London Mathematical Society
Lecture Note Series 126

Van der Corput's Method of Exponential Sums

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London Mathematical Society

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CAMBRIDGE UNIVERSITY PRESS
Cambridge
New York Port Chester Melbourne Sydney

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521339278

First published 1991
Re-issued in this digitally printed version 2008

A catalogue record for this publication is available from the British Library

ISBN 978-0-521-33927-8 paperback

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Acknowledgments

Several people read the preliminary drafts of this book, corrected errors, and made suggestions. It is our pleasure to thank Michael Filaseta, Mary Graham, Roger Heath-Brown, Martin Huxley, Matti Jutila, and Jeff Vaaler for their assistance. We also thank Hugh Montgomery for allowing us to use some unpublished material. We thank our editor, David Tranah, for his assistance and unflagging patience. Finally, the first author would like to thank Julian Gevirtz for introducing him to the world of electronic typesetting.

1. INTRODUCTION

1.1 BASIC DEFINITIONS

In this monograph, we will give an account of van der Corput's method in one and two dimensions. The purpose of this method is to obtain bounds for exponential sums, particularly those exponential sums that arise in number-theoretic problems.

An exponential sum is a sum of the form

$$\sum_{a<n\leq b} e(f(n)), \tag{1.1.1}$$

where $f(n)$ is a real-valued function, and the notation $e(x)$ is used to denote $e^{2\pi ix}$. Such sums arise in several different contexts in number theory. One of the simplest is in estimates for the Riemann-zeta function. The problem of obtaining an upper bound for $\zeta(\sigma+it)$ can be reduced to the problem of bounding

$$\sum_{N<n\leq 2N} n^{-\sigma+it}; \tag{1.1.2}$$

the details of the reduction will be given in Section 2.5. The $n^{-\sigma}$ can be removed via partial summation, and we are then left with a sum of the (1.1.1) with $f(n) = -\frac{t}{2\pi}\log n$.

Another simple example where exponential sums arise is in the Dirichlet divisor problem. Let $d(n)$ denote the numbers of divisors of the natural number n, and let

$$\Delta(x) = \sum_{n\leq x} d(n) - x\log x - (2\gamma-1)x, \tag{1.1.3}$$

where γ is Euler's constant. An elementary argument (see, for example, Chapter 18.2 of Hardy and Wright (1979)) can be used to show that $\Delta(x) \ll x^{1/2}$. Voronoï(1903) showed that the error term could be improved to $O(x^{1/3})$. As we shall see in Chapter 4,

$$\Delta(x) = -2\sum_{n\leq x^{1/2}} \psi(x/n) + O(1), \tag{1.1.4}$$

where $\psi(w) = w - [w] - 1/2$. Now ψ is a periodic function of period 1, and it can be expressed as a Fourier series. This gives a way of writing (1.1.4) in terms of exponential sums.

There are other problems that lead to consideration of sums of the form $\sum_n \psi(f(n))$, and we will discuss some of these in Chapter 4.

1.2 HISTORICAL OVERVIEW

The application of exponential sums to number theory began with the Weyl's paper, (Weyl 1916) "Über die Gleichverteilung von Zahlen mod. Eins." One of several new ideas that Weyl introduced was a useful transformation that arises upon squaring an exponential sum. Let S denote the sum in (1.1.1). Then

$$|S|^2 = \sum_{a<m,n\leq b} e(f(m)-f(n)) = \sum_{|h|<b-a} \sum_{n\in I_h} e(f(n+h)-f(n)) \tag{1.2.1}$$

where $I_r = \{n : a < n, n+r \leq b\}$. This is useful because the differenced function $f(n+r)-f(n)$ occurring in the inner sum is easier to handle than the original function $f(n)$. For example, if $f(n)$ is a polynomial of degree k, then $f(n+r)-f(n)$ is a polynomial of degree $k-1$. After $k-1$ applications of (1.2.1), the problem reduces to consideration of exponential sums of linear functions. The latter are easily handled because they are geometric series.

