London Mathematical Society Lecture Note Series 152

Oligomorphic Permutation Groups

Peter J. Cameron

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Oligomorphic Permutation Groups

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Preface

A permutation group G on an infinite set Ω is said to be *oligomorphic* if G has only finitely many orbits in its induced action on Ω^n for all n.

The class of oligomorphic permutation groups has many links with other topics, notably model theory. According to the theorem of Engeler, Ryll-Nardzewski and Svenonius, a countable first-order structure has $\aleph_0$-categorical theory if and only its automorphism group is oligomorphic. Moreover, oligomorphic permutation groups have special features: properties such as primitivity and preservation of a linear order are first-order. We may consider only countable domains Ω without significant loss. We can with advantage replace such a group by its closure in the symmetric group (in the topology of pointwise convergence). Many examples can be constructed, and subgroup theorems proved, by the methods of Baire category and measure. Many sequences of integers of great combinatorial interest count orbits on n-sets or n-tuples in oligomorphic groups; indeed, calculating such sequences is equivalent to certain combinatorial enumeration problems.

These notes are based on lectures I gave at a LMS Durham symposium on "Model Theory and Groups" in the summer of 1988, and include contributions from a number of the participants. The notes were written in a great rush immediately after the Durham symposium (and I thank my family for their tolerance during our summer holiday). Though it has been considerably revised since then, I have tried to retain the informal style of the lectures. Among many debts to them, I am grateful to Wilfrid Hodges and Dugald Macpherson for their detailed comments on the manuscript. I am also grateful to John Truss for saving me from making some rash conjectures (by disproving them); to colleagues at QMW (especially Francis Wright) and to Martyn Dryden (SWSL) for their help with TEX; and to Neill Cameron for the artwork. And finally, thanks to Roger Green for coining the unfamiliar word in the title.

I hope that these notes will encourage interest in a fascinating part of the rapidly developing theory of infinite permutation groups.

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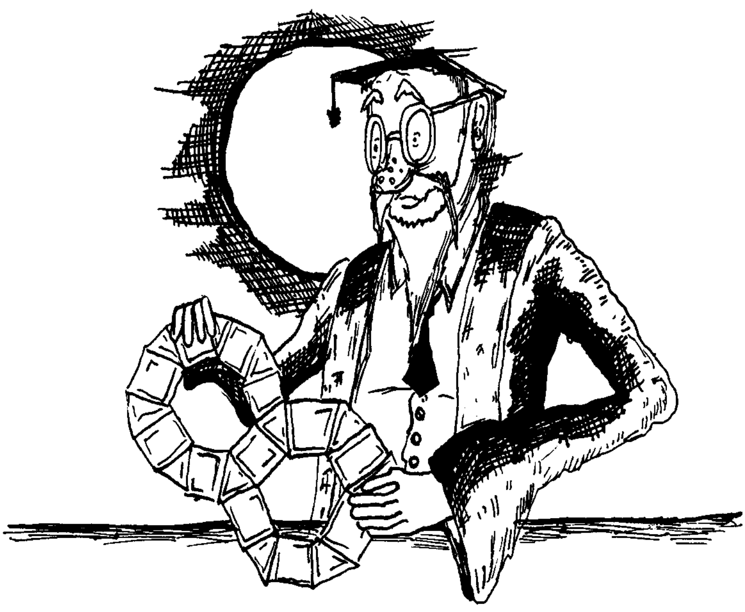
Background



1. Background

1.1. HISTORY AND NOTATION

In the summer of 1988, a London Mathematical Society symposium was held in Durham on "Model Theory and Groups", organised by Wilfrid Hodges, Otto Kegel and Πeter Neumann. This volume of lecture notes is based on the series of lectures I gave at the symposium, but is something more: since no Proceedings of the symposium was published, I have taken the opportunity to incorporate parts of the talks given by other participants, especially David Evans, Udi Hrushovski, Dugald Macpherson, Πeter Neumann, Simon Thomas and Boris Zil'ber. (A talk by Richard Kaye revealed new horizons to me which I have not fully assimilated; but Richard's own book should appear soon.) In addition, I have made use of parts of the proceedings of the Oxford–QMC seminar on the same subject which ran weekly in 1987–8 and continues once a term (now as the Oxford–QMW seminar!); contributions by Samson Adeleke, Jacinta Covington, Angus Macintyre and John Truss have been especially valuable to me.

Why model theory and groups? In particular, why the special class of permutation groups considered here?

In the middle 1970s, when my interests were entirely finite, John McDermott asked a question about the relationship between transitivity on ordered and unordered n-tuples for infinite permutation groups. The analogous question, and more besides, had been settled for finite permutation groups by Livingstone and Wagner (1965), with techniques which were largely combinatorial and representation-theoretic, and so not likely to be useful here. McDermott himself had constructed some examples showing that the infinite is very different from the finite.

