K}^{\mathrm{op}} \to \mathbf{Set}$, and to each morphism $f\colon K \to K'$ of $\mathcal{K}$ the natural transformation $\hom(-,f)\colon \hom(-,K) \to$

$\hom(-, K')$ defined via composites with f. The fact that Y is a full embedding follows from the *Yoneda lemma*: for each functor F in $\mathbf{Set}^{\mathcal{K}^{\mathrm{op}}}$ and each natural transformation $\tau\colon \hom(-, K) \to F$ there exists a unique element $x \in FK$ with $\tau_A(h) = Fh(x)$ for all $h \in \hom(A, K)$.

0.10 Cones. A set of morphisms with a common domain A is called a *cone* with domain A. Special cases: every morphism $f\colon A \to B$ is considered to be a cone with domain A, and every object A is considered to be the empty cone with domain A.

0.11 Cofinal Subdiagrams. A functor $H : \mathcal{D}_0 \to \mathcal{D}$ is said to be *cofinal* provided that for each object d in $\mathcal{D}$

(a) there exists a morphism $f\colon d \to Hd_0$ for some object d_0 in $\mathcal{D}_0$,

and

(b) given two such morphisms $f\colon d \to Hd_0$ and $f' : d \to Hd_0'$, there exist morphisms $g' : d_0 \to \bar{d}_0$ and $g\colon d_0' \to \bar{d}_0$ in $\mathcal{D}_0$ such that the square

$$\begin{array}{ccc} d & \xrightarrow{\;f\;} & Hd_0 \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle Hg'} \\ Hd_0' & \xrightarrow[Hg]{} & H\bar{d}_0 \end{array}$$

commutes.

Observation. For each cofinal functor $H : \mathcal{D}_0 \to \mathcal{D}$ the categories $\mathcal{D}_0$ and $\mathcal{D}$ are "equivalent as diagram schemes" w.r.t. colimits: (1) a category has colimits over $\mathcal{D}$ iff it has colimits over $\mathcal{D}_0$, and (2) a functor preserves colimits over $\mathcal{D}$ iff it preserves colimits over $\mathcal{D}_0$.

In more detail: let $D\colon \mathcal{D} \to \mathcal{K}$ be a diagram. There is a bijective correspondence between compatible cocones of D and those of $D \cdot H$ (hence, a bijective correspondence between colimits of D and $D \cdot H$). In fact:

(1) Given a compatible cocone $(Dd \xrightarrow{c_d} C)_{d \in \mathcal{D}^{\mathrm{obj}}}$ for D, then the cocone

$$\left(DHd_0 \xrightarrow{c_{Hd_0}} C\right)_{d_0 \in \mathcal{D}_0^{\mathrm{obj}}}$$

is compatible for $D \cdot H$.

(2) Given a compatible cocone $(DHd_0 \xrightarrow{c_{d_0}} C)_{d_0 \in \mathcal{D}_0^{\mathrm{obj}}}$ for $D \cdot H$, then choose a morphism $f\colon d \to Hd_0$ for each d in $\mathcal{D}$. It is clear that $\hat{c}_d = c_{d_0} \cdot Df : Dd \to C$ is independent of the choice of f and d_0, and that the cocone $(Dd \xrightarrow{\hat{c}_d} C)_{d \in \mathcal{D}^{\mathrm{obj}}}$ is compatible for D.

Remark. In particular, a *cofinal subdiagram* of a diagram $D\colon \mathcal{D} \to \mathcal{K}$ is a subdiagram $D_0\colon \mathcal{D}_0 \to \mathcal{K}$ (i.e., $\mathcal{D}_0$ is a subcategory of $\mathcal{D}$ and $D_0 = D/_{\mathcal{D}_0}$) such that the inclusion functor $\mathcal{D}_0 \hookrightarrow \mathcal{D}$ is cofinal.

0.12 Equivalence of Categories is a full and faithful functor $E\colon \mathcal{K} \to \mathcal{L}$ which is isomorphism-dense, i.e., each object of $\mathcal{L}$ is isomorphic to an object of $E(\mathcal{K})$. We call $\mathcal{K}$ and $\mathcal{L}$ equivalent; the notation for this is $\mathcal{K} \approx \mathcal{L}$. The notation for isomorphic categories is $\mathcal{K} \cong \mathcal{L}$.

