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Topics in Finite Groups

Terence M. Gagen

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Contents

Introduction

The following material is selected from a course of lectures given at the University of Florida in Gainesville, Florida during 1971/72. The reader is expected to have read both Gorensteins' Finite Groups and much of Huppert's Endliche Gruppen I. In particular he must be familiar with the concepts of p-constraint and p-stability in order to begin, although there is a short discussion of these concepts in an appendix here.

The topics covered are such that I feel rather diffident about publishing these notes at all. The title should perhaps be changed to something like 'Lectures on some results of Bender on finite groups'. No less than three of his major results are studied here and of course the classification of A*-groups depends on his 'strongly embedded subgroup' theorem - which is not studied here at all. I feel that the theorems and techniques of the papers 'On the uniqueness theorem' and 'On groups with abelian Sylow 2-subgroups' are too important for finite groups and much too original to remain, as at present, accessible only to a very few specialists. I think that I understand the motivation for the abbreviation of the published versions of these two results. However, though it is clear that a proof becomes considerably more readable when a two or three page induction can be replaced by the words 'By induction we have', these details must sometime be filled in. And unfortunately, I think Dr. Bender has sometimes disguised the deepest and most elegant arguments by this very brevity. I hope that these notes will serve to make more of the group theoretical public aware of these incredibly rich results.

I must thank here the audience at the University of Florida - Mark Hale, Karl Keppler, Ray Shepherd and Ernie Shult. The contribution of Ernie Shult in particular cannot be minimized. Without him, we would all have floundered very soon.

December, 1973

Terry Gagen
Sydney, Australia

Notations

The notation used here is more or less standard. The reader should refer to [12] or [15] when in doubt.

$\mathcal{SCN}(P)$	The set of all self centralizing normal subgroups of P.
$\mathcal{SCN}(p)$	The set of all self centralizing normal subgroups of a Sylow p-subgroup.
$\mathcal{U}_G(A, \pi)$	The set of all A-invariant π-subgroups of G where π is a set of primes.
$\mathcal{U}^*_G(A, \pi)$	The maximal elements of $\mathcal{U}_G(A, \pi)$.
$r(P)$	The number of generators of an elementary abelian subgroup of P of maximal order (amongst all elementary abelian subgroups of P).
A^B	$\langle A^b : b \in B\rangle$.
G_p	A Sylow p-subgroup of G.
$0_\pi(G)$	The maximal normal π-subgroup of G, π a set of primes.
$0_{\sigma,\pi}(G)$	$0_\pi(G \bmod 0_\sigma(G))$.
$0^\pi(G)$	The smallest normal subgroup of G such that $G/0^\pi(G)$ is a π-group.
$F(G)$	The Fitting subgroup of G.
$\Phi(G)$	The Frattini subgroup of G.

The following two results are absolutely basic.

1. The Three Subgroups Lemma
If $A, B, C \subseteq G$, $N \trianglelefteq G$ and $[A, B, C] \subseteq N$, $[B, C, A] \subseteq N$, then $[C, A, B] \subseteq N$.

2. If P is a p-group of class at most 2, then for all $n \in Z$ and for all $x, y \in P$,

$$(xy)^n = x^n y^n [y, x]^{n(n-1)/2}.$$