Van der Corput (1922) modified and improved Weyl's method as follows. Let H be an arbitrary positive integer. Then

$$HS = \sum_{h=1}^{H} \sum_{a-h<n\leq b-h} e(f(n+h)).$$

By squaring both sides of the above and applying Cauchy's inequality, one obtains

$$|S|^2 \leq \frac{|b-a|+H}{H} \sum_{|h|<H} \left(1-\frac{|h|}{H}\right) \sum_{n\in I_h} e(f(n+h)-f(n)). \tag{1.2.2}$$

The details of the derivation of (1.2.2) will be given in Section 2.3. Note that (1.2.2) is very similar to (1.2.1) when $H = b-a$.

Another innovation of van der Corput was a combination of the Poisson summation formula and the method of stationary phase. Under suitable conditions on f, it can be shown that

$$\sum_{n\in I} e(f(n)) = \sum_{\alpha\leq\nu\leq\beta} \frac{e(-\phi(\nu)-1/8)}{|f''(x_\nu)|^{1/2}} + \text{ error terms}, \tag{1.2.3}$$

where α and β are defined in terms of f', x_ν is defined by the relation $f'(x_\nu) = \nu$, and $\phi(\nu) = -f(x_\nu) + \nu x_\nu$. A more complete discussion of (1.2.3) will be given in Chapter 3.

Van der Corput used his method to prove that

$$\Delta(x) \ll x^{33/100}. \tag{1.2.4}$$

(Actually he proved (1.2.4) with the smaller exponent 163/494.) Walfisz (1924) used the same method to show that

$$\zeta(1/2+it) \ll t^{163/998}. \tag{1.2.5}$$

Van der Corput (1928) improved the exponent in the divisor problem to 27/82. Titchmarsh (1931) sharpened the exponent in (1.2.5) to 27/164, and Phillips (1933) further reduced the exponent to 229/1392.

A more important aspect of Phillips' 1933 paper was his refinement of exponent systems. Van der Corput (1922) defined an exponent system to be a set of ordered pairs $\{(k_1,l_1),\ldots,(k_r,l_r)\}$ such that, if $|f'| \approx y$ and $N_1 \le 2N$ then

$$\sum_{N<n\le N_1} e(f(n)) \ll y^{k_1}N^{l_1} + \ldots + y^{k_r}N^{l_r},$$

provided f satisfies a certain set of hypotheses. (A complete statement of the hypotheses is lengthy; see the discussion in Chapter 3.) Phillips gave a form of this theory in which only one pair (k,l) is required. Thus he refers to exponent pairs, rather than exponent systems. The trivial bound $\sum_{N<n\le N_1} e(f(n)) \ll N$ shows that $(0,1)$ is an exponent pair. Phillips showed that if (k,l) is an exponent pair then so are

$$A(k,l) = \left(\frac{k}{2k+2}, \frac{k+l+1}{2k+2}\right) \text{ and } B(k,l) = (l-1/2, k+1/2).$$

The A-result follows from (1.2.2), and the B-result follows from (1.2.3). The theory of exponent pairs will be discussed in Chapter 3.

It can be shown that if (k,l) is an exponent pair and $\theta(k,l) = (k+l-1/2)/2$, then

$$\zeta(1/2+it) \ll t^{\theta(k,l)}\log t.$$

For example, Phillips' exponent 229/1392 is $\theta(k,l)$ when

$$(k,l) = ABA^3BA^2BA^2B(0,1) = (97/696, 480/696).$$

This naturally leads to the question of finding the minimum of $\theta(k,l)$ for all exponent pairs that are obtainable from $(0,1)$ by application of the A and B processes. Rankin (1955) found this minimum. Graham (1985) gave further details and considered the corresponding problem when θ is a rational function of k and l. Graham's method will be presented in Chapter 5.

1.3 TWO DIMENSIONAL SUMS

In some applications, sums of the form

$$\sum_{(m,n)\in \mathbf{D}} e(f(m,n))$$