0.13 The Quasicategory of all Categories. We denote by **Cat** the category of all small categories and all functors. On two occasions we also refer to the quasicategory **CAT** of all categories and all functors. A quasicategory is defined as a category except that it lives beyond the universe of our set theory—thus, all objects may form a collection which fails to be a class, and the same is true about any $\hom(A, B)$. (We are vague here because the use is so infrequent and unimportant that an effort for axiomatization would be wasted. The reader may consult the monograph [Adámek, Herrlich, Strecker 1990].)

Chapter 1

Locally Presentable Categories

The first chapter is devoted to an important class of categories, the locally presentable categories, which is broad enough to encompass a great deal of mathematical life: varieties of algebras, implicational classes of relational structures, interesting cases of posets (domains, lattices), etc., and yet restricted enough to guarantee a number of completeness and smallness properties. Besides, locally presentable categories are closed under a number of categorical constructions (limits, comma-categories), see also Chapter 2. The basic concept, a finitely presentable object, can be regarded as a generalization of the concept of a finite (or compact) element in a Scott domain, i.e., an element a such that for each directed set $\{d_i \mid i \in I\}$ with $a \leq \bigvee_{i \in I} d_i$ it follows that $a \leq d_i$ for some $i \in I$. Now, an object A is finitely presentable if for each directed diagram $\{D_i \mid i \in I\}$ every morphism $A \to \operatorname{colim}_{i \in I} D_i$ factorizes (essentially uniquely) through D_i for some $i \in I$.

More generally, an object A is λ-presentable (for a cardinal λ) if every morphism from A to a λ-directed colimit $\operatorname{colim}_{i \in I} D_i$ factorizes (essentially uniquely) through some D_i. A category is locally λ-presentable iff it has colimits and is generated (in some strong sense) by a set of λ-presentable objects. We will see that there are many equivalent ways in which locally λ-presentable categories can be introduced: they are precisely

(1) the cocomplete categories in which every object is a λ-directed colimit of λ-presentable objects of a certain set (Definition 1.17);

(2) the cocomplete categories with a strongly generating set of λ-presentable objects (Theorem 1.20);

(3) the full, reflective subcategories of categories of relational structures closed under λ-directed colimits (Corollary 1.47);

(4) the categories of λ-continuous set-valued functors (Theorem 1.46);

(5) the λ-free cocompletions of small categories (Theorem 1.46);

(6) the λ-ary essentially algebraic categories (Theorem 3.36);

(7) the categories of models of λ-small limit-sketches (Corollary 1.52);

(8) the categories of models of λ-ary limit-theories (Theorem 5.30).

The role that the cardinal λ plays here is analogous to, say, an upper bound on the arity of operations in universal algebra. We begin with the important case of finitely presentable categories, i.e., with $\lambda = \aleph_0$.

The concept of a locally presentable category is closely related to that of a small-orthogonality class, i.e., the class $\mathcal{M}^{\perp}$ of all objects orthogonal to morphisms of a given set $\mathcal{M}$. We prove that (a) each locally presentable category is equivalent to a small-orthogonality class of some functor-category $\mathbf{Set}^{\mathcal{A}}$, and (b) every small-orthogonality class in a locally presentable category is locally presentable [Theorems 1.46 and 1.39].

A concept often easier to work with than λ-directed colimits is that of λ-directed unions. We show that locally presentable categories can equivalently be introduced by means of λ-directed unions—this is the local generation theorem 1.70 (which is, somewhat surprisingly, technically rather difficult).

1.A Locally Finitely Presentable Categories

The concept of a locally finitely presentable category can be viewed as a direct generalization of the concept of an algebraic lattice. Recall that a non-empty partially ordered set is called *directed* provided that each pair of elements has an upper bound. An element a of a partially ordered set $(K, \leq)$ is called *finite* (or compact) provided that for each directed set $D \subseteq K$ with $a \leq \bigvee D$ there exists $d \in D$ such that $a \leq d$. Now an *algebraic lattice* is a partially ordered set $(K, \leq)$ which is

(1) cocomplete, i.e., has all joins (and thus all meets)

and

(2) algebraic, i.e., every element is a directed join of finite elements.

When working with a category $\mathcal{K}$ rather than just a poset, directed joins have to be generalized to *directed colimits,* i.e., colimits of diagrams $D\colon (I, \leq) \to \mathcal{K}$, where $(I, \leq)$ is a directed poset (considered as a category).

Finitely Presentable Objects

1.1 Definition. An object K of a category $\mathcal{K}$ is called *finitely presentable* provided that its hom-functor

$$\hom(K, -) \colon \mathcal{K} \to \mathbf{Set}$$

preserves directed colimits.