At that time, "infinite permutation groups" could scarcely be described as a subject. In the Mathematical Reviews classification, permutation groups were explicitly finite.

The only work of substance was Wielandt's Tübingen lecture notes (1959), which was not readily available. Moreover, of the few results which were in the literature, a substantial proportion were by topologists (such as Anderson (1958) and Brown (1959)), and relatively unknown to group theorists.

Another symptom of the situation is illustrated by the following three theorems.

Tits (1952): There is no infinite 4-transitive group in which the stabiliser of 4 points is trivial.

Hall (1954): There is no infinite 4-transitive group in which the stabiliser of 4 points is finite of odd order.

Yoshizawa (1979): There is no infinite 4-transitive group in which the stabiliser of 4 points is finite.

Yoshizawa's theorem is not so much harder than the other two; why, then, the quarter-century gap? In the first two cases, the theorems were regarded as little more than footnotes to the complete determination of finite permutation groups with the same properties (viz. small symmetric, alternating and Mathieu groups). A finite version of Yoshizawa's theorem would be a list of all 4-transitive groups. This was out of reach in the 1950s and 1960s, and by the 1970s it was clear that it would be obtained as a corollary of the classification of finite simple groups; this duly happened in 1980. But the imminence of this classification had also made people think that interesting problems on infinite permutation groups might be waiting.

In addition, there was pressure from outside, especially from model theory. Questions that arise naturally in classification theory and enumeration of models lead to problems about structures with large automorphism groups. Work of Fraïssé and his school (notably Frasnay and Pouzet) leads in the same direction. (See Fraïssé (1986).) Another contribution was from Joyal (1981), who was developing a subject which included "Redfield-Pólya enumeration without groups".

To return to my personal narrative. I was able to answer John McDermott's question and give a classification of permutation groups which are transitive on unordered n-sets for all n. I spoke about this in Oxford, and in the pub afterwards Graham Higman said, "What about groups with finitely many orbits on n-sets for all n? That might be a good topic for a research student." I have researched and studied this topic on-and-off since then, and now I present my thesis.

A permutation group on an infinite set is called oligomorphic if it satisfies the condition of Higman's question (or, equivalently, if it has only finitely many orbits on n-tuples for all n). The main connections between oligomorphic permutation groups and the areas of model theory and combinatorics are provided by two key results which will be described further in Chapter 2, but which can be stated loosely as follows:

Ryll-Nardzewski's Theorem: A countable (first-order) structure is axiomatisable (that is, characterised, up to isomorphism, as a countable structure, by first-order sentences) if and only if its automorphism group is oligomorphic.

Fraïssé's Theorem: The problem of calculating the numbers of orbits, on n-sets or on n-tuples, of oligomorphic permutation groups is equivalent to that of enumerating the unlabelled or labelled structures in certain classes of finite structures (characterised, more-or-less, by the amalgamation property).

These two results provide the central theme of my lecture notes.

The remainder of this chapter tries to provide a crash course in some of the techniques needed later: first, the two areas principally involved, namely permutation groups and model theory; then, two areas which provide important tools, namely category and measure, and Ramsey theory. The relevant sections can safely be skipped by an expert. Chapter 2 presents the basic properties of oligomorphic permutation groups, and their connection with the theorems of Ryll-Nardzewski and Fraïssé. The third chapter discusses properties of the sequences enumerating orbits on n-sets or n-tuples, especially their growth rates. In Chapter 4 I turn to subgroup theorems explaining how techniques of measure and category, combined with Fraïssé's theorem, allow us to construct various interesting subgroups of closed oligomorphic groups. One of the theorems here has an application to the theory of measurement in mathematical psychology! The final chapter treats some important but miscellaneous topics.

For further reading on the topics of Chapter 1, see Wielandt (1964) for permutation groups, Chang and Keisler (1973) for model theory, Oxtoby (1980) for measure and category, and Graham, Rothschild and Spencer (1980) for Ramsey theory. Other useful books in related areas are those of Fraïssé (1986), Goulden and Jackson (1983) and Shelah (1978).

The exercises are a mixed bag, and should *not* be regarded as routine tests of comprehension. Some are very difficult, and I have given hints, of which the most detailed are outline proofs. Unsolved problems slipped in among the exercises are flagged as such; others are scattered through the text.

A few comments about terminology. As in logic, the natural numbers begin at 0. But I am not totally consistent: I use $\mathbb{N}$ rather than ω to denote the natural numbers, and $\aleph_0$ for their cardinality; and if I want to refer to just two objects, I usually number them 1 and 2 rather than 0 and 1. However, I use the term "ω-sequence" for an infinite sequence (of order type ω). Also, unlike the logicians, I don't insist that the domain of a structure be non-empty. As in model theory, it is convenient to treat n-tuples flexibly, regarding them as "ordered n-subsets". Thus, if I say that n-tuples $\bar{a} = (a_1, \ldots, a_n)$ and $\bar{b} = (b_1, \ldots, b_n)$ carry isomorphic substructures, I mean that the map $a_i \mapsto b_i$ ($i = 1, \ldots, n$) is an isomorphism between the induced substructures on $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$.