Explicitly: K is finitely presentable iff for each directed diagram

$$D \colon (I, \leq) \to \mathcal{K},$$

each colimit cocone $D_i \xrightarrow{c_i} C$ $(= \operatorname{colim} D)$, $i \in I$, and each morphism $f \colon K \to C$ there exists i such that

(1) f factorizes through c_i, i.e., $f = c_i \cdot g \quad (i \in I)$ for some $g \colon K \to D_i$,

and

(2) the factorization is essentially unique in the sense that if $f = c_i \cdot g = c_i \cdot g'$, then $D(i \to j) \cdot g = D(i \to j) \cdot g'$ for some $j \geq i$.

1.2 Examples

(1) A set K is finitely presentable in the category **Set** of sets and functions iff K is finite. In fact, every set K is a colimit of the directed diagram of all finite subsets of K; if K is finitely presentable, then id_K must factorize through the inclusion of one of the finite subsets—thus, K is finite. Conversely, let K be a finite set, and let $\left(D_i \xrightarrow{c_i} C\right)_{i \in I}$ be a directed colimit in **Set**. For each function $f \colon K \to C$ and each element $x \in K$ there exists $i_x \in I$ such that $f(x)$ lies in the image of c_{i_x}. Since K is finite, and I is directed, there exists an upper bound $i \in I$ of all i_x $(x \in K)$; thus, $f(K) \subseteq c_i(D_i)$. This implies that f factorizes through c_i. To show that the factorization is essentially unique, use the following property of directed colimits in **Set** (see Exercise 1.a): whenever elements $y, y' \in D_i$ fulfil $c_i(y) = c_i(y')$, then there exists $j \in I$ with $i \leq j$ such that $D(i \to j)(y) = D(i \to j)(y')$.

(2) For each set S (of *sorts*) let $\mathbf{Set}^S$ denote the category of *S-sorted sets*, i.e., collections $X = (X_s)_{s \in S}$ of sets X_s indexed by S, and *S-sorted functions* $f \colon X \to Y$, i.e., collections $f = (f_s)_{s \in S}$ of functions $f_s \colon X_s \to Y_s$ indexed by S. For each S-sorted set X we call the cardinal

$$\# X = \sum_{s \in S} \operatorname{card} X_s$$

the *power* of X. An S-sorted set is finitely presentable in $\mathbf{Set}^S$ iff it has finite power. The proof is analogous to (1) since (directed) colimits in $\mathbf{Set}^S$ are computed coordinate-wise.

(3) In the category **Pos** of posets (partially ordered sets) and order-preserving functions the finitely presentable objects are precisely the finite ones. The proof is analogous to (1) since directed colimits are computed on the level of **Set**.

Analogously in the category **Gra** of graphs (i.e., sets endowed with a binary relation) and homomorphisms (i.e., functions preserving the binary relation), the finitely presentable objects are precisely the finite graphs.

(4) Let S be a set of sorts, and let Σ be a (finitary, S-sorted, relational) *signature*. That is, with each symbol $\sigma \in \Sigma$ we are given an arity $\operatorname{ar}\sigma = (s_1, \ldots, s_n) \in S^n$. (If $n = 1$, then $\operatorname{ar}\sigma = s$ means that σ is a unary symbol of sort s, if $n = 2$, then σ is a binary symbol of sorts s_1, s_2; etc. The case $n = 0$ is denoted by $\operatorname{ar}\sigma = \emptyset$; this is a nullary symbol σ.)

A *relational structure* A of type Σ consists of an (*underlying*) S-sorted set $|A| = (A_s)_{s\in S}$ and, for each $\sigma \in \Sigma$, of a relation $\sigma_A \subseteq A_{s_1} \times A_{s_2} \times \cdots \times A_{s_n}$, where $\operatorname{ar}\sigma = (s_1, \ldots, s_n)$. (If $n = 0$, then $\sigma_A \subseteq A^\emptyset$, where $A^\emptyset$ is a terminal object. Thus, we just distinguish between two cases: $\sigma_A = \emptyset$ or $\sigma_A \neq \emptyset$.) Let $\mathbf{Rel}\,\Sigma$ denote the category of relational structures of type Σ, where morphisms $f\colon A \to B$ are the *homomorphisms*, i.e., S-sorted functions $f\colon |A| \to |B|$ such that for each $\sigma \in \Sigma$ of arity $(s_1, \ldots, s_n)$ with $n > 0$ we have that

$$(x_1, \ldots, x_n) \in \sigma_A \quad \text{implies} \quad \big(f_{s_1}(x_1), \ldots, f_{s_n}(x_n)\big) \in \sigma_B$$

and, for each σ of arity $\emptyset$, if $\sigma_A \neq \emptyset$ then $\sigma_B \neq \emptyset$. A relational structure A is finitely presentable in $\mathbf{Rel}\,\Sigma$ iff it has finitely many vertices (i.e., $|A|$ has finite power) and finitely many edges (i.e., $\sum_{\sigma\in\Sigma} \operatorname{card}\sigma_A$ is finite). The proof is analogous to (1) above.

(5) A group A is finitely presentable in **Grp**, the category of groups and homomorphisms, iff it can be presented by finitely many generators and finitely many equations in the usual algebraic sense. (That is, iff A is isomorphic to the quotient group of the free group $F\{x_i\}_{i=1}^n$ generated by $\{x_1, \ldots, x_n\}$ modulo a congruence generated by finitely many equations on $F\{x_i\}_{i=1}^n$.) For example, $(\mathbb{Z}, +)$ is finitely presentable and $(\mathbb{R}, +)$ is not.

In general, in each variety of finitary algebras, an algebra is finitely presentable iff it can be presented by finitely many generators and finitely many equations in the usual algebraic sense. A full proof will be presented later (see Theorem 3.12).

(6) Let **Aut** be the category of (deterministic, sequential) automata: objects are sixtuples $A = (Q, I, O, q_0, \delta, \beta)$ where

Q is a set of states,

I is a set of input symbols,

O is a set of output symbols,

$q_0 \in Q$ is the initial state,

$\delta\colon I \times Q \to Q$ is the next-state map,

and

$\beta\colon Q \to O$ is the output map.

Morphisms from A to $A' = (Q', I', O', q_0', \delta', \beta')$ are triples (f, i, o) of functions $f\colon Q \to Q'$, $i\colon I \to I'$, and $o\colon O \to O'$ satisfying

(i) $f(q_0) = q_0'$,

(ii) $f\big(\delta(q, x)\big) = \delta'\big(f(q), i(x)\big)$,

(iii) $\beta'\big(f(q)\big) = \beta\big(o(q)\big)$,

for all states $q \in Q$ and all inputs $x \in I$. Composition is defined coordinate-wise, and the identity morphisms are (id_Q, id_I, id_O).

An automaton is finitely presentable iff each of the sets Q, I, and O is finite. The proof is analogous to that in (1) since directed colimits in **Aut** are computed coordinate-wise.

(7) Let $\mathcal{A}$ be a small category. By the Yoneda lemma (see 0.9), every hom-functor is a finitely presentable object of $\mathbf{Set}^{\mathcal{A}}$.

(8) Let **CPO** denote the category of CPO's, i.e., *complete posets* (posets in which every directed set has a join) and *continuous functions* (i.e., functions preserving all directed joins). No non-empty object is finitely presentable in **CPO**. In fact, consider the following directed diagram D of inclusions of linearly ordered CPO's: $\{0\} \subseteq \{0,1\} \subseteq \{0,1,2\} \subseteq \ldots$. A colimit in **CPO** can be described by the inclusions of those CPO's into $\omega^T = \{0, 1, \ldots, n, \ldots\} \cup \{T\}$. Now let K be a non-empty CPO and let $f\colon K \to C$ be the constant map of value T. This is a continuous function which does not factorize through any of the colimit maps of D.

(9) In the category **CSLat** of complete semilattices (= complete lattices) and join-preserving homomorphisms no object of more than one element is finitely presentable—the proof is analogous to that in (8) above.

(10) A topological space is finitely presentable in **Top**, the category of topological spaces and continuous functions, iff it is finite and discrete. In fact, any topological space A with a non-open subset $M \subseteq A$ fails to be finitely presentable: consider the sequence D_n of topological spaces for $n \in \mathbb{N} = \{0, 1, 2, \dots\}$, where D_n is the space on the disjoint union $A + \mathbb{N}$ of A and $\mathbb{N}$ such that a subset of D_n is open iff its intersection with A is open in A and its intersection with $\mathbb{N}$ is disjoint with $\{0, 1, \dots, n-1\}$. The colimit of $D = (D_n)_{n \in \mathbb{N}}$ is the indiscrete space on the set $A + \mathbb{N}$. The canonical injection of A into $\operatorname{colim} D$ does not factorize through any of the colimit morphisms $D_n \hookrightarrow \operatorname{colim} D$.

1.3 Proposition. *A finite colimit of finitely presentable objects is finitely presentable.*

PROOF. Let $D\colon \mathcal{D} \to \mathcal{K}$ be a finite diagram with colimit $(Dd \xrightarrow{k_d} K)_{d \in \mathcal{D}^{\text{obj}}}$. Then we will prove that K is finitely presentable provided that each Dd is finitely presentable.