As usual, $\mathbf{Z}$, $\mathbf{Q}$, and $\mathbf{R}$ are the integers, rationals and real numbers.

1.2. PERMUTATION GROUPS

A *permutation group* G on a set Ω is simply a subgroup of the symmetric group on Ω (the group of all permutations of Ω). However, to allow us to consider a number of permutation groups isomorphic to (or homomorphic images of) a fixed group G, a more general concept is convenient. A *permutation representation* of G on Ω is a homomorphism from G to the symmetric group on Ω. Other terminology is often used: we say that G acts on Ω, or that Ω is a G-set or G-space. The image of the homomorphism is a permutation group, denoted G^Ω, and called the permutation group on Ω *induced* by G.

A permutation representation of G on Ω can be described by a function $\mu : \Omega \times G \to \Omega$, where $\mu(\alpha, g)$ is the image of α under the permutation corresponding to g. This function satisfies

(a) $\mu(\alpha, gh) = \mu(\mu(\alpha, g), h)$;

(b) $\mu(\alpha, 1) = \alpha$.

(These are translations of the closure and identity axioms for a group. Note that the other two group axioms do not have to be translated — composition of permutations is always associative; and the condition derived from the inverses axiom, namely

(c) $\mu(\alpha, g) = \beta \iff \mu(\beta, g^{-1}) = \alpha$,

is a consequence of (a) and (b).) Conversely, given a map μ satisfying (a) and (b), the function carrying g to the permutation $\alpha \mapsto \mu(\alpha, g)$ is a permutation representation of G.

I will from now on suppress the function μ and write αg for the image of α under g. (Notice that I have sneaked in the convention that permutations act on the right!)

Let G act on Ω. Set $\alpha \sim \beta$ if there exists $g \in G$ with $\alpha g = \beta$. This is an equivalence relation on Ω; the reflexive, symmetric and transitive laws correspond naturally to conditions (b), (c) and (a) above. Its equivalence classes are called *orbits*, and G is called *transitive* if it has but one orbit. If a subset Δ of Ω is a union of orbits, then we have an action of G on Δ. In the case when Δ is a single orbit, the permutation group G^Δ induced on Δ is called a *transitive constituent* of G.

The transitive constituents of a permutation group do not determine it uniquely; but we have:

(1.1) *Any permutation group is a subgroup of the cartesian product of its transitive constituents.*

The cartesian product of $(G_i : i \in I)$ is the set of functions $f : I \to \bigcup G_i$ such that $f(i) \in G_i$ for all $i \in I$; the group operation is componentwise. To each $g \in G$ corresponds the function f_g for which $f_g(i)$ is the restriction of g to the i^{th} orbit: this defines the embedding of G in the cartesian product. In fact, G is a *subcartesian product* of its transitive constituents. (This simply means that it projects onto each factor of the product.)

Note that we have the cartesian product here rather than the (restricted) direct product (which consists of those functions f for which $f(i) = 1$ for all but finitely many $i \in I$). Of course, if there are only finitely many orbits, then the two are indistinguishable.

Given any family $(G_i : i \in I)$ of transitive permutation groups, their cartesian product has a natural action for which the G_i are the transitive constituents. When I refer to the cartesian (or direct) product of permutation groups, this action is intended. There are other actions, which will sometimes be needed; for example, there is an action on the cartesian product of the domains (rather than the disjoint union).

Let G act on Ω. The *stabiliser* G_α of a point $\alpha \in \Omega$ is the set $\{g \in G : \alpha g = \alpha\}$. It is a subgroup of G. Similarly, if $\Delta \subseteq \Omega$, the setwise stabiliser G_Δ of Δ consists of all permutations $g \in G$ which map Δ onto itself; and the pointwise stabiliser $G_{(\Delta)}$ is the set of permutations which fix every point of Δ. Often, $\bar{\alpha}$ will denote an ordered tuple of elements of Ω, and then $G_{\bar{\alpha}}$ will denote the pointwise stabiliser of $\bar{\alpha}$.

Let H be a subgroup of the abstract group G. The *coset space* of H in G is the set of right cosets of H in G; there is an action of G on it given by $(Hx)g = Hxg$ (or, more pedantically, $\mu(Hx, g) = Hxg$). Coset spaces provide "canonical" transitive G-spaces:

(1.2) *If G acts transitively on Ω, then Ω is isomorphic to the coset space of G_α in G, for $\alpha \in \Omega$.*