Suppose $D^*: (I, \leq) \to \mathcal{K}$ is a directed diagram with a colimit

$$(D_i^* \xrightarrow{c_i} C)_{i \in I}.$$

For each morphism $f\colon K \to C$ and every object d of $\mathcal{D}$ the morphism $f \cdot k_d$ factorizes through $c_{i(d)}$ for some $i(d) \in I$ (because Dd is finitely presentable). Since $\mathcal{D}$ has finitely many objects, there exists an upper bound $i_0 \in I$ of all $i(d)$, $d \in \mathcal{D}^{\text{obj}}$. Then for each d there exists a factorization $f \cdot k_d = c_{i_0} \cdot g_d$ for some $g_d\colon Dd \to D^*_{i_0}$. Next, for each morphism $\delta\colon d \to d'$ in $\mathcal{D}$ we have two factorizations of $f \cdot k_d$: since $k_d = k_{d'} \cdot D\delta$ we get

$$f \cdot k_d = c_{i_0} \cdot g_d = c_{i_0} \cdot g_{d'} \cdot D\delta.$$

Thus, there exists $j(\delta) \geq i_0$ such that

$$D^*(i_0 \to j(\delta)) \cdot g_d = D^*(i_0 \to j(\delta)) \cdot g_{d'} \cdot D\delta.$$

Since $\mathcal{D}$ has finitely many morphisms, there exists an upper bound $i_1 \in I$ of all $j(\delta)$, $\delta \in \mathcal{D}^{\text{mor}}$. The cocone

$$D(i_0 \to i_1) \cdot g_d\colon Dd \to D^*_{i_1} \qquad (d \in \mathcal{D}^{\text{obj}})$$

is compatible for D. Thus, there exists $g\colon K \to D^*_{i_1}$ with $D(i_0 \to i_1) \cdot g_d = g \cdot k_d$ ($d \in \mathcal{D}^{\mathrm{obj}}$). Consequently,

$$f \cdot k_d = c_{i_0} \cdot g_d = c_{i_1} \cdot D(i_0 \to i_1) \cdot g_d = c_{i_1} \cdot g \cdot k_d$$

for all $d \in \mathcal{D}^{\mathrm{obj}}$, which implies $f = c_{i_1} \cdot g$.

To prove that f factorizes through c_{i_1} essentially uniquely, consider $g'\colon K \to D^*_{i_1}$ with $c_{i_1} \cdot g = c_{i_1} \cdot g'$. For each $d \in \mathcal{D}^{\mathrm{obj}}$ we have two factorizations of $f \cdot k_d$:

$$f \cdot k_d = c_{i_1} \cdot g \cdot k_d = c_{i_1} \cdot g' \cdot k_d$$

which (since Dd is finitely presentable) implies that there exists $j(d)$ with $D\big(i_1 \to j(d)\big) \cdot g \cdot k_d = D\big(i_1 \to j(d)\big) \cdot g' \cdot k_d$. Finally, let j be an upper bound of all $j(d)$, $d \in \mathcal{D}^{\mathrm{obj}}$, then $D(i_1 \to j) \cdot g \cdot k_d = D(i_1 \to j) \cdot g' \cdot k_d$ (for all d). This implies $D(i_1 \to j) \cdot g = D(i_1 \to j) \cdot g'$. □

Remark. Consequently, a split subobject (or a split quotient) of a finitely presentable object A is finitely presentable: we can express it by a coequalizer of two endomorphisms of A.

In contrast, a quotient (or, dually, a subobject) of a finitely presentable object need not be finitely presentable: consider an algebraic lattice as a category, then every object is a quotient of the initial object. Even a regular subobject can fail to be finitely presentable. For example, in the category of lattices the free lattice on three generators contains sublattices which are not finitely presentable, see [Whitman 1941].

Directed and Filtered Colimits

A number of authors prefer working with filtered rather than directed, colimits. (The obvious reason is that canonical diagrams are often filtered, but not directed.) In this subsection we will show that those two concepts are equivalent.

1.4 Definition. A category $\mathcal{D}$ is called *filtered* provided that every finite subcategory of $\mathcal{D}$ has a compatible cocone in $\mathcal{D}$. In other words,

(1) $\mathcal{D}$ is non-empty,

(2) for each pair D_1, D_2 of objects there exists an object D and morphisms $f_1\colon D_1 \to D$ and $f_2\colon D_2 \to D$ in $\mathcal{D}$,

(3) for each pair $g, g'\colon D_1 \to D_2$ of morphisms in $\mathcal{D}$ there exists a morphism $f\colon D_2 \to D$ in $\mathcal{D}$ with $f \cdot g = f \cdot g'$.

Observe that (2) and (3) imply that every pair of morphisms with a common domain can be completed to a commuting square.

Every directed poset, considered as a category, is filtered. The category with one object and two morphisms id, f satisfying $f \cdot f = f$ is filtered.

A *filtered diagram* in a category $\mathcal{K}$ is a diagram $D\colon \mathcal{D} \to \mathcal{K}$ whose scheme $\mathcal{D}$ is filtered. Colimits of such diagrams are called *filtered colimits*. We will show how to "reduce" every filtered diagram to a directed diagram. To make the concept of reduction precise, we use cofinality (see 0.11):

1.5 Theorem. *For every (small) filtered category $\mathcal{D}$ there exists a (small) directed poset $\mathcal{D}_0$ and a cofinal functor $H : \mathcal{D}_0 \to \mathcal{D}$.*

Remark. The following proof is somewhat more technical than the reader might expect at first sight. To realize the difficulty, consider the simple case where $\mathcal{D}$ has just one object d and two morphisms id and $f = f \cdot f$. Here $\mathcal{D}_0$ has to be an infinite directed category such as $d \xrightarrow{f} d \xrightarrow{f} d \xrightarrow{f} d \ldots$.

PROOF. I. Let us first suppose that $\mathcal{D}$ has the following property: every finite subcategory of $\mathcal{D}$ can be extended into a finite subcategory with a unique terminal object. Then the set I of all subcategories of $\mathcal{D}$ with a unique terminal object, ordered by inclusion, is obviously directed: given two such subcategories $\mathcal{A}_1, \mathcal{A}_2$, we extend $\mathcal{A}_1 \cup \mathcal{A}_2$ to a subcategory $\mathcal{A}$ with a unique terminal object, and we have an upper bound of $\mathcal{A}_1$, $\mathcal{A}_2$ in I. The functor $H : (I, \subseteq) \to \mathcal{D}$ defined by

$$H(\mathcal{A}) = \text{the terminal object of } \mathcal{A}$$
$$H(\mathcal{A} \to \mathcal{A}') = \text{the unique } \mathcal{A}'\text{-morphism from } H(\mathcal{A}) \text{ to } H(\mathcal{A}')$$

is cofinal. In fact, for each object d we have $id_d\colon d \to H\{d\}$. Given $f_i : d \to H\mathcal{A}_i$ $(i = 1, 2)$, let $\mathcal{A} \in I$ contain $\mathcal{A}_1 \cup \mathcal{A}_2 \cup \{f_1, f_2\}$, then $H(\mathcal{A}_1 \to \mathcal{A}) \cdot f_1 = H(\mathcal{A}_2 \to \mathcal{A}) \cdot f_2$.

II. Let $\mathcal{D}$ be an arbitrary filtered category. Then the category $\mathcal{D} \times \omega$ (where ω is the linearly ordered category of natural numbers) is also filtered, and it has the property required in I above. In fact, if a subcategory $\mathcal{A}$ of $\mathcal{D} \times \omega$ has a compatible cocone with codomain (A, n), then the object $(A, n+1)$ is the unique terminal object of the following extension of $\mathcal{A}$: add to $\mathcal{A}$ the object $(A, n+1)$, its identity morphism and all composites of the given cocone and the canonical morphism $(A, n) \to (A, n+1)$. The projection functor $\mathcal{D} \times \omega \to \mathcal{D}$ is, obviously, cofinal. By I, we have a cofinal functor from a directed poset into $\mathcal{D} \times \omega$. It is clear that the composite of two cofinal functors is cofinal. □

Corollary. *A category has filtered colimits iff it has directed colimits. For such categories $\mathcal{K}$, a functor $F\colon \mathcal{K} \to \mathcal{L}$ preserves filtered colimits iff it preserves directed colimits.*

1.6. We have reduced filtered colimits to directed colimits. We will make a further step, and reduce directed colimits to colimits of chains (= well-ordered diagrams, or, diagrams whose schemes are ordinals).

Lemma. *Every infinite directed poset $(I, \leq)$ can be expressed as a union of a chain of directed subposets each of which has a smaller cardinality than* card I. *In more detail, if* card $I = \lambda$, *then there exist directed subposets $I_k \subseteq I$ $(k < \lambda)$ such that*

(i) $I = \bigcup_{k<\lambda} I_k$,

(ii) $I_k \subseteq I_{k'}$ *for* $k \leq k'$,

(iii) card $I_k < \lambda$ *for each* k,

and

(iv) $I_k = \bigcup_{k'<k} I_{k'}$ *for every limit ordinal* $k < \lambda$.

PROOF. Express I as

$$I = \{ i_k \mid k < \lambda \}.$$

For each finite set $J \subseteq I$ choose an upper bound $j \in I$ and put $J^* = J \cup \{j\}$; for each infinite subset $J \subseteq I$ there exists a directed set $J^* \subseteq I$ of the same cardinality containing J. In fact, put $J^* = \bigcup_{n<\omega} J_n$, where $J_0 = J$ and J_{n+1} is obtained from J_n by adding, for each pair of elements in J_n, an upper bound. The following subposets I_k $(k < \lambda)$ of I have the required properties:

$$I_0 = \emptyset;$$
$$I_{k+1} = \left(I_k \cup \{ i_k \}\right)^*;$$
$$I_k = \bigcup_{k'<k} I_{k'} \qquad \text{for limit ordinals } k < \lambda. \qquad \square$$

1.7 Corollary. *A category has directed colimits iff it has colimits of chains. For such categories $\mathcal{K}$, a functor $F\colon \mathcal{K} \to \mathcal{L}$ preserves directed colimits iff it preserves colimits of chains.*

PROOF. Let $\mathcal{K}$ be a category with colimits of chains. We will prove that $\mathcal{K}$ has colimits of diagrams D indexed by an arbitrary directed poset $(I, \leq)$. We proceed by transfinite induction on the cardinality $\lambda = \operatorname{card} I$.

First step: If λ is a finite cardinal, then I has a largest element, so there is nothing to prove.

Induction step: If the statement holds for all directed posets of cardinality less than λ, then we use Lemma 1.6: since $I = \bigcup_{k<\lambda} I_k$, a colimit of D can be simply constructed as a colimit of the λ-chain of $\operatorname{colim} D_k$, $k < \lambda$, where D_k is the diagram D restricted to I_k (and $\operatorname{colim} D_k$ exists by induction hypothesis).

Analogously for the statement on functors. □

Remark. Let us call a chain $D\colon \lambda \to \mathcal{K}$ (λ an infinite ordinal) *smooth* provided that for each limit ordinal $i < \lambda$ we have $D_i = \operatorname{colim}_{j<i} D_j$ with the colimit cocone formed by the D-arrows $D_j \to D_i$ ($j < i$). The above corollary can be sharpened by substituting colimits of chains by colimits of smooth chains. This follows from (iv) in Lemma 1.6.

1.8 Example. Unlike the situation for filtered and directed colimits, it is *not* true that every directed diagram is cofinal with a chain! For example, the directed poset of all finite sets of real numbers (ordered by inclusion) does not have a cofinal chain.

Locally Finitely Presentable Categories

1.9 Definition. A category $\mathcal{K}$ is called *locally finitely presentable* provided that it is cocomplete and has a set $\mathcal{A}$ of finitely presentable objects such that every object is a directed colimit of objects from $\mathcal{A}$.

Remark. The condition concerning $\mathcal{A}$ says that for every object K there exists a directed diagram in $\mathcal{A}$ (considered as a full subcategory of $\mathcal{K}$) such that K is a colimit object of that diagram.

The condition on the existence of $\mathcal{A}$ can be reformulated by the following two conditions (which are often easier to verify):

(1) every object is a directed colimit of finitely presentable objects,

and

(2) there exists, up-to isomorphism, only a set of finitely presentable objects.

In fact, assuming (1) and (2), any set $\mathcal{A}$ of representatives of finitely presentable objects has the property in the definition above. Conversely, given $\mathcal{A}$ as above, it is clear that (1) holds. To verify (2), observe first that $\mathcal{A}$ is a strong generator of $\mathcal{K}$; thus, $\mathcal{K}$ is a wellpowered category (see 0.6). It is sufficient to verify that every finitely presentable object K is a subobject of an object in $\mathcal{A}$. In fact, let D be a directed diagram in $\mathcal{A}$ with a colimit $(D_i \xrightarrow{k_i} K)_{i\in I}$. Since K is finitely presentable, id_A factorizes through some k_i, i.e., there exists $m\colon K \to D_i$ $(i \in I)$ with $k_i \cdot m = id_K$. Thus, K is a split subobject of $D_i \in \mathcal{A}$.

1.10 Examples

(1) **Set** is locally finitely presentable. In fact (i) every set is a directed colimit of the diagram of all of its finite subsets (ordered by inclusion), and (ii) there exists, up to isomorphism, only a (countable) set of finite sets.

Analogously, **Pos**, **Rel** Σ, **Grp**, and **Aut** are locally finitely presentable categories.

(2) Every variety of finitary (many-sorted) algebras is locally finitely presentable, as will be proved in Chapter 3.

(3) **CPO** and **Top** are not locally finitely presentable.

(4) The category of finite sets is not locally finitely presentable since it is not cocomplete.

(5) A poset, considered as a category, is locally finitely presentable iff it is a complete lattice which is algebraic (i.e., each element is a directed join of finite elements).

A Criterion for Local Finite Presentability

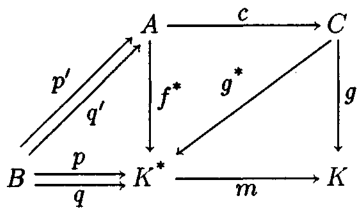
In the definition of locally finitely presentable category $\mathcal{K}$ an important weakening is possible: instead of a set $\mathcal{A}$ of finitely presentable objects which "generates" all of $\mathcal{K}$ via directed colimits, it is sufficient to require that $\mathcal{A}$ be a strong generator (see 0.6):

1.11 Theorem. *A category is locally finitely presentable iff it is cocomplete, and has a strong generator formed by finitely presentable objects.*

PROOF. The necessity is clear. To prove the sufficiency, let $\mathcal{K}$ be a cocomplete category with a strong generator $\mathcal{A}$ formed by finitely presentable objects. Let $\overline{\mathcal{A}}$ be a closure of $\mathcal{A}$ under finite colimits (i.e., the smallest subcategory of $\mathcal{K}$ closed under finite colimits and containing $\mathcal{A}$). It is clear that $\overline{\mathcal{A}}$ is essentially small and, by Proposition 1.3, objects of $\overline{\mathcal{A}}$ are finitely presentable. (In fact, the collection of all finitely presentable objects is closed under finite colimits and, since it contains $\mathcal{A}$, it must contain $\overline{\mathcal{A}}$.) It is sufficient to prove that every object of $\mathcal{K}$ is a filtered colimit of objects of $\overline{\mathcal{A}}$.

For each object K we can form the canonical diagram D w.r.t. $\overline{\mathcal{A}}$ (see Definition 0.4). Since $\overline{\mathcal{A}}$ is closed under finite colimits, D is clearly filtered. Put $K^* = \operatorname{colim} D$ and for each morphism $f\colon A \to K$ with A in $\overline{\mathcal{A}}$ denote the corresponding colimit morphism by $f^*\colon A \to K^*$. Let $m\colon K^* \to K$ be the unique morphism with $f = m \cdot f^*$ for each f. We are going to show that m is an isomorphism. It is sufficient to verify that m is a monomorphism: since $\mathcal{A}$ is a strong generator and each morphism from an $\mathcal{A}$-object into K factorizes through m, it then follows that m is an isomorphism.

Given $p, q\colon B \to K^*$ with $m \cdot p = m \cdot q$, we will prove that $p = q$; it is sufficient to prove this in the case $B \in \mathcal{A}$, since the general case follows from the fact that $\mathcal{A}$ is a (strong) generator. The diagram D is filtered, and B is finitely presentable. Thus, there exists $f\colon A \to K$, for A in $\overline{\mathcal{A}}$, such that both p and q factorize through f^*. That is, we have $p', q'\colon B \to A$ with $p = f^* \cdot p'$ and $q = f^* \cdot q'$.



Let $c\colon A \to C$ be a coequalizer of p' and q'. Since A and B lie in $\overline{\mathcal{A}}$, it follows that C also lies in $\overline{\mathcal{A}}$, thus, the unique morphism $g\colon C \to K$ with $f = g \cdot c$ belongs to the diagram D. Since $f^* = (g \cdot c)^* = g^* \cdot c$, we have $p = g^* \cdot c \cdot p' = g^* \cdot c \cdot q' = q$. □

1.12 Example. For each small category $\mathcal{A}$ the category $\mathbf{Set}^{\mathcal{A}}$ of all functors from $\mathcal{A}$ to $\mathbf{Set}$ is locally finitely presentable: it is cocomplete, and the set of all hom-functors (which are finitely presentable objects) is a strong generator.

1.B Locally Presentable Categories

Analogously to the transition from finitary to infinitary algebras, we now generalize the concept of a locally finitely presentable category. We use